

# D2-brane Chern-Simons theories: $F$ -maximization = $a$ -maximization

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**ABSTRACT:** We study a system of  $N$  D2-branes probing a generic Calabi-Yau three-fold singularity in the presence of a non-zero quantized Romans mass  $n$ . We argue that the low-energy effective  $\mathcal{N} = 2$  Chern-Simons quiver gauge theory flows to a superconformal fixed point in the IR, and construct the dual AdS<sub>4</sub> solution in massive IIA supergravity. We compute the free energy  $F$  of the gauge theory on  $S^3$  using localization. In the large  $N$  limit we find  $F = c(nN)^{1/3}a^{2/3}$ , where  $c$  is a universal constant and  $a$  is the  $a$ -function of the “parent” four-dimensional  $\mathcal{N} = 1$  theory on  $N$  D3-branes probing the same Calabi-Yau singularity. It follows that maximizing  $F$  over the space of admissible R-symmetries is equivalent to maximizing  $a$  for this class of theories. Moreover, we show that the gauge theory result precisely matches the holographic free energy of the supergravity solution, and provide a similar matching of the VEV of a BPS Wilson loop operator.

**KEYWORDS:** Gauge-gravity correspondence, AdS-CFT Correspondence, Chern-Simons Theories, Supergravity Models

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## 1 Introduction and summary

Superconformal field theories (SCFTs) form an interesting class of supersymmetric QFTs. They arise naturally at fixed points of renormalization group flows, and the additional symmetry provides important constraints on the theory. SCFTs also arise on one side of the AdS/CFT correspondence, giving dual descriptions of quantum gravity in AdS space.

Since the advent of AdS/CFT [1] there has been enormous progress in the understanding and construction of SCFTs, often through their embedding in string theory. There are many classes of such constructions. One approach is via the low-energy effective field theories on branes probing singular spaces. The geometry of the singularity determines the amount of supersymmetry, gauge group, matter content and interactions of this low-energy theory.

For example, one can engineer a four-dimensional  $\mathcal{N} = 1$  quiver gauge theory as the worldvolume theory on a stack of  $N$  D3-branes probing a Calabi-Yau three-fold singularity  $X$ . When  $X$  admits a conical metric of the form  $g_X = dr^2 + r^2 g_Y$ , such gauge theories are expected to flow to (in general strongly interacting) SCFTs in the IR, with large  $N$  type IIB supergravity duals of the form  $\text{AdS}_5 \times Y$ . This is by now a well-established story. An important development on the field theory side was  $a$ -maximization [2]. Every four-dimensional  $\mathcal{N} = 1$  SCFT has a conserved  $U(1)_R$  symmetry, which for example determines the scaling dimensions of chiral primary operators. However, due to potential mixing with non-R Abelian flavour symmetries, in general symmetry principles alone do not determine the superconformal  $U(1)_R$ . In [2] it was shown that the latter is determined uniquely as the local maximum of the  $a$ -function  $a = a(R)$ . Here  $a(R)$  is a certain cubic combination of 't Hooft anomalies, which may thus be computed in the UV theory. Evaluated on the superconformal R-symmetry the  $a$ -function is simply the  $a$  central charge, which for theories with AdS gravity duals of the above type is related to the volume of  $Y$  [3, 4]:

$$a = \frac{\pi^3}{4 \text{Vol}(Y)} N^2 . \tag{1.1}$$

This allowed for precision tests of the AdS/CFT correspondence, notably for the infinite family of  $Y^{p,q}$  models in [5, 6]. This work led to many related developments and generalizations.

There is a parallel, but more recent, story for three-dimensional SCFTs, starting with the seminal work of [7]. Here one can for example engineer an  $\mathcal{N} = 2$  superconformal theory on the worldvolume of  $N$  M2-branes probing a Calabi-Yau four-fold singularity. These have large  $N$  M-theory duals of the form  $\text{AdS}_4 \times Y$ , with the UV gauge theory typically being a Chern-Simons quiver theory. Three-dimensional  $\mathcal{N} = 2$  superconformal field theories similarly have a conserved  $U(1)_R$  symmetry, but there is of course no central charge in three dimensions. However, it turns out that a closely analogous role is played by the free energy  $F = -\log Z_{S^3}$ , where  $Z_{S^3}$  denotes the partition function on the three-sphere. For  $\mathcal{N} = 2$  theories this may be computed using localization techniques, and depends on a choice of R-symmetry for the UV theory [8]. The superconformal  $U(1)_R$  locally maximizes the real part of  $F = F(R)$  [9, 10].

In this note we show that there is a class of theories which link these two developments. We begin with a four-dimensional  $\mathcal{N} = 1$  “parent” quiver gauge theory, arising on  $N$  D3-branes probing a Calabi-Yau three-fold singularity  $X$ . On T-dualizing/dimensional reduction one obtains a three-dimensional  $\mathcal{N} = 2$  theory on  $N$  D2-branes probing  $\mathbb{R} \times X$ . We then add a Romans mass  $F_0 = n/(2\pi\ell_s)$ , where  $n$  is quantized to be an integer. This is well-known to generate a Chern-Simons coupling on the D2-branes, and we conjecture that the resulting theory flows to a superconformal fixed point in the IR, with very closely related properties to the parent four-dimensional theory. In fact this construction was recently applied to  $N$  D2-branes in flat space in [11], where the dual  $\mathcal{N} = 2$   $\text{AdS}_4 \times S^6$  solution in massive IIA supergravity was constructed. For D2-branes probing  $\mathbb{R} \times X$ , where the conical singularity  $X$  has link  $Y = \{r = 1\}$ , this supergravity solution generalizes to  $\text{AdS}_4 \times M_6$ , where  $M_6 \cong \mathcal{S}Y$  is topologically the suspension of  $Y$ . We find that the gravitational free energy of this solution is

$$F_{\text{gravity}} = \frac{2^{1/3} 3^{1/6} \pi^3}{5 \text{Vol}(Y)^{2/3}} n^{1/3} N^{5/3} . \tag{1.2}$$

