Two-loop sunset integrals at finite volume

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ABSTRACT: We show how to compute the two-loop sunset integrals at finite volume, for non-degenerate masses and non-zero momentum. We present results for all integrals that appear in the Chiral Perturbation Theory (χPT) calculation of the pseudoscalar meson masses and decay constants at next-to-next-to-leading order, including the case of Partially Quenched χPT. We also provide numerical implementations of the finite-volume sunset integrals, and review the results for one-loop integrals at finite volume.

Contents

1 Introduction

An analytical, *ab initio* description of Quantum Chromodynamics (QCD) in the hadronic low-energy regime remains elusive. One of the most promising alternatives involves numerical evaluation of the functional integral of QCD on a discretized space-time lattice. Known as Lattice QCD, this approach has long been restricted for computational reasons to large and unphysical values of the light quark masses. Recently, due to improvements in computing power and algorithmics, calculations with significantly smaller quark masses have become possible. A side effect of the lowered quark masses is an increase in the size of finite-volume corrections, and a detailed treatment of such effects is thus called for.

Fortunately, in many cases the finite-volume corrections can be evaluated analytically using Chiral Perturbation Theory (χPT) [1–3], which is the low-energy effective theory of QCD. The application of χPT at finite volume was first performed by Gasser and Leutwyler in ref. [4], and a review of recent work in this area can be found in ref. [5]. As many Lattice QCD simulations are performed with unequal valence and sea-quark masses [6], the properties of the light pseudoscalar mesons have also been calculated to next-to-next-to-leading order (NNLO) in Partially Quenched χPT (PQχPT) in refs. [7–9]. Therefore, it is also of interest to extend the finite-volume description of the relevant loop integrals to account for the appearance of double poles in the PQχPT propagators. It should be noted that χPT is applicable at finite volume as soon as the typical momenta of a given process are sufficiently small. This imposes the restriction $F_\pi L > 1$, where F_π is the pion decay constant and the volume $V \equiv L^3$. This study deals with the so-called p-regime, in which V is sufficiently large for zero-momentum fluctuations of the meson fields to be treated perturbatively, which introduces the additional requirement $m_\pi^2 F_\pi^2 V \gg 1$, where m_π is the pion mass. A multitude of finite-volume calculations exist at one-loop or next-to-leading order (NLO), and it should also be noted that some work at NNLO has recently appeared. This includes ref. [10], where the finite-volume corrections to the quark condensate were calculated, and ref. [11] which considered m_π for the case of degenerate quark masses.

Our main objective is to show how the integrals needed in χPT calculations of pseudoscalar meson properties at NNLO and finite volume can be performed. As a starting point, the known results at one-loop order are reviewed, and we also show how these can be extended to higher order in $d - 4$. The methods for the one-loop integrals are then applied to the two-loop "sunset" integrals for arbitrary masses and momenta. We focus here on the integrals necessary for the calculation of form factors to NLO, and for the calculation of masses, decay constants and two-point functions to two loops (NNLO).

This paper is structured as follows: section 2 discusses a few preliminaries. In section 3, the derivation of the one-loop integrals at finite volume is revisited, with emphasis on the treatment of PQχPT calculations at NLO. In section 4, the two-loop sunset integrals are considered, and explicit expressions are given for the finite and divergent parts, for arbitrary values of the quark masses and with the propagator structure of PQχPT fully accounted for. Section 5 contains a numerical overview of the integrals presented in this study, along with a concluding discussion in section 6. The appendices summarize the ingredients involving modified Bessel functions and theta functions, along with basic integrals in d dimensions

and comments on the notational conventions in earlier work. Some preliminary results related to this study have been presented in refs. [12, 13].

2 Preliminaries

2.1 Finite-volume sums

At finite volume in a cubic box, integrals over momenta should be replaced by sums over the allowed momenta. In one dimension of length L, with periodic boundary conditions,[1] this entails a summation over the allowed momenta $p_n \equiv 2\pi n/L$, with $n \in \mathbf{Z}$ integer. The integrals over momenta should thus be replaced according to

$$\int \frac{dp}{2\pi} F(p) \rightarrow \frac{1}{L} \sum_{n \in \mathbf{Z}} F(p_n) \equiv \int_V \frac{dp}{2\pi} F(p), \tag{2.1}$$

where the latter notation will be used to indicate a finite-volume summation in the remainder of this paper. Infinities will be treated by dimensional regularization, using the convention $d \equiv 4 - 2\varepsilon$. The infinite-volume integrals have been treated extensively in the literature, see e.g. ref. [15] including appendices and references therein.

In practice, it is often desirable to study deviations from the infinite-volume limit, and we shall therefore use a framework in which the infinite-volume contribution can be easily identified. This can be achieved by application of the Poisson summation formula to eq. (2.1), yielding

$$\frac{1}{L} \sum_{n \in \mathbf{Z}} F(p_n) = \sum_{l_p} \int \frac{dp}{2\pi} e^{il_p p} F(p), \tag{2.2}$$

where the summation over l_p spans a set of vectors of length nL such that $n \in \mathbf{Z}$. The term with $n = 0$ then represents the infinite-volume result, while the sum of all the other terms is the finite-volume correction.

In the case of loop integrals over momenta in higher dimensions, eq. (2.2) should be applied to all dimensions which have a finite extent. The four-vector $l_{p\mu}$ then has components $(0, n_1 L, n_2 L, n_3 L)$ when three of the dimensions have a finite extent L. The loop integrals in this paper are performed throughout in Euclidean space, with metric $g_{\mu\nu} = \delta_{\mu\nu}$ and signature $(+, +, +, +)$. Throughout this paper, one of the dimensions (the "time" dimension) is assumed to be much larger in extent than the other three dimensions, which is the usual situation encountered in Lattice QCD.

2.2 Passarino-Veltman reduction

At infinite volume, a general method was developed by Passarino and Veltman [16] to obtain a minimal set of integrals by reduction of the tensor integrals $H_{\mu\nu}$ to a set of scalar integrals. This method relies on separation of the integrals into components that are scalars under Lorentz transformations and prefactors that contain $\delta_{\mu\nu}$ and various momenta. Although Lorentz-invariance is explicitly broken by the introduction of a finite size, it is still possible,

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[1] We do not consider twisted boundary conditions as discussed in ref. [14]. These can be treated by adding a shift to the allowed momenta, relative to the summations used here.

in the frame where $p \cdot l_p = 0$, to rewrite the integrals in scalar components, provided that a four-vector

$$t_\mu \equiv (1, 0, 0, 0) \tag{2.3}$$

is introduced. The situation $p \cdot l_p = 0$ is referred to as the "center-of-mass" (cms) frame, which is a situation often realized in Lattice QCD. Because of the remaining symmetries, t_μ is the only additional object required to rewrite the integrals in scalar components, but we also introduce the tensor

$$t_{\mu\nu} \equiv \delta_{\mu\nu} - t_\mu t_\nu = \mathrm{diag}(0, 1, 1, 1) \tag{2.4}$$

as a convenient additional abbreviation.

3 One-loop integrals at finite volume

In general, the one-loop integrals in the NNLO expressions for the pseudoscalar meson masses and decay constants contain a maximum of two propagators with distinct masses. The simplest case with one propagator is denoted A, whereas the case with two distinct propagators is denoted B. In PQχPT, some three-propagator integrals denoted C also appear. These are due to the mixing of different lowest-order states in PQχPT, and they can always be re-expressed in terms of the B integrals.

All of the integrals mentioned above have been extensively treated in the literature, see e.g. refs. [4, 17–19]. However, it is instructive to review certain aspects of their derivation and numerical evaluation here, since they form building blocks in the calculation of the two-loop sunset integrals at finite volume.

3.1 One-propagator integrals

The basic one-loop, one-propagator integrals are

$$\lfloor X \rfloor = \int_V \frac{d^d r}{(2\pi)^d} \frac{X}{(r^2 + m^2)^n}, \tag{3.1}$$

where $X = 1, r_\mu$ and $r_\mu r_\nu$. By application of the Poisson summation formula for the finite dimensions, eq. (3.1) may be written as

$$\lfloor X \rfloor = \sum_{l_r} \int \frac{d^d r}{(2\pi)^d} \frac{X \, e^{i l_r \cdot r}}{(r^2 + m^2)^n}, \tag{3.2}$$

where the term with $l_r = 0$ represents the infinite-volume contribution. In order to isolate the finite-volume part, eq. (3.2) is decomposed according to

$$\lfloor X \rfloor \equiv \lfloor X \rfloor^\infty + \lfloor X \rfloor^V, \tag{3.3}$$

where the first term represents the infinite-volume result and will not be considered further. The second term represents the finite-volume correction, and is free from divergences.

First, we consider the case of $X = 1$. We rewrite eq. (3.1) using eq. (A.1) as

$$\lfloor 1 \rfloor^V = \frac{1}{\Gamma(n)} \sum_{l_r}' \int \frac{d^d r}{(2\pi)^d} \int_0^\infty d\lambda \, \lambda^{n-1} \, e^{i l_r \cdot r} e^{-\lambda(r^2 + m^2)}, \tag{3.4}$$

where the primed sum indicates that the term with $l_r = 0$ is excluded. We next substitute $r \equiv \bar{r} + i l_r / (2\lambda)$, and obtain

$$\lfloor 1 \rfloor^V = \frac{1}{\Gamma(n)} \sum_{l_r}' \int_0^\infty d\lambda \, \lambda^{n-1} \, e^{-\lambda m^2 - \frac{l_r^2}{4\lambda}} \int \frac{d^d \bar{r}}{(2\pi)^d} \, e^{-\lambda \bar{r}^2}, \tag{3.5}$$

where the $\bar{r}$ integral can be performed using eq. (C.3) and by rescaling $\bar{r} \equiv \tilde{r}/\sqrt{\lambda}$, which gives

$$\lfloor 1 \rfloor^V = \frac{1}{(4\pi)^{d/2}\Gamma(n)} \sum_{l_r}' \int_0^\infty d\lambda \, \lambda^{n-\frac{d}{2}-1} \, e^{-\lambda m^2 - \frac{l_r^2}{4\lambda}}. \tag{3.6}$$

The (triple) sum and integral can be evaluated in different ways. The technique used in refs. [4, 17] is to employ eq. (A.2), which yields

$$\lfloor 1 \rfloor^V = \frac{1}{(4\pi)^{d/2}\Gamma(n)} \sum_{l_r}' \mathcal{K}_{n-\frac{d}{2}} \left(\frac{l_r^2}{4}, m^2 \right), \tag{3.7}$$

where the modified Bessel functions $\mathcal{K}_\nu$ are defined in appendix A. The triple sum can be simplified by observing that $l_r^2 = k L^2$, with k integer. We further define the factor $x(k)$, which indicates the number of times each value of $k \equiv n_1^2 + n_2^2 + n_3^2$ occurs in the triple sum. We then find

$$\sum_{l_r}' f(l_r^2) = \sum_{k>0} x(k) f(k), \tag{3.8}$$

which reduces the triple sum to a single sum. The final result is

$$\lfloor 1 \rfloor^V = \frac{1}{(4\pi)^{d/2}\Gamma(n)} \sum_{k>0} x(k) \, \mathcal{K}_{n-\frac{d}{2}} \left(\frac{k L^2}{4}, m^2 \right), \tag{3.9}$$

where the arguments of $\mathcal{K}_\nu$ can be modified by rescaling λ before eq. (A.2) is applied. Also, the sum over modified Bessel functions is found to converge fairly slowly.