The field theory free energy  $F = F(R)$  localizes to a matrix model, and the large  $N$  limit may be computed. Remarkably, we find that to leading order at large  $N$

$$\text{Re } F(R) = \frac{2^{5/3} 3^{1/6} \pi}{5} (nN)^{1/3} a(R)^{2/3} , \tag{1.3}$$

where  $a(R)$  is the  $a$ -function of the parent four-dimensional theory! It immediately follows that locally maximizing  $\text{Re } F(R)$  is equivalent to locally maximizing  $a(R)$ . Moreover, substituting (1.1) into (1.3) shows that the field theory free energy agrees precisely with the supergravity result (1.2), for a generic Calabi-Yau three-fold singularity. We also compute the VEV of a certain 1/2 BPS Wilson loop operator  $\mathcal{W}$  in the matrix model, finding the leading order large  $N$  result

$$\text{Re } \log \langle \mathcal{W} \rangle = \frac{2^{4/3} \pi}{3^{1/6}} (nN)^{-1/3} a^{1/3} . \tag{1.4}$$

We show that this precisely agrees with minus the action of a dual fundamental string in AdS<sub>4</sub>, where the string sits at either of the two points of suspension in the internal space  $M_6 \cong \mathcal{S}Y$ .

The outline of this note is as follows. In section 2 we describe the string theory origin of the Chern-Simons gauge theories of interest, and write down the dual AdS<sub>4</sub> solution of massive type IIA supergravity. In section 3 we compute the large  $N$  limit of the field theory matrix model, and compare to the supergravity results.

## 2 D2-brane Chern-Simons theories

### 2.1 String theory set-up

Consider a system of  $N$  D2-branes on the background  $\mathbb{R}^{1,2} \times \mathbb{R} \times X$ , where  $X$  is a local Calabi-Yau three-fold singularity. The metric on  $\mathbb{R} \times X$  is given by

$$g_{\mathbb{R} \times X} = dz^2 + dr^2 + r^2 g_Y, \tag{2.1}$$

where  $z \in \mathbb{R}$ ,  $r \geq 0$  and  $(Y, g_Y)$  is a Sasaki-Einstein five-manifold. There are many constructions of such Calabi-Yau cones  $X$ , including infinite families of explicit metrics, as well as abstract existence results — see [12] for a review. Taking  $Y = S^5$  with the round metric leads to flat space, while other simple examples include the homogeneous space  $Y = T^{1,1}$  and the infinite family  $Y = Y^{p,q}$  [5].

The low-energy effective theory on  $N$  D2-branes placed at  $z = r = 0$  is in general a three-dimensional  $\mathcal{N} = 2$  field theory. When the singularity  $X$  admits a Calabi-Yau (crepant) resolution, one expects this effective theory to be a quiver gauge theory, with superpotential. For example this is the case for toric Calabi-Yau singularities, for which the gauge group will be  $U(N)^G$  where  $G$  is the Euler number of the resolved space. Each copy of  $U(N)$  arises as the gauge group on a fractional D-brane, wrapping various collapsed cycles at  $r = 0$ , and the bifundamental matter fields in the quiver are massless strings between these branes. This set-up is perhaps more familiar in the context of  $N$  D3-branes probing  $X$ , which leads to a four-dimensional  $\mathcal{N} = 1$  theory. A simple T-duality/dimensional reduction relates the two, and since the D3-brane theory will play a role later in the paper we shall refer to it as the “parent” theory.

The Yang-Mills gauge coupling is dimensionful in three dimensions, but an alternative gauge kinetic term is provided by the Chern-Simons three-form. Suppose we have a three-dimensional  $\mathcal{N} = 2$   $U(N)^G$  quiver gauge theory, engineered as above. The  $\mathcal{N} = 2$  vector multiplet contains the gauge field  $\mathcal{A}$ , two real scalars  $D$  and  $\sigma$ , and a two-component spinor  $\lambda$ , all in the adjoint representation of the gauge group. Labelling the gauge groups by  $a = 1, \dots, G$ , we can consider adding the  $\mathcal{N} = 2$  Chern-Simons interaction

$$\mathcal{L}_{\text{CS}} = \sum_{a=1}^G \frac{k_a}{4\pi} \int_{\mathbb{R}^{1,2}} \text{Tr} \left[ \mathcal{A}_a \wedge d\mathcal{A}_a + \frac{2}{3} \mathcal{A}_a \wedge \mathcal{A}_a \wedge \mathcal{A}_a + (2D_a \sigma_a - \lambda_a^\dagger \lambda_a) \text{vol}_3 \right]. \tag{2.2}$$

The Chern-Simons levels  $k_a$  for a  $U(N)$  or  $SU(N)$  gauge group should be integer in order for the theory to be gauge invariant. Provided the sum of levels  $k_a$  is zero,  $\sum_{a=1}^G k_a = 0$ , such

gauge theories generically describe the low-energy limit of  $N$  M2-branes probing certain Calabi-Yau four-fold singularities. Such a relation to parent four-dimensional  $\mathcal{N} = 1$  gauge theories was first suggested in [13], and later given a string theory derivation in [14]. The idea is that M-theory on a Calabi-Yau four-fold singularity may often be reduced to type IIA string theory on a Calabi-Yau three-fold  $X$  fibered over  $\mathbb{R}$ , with RR two-form flux arising as the curvature of the corresponding  $S^1_{\text{M-theory}}$  bundle. The M2-branes reduce to  $N$  D2-branes probing  $\mathbb{R} \times X$ , whose  $\mathcal{N} = 2$  worldvolume theories are essentially just the dimensional reduction of the four-dimensional  $\mathcal{N} = 1$  D3-brane theory. Turning on the RR flux and fibering  $X$  over  $\mathbb{R}$  induces Chern-Simons couplings in this theory, via the Wess-Zumino couplings on a D-brane.