The second method considered here involves performing the summation, and leaving the integral to be evaluated numerically, see ref. [19]. We observe that

$$\sum_{l_r}' e^{-\frac{l_r^2}{4\lambda}} = \left[\sum_{l_1} e^{-\frac{L^2}{4\lambda} l_1^2} \right]^3 - 1, \tag{3.10}$$

using the relation $l_r^2 = (l_1^2 + l_2^2 + l_3^2) L^2$. The cubic power accounts for the summations over l_1, l_2 and l_3. The remaining sum in eq. (3.10) can be performed in terms of the theta function θ_{30}, which is defined in appendix B. This gives

$$\lfloor 1 \rfloor^V = \frac{1}{(4\pi)^{d/2}\Gamma(n)} \int_0^\infty d\lambda \, \lambda^{n-\frac{d}{2}-1} \left[\theta_{30} \left(e^{-L^2/(4\lambda)} \right)^3 - 1 \right] e^{-\lambda m^2}, \tag{3.11}$$

where, as a final step, we rescale λ to obtain

$$\lfloor 1 \rfloor^V = \frac{1}{(4\pi)^{d/2}\Gamma(n)} \left(\frac{L^2}{4}\right)^{n-\frac{d}{2}} \int_0^\infty d\lambda\,\lambda^{n-\frac{d}{2}-1} \left[\theta_{30}(e^{-1/\lambda})^3 - 1\right] e^{-\lambda\frac{m^2 L^2}{4}}, \qquad (3.12)$$

which is also valid for $mL \sim 1$.

Integrals with factors of r_μ in the numerator are also required. Up to NNLO, these are $\lfloor r_\mu \rfloor$ and $\lfloor r_\mu r_\nu \rfloor$. Proceeding as above, we obtain

$$\lfloor (r_\mu; r_\mu r_\nu) \rfloor^V = \frac{1}{\Gamma(n)} \sum_{l_r}' \int_0^\infty d\lambda\,\lambda^{n-1}\,e^{-\lambda m^2 - \frac{l_r^2}{4\lambda}}$$
$$\times \int \frac{d^d\bar{r}}{(2\pi)^d} \left[\bar{r}_\mu + \frac{il_{r\mu}}{2\lambda}; \left(\bar{r}_\mu + \frac{il_{r\mu}}{2\lambda}\right)\left(\bar{r}_\nu + \frac{il_{r\nu}}{2\lambda}\right)\right] e^{-\lambda\bar{r}^2}, \qquad (3.13)$$

where we note that integrals odd in $\bar{r}$ vanish, and that

$$\int d^d\bar{r}\,\bar{r}_\mu\bar{r}_\nu\,f(\bar{r}^2) = \frac{\delta_{\mu\nu}}{d}\int d^d\bar{r}\,\bar{r}^2 f(\bar{r}^2). \qquad (3.14)$$

The summations over the components of l_r include both positive and negative contributions, and are symmetric under interchange of spatial directions. The sums which are odd in the components of l_r then vanish, and

$$\sum_{l_r} l_{r\mu}l_{r\nu}f(l_r^2) = \frac{1}{3}t_{\mu\nu}\sum_{l_r} l_r^2 f(l_r^2). \qquad (3.15)$$

Thus, the final results for the $\lfloor r_\mu \rfloor$ and $\lfloor r_\mu r_\nu \rfloor$ integrals are

$$\lfloor r_\mu \rfloor^V = 0,$$
$$\lfloor r_\mu r_\nu \rfloor^V = \frac{1}{\Gamma(n)} \sum_{l_r}' \int_0^\infty d\lambda\,\lambda^{n-1}\,e^{-\lambda m^2 - \frac{l_r^2}{4\lambda}} \int \frac{d^d\bar{r}}{(2\pi)^d} \left(\frac{\delta_{\mu\nu}}{d}\bar{r}^2 - \frac{t_{\mu\nu}}{12\lambda^2}l_r^2\right) e^{-\lambda\bar{r}^2},$$
$$= \frac{1}{(4\pi)^{d/2}\Gamma(n)} \sum_{l_r}' \int_0^\infty d\lambda\,\lambda^{n-\frac{d}{2}-1} \left(\frac{\delta_{\mu\nu}}{2\lambda} - \frac{t_{\mu\nu}}{12\lambda^2}l_r^2\right) e^{-\lambda m^2 - \frac{l_r^2}{4\lambda}}, \qquad (3.16)$$

where the remaining integration can again be performed in terms of the modified Bessel functions, giving

$$\lfloor r_\mu r_\nu \rfloor^V = \frac{1}{(4\pi)^{d/2}\Gamma(n)} \sum_{k>0} x(k) \left[\frac{\delta_{\mu\nu}}{2}\mathcal{K}_{n-\frac{d}{2}-1}\left(\frac{kL^2}{4}, m^2\right) - \frac{t_{\mu\nu}}{12}kL^2\mathcal{K}_{n-\frac{d}{2}-2}\left(\frac{kL^2}{4}, m^2\right)\right]. \qquad (3.17)$$

For the second method which involves the theta functions, we rewrite the sum using the identity

$$\sum_{n\in\mathbb{Z}^3} n^2 q^{(n^2)} = q\frac{\partial}{\partial q}\left[\sum_{n\in\mathbb{Z}^3} q^{(n^2)}\right] = q\frac{\partial}{\partial q}\left[\theta_{30}(q)^3\right] = 3\theta_{32}(q)\theta_{30}(q)^2, \qquad (3.18)$$

which is also valid for the primed sums, as the term with $n = 0$ does not contribute. After rescaling λ, this gives

$$
\lfloor r_\mu r_\nu \rfloor^V = \frac{1}{(4\pi)^{d/2}\Gamma(n)} \left(\frac{L^2}{4}\right)^{n-\frac{d}{2}-1} \int_0^\infty d\lambda\, \lambda^{n-\frac{d}{2}-2}\, e^{-\lambda\frac{m^2 L^2}{4}}
$$
$$
\times \left\{ \frac{\delta_{\mu\nu}}{2} \left[\theta_{30}\left(e^{-1/\lambda}\right)^3 - 1\right] - \frac{t_{\mu\nu}}{\lambda}\left[\theta_{32}\left(e^{-1/\lambda}\right)\theta_{30}\left(e^{-1/\lambda}\right)^2\right]\right\}, \qquad (3.19)
$$

and following the same steps as before, we also find

$$
\lfloor r_\mu r_\nu r_\alpha \rfloor^V = 0. \qquad (3.20)
$$

3.2 Two-propagator integrals

The basic one-loop, two-propagator integrals are

$$
\langle X \rangle = \int_V \frac{d^d r}{(2\pi)^d} \frac{X}{(r^2 + m_1^2)^{n_1}((r-p)^2 + m_2^2)^{n_2}}, \qquad (3.21)
$$

where $X = 1, r_\mu, r_\mu r_\nu$ and $r_\mu r_\nu r_\alpha$. By application of the Poisson summation formula for the finite dimensions, eq. (3.21) may be written as

$$
\langle X \rangle = \sum_{l_r} \int \frac{d^d r}{(2\pi)^d} \frac{X\, e^{il_r \cdot r}}{(r^2 + m_1^2)^{n_1}((r-p)^2 + m_2^2)^{n_2}}, \qquad (3.22)
$$

where the term with $l_r = 0$ represents the infinite-volume contribution. We again decompose eq. (3.22) into the infinite-volume part and the finite-volume correction using

$$
\langle X \rangle \equiv \langle X \rangle^\infty + \langle X \rangle^V, \qquad (3.23)
$$

where the latter term is obtained from eq. (3.22) by replacing the unprimed sum with the primed one, indicating that the term with $l_r = 0$ is excluded.

The methods of section 3.1 also apply here. We begin by introducing Gaussian parameterizations for both propagators in eq. (3.22) in terms of the integration variables λ_1 and λ_2. In a second step, we switch to a new set of variables (λ, x) with $\lambda_1 \equiv x\lambda$ and $\lambda_2 \equiv (1-x)\lambda$. Alternatively, we may first combine the two propagators using the Feynman parameterization

$$
\frac{1}{a^m b^n} = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \int_0^1 dx\, \frac{x^{m-1} y^{n-1}}{(ax + yb)^{m+n}}, \qquad (3.24)
$$

where $y = 1 - x$, and then treat the denominator according to eq. (A.1). In both cases, the result is

$$
\langle X \rangle^V = \frac{1}{\Gamma(n_1)\Gamma(n_2)} \sum_{l_r}{}' \int_0^1 dx \int \frac{d^d r}{(2\pi)^d}
$$
$$
\times \int_0^\infty d\lambda\, \lambda^{n_1+n_2-1}\, x^{n_1-1} y^{n_2-1} X\, e^{il_r \cdot r} e^{-\lambda[x(r^2+m_1^2)+y((r-p)^2+m_2^2)]}, \qquad (3.25)
$$

which is the equivalent of eq. (3.4). We now shift the integration variable to $r \equiv \bar{r} + il/(2\lambda) + yp$ and obtain for the simplest case

$$\langle 1 \rangle^V = \frac{1}{\Gamma(n_1)\Gamma(n_2)} {\sum_{l_r}}' \int_0^1 dx \int_0^\infty d\lambda\, \lambda^{n_1+n_2-1} x^{n_1-1} y^{n_2-1}\, e^{iyl_r \cdot p} e^{-\lambda \tilde{m}^2 - \frac{l_r^2}{4\lambda}} \int \frac{d^d r}{(2\pi)^d}\, e^{-\lambda \bar{r}^2}$$

$$= \frac{1}{(4\pi)^{d/2}\Gamma(n_1)\Gamma(n_2)} {\sum_{l_r}}' \int_0^1 dx \int_0^\infty d\lambda\, \lambda^{n_1+n_2-\frac{d}{2}-1}\, x^{n_1-1} y^{n_2-1}\, e^{iyl_r \cdot p} e^{-\lambda \tilde{m}^2 - \frac{l_r^2}{4\lambda}}, \quad (3.26)$$

where

$$\tilde{m}^2 = xm_1^2 + ym_2^2 + xyp^2, \qquad (3.27)$$

which differs from eq. (3.6) by the integration over x and the factor $e^{iyl_r \cdot p}$. Due to the summation over components of $l_{r\mu}$ with alternating signs, this factor always produces real-valued results. For the remaining integrals, we obtain

$$\langle X \rangle^V = \frac{1}{(4\pi)^{d/2}\Gamma(n_1)\Gamma(n_2)} {\sum_{l_r}}' \int_0^1 dx \int_0^\infty d\lambda\, \lambda^{n_1+n_2-\frac{d}{2}-1} x^{n_1-1} y^{n_2-1} \lceil X \rceil\, e^{iyl_r \cdot p} e^{-\lambda \tilde{m}^2 - \frac{l_r^2}{4\lambda}},$$

$$(3.28)$$

with

$$\lceil r_\mu \rceil = yp_\mu + \frac{il_{r\mu}}{2\lambda},$$

$$\lceil r_\mu r_\nu \rceil = \frac{\delta_{\mu\nu}}{2} + y^2 p_\mu p_\nu + \frac{iy}{2\lambda}\{l_r, p\}_{\mu\nu} - \frac{l_{r\mu}l_{r\nu}}{4\lambda^2},$$

$$\lceil r_\mu r_\nu r_\alpha \rceil = \frac{1}{2}\left[\delta_{\mu\nu}\left(yp_\alpha + \frac{il_{r\alpha}}{2\lambda}\right) + \delta_{\mu\alpha}\left(yp_\nu + \frac{il_{r\nu}}{2\lambda}\right) + \delta_{\nu\alpha}\left(yp_\mu + \frac{il_{r\mu}}{2\lambda}\right)\right]$$

$$+ \left(yp_\mu + \frac{il_{r\mu}}{2\lambda}\right)\left(yp_\nu + \frac{il_{r\nu}}{2\lambda}\right)\left(yp_\alpha + \frac{il_{r\alpha}}{2\lambda}\right), \qquad (3.29)$$

where $\{a, b\}_{\mu\nu} \equiv a_\mu b_\nu + b_\mu a_\nu$.

3.2.1 Center-of-mass frame

In the cms frame, $p = (p, 0, 0, 0)$ such that $p \cdot l_r = 0$ for all l_r. The integrals in the cms frame can be computed similarly to the one-propagator integrals, giving

$$\langle 1 \rangle^V_{n_1 n_2} = \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 dx\, x^{n_1-1} y^{n_2-1} \lfloor 1 \rfloor^V_{n_1+n_2},$$

$$\langle r_\mu \rangle^V_{n_1 n_2} = \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 dx\, x^{n_1-1} y^{n_2} p_\mu \lfloor 1 \rfloor^V_{n_1+n_2},$$

$$\langle r_\mu r_\nu \rangle^V_{n_1 n_2} = \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 dx\, x^{n_1-1} y^{n_2-1}\left(\lfloor r_\mu r_\nu \rfloor^V_{n_1+n_2} + y^2 p_\mu p_\nu \lfloor 1 \rfloor^V_{n_1+n_2}\right),$$

$$\langle r_\mu r_\nu r_\alpha \rangle^V_{n_1 n_2} = \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 dx\, x^{n_1-1} y^{n_2-1}\left(yp_\alpha \lfloor r_\mu r_\nu \rfloor^V_{n_1+n_2} + yp_\mu \lfloor r_\nu r_\alpha \rfloor^V_{n_1+n_2}\right.$$

$$\left. + yp_\nu \lfloor r_\alpha r_\mu \rfloor^V_{n_1+n_2} + y^3 p_\mu p_\nu p_\alpha \lfloor 1 \rfloor^V_{n_1+n_2}\right), \qquad (3.30)$$

where the subscripts of the $\lfloor X \rfloor^V$ indicate the value of n in the one-propagator integrals given in section 3.1. Also, the one-propagator integrals in the above expressions are functions of $\tilde{m}^2$ rather than m^2. We may then compute the integral over λ in $\lfloor X \rfloor^V$ and obtain a sum over modified Bessel functions. We are finally left with a single summation and an integral over x, to be performed numerically.

The method introduced in section 3.1 where the summations are performed in terms of theta functions is also applicable here, and yields a double integral over λ and x. In that case, the integral over x can be performed analytically. By setting

$$
\begin{aligned}
\tilde{m}^2 &= -p^2 \left(x - \frac{m_1^2 - m_2^2 + p^2}{2p^2} \right)^2 + m_2^2 + \frac{\left(m_1^2 - m_2^2 + p^2 \right)^2}{4p^2}, \\
z &= x - \frac{m_1^2 - m_2^2 + p^2}{2p^2},
\end{aligned}
\tag{3.31}
$$

the resulting integral with no additional powers of z is related to Dawson's integral or the error function (erf), depending on the sign of p^2. The other cases are related to the (complex-valued) incomplete Gamma function by the substitution $z^2 = u$. However, a straightforward numerical evaluation of the double integral converges sufficiently fast for practical purposes.

3.2.2 Moving frame

In a general "moving frame", p can have non-zero components in the dimensions of finite length. In this case, the sums with odd powers of components of l_r no longer vanish. In general, the finite-volume corrections can depend on all components of p, and no simple way of writing the result in terms of scalar functions of p^2 exists, as only a discrete subgroup of the three-dimensional rotation group remains as a symmetry in a finite cubic volume.

Nevertheless, the relevant expressions can be evaluated numerically, albeit with some additional complications. For the formulation in terms of modified Bessel functions, the summation is no longer exclusively dependent on l_r^2, and thus the reduction of the triple sums using eq. (3.8) is no longer possible. For the formulation in terms of theta functions, the summation over l_r can still be performed separately for each dimension, provided that the factors of θ_{30}^3 are replaced by the product $\theta_3(u_1, q) \, \theta_3(u_2, q) \theta_3(u_3, q)$, where $u_i \equiv y p_i L / (2\pi)$ and $q \equiv e^{-1/\lambda}$. When factors of r_μ appear in the integrands, derivatives w.r.t. u and q, as well as uncontracted factors of $l_{r\mu}$, also need to be accounted for.

3.3 Summary of one-loop results

Next, we discuss the relations between the various one-loop integrals and summarize the explicit expressions in a concise form. With the definition of eq. (3.1) in mind, we introduce the more conventional notation

$$
\begin{aligned}
\lfloor 1 \rfloor^V &= A^V, \\
\lfloor r_\mu \rfloor^V &= 0, \\
\lfloor r_\mu r_\nu \rfloor^V &= \delta_{\mu\nu} A_{22}^V + t_{\mu\nu} A_{23}^V, \\
\lfloor r_\mu r_\nu r_\alpha \rfloor^V &= 0,
\end{aligned}
\tag{3.32}
$$

where only the finite-volume correction has been retained. As discussed above, no simple rewriting in scalar components is possible for the momentum-dependent integrals, except in the cms frame with $p = (p, 0, 0, 0)$. In that frame, we define

$$
\begin{aligned}
\langle 1 \rangle^V \big|_{\text{cms}} &= B^V, \\
\langle r_\mu \rangle^V \big|_{\text{cms}} &= p_\mu B_1^V, \\
\langle r_\mu r_\nu \rangle^V \big|_{\text{cms}} &= p_\mu p_\nu B_{21}^V + \delta_{\mu\nu} B_{22}^V + B_{23}^V t_{\mu\nu}, \\
\langle r_\mu r_\nu r_\alpha \rangle^V \big|_{\text{cms}} &= p_\mu p_\nu p_\alpha B_{31}^V + (\delta_{\mu\nu} p_\alpha + \delta_{\mu\alpha} p_\nu + \delta_{\nu\alpha} p_\mu) B_{32}^V \\
&\quad + (t_{\mu\nu} p_\alpha + t_{\mu\alpha} p_\nu + t_{\nu\alpha} p_\mu) B_{33}^V,
\end{aligned}
\tag{3.33}
$$

which correspond to the usual definitions at infinite volume, except for the terms involving $t_{\mu\nu}$, which appear only in the finite-volume contribution.

The Passarino-Veltman construction [16] produces relations between the various integrals upon multiplication with p_μ or $\delta_{\mu\nu}$. Using

$$
2p \cdot r = (r^2 + m_1^2) - [(r - p)^2 + m_2^2] - m_1^2 + m_2^2,
\tag{3.34}
$$

a number of relations can be obtained. These are

$$
dA_{22}^V(n) + 3A_{23}^V(n) + m^2 A^V(n) = A^V(n-1),
\tag{3.35}
$$
$$
p^2 B_1^V(n_1, n_2) + \frac{1}{2}(m_1^2 - m_2^2 - p^2) B^V(n_1, n_2) = \frac{1}{2} B^V(n_1 - 1, n_2) - \frac{1}{2} B^V(n_1, n_2 - 1),
$$
$$
p^2 B_{21}^V(n_1, n_2) + d B_{22}^V(n_1, n_2) + 3 B_{23}^V(n_1, n_2) + m_1^2 B^V(n_1, n_2) = B^V(n_1 - 1, n_2),
$$
$$
p^2 B_{21}^V(n_1, n_2) + B_{22}^V(n_1, n_2) + \frac{1}{2}(m_1^2 - m_2^2 - p^2) B_1^V(n_1, n_2)
$$
$$
= \frac{1}{2} B_1^V(n_1 - 1, n_2) - \frac{1}{2} B_1^V(n_1, n_2 - 1),
$$

and

$$
p^2 B_{31}^V(n_1, n_2) + (d+2) B_{32}^V(n_1, n_2) + 3 B_{33}^V(n_1, n_2) + m_1^2 B_1^V(n_1, n_2) = B_1^V(n_1 - 1, n_2),
$$
$$
p^2 B_{31}^V(n_1, n_2) + 2 B_{32}^V(n_1, n_2) + \frac{1}{2}(m_1^2 - m_2^2 - p^2) B_{21}^V(n_1, n_2)
\tag{3.36}
$$
$$
= \frac{1}{2} B_{21}^V(n_1 - 1, n_2) - \frac{1}{2} B_{21}^V(n_1, n_2 - 1),
$$
$$
p^2 B_{32}^V(n_1, n_2) + \frac{1}{2}(m_1^2 - m_2^2 - p^2) B_{22}^V(n_1, n_2) = \frac{1}{2} B_{22}^V(n_1 - 1, n_2) - \frac{1}{2} B_{22}^V(n_1, n_2 - 1),
$$
$$
p^2 B_{33}^V(n_1, n_2) + \frac{1}{2}(m_1^2 - m_2^2 - p^2) B_{23}^V(n_1, n_2) = \frac{1}{2} B_{23}^V(n_1 - 1, n_2) - \frac{1}{2} B_{23}^V(n_1, n_2 - 1),
$$

where we note that the relations in eq. (3.36) are linearly dependent. Up to the order considered here, this leaves A^V, A_{23}^V, B^V and B_{23}^V as independent functions. We have checked the validity of the above relations numerically for $n_1, n_2 = 1, 2$.