In this paper we would like to consider precisely the opposite type of Chern-Simons deformation, setting instead

$$k_a = k, \quad a = 1, \dots, G. \tag{2.3}$$

As explained in [15], in type IIA string theory this corresponds to adding a Romans mass

$$F_0 = \frac{n}{2\pi\ell_s}, \tag{2.4}$$

where  $\ell_s$  denotes the string length and

$$n = \sum_{a=1}^G k_a = Gk. \tag{2.5}$$

For  $N$  D2-branes in flat space the low-energy effective theory is of course  $\mathcal{N} = 8$   $U(N)$  super-Yang-Mills, and turning on the Romans mass  $n$  induces a Chern-Simons term via the Wess-Zumino coupling

$$\begin{aligned} S_{\text{WZ}} &= (2\pi\ell_s^2)^2 \mu_{\text{D2}} \int_{\mathbb{R}^{1,2}} F_0 \cdot \frac{1}{2} \text{Tr} \left( \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) \\ &= \frac{n}{4\pi} \int_{\mathbb{R}^{1,2}} \text{Tr} \left( \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right), \end{aligned} \tag{2.6}$$

where  $\mu_{\text{D2}}$  is the D2-brane charge. For our more general set-up of D2-branes at a Calabi-Yau singularity, a similar mechanism induces Chern-Simons couplings for each fractional brane, resulting in the relation (2.5).

Provided the Romans mass can be turned on preserving  $\mathcal{N} = 2$  supersymmetry, the Chern-Simons terms above will be completed to the  $\mathcal{N} = 2$  couplings (2.2). At the level of the corresponding D2-brane supergravity solution this is not immediately clear, since turning on the Romans mass modifies the stress energy tensor and supersymmetry transformations, potentially sourcing other supergravity fields and deforming the D2-brane geometry. It would be interesting to try to construct such a solution explicitly, but we shall content ourselves in this note with instead writing down the  $\text{AdS}_4$  near horizon limit. We turn to this now.

## 2.2 Dual supergravity solution

We would like to argue that the above D2-brane Chern-Simons theories flow in the IR to a superconformal fixed point with an AdS dual. Such a relation between superconformal Chern-Simons gauge theories and massive IIA was first suggested in [16], and was recently reconsidered in [11]. In the latter reference the authors considered  $N$  D2-branes in flat space, together with an  $\mathcal{N} = 2$  Chern-Simons interaction generated by the Romans mass. The dual supergravity solution is a warped product  $\text{AdS}_4 \times S^6$ . In our more general set-up, topologically the near horizon geometry should be  $\text{AdS}_4 \times \mathcal{S}Y$ , where  $\mathcal{S}Y$  denotes the suspension of  $Y$ . This is because the link of the singularity  $z = r = 0$  in (2.1) is  $z^2 + r^2 = 1$ , which has induced metric

$$ds_{\mathcal{S}Y}^2 = d\alpha^2 + \sin^2 \alpha ds_Y^2, \tag{2.7}$$

where  $r = \sin \alpha, z = \cos \alpha$ . The metric (2.7) is called the *sine cone* over  $Y$ , and is itself an Einstein metric admitting a Killing spinor. There are isolated conical singularities at  $\alpha = 0, \alpha = \pi$ , inherited from the line of singularities at  $r = 0$  parametrized by  $z$ . However, as mentioned in the last section, we expect the  $\mathcal{N} = 2$  preserving Romans mass deformation to also source other supergravity fields, and for the metric (2.7) to correspondingly be deformed, but with the same topology.

$\mathcal{N} = 2$  supersymmetric  $\text{AdS}_4 \times M_6$  solutions to massive IIA supergravity have been constructed in the literature, notably in [17–19]. In particular in the latter reference  $M_6$  is constructed from a generic Sasaki-Einstein manifold  $(Y, g_Y)$ . However, these solutions are globally well-defined only when  $Y$  is a regular circle bundle over a Kähler-Einstein manifold  $M_4$ , so that  $M_6$  is the total space of an  $S^2$  bundle over  $M_4$ . This is not the topology we want, and the restriction on  $Y$  being regular is too strong. We conjecture that the relevant  $\mathcal{N} = 2$   $\text{AdS}_4$  supergravity solution in massive IIA supergravity is the following solution, constructed recently in [11]:

$$\begin{aligned} g &= e^{2A} \left( g_{\text{AdS}_4} + \frac{3}{2} d\alpha^2 + \frac{9 \sin^2 \alpha}{5 + \cos 2\alpha} \eta^2 + \frac{6 \sin^2 \alpha}{3 + \cos 2\alpha} g_T \right), \\ e^\Phi &= e^{\Phi_0} \frac{(5 + \cos 2\alpha)^{3/4}}{3 + \cos 2\alpha}, & F_0 &= \frac{e^{-5\Phi_0/4}}{\sqrt{3}L}, \\ B &= 6\sqrt{2} L^2 e^{\Phi_0/2} d \left( \frac{\cos \alpha}{3 + \cos 2\alpha} \right) \wedge \eta, \\ F_2 &= -\sqrt{6} L e^{-3\Phi_0/4} \left[ \frac{4 \sin^2 \alpha \cos \alpha}{(3 + \cos 2\alpha)(5 + \cos 2\alpha)} \omega_T - 3 \frac{(3 - \cos 2\alpha)}{(5 + \cos 2\alpha)^2} d \cos \alpha \wedge \eta \right], \\ A_3 &= 6L^3 e^{-\Phi_0/4} \Gamma + 3\sqrt{3} L^3 e^{-\Phi_0/4} \frac{7 + 3 \cos 2\alpha}{(3 + \cos 2\alpha)^2} \sin^4 \alpha \omega_T \wedge \eta, \end{aligned} \tag{2.8}$$

where  $d\Gamma = \text{vol}_{\text{AdS}_4}$  and the warp factor is

$$e^{2A} = L^2 (3 + \cos 2\alpha)^{1/2} (5 + \cos 2\alpha)^{1/8}. \tag{2.9}$$

Here  $g$  is the ten-dimensional Einstein frame metric,  $\Phi$  is the dilaton,  $B$  is the  $B$ -field, while  $F_2$  is the RR two-form flux and  $A_3$  is the RR three-form potential. All Sasaki-Einstein

metrics take the form

$$g_Y = \eta^2 + g_T, \tag{2.10}$$

where  $\eta$  is a contact one-form and  $g_T$  is a transversely Kähler-Einstein metric. The corresponding transverse Kähler form has been denoted  $\omega_T$ , and  $d\eta = 2\omega_T$ . The  $\text{AdS}_4$  metric  $g_{\text{AdS}_4}$  has unit AdS radius. Finally  $L$  and  $\Phi_0$  are constants, that we shall determine shortly.

The ten-dimensional metric takes a warped product form  $g = e^{2A}(g_{\text{AdS}_4} + g_{M_6})$ , where

$$g_{M_6} = \frac{3}{2}d\alpha^2 + \frac{9 \sin^2 \alpha}{5 + \cos 2\alpha} \eta^2 + \frac{6 \sin^2 \alpha}{3 + \cos 2\alpha} g_T. \tag{2.11}$$

Here  $\alpha \in [0, \pi]$ , and this metric may be compared to the sine cone (2.7). The topology is precisely  $M_6 = \mathcal{S}Y$ , with  $\alpha = 0$  and  $\alpha = \pi$  being isolated conical singularities. However, compared to (2.7) the Sasaki-Einstein metric at fixed  $\alpha \in (0, \pi)$  has been squashed, with the relative sizes of  $\eta^2$  and  $g_T$  varying along the polar direction  $\alpha$ .<sup>1</sup>

Recall that the Romans mass  $F_0$  is quantized as in (2.4), while  $N$  D2-branes source  $N$  units of six-form flux over  $M_6$ . These Dirac flux quantization conditions lead to the unlikely expressions

$$L = \frac{\pi \ell_s n^{1/24}}{27^{1/48} 3^{7/24}} \left( \frac{N}{\text{Vol}(Y)} \right)^{5/24}, \quad e^{\Phi_0} = \frac{2^{11/12}}{3^{1/6} n^{5/6}} \left( \frac{N}{\text{Vol}(Y)} \right)^{-1/6}, \tag{2.12}$$

where  $\text{Vol}(Y)$  denotes the volume of the Sasaki-Einstein metric on  $Y$ . The holographic free energy is given by the effective four-dimensional Newton constant,  $F = \pi/(2G_N)$ , which we calculate as

$$F_{\text{gravity}} = \frac{16\pi^3}{(2\pi\ell_s)^8} \int_{M_6} e^{8A} \text{vol}_{M_6} = \frac{2^{1/3} 3^{1/6} \pi^3}{5 \text{Vol}(Y)^{2/3}} n^{1/3} N^{5/3}. \tag{2.13}$$

Setting  $Y = S^5$  equipped with its round metric, topologically  $M_6 = \mathcal{S}S^5 \cong S^6$  and  $\text{Vol}(S^5) = \pi^3$ , and (2.13) reduces to the result in [11].

The metric (2.11) has Calabi-Yau conical singularities at  $\alpha = 0$ ,  $\alpha = \pi$ , for generic  $Y$ . Nevertheless, these singularities do not lead to any divergences in the holographic free energy, and we believe the supergravity solution (2.8) is the correct gravity dual. More precisely, although one expects some stringy degrees of freedom to be supported at the Calabi-Yau singularities, in addition to the massive IIA supergravity fields, the supergravity solution captures the leading order large  $N$  behaviour. We shall confirm this in the next section, by computing the free energy directly in field theory.

### 3 Field theory

Let us now turn to the dual field theory. The three-dimensional  $\mathcal{N} = 2$  superconformal field theories of interest in this paper have UV descriptions as Chern-Simons quiver gauge theories. There is some number  $G$  of  $U(N)$  gauge groups, with equal Chern-Simons couplings  $k$ , together with various chiral multiplets in bifundamental representations  $(\mathbf{N}, \bar{\mathbf{N}})$  of

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<sup>1</sup>These are called  $\eta$ -Sasaki-Einstein metrics in the mathematics literature.