At NNLO in χPT, all one-loop integrals should be expanded around $d = 4$ up to and including terms of $\mathcal{O}(\varepsilon)$. This is necessary, since products of two one-loop integrals appear

throughout the NNLO expressions, including the factorizable parts of the two-loop sunset integrals. We thus define

$$
\begin{aligned}
A^V &\equiv \bar{A}^V + \varepsilon \bar{A}^{V\varepsilon} + \mathcal{O}(\varepsilon^2), \\
B^V &\equiv \bar{B}^V + \varepsilon \bar{B}^{V\varepsilon} + \mathcal{O}(\varepsilon^2),
\end{aligned}
\tag{3.37}
$$

with similar expansions for all functions A_i^V and B_i^V in eqs. (3.32) and (3.33). The one-propagator integrals can then be written as

$$
\bar{A}^V = \frac{1}{16\pi^2 \Gamma(n)} \sum_{k>0} x(k)\hat{A}^V = \frac{1}{16\pi^2 \Gamma(n)} \left(\frac{L^2}{4}\right)^{n-2} \int_0^\infty d\lambda\, \lambda^{n-3} e^{-\lambda \frac{m^2 L^2}{4}} \tilde{A}^V,
\tag{3.38}
$$

using eqs. (3.9), (3.12), (3.17) and (3.19). The integrands can be expressed either in terms of modified Bessel functions or theta functions, and are in each case given by

$$
\begin{aligned}
\hat{A}^V &= \mathcal{K}_{n-2}\left(\frac{kL^2}{4}, m^2\right), & \tilde{A}^V &= \theta_{30}\left(e^{-1/\lambda}\right)^3 - 1, \\
\hat{A}_{22}^V &= \frac{1}{2}\,\mathcal{K}_{n-3}\left(\frac{kL^2}{4}, m^2\right), & \tilde{A}_{22}^V &= \frac{2}{\lambda L^2}\left[\theta_{30}\left(e^{-1/\lambda}\right)^3 - 1\right], \\
\hat{A}_{23}^V &= -\frac{1}{12}kL^2\,\mathcal{K}_{n-4}\left(\frac{kL^2}{4}, m^2\right), & \tilde{A}_{23}^V &= -\frac{4}{\lambda^2 L^2}\,\theta_{32}\left(e^{-1/\lambda}\right)\theta_{30}\left(e^{-1/\lambda}\right)^2.
\end{aligned}
\tag{3.39}
$$

The expansion in $\varepsilon = (4-d)/2$ can be performed using

$$
\begin{aligned}
(4\pi)^\varepsilon &= 1 + \varepsilon \log(4\pi) + \mathcal{O}(\varepsilon^2), \\
\mathcal{K}_{m+\varepsilon} &= \mathcal{K}_m + \varepsilon\tilde{\mathcal{K}}_m + \mathcal{O}(\varepsilon^2), \\
(4\pi\lambda L^2)^\varepsilon &= 1 + \varepsilon \log(4\pi\lambda L^2) + \mathcal{O}(\varepsilon^2),
\end{aligned}
\tag{3.40}
$$

where the functions $\tilde{\mathcal{K}}_m$ are related to the modified Bessel functions and are defined in appendix A. For all quantities in eq. (3.39), the above results lead to

$$
\begin{aligned}
\hat{A}^{V\varepsilon} &= \log(4\pi)\,\hat{A}^V + A^V(\mathcal{K}_m \to \tilde{\mathcal{K}}_m), \\
\tilde{A}^{V\varepsilon} &= [\log(4\pi) + \log(\lambda) + 2\log(L)]\,\tilde{A}^V,
\end{aligned}
\tag{3.41}
$$

where $\mathcal{K}_m \to \tilde{\mathcal{K}}_m$ indicates that the functions $\mathcal{K}_m$ should be replaced by the corresponding expressions for $\tilde{\mathcal{K}}_m$.

For the one-loop two-propagator integrals, we find similar results, given by

$$
\begin{aligned}
\bar{B}^V &= \frac{1}{16\pi^2 \Gamma(n_1)\Gamma(n_2)} \sum_{k>0} x(k) \int_0^1 dx\, x^{n_1-1} y^{n_2-1} \hat{B}^V \\
&= \frac{1}{16\pi^2 \Gamma(n_1)\Gamma(n_2)} \int_0^1 dx\, x^{n_1-1} y^{n_2-1} \left(\frac{L^2}{4}\right)^{n_1+n_2-2} \int_0^\infty d\lambda\, \lambda^{n_1+n_2-3} e^{-\lambda \frac{\tilde{m}^2 L^2}{4}} \tilde{B}^V,
\end{aligned}
\tag{3.42}
$$

with $\tilde{m}^2 = xm_1^2 + (1-x)m_2^2 + xyp^2$ and $y = 1 - x$, where $x(k)$ is defined in eq. (3.8). The explicit expressions for the integrands are

$$\hat{B}^V = \mathcal{K}_{n_1+n_2-2}\left(\frac{kL^2}{4}, \tilde{m}^2\right), \qquad \tilde{B}^V = \theta_{30}\left(e^{-1/\lambda}\right)^3 - 1,$$

$$\hat{B}_1^V = y\,\mathcal{K}_{n_1+n_2-2}\left(\frac{kL^2}{4}, \tilde{m}^2\right), \qquad \tilde{B}_1^V = y\left[\theta_{30}\left(e^{-1/\lambda}\right)^3 - 1\right],$$

$$\hat{B}_{21}^V = y^2\,\mathcal{K}_{n_1+n_2-2}\left(\frac{kL^2}{4}, \tilde{m}^2\right), \qquad \tilde{B}_{21}^V = y^2\left[\theta_{30}\left(e^{-1/\lambda}\right)^3 - 1\right],$$

$$\hat{B}_{22}^V = \frac{1}{2}\mathcal{K}_{n_1+n_2-3}\left(\frac{kL^2}{4}, \tilde{m}^2\right), \qquad \tilde{B}_{22}^V = \frac{2}{\lambda L^2}\left[\theta_{30}\left(e^{-1/\lambda}\right)^3 - 1\right],$$

$$\hat{B}_{23}^V = -\frac{1}{12}kL^2\mathcal{K}_{n_1+n_2-4}\left(\frac{kL^2}{4}, \tilde{m}^2\right), \quad \tilde{B}_{23}^V = -\frac{4}{\lambda^2 L^2}\theta_{32}\left(e^{-1/\lambda}\right)\theta_{30}\left(e^{-1/\lambda}\right)^2.$$

$$\hat{B}_{31}^V = y^3\mathcal{K}_{n_1+n_2-2}\left(\frac{kL^2}{4}, \tilde{m}^2\right), \qquad \tilde{B}_{31}^V = y^3\left[\theta_{30}\left(e^{-1/\lambda}\right)^3 - 1\right],$$

$$\hat{B}_{32}^V = \frac{y}{2}\mathcal{K}_{n_1+n_2-3}\left(\frac{kL^2}{4}, \tilde{m}^2\right), \qquad \tilde{B}_{32}^V = y\frac{2}{\lambda L^2}\left[\theta_{30}\left(e^{-1/\lambda}\right)^3 - 1\right],$$

$$\hat{B}_{33}^V - \frac{y}{12}kL^2\mathcal{K}_{n_1+n_2-4}\left(\frac{kL^2}{4}, \tilde{m}^2\right), \qquad \tilde{B}_{33}^V = -\frac{4y}{\lambda^2 L^2}\theta_{32}\left(e^{-1/\lambda}\right)\theta_{30}\left(e^{-1/\lambda}\right)^2, \quad (3.43)$$

where each case has again been given in terms of modified Bessel functions or theta functions. The functions $\bar{B}^{V\varepsilon}$ can be obtained from the above expressions using the equivalent of eq. (3.40), along with corresponding changes in eq. (3.41). However, the functions $\bar{A}^{V\varepsilon}$ and $\bar{B}^{V\varepsilon}$ are expected to cancel completely in a full calculation within the $\overline{MS}$ scheme. This cancellation has already been demonstrated at NNLO for the scalar condensate in ref. [10], and for m_π in two-flavour ChPT in ref. [11].

4 Two-loop sunset integrals at finite volume

First, we recall that some NNLO work at finite volume already exists. In ref. [10], the finite-volume corrections were calculated for the quark condensate, and in ref. [11] for m_π. The former only involved products of one-loop integrals, while the latter only required consideration of the sunset integrals with degenerate masses. In this section, we provide completely general expressions for the sunset integrals, for arbitrary, non-degenerate masses. At finite volume, we define the basic sunset integral as

$$\langle\langle X \rangle\rangle \equiv \int_V \frac{d^d r}{(2\pi)^d} \frac{d^d s}{(2\pi)^d} \frac{X}{(r^2 + m_1^2)^{n_1}(s^2 + m_2^2)^{n_2}((r+s-p)^2 + m_3^2)^{n_3}}, \qquad (4.1)$$

where the required operators X are $1, r_\mu, s_\mu, r_\mu r_\nu, r_\mu s_\nu$ and $s_\mu s_\nu$. In eq. (4.1), the n_i are always non-zero and positive. If one of the n_i is zero or negative, the integral becomes separable into a product of two one-loop integrals, which we have already dealt with in section 3.

Application of the Poisson summation formula for all momenta in a finite dimension yields

$$\langle\langle X\rangle\rangle = \sum_{l_r,l_s} \int \frac{d^d r}{(2\pi)^d} \frac{d^d s}{(2\pi)^d} \frac{X\, e^{il_r\cdot r} e^{il_s\cdot s}}{(r^2+m_1^2)^{n_1}(s^2+m_2^2)^{n_2}((r+s-p)^2+m_3^2)^{n_3}}, \tag{4.2}$$

where $\langle\langle X\rangle\rangle(1,2,3)$ will be used as a short-hand notation indicating which of the arguments (n_i,m_i^2) are associated with the first, second and third propagators in eq. (4.2), respectively. The vectors l_r, l_s are of the form $(0, k_1 L, k_2 L, k_3 L)$ with $k_i \in \mathbf{Z}$. Eq. (4.2) can then be decomposed according to

$$\langle\langle X\rangle\rangle \equiv \langle\langle X\rangle\rangle^{\infty} + \langle\langle X\rangle\rangle^{V}, \tag{4.3}$$

where $\langle\langle X\rangle\rangle^{\infty}$ denotes the infinite-volume result with $l_r = l_s = 0$. The sunset integrals at infinite volume have been evaluated in several different ways (see e.g. refs. [20–24]) and will not be considered further here. The second term in eq. (4.3) represents the finite-volume correction. The present approach to the finite-volume correction is along the lines of refs. [20, 21], combined with an extension of the methods for the one-loop integrals in section 3.