$U(N)_a \times U(N)_b$ , and adjoint representations of  $U(N)_c$ , specified by the quiver diagram. We assign the matter fields R-charges  $\Delta^{ab}$ ,  $\Delta^c$  respectively, consistent with the superpotential having R-charge 2, which is necessary in order to define the theory on  $S^3$  [8]. The partition function for such theories is given by [8, 20, 21]

$$Z_{S^3} = \frac{1}{(N!)^G} \int \prod_{a=1}^G \left[ \prod_{i=1}^N \frac{d\lambda_i^a}{2\pi} \right] \exp \left[ \frac{ik}{4\pi} \sum_{i=1}^N (\lambda_i^a)^2 \right] \prod_{i \neq j}^N \sinh^2 \left( \frac{\lambda_i^a - \lambda_j^a}{2} \right) e^{-F_{\text{matter}}}, \quad (3.1)$$

where  $\lambda_i^a$ ,  $i = 1, \dots, N$ , are the eigenvalues of  $2\pi\sigma_a$  and the matter part is determined by the precise quiver data. Here a single bifundamental chiral multiplet transforming in a representation  $(\mathbf{N}, \bar{\mathbf{N}})$  of  $U(N)_a \times U(N)_b$  and with R-charge  $\Delta^{ab}$  contributes as

$$F_{\text{matter}}^{ab} = - \sum_{i,j=1}^N \ell \left[ 1 - \Delta^{ab} + \frac{i}{2\pi} (\lambda_i^a - \lambda_j^b) \right], \quad (3.2)$$

and a chiral multiplet transforming in the adjoint representation of  $U(N)_c$  and with R-charge  $\Delta^c$  contributes as

$$F_{\text{matter}}^{\text{adj},c} = - \sum_{i,j=1}^N \ell \left[ 1 - \Delta^c + \frac{i}{2\pi} (\lambda_i^c - \lambda_j^c) \right]. \quad (3.3)$$

Here we have used the common definition

$$\ell(z) = -z \log(1 - e^{2\pi iz}) + \frac{i}{2} \left[ \pi z^2 + \frac{1}{\pi} \text{Li}_2(e^{2\pi iz}) \right] - \frac{i\pi}{12}. \quad (3.4)$$

### 3.1 Matrix model large $N$ limit

We next compute the large  $N$  limit of a rather generic such matrix model, following the saddle point method of [22]. Based on numerical simulations in a variety of examples, including both non-chiral and chiral models, we conjecture the following leading order ansatz for the large  $N$  saddle point eigenvalue distribution:

$$\lambda_i^a = N^\nu (x_i + iy_i). \quad (3.5)$$

Here  $x_i, y_i \in \mathbb{R}$  are  $O(1)$  in the large  $N$  expansion, and we shall determine the exponent  $\nu > 0$  analytically later. Notice that crucially all the  $U(N)$  gauge groups have the same behaviour, i.e. the right hand side of (3.5) is independent of  $a = 1, \dots, G$ .

Following [22] to compute the large  $N$  limit of the free energy, we define a density

$$\rho(x) = \frac{1}{N} \sum_{i=1}^N \delta(x - x_i), \quad (3.6)$$

with support on the finite interval  $[-x_*, x_*]$ . In the continuum limit,  $\rho(x)$  becomes an integrable function satisfying

$$\int_{-x_*}^{x_*} dx \rho(x) = 1. \quad (3.7)$$



Furthermore a discrete sum over eigenvalues converges to a Riemann integral

$$\frac{1}{N} \sum_{i=1}^N \longrightarrow \int_{-x_\star}^{x_\star} dx \rho(x) \tag{3.8}$$

in the continuum limit  $N \rightarrow \infty$ .

**Classical contribution.** Given the large  $N$  behaviour of the eigenvalues (3.5), the classical contribution to the large  $N$  free energy  $F = -\log Z_{G3}$  for a theory with  $G$   $U(N)$  gauge groups is

$$F_{\text{classical}} = -\frac{in}{4\pi} N^{1+2\nu} \int_{-x_\star}^{x_\star} dx \rho(x) [2ixy(x) + (x^2 - y(x)^2)] + o(N^{1+2\nu}), \tag{3.9}$$

where  $n = Gk$ .

**Vector multiplet contribution.** A single vector multiplet appears as

$$F_{\text{vector}} = -\sum_{i<j}^N 2 \log 2 \sinh \left( \frac{\lambda_i - \lambda_j}{2} \right) \tag{3.10}$$

in the matrix model. In order to obtain the continuum limit we use the expansion

$$\log 2 \sinh z = \text{sgn}[\text{Re } z] z - \sum_{m \geq 1} \frac{e^{-\text{sgn}[\text{Re } z] 2mz}}{m}. \tag{3.11}$$

The first term will cancel against the chiral multiplet contribution for the models of interest (see below). The second term can be evaluated by a repeated application of integration by parts

$$\begin{aligned} & 2N^2 \int_{-x_\star}^{x_\star} dx' \int_{-x_\star}^{x'} dx \rho(x') \rho(x) \sum_{m \geq 1} \frac{1}{m} e^{-N^\nu m[x'-x+i(y(x')-y(x))]} \\ &= 2N^{2-\nu} \int_{-x_\star}^{x_\star} dx' \rho(x') \rho(x) \sum_{m \geq 1} \frac{1}{m^2} e^{-N^\nu m[x'-x+i(y(x')-y(x))]} \Big|_{-x_\star}^{x'} - 2N^2 \int_{-x_\star}^{x_\star} dx' \\ & \quad \cdot \int_{-x_\star}^{x'} dx \rho(x') \sum_{m \geq 1} \frac{1}{m} \left[ \frac{N^{-\nu}}{m} \rho'(x) + i\rho(x)y'(x) \right] e^{-N^\nu m[x'-x+i(y(x')-y(x))]} . \end{aligned} \tag{3.12}$$