We further decompose $\langle\langle X\rangle\rangle^{V}$ into terms where one of the possible loop momenta is not quantized and a contribution where both are quantized, according to

$$\langle\langle X\rangle\rangle^{V} \equiv \langle\langle X\rangle\rangle_r + \langle\langle X\rangle\rangle_s + \langle\langle X\rangle\rangle_t + \langle\langle X\rangle\rangle_{rs}, \tag{4.4}$$

with

$$\langle\langle X\rangle\rangle_r = \sum_{l_r}' \int \frac{d^d r}{(2\pi)^d} \frac{d^d s}{(2\pi)^d} \frac{X\, e^{il_r\cdot r}}{(r^2+m_1^2)^{n_1}(s^2+m_2^2)^{n_2}((r+s-p)^2+m_3^2)^{n_3}},$$

$$\langle\langle X\rangle\rangle_s = \sum_{l_s}' \int \frac{d^d r}{(2\pi)^d} \frac{d^d s}{(2\pi)^d} \frac{X\, e^{il_s\cdot s}}{(r^2+m_1^2)^{n_1}(s^2+m_2^2)^{n_2}((r+s-p)^2+m_3^2)^{n_3}},$$

$$\langle\langle X\rangle\rangle_t = \sum_{l_t}' \int \frac{d^d r}{(2\pi)^d} \frac{d^d s}{(2\pi)^d} \frac{X\, e^{il_t\cdot(p-r-s)}}{(r^2+m_1^2)^{n_1}(s^2+m_2^2)^{n_2}((r+s-p)^2+m_3^2)^{n_3}},$$

$$\langle\langle X\rangle\rangle_{rs} = \sum_{l_r,l_s}'' \int \frac{d^d r}{(2\pi)^d} \frac{d^d s}{(2\pi)^d} \frac{X\, e^{il_r\cdot r} e^{il_s\cdot s}}{(r^2+m_1^2)^{n_1}(s^2+m_2^2)^{n_2}((r+s-p)^2+m_3^2)^{n_3}}, \tag{4.5}$$

where a "singly primed" sum indicates that the term with $l = 0$ has been excluded. For the "doubly primed" sums, all contributions with $l_r = 0$, $l_s = 0$ or $l_r = l_s$ have been removed, i.e. the retained terms satisfy $l_r \neq 0, l_s \neq 0$ and $l_r \neq l_s$. The sum of all the terms in eq. (4.5) reproduces the full finite volume correction in eq. (4.2). Here, it should be taken into account that p is also quantized in the finite dimensions, such that the spatial momentum components satisfy

$$p_i \equiv \frac{2\pi j_i}{L}, \qquad e^{il_r\cdot p} = e^{il_s\cdot p} = e^{il_t\cdot p} = 1. \tag{4.6}$$

We note that $\langle\langle X\rangle\rangle_{rs}$ is always finite, whereas $\langle\langle X\rangle\rangle_r$, $\langle\langle X\rangle\rangle_s$ and $\langle\langle X\rangle\rangle_t$ may contain a non-local divergence, depending on the operator X and the values of the n_i. If these integrals should be finite, they can be included in $\langle\langle X\rangle\rangle_{rs}$ by summation over all values of l_r and l_s (except of course $l_r = l_s = 0$).

4.1 Simplest sunset integral

We first restrict ourselves to the simplest case of $\langle\langle 1\rangle\rangle$ with $n_1 = n_2 = n_3 = 1$, which allows us to outline our procedure in a straightforward way. We will then proceed to give the expressions for the general case using the formalism established here.

From eqs. (4.1), (4.2) and (4.5), and keeping in mind eq. (4.6), we find that the sunset integrals exhibit a high degree of symmetry with respect to interchanges of r, s and $t = p - r - s$, together with l_r, l_s and l_t. Substituting $(r, s) \to (s, r)$ and $(r, t) \to (t, r)$, including the respective l_i, leads to the relations

$$\langle\langle 1\rangle\rangle(1, 2, 3) = \langle\langle 1\rangle\rangle(2, 1, 3) = \langle\langle 1\rangle\rangle(3, 2, 1),$$
$$\langle\langle 1\rangle\rangle^\infty(1, 2, 3) = \langle\langle 1\rangle\rangle^\infty(2, 1, 3) = \langle\langle 1\rangle\rangle^\infty(3, 2, 1),$$
$$\langle\langle 1\rangle\rangle^V(1, 2, 3) = \langle\langle 1\rangle\rangle^V(2, 1, 3) = \langle\langle 1\rangle\rangle^V(3, 2, 1),$$
$$\langle\langle 1\rangle\rangle_{rs}(1, 2, 3) = \langle\langle 1\rangle\rangle_{rs}(2, 1, 3) = \langle\langle 1\rangle\rangle_{rs}(3, 2, 1),$$
$$\langle\langle 1\rangle\rangle_r(1, 2, 3) = \langle\langle 1\rangle\rangle_r(1, 3, 2),$$
$$\langle\langle 1\rangle\rangle_r(1, 2, 3) = \langle\langle 1\rangle\rangle_s(2, 1, 3) = \langle\langle 1\rangle\rangle_t(3, 2, 1), \tag{4.7}$$

where we recall that the notation $(1, 2, 3)$ refers to the propagators, as exhibited in eq. (4.2). From the last relation in eq. (4.7), we find that the evaluation of $\langle\langle 1\rangle\rangle_r$ and $\langle\langle 1\rangle\rangle_{rs}$ suffices to obtain the full result.

4.1.1 Simplest sunset integral with one quantized loop momentum

First, we calculate $\langle\langle 1\rangle\rangle_r$. We begin by combining two of the propagators with a Feynman parameter x, giving

$$\langle\langle 1\rangle\rangle_r = \sideset{}{'}\sum_{l_r} \int \frac{d^d r}{(2\pi)^d} \frac{d^d s}{(2\pi)^d} \frac{e^{il_r \cdot r}}{(r^2 + m_1^2)(s^2 + m_2^2)((r + s - p)^2 + m_3^2)}$$
$$= \sideset{}{'}\sum_{l_r} \int \frac{d^d r}{(2\pi)^d} \frac{e^{il_r \cdot r}}{(r^2 + m_1^2)} \int_0^1 dx \int \frac{d^d \tilde{s}}{(2\pi)^d} \frac{1}{\left(\tilde{s}^2 + \overline{m}^2\right)^2}, \tag{4.8}$$

where we have shifted the integration variable according to $s_\mu \equiv \tilde{s}_\mu - x(r - p)_\mu$, and defined

$$\overline{m}^2 \equiv (1 - x)m_2^2 + x m_3^2 + x(1 - x)(r - p)^2. \tag{4.9}$$

The integration over $\tilde{s}$ may then be performed in terms of standard d-dimensional integrals in Euclidean space, given in appendix C.1. This gives

$$\langle\langle 1\rangle\rangle_r = \sideset{}{'}\sum_{l_r} \int \frac{d^d r}{(2\pi)^d} \frac{e^{il_r \cdot r}}{(r^2 + m_1^2)} \int_0^1 dx \frac{\Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \left(\overline{m}^2\right)^{\frac{d}{2} - 2}, \tag{4.10}$$

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where the expansion to $\mathcal{O}(\varepsilon)$ may be performed using

$$\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}(\overline{m}^2)^{\frac{d}{2}-2} = \frac{1}{16\pi^2}\left[\lambda_0 - 1 - \log(\overline{m}^2)\right] + \mathcal{O}(\varepsilon), \tag{4.11}$$

where $\lambda_0 \equiv 1/\varepsilon + \log(4\pi) + 1 - \gamma$. The term proportional to λ_0 involves the one-loop integral A^V, which has been treated in section 3. This also contains the nonlocal divergence, and contributes

$$\langle\langle 1\rangle\rangle_{r,A} = \frac{\lambda_0}{16\pi^2}\lfloor 1\rfloor^V(1, m_1^2) \tag{4.12}$$

to $\langle\langle 1\rangle\rangle_r$. For clarity, we have added the arguments $n_1 = 1$ and m_1^2 to the notation for the one-loop integral. The remaining terms in eq. (4.11) contribute

$$\langle\langle 1\rangle\rangle_{r,F} = -\frac{1}{16\pi^2}\sum_{l_r}'\int\frac{d^d r}{(2\pi)^d}\frac{e^{il_r\cdot r}}{(r^2+m_1^2)}\int_0^1 dx\left[1+\log(\overline{m}^2)\right], \tag{4.13}$$

where we can set $d = 4$ directly. In order to deal with the dependence of $\overline{m}^2$ or r, we perform a partial integration in x to obtain

$$\langle\langle 1\rangle\rangle_{r,F} = -\frac{1}{16\pi^2}\sum_{l_r}'\int\frac{d^4 r}{(2\pi)^4}\frac{e^{il_r\cdot r}}{(r^2+m_1^2)}\left[1+\log(m_3^2)\right.$$
$$\left. -\int_0^1 dx\, x\,\frac{m_3^2 - m_2^2 + (1-2x)(r-p)^2}{\overline{m}^2}\right]. \tag{4.14}$$

Here, the first two terms once more contain a one-loop integral, and we refer to this part as $\langle\langle 1\rangle\rangle_{r,G}$, with the remainder labeled $\langle\langle 1\rangle\rangle_{r,H}$. Further, we introduce the Gaussian parameters λ_1 and λ_4 according to eq. (A.1) for the denominators $(r^2 + m_1^2)$ and $\overline{m}^2$, respectively. This gives

$$\langle\langle 1\rangle\rangle_{r,F} \equiv \langle\langle 1\rangle\rangle_{r,G} + \langle\langle 1\rangle\rangle_{r,H},$$
$$\langle\langle 1\rangle\rangle_{r,G} = -\frac{1+\log(m_3^2)}{16\pi^2}\lfloor 1\rfloor^V(1, m_1^2),$$
$$\langle\langle 1\rangle\rangle_{r,H} = \frac{1}{16\pi^2}\sum_{l_r}'\int\frac{d^4 r}{(2\pi)^4}\int_0^\infty d\lambda_1 d\lambda_4\int_0^1 dx$$
$$\times x\left[m_3^2 - m_2^2 + (1-2x)(r-p)^2\right]e^{il_r\cdot r - \lambda_1(r^2+m_1^2)-\lambda_4\overline{m}^2}, \tag{4.15}$$

where we may complete the square in the exponential factor by substituting

$$r \equiv \frac{1}{\sqrt{\lambda_5}}\tilde{r} + \frac{il_r}{2\lambda_5} + \frac{x(1-x)\lambda_4}{\lambda_5}p,$$
$$\lambda_5 \equiv \lambda_1 + x(1-x)\lambda_4. \tag{4.16}$$

The $\tilde{r}$ integral can then be performed using eq. (C.3), which gives

$$
\begin{aligned}
\langle\langle 1\rangle\rangle_{r,H} = \frac{1}{(16\pi^2)^2} &\sum_{l_r}^{\prime} \int_0^\infty d\lambda_1 d\lambda_4 \int_0^1 dx\, \frac{x}{\lambda_5^2} \\
&\times \left[m_3^2 - m_2^2 + \frac{1-2x}{\lambda_5^2}\left(2\lambda_5 + \lambda_1^2 p^2 - \frac{l_r^2}{4} - i\lambda_1 l_r \cdot p\right)\right] \\
&\times e^{-\left(\lambda_1 m_1^2 + \lambda_4(1-x)m_2^2 + \lambda_4 x m_3^2 + \frac{\lambda_1\lambda_4 x(1-x)}{\lambda_5}p^2 + \frac{l_r^2}{4\lambda_5} - i\frac{\lambda_4 x(1-x)}{\lambda_5}l_r\cdot p\right)}.
\end{aligned} \tag{4.17}
$$

Here, a more symmetric form can be obtained by substituting $\lambda_2 \equiv (1-x)\lambda_4$ and $\lambda_3 \equiv x\lambda_4$ as integration variables, giving

$$
\langle\langle 1\rangle\rangle_{r,H} = \frac{1}{(16\pi^2)^2} \sum_{l_r}^{\prime} \int_0^\infty d\lambda_1 d\lambda_2 d\lambda_3\, \frac{\lambda_3}{\tilde{\lambda}^2}\left[m_3^2 - m_2^2 + \frac{\lambda_2 - \lambda_3}{\tilde{\lambda}}\left(2 + \frac{\lambda_3 + \lambda_2}{\tilde{\lambda}}\tilde{p}^2\right)\right]e^{-M^2}, \tag{4.18}
$$

with

$$
\begin{aligned}
M^2 &\equiv \lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2 + \frac{\lambda_1\lambda_2\lambda_3}{\tilde{\lambda}}p^2 + \frac{\lambda_2+\lambda_3}{\tilde{\lambda}}\frac{l_r^2}{4} - i\frac{\lambda_2\lambda_3}{\tilde{\lambda}}l_r\cdot p, \\
\tilde{\lambda} &\equiv \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \\
\tilde{p} &\equiv \frac{il_r}{2} - \lambda_1 p,
\end{aligned} \tag{4.19}
$$

which can be evaluated numerically with the methods discussed in section 4.1.3.

4.1.2 Simplest sunset integral with two quantized loop momenta

Second, we calculate $\langle\langle 1\rangle\rangle_{rs}$. We introduce Gaussian parameterizations for all three propagators using eq. (A.1) and set $d = 4$, giving

$$
\begin{aligned}
\langle\langle 1\rangle\rangle_{rs} = \sum_{l_r,l_s}^{\prime\prime} &\int_0^\infty d\lambda_1 d\lambda_2 d\lambda_3 \int \frac{d^4r}{(2\pi)^4}\frac{d^4s}{(2\pi)^4} \\
&\times e^{-\left(\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2 - il_r\cdot r - l_s\cdot s + \lambda_1 r^2 + \lambda_2 s^2 + \lambda_3(r+s-p)^2\right)},
\end{aligned} \tag{4.20}
$$

after which we perform the redefinition

$$
r \equiv \frac{1}{\sqrt{\lambda_1 + \lambda_3}}\tilde{r} - \frac{\lambda_3}{\lambda_1 + \lambda_3}(s - p) + \frac{i}{2(\lambda_1 + \lambda_3)}l_r, \tag{4.21}
$$

and shift s by

$$
s \equiv \frac{\sqrt{\lambda_1 + \lambda_3}}{\sqrt{\tilde{\lambda}}}\tilde{s} + \frac{\lambda_1\lambda_3}{\tilde{\lambda}}p + \frac{i(\lambda_1 + \lambda_3)}{2\tilde{\lambda}}l_s - \frac{i\lambda_3}{2\tilde{\lambda}}l_r, \tag{4.22}
$$

where we have again made use of $\tilde{\lambda} \equiv \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$. We note that an analogous transformation results by first redefining s and then shifting r. The result is

$$
\begin{aligned}
\langle\langle 1\rangle\rangle_{rs} &= \sum_{l_r,l_s}^{\prime\prime} \int_0^\infty d\lambda_1 d\lambda_2 d\lambda_3 \int \frac{d^4\tilde{r}}{(2\pi)^4}\frac{d^4\tilde{s}}{(2\pi)^4}\, \tilde{\lambda}^{-2} e^{-\tilde{r}^2 - \tilde{s}^2 - \tilde{M}^2} \\
&= \frac{1}{(16\pi^2)^2}\sum_{l_r,l_s}^{\prime\prime} \int_0^\infty d\lambda_1 d\lambda_2 d\lambda_3\, \tilde{\lambda}^{-2} e^{-\tilde{M}^2},
\end{aligned} \tag{4.23}
$$

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with

$$\tilde{M}^2 \equiv \lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2 + \frac{\lambda_1 \lambda_2 \lambda_3}{\tilde{\lambda}} p^2 + \frac{\lambda_2}{\tilde{\lambda}} \frac{l_r^2}{4} + \frac{\lambda_1}{\tilde{\lambda}} \frac{l_s^2}{4} + \frac{\lambda_3}{\tilde{\lambda}} \frac{(l_r - l_s)^2}{4}$$
$$- i \frac{\lambda_2 \lambda_3}{\tilde{\lambda}} l_r \cdot p - i \frac{\lambda_1 \lambda_3}{\tilde{\lambda}} l_s \cdot p. \qquad (4.24)$$

We note that the arguments of the exponential functions in eqs. (4.18) and (4.23) coincide when $l_s = 0$.

4.1.3 Numerical evaluation

Next, we discuss the numerical evaluation of eq. (4.23). For this purpose, it is convenient to switch to the variables x, y, z and λ,

$$\lambda_1 \equiv x\lambda, \quad \lambda_2 \equiv y\lambda, \quad \lambda_3 \equiv (1 - x - y)\lambda = z\lambda, \quad \tilde{\lambda} = \lambda^2 (xy + yz + zx) \equiv \lambda^2 \sigma, \quad (4.25)$$

where $\sigma \equiv xy + yz + zx$ and $x + y + z = 1$. We also introduce the quantities

$$\begin{aligned}
l_n &\equiv l_r - l_s, \\
S_{rs} &\equiv -\frac{yz}{\sigma} l_r \cdot p - \frac{xz}{\sigma} l_s \cdot p, \\
Y_{rs} &\equiv \frac{y}{4\sigma} l_r^2 + \frac{x}{4\sigma} l_s^2 + \frac{z}{4\sigma} l_n^2, \\
Z_{rs} &\equiv x m_1^2 + y m_2^2 + z m_3^2 + \frac{xyz}{\sigma} p^2,
\end{aligned} \qquad (4.26)$$

which brings eq. (4.23) into the form

$$\langle\langle 1 \rangle\rangle_{rs} = \frac{1}{(16\pi^2)^2} \sum_{l_r, l_s}'' \int_0^\infty d\lambda \int_0^1 dx \int_0^{1-x} dy \, \sigma^{-2} \lambda^{-2} e^{-\lambda Z_{rs} - \frac{Y_{rs}}{\lambda}} e^{iS_{rs}}. \qquad (4.27)$$

As for the one-loop integrals, we may either perform the summations in terms of theta functions, or the λ integration in terms of modified Bessel functions. In terms of the latter, the result is

$$\langle\langle 1 \rangle\rangle_{rs} = \frac{1}{(16\pi^2)^2} \sum_{l_r, l_s}'' \int_0^1 dx \int_0^{1-x} dy \, \sigma^{-2} \mathcal{K}_{-1} (Y_{rs}, Z_{rs}) e^{iS_{rs}}, \qquad (4.28)$$

where we note that in the cms frame where $S_{rs} = 0$, we may write

$$\sum_{l_r, l_s}'' f(l_r^2, l_s^2, l_n^2) = \sum_{k_r, k_s, k_n = 1}^\infty x(k_r, k_s, k_n) \times f(k_r L^2, k_s L^2, k_n L^2), \qquad (4.29)$$

similarly to eq. (3.8). Here, the factor $x(k_r, k_s, k_n)$ denotes the number of times a given triplet of squares appears when the components of l_r and l_s are varied over all positive and negative integer values. In terms of theta functions, we find in the cms frame

$$\langle\langle 1 \rangle\rangle_{rs} = \frac{1}{(16\pi^2)^2} \int_0^1 dx \int_0^{1-x} dy \int_0^\infty d\lambda \, \frac{e^{-\lambda Z_{rs}}}{(\sigma\lambda)^2} \left[\theta_0^{(2)} \left(\frac{yL^2}{4\sigma\lambda}, \frac{xL^2}{4\sigma\lambda}, \frac{zL^2}{4\sigma\lambda} \right)^3 \right.$$
$$\left. - \theta_{30} \left(e^{-\frac{(x+z)L^2}{4\sigma\lambda}} \right)^3 - \theta_{30} \left(e^{-\frac{(y+z)L^2}{4\sigma\lambda}} \right)^3 - \theta_{30} \left(e^{-\frac{(x+y)L^2}{4\sigma\lambda}} \right)^3 + 2 \right], \qquad (4.30)$$

where the contributions with l_r^2, l_s^2 or l_n^2 equal to zero have been subtracted. The Jacobi and Riemann theta functions are defined in appendix B, see also eq. (4.91) and the accompanying discussion.

The expression for $\langle\langle 1 \rangle\rangle_{r,H}$ in eq. (4.18) is clearly similar and can be treated along the same lines. In terms of modified Bessel functions, the terms with a single sum over l_r may be treated similarly to the one-loop integrals using eq. (3.8). Alternatively, the summation can be performed in terms of theta functions. The relevant expressions will be given when we summarize the full results for the sunset integrals.

4.2 Permutation properties

The finite-volume sunset integrals satisfy a number of relations which simplify the calculations, and provide useful checks on the numerics. These are the more general versions of eq. (4.7). When applied to the full sunset integrals $\langle\langle X \rangle\rangle$, the variable interchanges (s, r), (r, t) and (s, t), with $t = p - r - s$, yield the relations

$$
\begin{aligned}
\langle\langle 1 \rangle\rangle(1,2,3) &= \langle\langle 1 \rangle\rangle(2,1,3) = \langle\langle 1 \rangle\rangle(3,2,1), \\
\langle\langle r_\mu \rangle\rangle(1,2,3) &= \langle\langle r_\mu \rangle\rangle(1,3,2), \\
\langle\langle s_\mu \rangle\rangle(1,2,3) &= \langle\langle r_\mu \rangle\rangle(2,1,3), \\
\langle\langle r_\mu r_\nu \rangle\rangle(1,2,3) &= \langle\langle r_\mu r_\nu \rangle\rangle(1,3,2), \\
\langle\langle s_\mu s_\nu \rangle\rangle(1,2,3) &= \langle\langle r_\mu r_\nu \rangle\rangle(2,1,3), \\
\langle\langle r_\mu s_\nu \rangle\rangle(1,2,3) &= \langle\langle r_\mu s_\nu \rangle\rangle(2,1,3),
\end{aligned}
\tag{4.31}
$$

where the notation $(1, 2, 3)$ is explained in the context of eq. (4.2), and refers to the masses m_i^2 and powers n_i of the propagators in eq. (4.1).