The contribution in the second line coming from  $x = -x_\star$  is exponentially suppressed and only the term with  $x = x'$  contributes to leading order. We again perform integration by parts on the last line in (3.12) and make sure to only extract the leading order terms for large  $N$  (in particular the  $\rho'(x)$  term is subleading). Repeated application of this procedure leads to a geometric series

$$2N^{2-\nu} \int_{-x_\star}^{x_\star} dx' \rho^2(x') \sum_{m \geq 1} \frac{1}{m^2} [1 - iy'(x') - y'(x')^2 + \dots], \tag{3.13}$$

which, using  $\sum_{\ell \geq 1} \frac{1}{\ell^2} = \frac{\pi^2}{6}$ , gives the leading order behaviour of a single vector multiplet<sup>2</sup>

$$F_{\text{vector}} = N^{2-\nu} \frac{\pi^2}{3} \int_{-x_*}^{x_*} dx \frac{\rho^2(x)}{1 + iy'(x)} + o(N^{2-\nu}). \quad (3.14)$$

**Chiral multiplet contribution.** A bifundamental chiral multiplet between  $U(N)_a$  and  $U(N)_b$ , with R-charge  $\Delta = \Delta^{ab}$ , contributes to the matrix model via (3.2). We use a convenient expansion of the  $\ell$ -function (see for example [23])

$$\ell(z) = \text{sgn}[\text{Im} z] \frac{i\pi}{2} \left( z^2 - \frac{1}{6} \right) + \sum_{m \geq 1} \left( \frac{z}{m} + i \frac{\text{sgn}[\text{Im} z]}{2\pi m^2} \right) e^{\text{sgn}[\text{Im} z] 2\pi i m z}. \quad (3.15)$$

Substituting in our ansatz (3.5) and taking the continuum limit, the first term in the expansions leads to a contribution of

$$\begin{aligned} & -N^\nu \sum_{i < j}^N (1 - \Delta) \text{sgn}(x_i - x_j) [x_i - x_j + i(y_i - y_j)] \\ & \longrightarrow -N^{2+\nu} (1 - \Delta) \int_{-x_*}^{x_*} dx' \rho(x') \int_{-x_*}^{x'} dx \rho(x) [x - x' + i(y(x) - y(x'))]. \end{aligned} \quad (3.16)$$

This precisely cancels the contribution coming from the vector multiplets. More precisely, it cancels against the first term in the expansion (3.11) provided that

$$G = \sum_{I \in \text{matter fields}} (1 - \Delta^I), \quad (3.17)$$

where the sum is taken over all matter fields (bifundamental and adjoint) in the quiver and  $\Delta^I$  are their respective R-charges. Equation (3.17) is simply  $\text{Tr} R = 0$ , where the trace is over all fermions in the theory (compare to (3.29) below). Precisely the same constraint arose in the Chern-Simons quiver theories in [23], for essentially the same reason. The parent four-dimensional  $\mathcal{N} = 1$  superconformal field theory certainly satisfies (3.17), as it is implied by the vanishing of the NSVZ beta functions for all the gauge groups (see for example [24]). As in [23],  $\text{Tr} R = 0$  is an additional constraint on the three-dimensional theories under consideration.

Focusing on the second part in the expansion (3.15), and using exactly the same integration-by-parts argument as for the vector multiplet, a careful analysis shows that

$$\begin{aligned} & \sum_{i,j=1}^N \sum_{m \geq 1} \frac{i \text{sgn}[x_i - x_j]}{2\pi m^2} e^{\text{sgn}[x_i - x_j] 2\pi i m [1 - \Delta + \frac{i}{2\pi} N^\nu (x_i + iy_i - (x_j + iy_j))]} \\ & \longrightarrow -N^{2-\nu} \sum_{m \geq 1} \frac{\sin 2\pi m (1 - \Delta)}{\pi m^3} \int_{-x_*}^{x_*} dx \frac{\rho^2(x)}{1 + iy'(x)}, \end{aligned} \quad (3.18)$$

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<sup>2</sup>Recall that we are neglecting the linear term in (3.11), as it will cancel for the models considered in this paper.

for large  $N$ . For the remaining part of the expansion we compute the large  $N$  contribution

$$\begin{aligned} & \sum_{i,j=1}^N \sum_{m \geq 1} \frac{1 - \Delta + \frac{i}{2\pi} N^\nu [x_i + iy_i - (x_j + iy_j)]}{m} e^{\text{sgn}[x_i - x_j] 2\pi i m [1 - \Delta + \frac{i}{2\pi} N^\nu (x_i + iy_i - (x_j + iy_j))]} \\ & \rightarrow N^{2-\nu} \sum_{m \geq 1} \left[ \frac{2(1 - \Delta) \cos 2\pi m(1 - \Delta)}{m^2} - \frac{\sin 2\pi m(1 - \Delta)}{\pi m^3} \right] \int_{-x_\star}^{x_\star} dx \frac{\rho^2(x)}{1 + iy'(x)}. \end{aligned} \quad (3.19)$$

It remains to evaluate the Fourier series. For  $\Delta \in [0, 1]$  these are given by

$$\begin{aligned} \sum_{m \geq 1} \frac{\sin 2\pi m \Delta}{m^3} &= \frac{\pi^3}{3} \Delta(1 - \Delta)(1 - 2\Delta), \\ \sum_{m \geq 1} \frac{\cos 2\pi m \Delta}{m^2} &= \pi^2 \left( \Delta - \frac{1}{2} \right)^2 - \frac{\pi^2}{12}. \end{aligned} \quad (3.20)$$

Putting everything together we arrive at the final form of the large  $N$  chiral multiplet contributions<sup>3</sup>

$$\begin{aligned} F_{\text{matter}}^{ab} &= N^{2-\nu} \frac{\pi^2}{3} (1 - \Delta^{ab}) \left[ 1 - 2(1 - \Delta^{ab})^2 \right] \int_{-x_\star}^{x_\star} dx \frac{\rho^2(x)}{1 + iy'(x)}, \\ F_{\text{matter}}^{\text{adj},c} &= N^{2-\nu} \frac{\pi^2}{3} (1 - \Delta^c) \left[ 1 - 2(1 - \Delta^c)^2 \right] \int_{-x_\star}^{x_\star} dx \frac{\rho^2(x)}{1 + iy'(x)}, \end{aligned} \quad (3.21)$$

for bifundamental and adjoint chiral multiplets of R-charges  $\Delta^{ab}$ ,  $\Delta^c$ , respectively. The contribution of an adjoint chiral multiplet is derived in exactly the same fashion.

**Large  $N$  free energy.** We consider a generic three-dimensional  $\mathcal{N} = 2$  Chern-Simons quiver theory, with  $G$   $U(N)$  gauge groups and some number of bifundamental and adjoint chiral multiplets, giving the R-charge spectrum  $\{\Delta^I : I \in \text{matter fields}\}$ . Given the expressions obtained above, the saddle point large  $N$  free energy for such a model is obtained by extremizing

$$\begin{aligned} F &= \frac{n}{4\pi} N^{1+2\nu} \int_{-x_\star}^{x_\star} dx \rho(x) [2xy(x) - i(x^2 - y(x)^2)] \\ &+ N^{2-\nu} \frac{\pi^2}{3} \left\{ G + \sum_{I \in \text{matter fields}} (1 - \Delta^I) [1 - 2(1 - \Delta^I)^2] \right\} \int_{-x_\star}^{x_\star} dx \frac{\rho^2(x)}{1 + iy'(x)} \\ &+ o(N^{2-\nu}). \end{aligned} \quad (3.22)$$

In order to find a non-trivial saddle point both contributions have to be of the same order, which determines  $\nu = 1/3$ . This precisely agrees with numerical simulations of a variety of models.

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<sup>3</sup>Again recall that we are neglecting the linear term in (3.15), since we showed that it will cancel for the models considered in this paper.

Noting that  $y(x)$  is real, computing the leading order behaviour is now a simple variational problem, with the following general solution

$$y(x) = \frac{1}{\sqrt{3}}x, \tag{3.23}$$

$$\rho(x) = \frac{3 \cdot 2^{2/3} \pi^2 n^{1/3} \{G + \sum_I (1 - \Delta^I) [1 - 2(1 - \Delta^I)^2]\}^{2/3} - 4nx^2}{4\sqrt{3}\pi^3 \{G + \sum_I (1 - \Delta^I) [1 - 2(1 - \Delta^I)^2]\}}, \tag{3.24}$$

$$x_\star = \frac{\sqrt{3}\pi \{G + \sum_I (1 - \Delta^I) [1 - 2(1 - \Delta^I)^2]\}^{1/3}}{2^{2/3} n^{1/3}}. \tag{3.25}$$

Substituting these expressions back into (3.22), we arrive at the final form of the large  $N$  free energy

$$F = \frac{3\sqrt{3}\pi}{20 \cdot 2^{1/3}} \left(1 - \frac{i}{\sqrt{3}}\right) \left\{G + \sum_{I \in \text{matter fields}} (1 - \Delta^I) [1 - 2(1 - \Delta^I)^2]\right\}^{2/3} n^{1/3} N^{5/3} + o(N^{5/3}). \tag{3.26}$$

### 3.2 Comparison to supergravity

Let us compare this result to our supergravity prediction (2.13). To do so, we start by making an important observation. The part of (3.26) which depends on the quiver data can in fact be rewritten in terms of the  $a$ -function of the four-dimensional parent theory. In particular, using the relation (3.17), which as mentioned above is implied by the four-dimensional theory having vanishing beta functions, one finds that

$$GN^2 + \sum_{I \in \text{matter fields}} (1 - \Delta^I) (1 - 2(1 - \Delta^I)^2) N^2 = \frac{64}{9} a. \tag{3.27}$$

Here  $a$  is the  $a$ -function of the four-dimensional  $\mathcal{N} = 1$  superconformal parent theory. It can be expressed as [25, 26]

$$a = \frac{3}{32} (\text{Tr } R^3 - 5\text{Tr } R), \tag{3.28}$$

where  $R$  is a choice of  $U(1)_R$  symmetry. In terms of the quiver data we have

$$\text{Tr } R^\gamma = G \dim U(N) + \sum_{I \in \text{matter fields}} \dim \mathcal{R}^I (\Delta^I - 1)^\gamma, \tag{3.29}$$

where  $\dim \mathcal{R}^I$  is the dimension of the respective matter representation with R-charge  $\Delta^I$ . (Notice then that (3.17) is simply  $\text{Tr } R = 0$ .) Given this remarkable relation, we can express the free energy (3.26) in terms of the  $a$ -function

$$F(R) = \frac{2^{5/3} 3^{1/6} \pi}{5} \left(1 - \frac{i}{\sqrt{3}}\right) (nN)^{1/3} a(R)^{2/3} + o(N^{5/3}), \tag{3.30}$$

and we see that  $F$ -maximization is in fact *equivalent* to  $a$ -maximization for the three-dimensional SCFTs considered here. Evaluating the  $a$ -function on the superconformal

$U(1)_R$  gives the  $a$  central charge. This in turn is related by AdS<sub>5</sub>/CFT<sub>4</sub> to the volume of the internal space  $\text{Vol}(Y)$  (1.1). Putting all this together, we see that the real part of the large  $N$  free energy precisely agrees with the supergravity result (2.13).