Further, we may derive the relations

$$
p_\mu \langle\langle 1 \rangle\rangle(1,2,3) = \langle\langle r_\mu \rangle\rangle(1,2,3) + \langle\langle r_\mu \rangle\rangle(2,1,3) + \langle\langle r_\mu \rangle\rangle(3,1,2),
$$

$$
\begin{aligned}
\langle\langle r_\mu s_\nu + s_\mu r_\nu \rangle\rangle(1,2,3) = {} &\langle\langle r_\mu r_\nu \rangle\rangle(3,1,2) - \langle\langle r_\mu r_\nu \rangle\rangle(1,2,3) - \langle\langle r_\mu r_\nu \rangle\rangle(2,1,3) \\
&- p_\mu \langle\langle r_\nu \rangle\rangle(3,1,2) - p_\nu \langle\langle r_\mu \rangle\rangle(3,1,2) + p_\mu p_\nu \langle\langle 1 \rangle\rangle(1,2,3), \quad (4.32)
\end{aligned}
$$

where the latter one follows from the identity

$$
\begin{aligned}
r_\mu s_\nu + s_\mu r_\nu = {} &(r + s - p)_\mu (r + s - p)_\nu - r_\mu r_\nu - s_\mu s_\nu - p_\mu (-r - s + p)_\nu \\
&- (-r - s + p)_\mu p_\nu + p_\mu p_\nu,
\end{aligned}
\tag{4.33}
$$

from which it also follows that all parts of $\langle\langle r_\mu s_\nu \rangle\rangle$ that are symmetric in μ and ν can be rewritten in terms of other integrals. In particular, at infinite volume $\langle\langle r_\mu s_\nu \rangle\rangle$ can be expressed in terms of $\langle\langle r_\mu r_\nu \rangle\rangle$ using various permutations of the m_i^2 and n_i. This also holds for the case of $m_1 = m_2$ and $n_1 = n_2$. The relations (4.31) and (4.32) are also separately valid for $\langle\langle X \rangle\rangle^\infty$, $\langle\langle X \rangle\rangle^V$ and $\langle\langle X \rangle\rangle_{rs}$, but not for the other components of eq. (4.4).

From the above considerations, we can deduce what integrals should be calculated in order to obtain a complete description. As $\langle\langle X \rangle\rangle_r$, $\langle\langle X \rangle\rangle_s$ and $\langle\langle X \rangle\rangle_t$ are closely related,

we can obtain the required cases of $\langle\langle X \rangle\rangle_s$ using

$$\langle\langle 1 \rangle\rangle_s(1,2,3) = \langle\langle 1 \rangle\rangle_r(2,1,3; l_r \to l_s),$$
$$\langle\langle r_\mu \rangle\rangle_s(1,2,3) = \langle\langle s_\mu \rangle\rangle_r(2,1,3; l_r \to l_s),$$
$$\langle\langle r_\mu r_\nu \rangle\rangle_s(1,2,3) = \langle\langle s_\mu s_\nu \rangle\rangle_r(2,1,3; l_r \to l_s),$$
$$\langle\langle r_\mu s_\nu \rangle\rangle_s(1,2,3) = \langle\langle s_\mu r_\nu \rangle\rangle_r(2,1,3; l_r \to l_s), \tag{4.34}$$

and for the $\langle\langle X \rangle\rangle_t$ we find[2]

$$\langle\langle 1 \rangle\rangle_t(1,2,3) = \langle\langle 1 \rangle\rangle_r(3,2,1; l_r \to -l_t),$$
$$\langle\langle r_\mu \rangle\rangle_t(1,2,3) = \langle\langle -r_\mu - s_\mu + p_\mu \rangle\rangle_r(3,2,1; l_r \to -l_t),$$
$$\langle\langle r_\mu r_\nu \rangle\rangle_t(1,2,3) = \langle\langle (r+s-p)_\mu (r+s-p)_\nu \rangle\rangle_r(3,2,1; l_r \to -l_t),$$
$$\langle\langle r_\mu s_\nu \rangle\rangle_t(1,2,3) = \langle\langle -r_\mu s_\nu - s_\mu s_\nu + p_\mu s_\nu \rangle\rangle_r(3,2,1; l_r \to -l_t), \tag{4.35}$$

from which we conclude that a complete description entails the calculation of $\langle\langle X \rangle\rangle_{rs}$ for $X = 1, r_\mu, r_\mu r_\nu$ and $r_\mu s_\nu$, and of $\langle\langle X \rangle\rangle_r$ for $X = 1, r_\mu, s_\mu, r_\mu r_\nu, r_\mu s_\nu$ and $s_\mu s_\nu$. We also note that the $\langle\langle X \rangle\rangle_r$ are symmetric under the interchange $(m_2, n_2) \leftrightarrow (m_3, n_3)$ for $X = 1$, r_μ and $r_\mu r_\nu$.

For conciseness, we now introduce a set of functions to be used in the remainder of the text. In an arbitrary frame, we define

$$\langle\langle 1 \rangle\rangle^V \equiv H^V,$$
$$\langle\langle r_\mu \rangle\rangle^V \equiv H_1^V p_\mu + H_{3\mu}^V,$$
$$\langle\langle s_\mu \rangle\rangle^V \equiv H_2^V p_\mu + H_{4\mu}^V,$$
$$\langle\langle r_\mu r_\nu \rangle\rangle^V \equiv H_{21}^V p_\mu p_\nu + H_{22}^V \delta_{\mu\nu} + H_{27\mu\nu}^V,$$
$$\langle\langle r_\mu s_\nu \rangle\rangle^V \equiv H_{23}^V p_\mu p_\nu + H_{24}^V \delta_{\mu\nu} + H_{28\mu\nu}^V,$$
$$\langle\langle s_\mu s_\nu \rangle\rangle^V \equiv H_{25}^V p_\mu p_\nu + H_{26}^V \delta_{\mu\nu} + H_{29\mu\nu}^V, \tag{4.36}$$

where the $H_{3\mu}^V$, $H_{4\mu}^V$, $H_{27\mu\nu}^V$, $H_{26\mu\nu}^V$ and $H_{28\mu\nu}^V$ contain instances of the vectors l_r or l_s with uncontracted Lorentz indices. In the cms frame, such contributions with one Lorentz index vanish, and the bilinear ones become proportional to $t_{\mu\nu}$. In the cms frame, we therefore have a simplified set of functions

$$\langle\langle 1 \rangle\rangle^V \equiv H^V,$$
$$\langle\langle r_\mu \rangle\rangle^V \equiv H_1^V p_\mu,$$
$$\langle\langle s_\mu \rangle\rangle^V \equiv H_2^V p_\mu,$$
$$\langle\langle r_\mu r_\nu \rangle\rangle^V \equiv H_{21}^V p_\mu p_\nu + H_{22}^V \delta_{\mu\nu} + H_{27}^V t_{\mu\nu},$$
$$\langle\langle r_\mu s_\nu \rangle\rangle^V \equiv H_{23}^V p_\mu p_\nu + H_{24}^V \delta_{\mu\nu} + H_{28}^V t_{\mu\nu},$$
$$\langle\langle s_\mu s_\nu \rangle\rangle^V \equiv H_{25}^V p_\mu p_\nu + H_{26}^V \delta_{\mu\nu} + H_{29}^V t_{\mu\nu}. \tag{4.37}$$

---

[2]Here, we used the fact that the spatial components of p satisfy periodic boundary conditions, and hence $e^{i l_t \cdot p} = 1$.

Because of this structure, $\langle\langle r_\mu s_\nu \rangle\rangle$ is symmetric in μ, ν and can be obtained using eq. (4.32). Still, we include $\langle\langle r_\mu s_\nu \rangle\rangle$ as a useful check on our numerics, and because it appears in the expressions for the sunset integrals with one quantized loop momentum. Our numbering scheme for the sunset integrals has been chosen to be consistent with ref. [20]. We also refer to the components of the functions H_i by appending the indices (r, G), (r, H) etc., which were introduced in the detailed treatment of the simplest sunset integral.

4.3 Sunset integrals with one quantized loop momentum

Here, we follow along the lines of section 4.1.1 and account for all needed cases of $\langle\langle X \rangle\rangle_r$ with $X = 1, r_\mu, s_\mu, r_\mu r_\nu, r_\mu s_\nu$ and $s_\mu s_\nu$. Again, the first step is to combine the last two propagators with a Feynman parameter x and shift the integration variable by $s_\mu \equiv \tilde{s}_\mu - x(r-p)_\mu$. The integral over $\tilde{s}$ can then be performed using eq. (C.1). Using the notation $f(r_\alpha)$ for additional factors of r_μ, r_ν, this gives

$$\langle\langle f(r_\alpha) \rangle\rangle_r = \int \frac{d^d r}{(2\pi)^d} \frac{e^{il_r \cdot r} f(r_\alpha)}{(r^2 + m_1^2)^{n_1}} \int_0^1 dx \, \frac{\Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} (\overline{m}^2)^{\frac{d}{2}-2}, \tag{4.38}$$

$$\langle\langle f(r_\alpha) s_\mu \rangle\rangle_r = \int \frac{d^d r}{(2\pi)^d} \frac{e^{il_r \cdot r} f(r_\alpha)}{(r^2 + m_1^2)^{n_1}} \int_0^1 dx \, \frac{\Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} (\overline{m}^2)^{\frac{d}{2}-2}(-x)(r-p)_\mu, \tag{4.39}$$

$$\langle\langle s_\mu s_\nu \rangle\rangle_r = \int \frac{d^d r}{(2\pi)^d} \frac{e^{il_r \cdot r}}{(r^2 + m_1^2)^{n_1}} \int_0^1 dx \left[\frac{\Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} (\overline{m}^2)^{\frac{d}{2}-2} x^2 (r-p)_\mu (r-p)_\nu \right.$$
$$\left. + \frac{\Gamma\left(1 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} (\overline{m}^2)^{\frac{d}{2}-1} \frac{\delta_{\mu\nu}}{2} \right], \tag{4.40}$$

where the remaining integral over r is always finite because of the factor $e^{il_r \cdot r}$. It is then sufficient to expand the $\tilde{s}$ integral in ε, while keeping only the singular and $\mathcal{O}(1)$ terms as in eq. (4.11). We rewrite the singular terms using $\lambda_0 \equiv 1/\varepsilon + \ln(4\pi) + 1 - \gamma$, and define the components of the sunset integrals proportional to λ_0 with the subscript A as in eq. (4.12). In terms of the one-loop integrals defined in section 3, we find for the non-zero cases with $n_2, n_3 = 1, 2$ the expressions

$$\langle\langle 1 \rangle\rangle_{r,A}^{n_1 11} = \frac{\lambda_0}{16\pi^2} \lfloor 1 \rfloor^V (n_1, m_1^2),$$

$$\langle\langle s_\mu \rangle\rangle_{r,A}^{n_1 11} = \frac{\lambda_0}{16\pi^2} \frac{p_\mu}{2} \lfloor 1 \rfloor^V (n_1, m_1^2),$$

$$\langle\langle r_\mu r_\nu \rangle\rangle_{r,A}^{n_1 11} = \frac{\lambda_0}{16\pi^2} \lfloor r_\mu r_\nu \rfloor^V (n_1, m_1^2),$$

$$\langle\langle r_\mu s_\nu \rangle\rangle_{r,A}^{n_1 11} = \frac{\lambda_0}{16\pi^2} \frac{-1}{2} \lfloor r_\mu r_\nu \rfloor^V (n_1, m_1^2),$$

$$\langle\langle s_\mu s_\nu \rangle\rangle_{r,A}^{n_1 11} = \frac{\lambda_0}{16\pi^2} \left\{ \left[\delta_{\mu\nu} \left(-\frac{m_2^2}{4} - \frac{m_3^2}{4} - \frac{p^2}{12} \right) + \frac{p_\mu p_\nu}{3} \right] \lfloor 1 \rfloor^V (n_1, m_1^2) \right.$$
$$\left. + \left(\frac{1}{3} \delta_{\mu\alpha} \delta_{\nu\beta} - \frac{1}{12} \delta_{\mu\nu} \delta_{\alpha\beta} \right) \lfloor r_\alpha r_\beta \rfloor^V (n_1, m_1^2) \right\},$$

$$\langle\langle s_\mu s_\nu \rangle\rangle_{r,A}^{n_1 21} = \langle\langle s_\mu s_\nu \rangle\rangle_{r,A}^{n_1 12} = \frac{\lambda_0}{16\pi^2} \frac{\delta_{\mu\nu}}{4} \lfloor 1 \rfloor^V (n_1, m_1^2), \tag{4.41}$$

where the superscripts denote the n_i in the sunset integrals. Also, the one-loop integrals now show explicitly the m_i^2 and n_i of the denominator they involve. As the above integrals contain a non-local divergence, they should always cancel in physical results.