We conclude this subsection with a brief comment on the imaginary part of the free energy (3.30). This is well-defined only modulo  $2\pi$ , and so is effectively  $O(1)$  in the large  $N$  expansion. Its value is also altered by adding a Chern-Simons term for a background Abelian gauge field [10]. One therefore doesn't expect to be able to match it to a leading order supergravity calculation.

### 3.3 Wilson loop

We may decorate our three-dimensional field theory with Wilson loops. A BPS Wilson loop in a representation  $\mathcal{R}$  of a single  $U(N)$  gauge group is given by

$$\mathcal{W}_{\mathcal{R}} = \frac{1}{\dim \mathcal{R}} \text{Tr}_{\mathcal{R}} \left\{ \mathcal{P} \exp \left( \oint_{\gamma} ds (iA_{\mu} \dot{x}^{\mu} + \sigma|x|) \right) \right\}, \quad (3.31)$$

where  $\mathcal{P}$  denotes the path-ordering operator,  $A_{\mu}$  is the gauge field with  $\sigma$  the corresponding scalar in the  $\mathcal{N} = 2$  vector multiplet, and  $x^{\mu}(s)$  parametrizes the Wilson line  $\gamma \subset S^3$ . In order to preserve some supersymmetry,  $\gamma$  lies on a Hopf circle in  $S^3$ . In particular this gives rise to 1/2 BPS Wilson loops in the three-dimensional theory. The VEV of such a 1/2 BPS Wilson loop can be computed by insertion of  $\mathcal{W}_{\mathcal{R}}$  into the path integral, and so expressed in terms of the matrix model via localization [20]. This amounts to an additional factor of

$$\text{Tr}_{\mathcal{R}} e^{2\pi\sigma}, \quad (3.32)$$

where  $\gamma$  has length  $2\pi$ .

We now focus on Wilson loops in the fundamental representation of a quiver with  $G$   $U(N)$  gauge groups and some generic matter. In the large  $N$  limit the VEV is then given by

$$\langle \mathcal{W}_{\text{fund}} \rangle \longrightarrow GN \int_{-x_{\star}}^{x_{\star}} dx \rho(x) e^{N\nu(x+iy(x))}. \quad (3.33)$$

The leading order saddle point configurations in (3.23), (3.24) and (3.25) are not affected by the addition of a Wilson loop, since it is subleading. The large  $N$  VEV of a 1/2 BPS fundamental Wilson loop is therefore simply given by

$$\log \langle \mathcal{W}_{\text{fund}} \rangle = \left( 1 + \frac{i}{\sqrt{3}} \right) x_{\star} N^{1/3} + o(N^{1/3}). \quad (3.34)$$

Substituting for the saddle point configuration of  $x_{\star}$  as written in (3.25), we end up with

$$\log \langle \mathcal{W}_{\text{fund}} \rangle = \frac{2^{4/3}\pi}{3^{1/6}} \left( 1 + \frac{i}{\sqrt{3}} \right) (nN)^{-1/3} a(R)^{1/3}, \quad (3.35)$$

where we have used equation (3.27). Again appealing to the relation (1.1) leads to a precise supergravity prediction.

**Dual fundamental string.** On general grounds the 1/2 BPS fundamental Wilson loop should map to a fundamental string in  $\text{AdS}_4$  [27]. This wraps a disk  $\Sigma_2 \subset \text{AdS}_4$ , where  $\Sigma_2 \cong \text{AdS}_2$  has boundary  $\partial\Sigma_2 = \gamma \subset S^3$ . It is natural for this to sit at one of the two suspension points in the internal space, with  $\alpha = 0$  or  $\alpha = \pi$ . Computing the fundamental string action simply reduces to computing the volume of  $\Sigma_2$  in the string frame metric. Converting from Einstein to string frame introduces an additional factor of the ten-dimensional dilaton,  $e^{\Phi/2}$ . Putting everything together, we find

$$S_{\text{string}} = \frac{6^{1/8}}{\pi \ell_s^2} e^{\Phi/2}|_{\alpha=0,\pi} L^2 \text{Vol}(\Sigma_2). \quad (3.36)$$

The divergent area of  $\Sigma_2 \cong \text{AdS}_2$  is here renormalized via a local counterterm, namely the length of the boundary, leading to the standard result

$$\text{Vol}(\text{AdS}_2) = -2\pi. \quad (3.37)$$

We hence find that

$$S_{\text{string}} = -\frac{2^{2/3} \pi^2}{3^{1/6} n^{1/3} \text{Vol}(Y)^{1/3}} N^{1/3}. \quad (3.38)$$

As expected, this precisely agrees with minus the real part of  $\log \langle \mathcal{W}_{\text{fund}} \rangle$  as given in (3.35), if we use the relation (1.1) to express the  $a$  central charge in terms of the internal space volume  $\text{Vol}(Y)$ .

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