We now proceed to treat the terms containing $\log(\overline{m}^2)$. As before, we first perform a partial integration in x, giving

$$\int_0^1 dx\, x^n \log(\overline{m}^2) = \frac{1}{n+1} \log(m_3^2)$$
$$- \frac{1}{n+1} \int_0^1 dx\, x^{n+1} \Big[m_3^2 - m_2^2 + (1-2x)(r-p)^2 \Big] \frac{1}{\overline{m}^2}, \qquad (4.42)$$

after which we denote the terms with negative powers of $\overline{m}^2$ as $\langle\langle X \rangle\rangle_{r,H}$, and the others as $\langle\langle X \rangle\rangle_{r,G}$, as defined in eq. (4.15) for the case of the simplest sunset integral. We note that the $\langle\langle X \rangle\rangle_{r,G}$ can again be expressed in terms of one-loop integrals. For $n_2, n_3 = 1, 2$, the non-zero cases are

$$\langle\langle 1 \rangle\rangle_{r,G}^{n_1 11} = \frac{1}{16\pi^2} \left(-1 - \log(m_3^2) \right) \lfloor 1 \rfloor^V(n_1, m_1^2),$$

$$\langle\langle s_\mu \rangle\rangle_{r,G}^{n_1 11} = \frac{1}{16\pi^2} \frac{p_\mu}{2} \left(-1 - \log(m_3^2) \right) \lfloor 1 \rfloor^V(n_1, m_1^2),$$

$$\langle\langle r_\mu r_\nu \rangle\rangle_{r,G}^{n_1 11} = \frac{1}{16\pi^2} \left(-1 - \log(m_3^2) \right) \lfloor r_\mu r_\nu \rfloor^V(n_1, m_1^2),$$

$$\langle\langle r_\mu s_\nu \rangle\rangle_{r,G}^{n_1 11} = \frac{1}{16\pi^2} \frac{1}{2} \left(1 + \log(m_3^2) \right) \lfloor r_\mu r_\nu \rfloor^V(n_1, m_1^2),$$

$$\langle\langle s_\mu s_\nu \rangle\rangle_{r,G}^{n_1 11} = \frac{1}{16\pi^2} \left\{ \left[\delta_{\mu\nu} \log(m_3^2) \left(\frac{m_2^2}{4} + \frac{m_3^2}{4} + \frac{p^2}{12} \right) - \frac{p_\mu p_\nu}{3} \left(1 + \log(m_3^2) \right) \right] \lfloor 1 \rfloor^V(n_1, m_1^2) \right.$$
$$\left. + \left[\frac{-1}{3} \delta_{\mu\alpha} \delta_{\nu\beta} \left(1 + \log(m_3^2) \right) + \frac{1}{12} \delta_{\mu\nu} \delta_{\alpha\beta} \log(m_3^2) \right] \lfloor r_\alpha r_\beta \rfloor^V(n_1, m_1^2) \right\},$$

$$\langle\langle s_\mu s_\nu \rangle\rangle_{r,G}^{n_1 21} = \langle\langle s_\mu s_\nu \rangle\rangle_{r,G}^{n_1 12} = \frac{1}{16\pi^2} \frac{-\delta_{\mu\nu}}{4} \left(1 + \log(m_3^2) \right) \lfloor 1 \rfloor^V(n_1, m_1^2). \qquad (4.43)$$

We note that the decomposition of the parts of the sunset integrals which do not depend on λ_0 into $\langle\langle X \rangle\rangle_{r,G}$ and $\langle\langle X \rangle\rangle_{r,H}$ is clearly not unique, as it depends on the choice of Feynman parameterization. For example, had we chosen $y = 1 - x$ instead of x as the Feynman parameter, we would have obtained terms containing $\log(m_2^2)$ in the $\langle\langle X \rangle\rangle_{r,G}$. Also, the decomposition does not commute with derivatives w.r.t. masses, note e.g. that $\langle\langle 1 \rangle\rangle_{r,G}^{n_1 12} = 0 \neq -(\partial/\partial m_3^2) \langle\langle 1 \rangle\rangle_{r,G}^{n_1 11}$.

The remaining part $\langle\langle X \rangle\rangle_{r,H}$ is algebraically the most complicated, but again follows exactly the procedure for the simplest sunset integral. First, we introduce Gaussian parameterizations for the negative powers of $\overline{m}^2$ and $(r^2 + m_1^2)$ using eq. (A.1) with parameters λ_4 and λ_1, respectively. While the expressions corresponding to $\langle\langle 1 \rangle\rangle_{r,H}$ in eq. (4.15) are relatively lengthy, they all share the same basic structure. In particular, they all contain the same exponential factor, for which we may complete the square using the substitutions of eq. (4.16). The resulting integrals can then be performed by means of eq. (C.3). Finally, we define $\lambda_2 \equiv (1-x)\lambda_4$ and $\lambda_3 \equiv x\lambda_4$ and perform the substitutions of eq. (4.25) to obtain an integral in terms of x, y, z and λ.

Before we give explicit expressions for $\langle\langle X \rangle\rangle_{r,H}$, we briefly discuss the methods used to obtain them. Due to the complexity of the required analytical manipulations, we have found it convenient to use FORM [25] according to the procedure outlined above. Alternatively, as described in ref. [12], a number of tricks can be used to considerably simplify the task. For example, powers of r_μ can be introduced into the numerators of the sunset integrals by taking derivatives w.r.t. l_r, giving

$$\langle\langle r_\mu \rangle\rangle_r = -i\frac{\partial}{\partial l_{r\mu}}\langle\langle 1 \rangle\rangle_r. \tag{4.44}$$

It is also noteworthy that integrals such as $\langle\langle s_\mu \rangle\rangle_r$ are very similar to the case of $\langle\langle 1 \rangle\rangle_r$, differing only in an additional factor of $x(r-p)_\mu$. This leads to relations such as

$$\langle\langle s_\mu \rangle\rangle = \langle\langle x r_\mu \rangle\rangle - p_\mu\langle\langle x 1 \rangle\rangle, \tag{4.45}$$

where the factor of x is understood to be included in the respective integrals. Due to the length and complexity of the resulting expressions for $\langle\langle X \rangle\rangle_{r,H}$, we make use of the auxiliary quantities

$$
\begin{aligned}
\delta &\equiv \frac{y-z}{\sigma}, & A &\equiv m_3^2 - m_2^2 + \delta\rho x^2 p^2, \\
\rho &\equiv \frac{y+z}{\sigma}, & B &\equiv ix\delta\rho\, l_r \cdot p, \\
\sigma &\equiv xy + yz + zx, & C &\equiv \frac{\delta\rho}{4}l_r^2, \\
\tau &\equiv \frac{yz}{\sigma}, & D &\equiv A - \frac{B}{\lambda} - \frac{C}{\lambda^2}, \tag{4.46}
\end{aligned}
$$

and

$$Y \equiv \frac{\rho}{4}l_r^2, \qquad Z \equiv xm_1^2 + ym_2^2 + zm_3^2 + \frac{xyz}{\sigma}p^2, \tag{4.47}$$

and we also introduce the notation

$$
\begin{aligned}
\langle\langle X \rangle\rangle_{r,H}^{n_1 n_2 n_3} &= \frac{1}{\Gamma(n_1)(16\pi^2)^2}\sum_{l_r}{}'\int_0^1 dx \int_0^{1-x} dy \int_0^\infty d\lambda \\
&\quad \times \frac{(x\lambda)^{n_1-1}}{\lambda\sigma^2}[[X]]_{r,H}^{n_1 n_2 n_3}\, e^{-\lambda Y - \frac{Z}{\lambda} + \frac{iyz}{\sigma}l_r \cdot p}. \tag{4.48}
\end{aligned}
$$

With these abbreviations, we obtain

$$
\begin{aligned}
[[1]]_{r,H}^{n_1 11} &= z\left(D + \frac{2\delta}{\lambda}\right), \\
[[1]]_{r,H}^{n_1 21} &= y, \\
[[1]]_{r,H}^{n_1 12} &= z, \\
[[1]]_{r,H}^{n_1 22} &= yz\lambda, \tag{4.49}
\end{aligned}
$$

for the simplest sunset integral, and

$$[[r_\mu]]^{n_1 11}_{r,H} = [[1]]^{n_1 11}_{r,H} \left(\frac{i\rho l_{r\mu}}{2\lambda} + \tau p_\mu \right) + \frac{z\rho\delta}{\lambda} \left(\frac{i l_{r\mu}}{2\lambda} - x p_\mu \right),$$

$$[[r_\mu]]^{n_1 21}_{r,H} = y \left(\frac{i\rho l_{r\mu}}{2\lambda} + \tau p_\mu \right),$$

$$[[r_\mu]]^{n_1 12}_{r,H} = z \left(\frac{i\rho l_{r\mu}}{2\lambda} + \tau p_\mu \right),$$

$$[[r_\mu]]^{n_1 22}_{r,H} = yz\lambda \left(\frac{i\rho l_{r\mu}}{2\lambda} + \tau p_\mu \right), \tag{4.50}$$

$$[[s_\mu]]^{n_1 11}_{r,H} = \frac{-z^2}{2\sigma} \left(D + \frac{3\delta}{\lambda} \right) \left(\frac{i l_{r\mu}}{2\lambda} - x p_\mu \right),$$

$$[[s_\mu]]^{n_1 21}_{r,H} = \frac{-zy}{\sigma} \left(\frac{i l_{r\mu}}{2\lambda} - x p_\mu \right),$$

$$[[s_\mu]]^{n_1 12}_{r,H} = \frac{-z^2}{\sigma} \left(\frac{i l_{r\mu}}{2\lambda} - x p_\mu \right),$$

$$[[s_\mu]]^{n_1 22}_{r,H} = \frac{-yz^2\lambda}{\sigma} \left(\frac{i l_{r\mu}}{2\lambda} - x p_\mu \right), \tag{4.51}$$

for $X = r_\mu, s_\mu$. With $\{a,b\}_{\mu\nu} = a_\mu b_\nu + a_\nu b_\mu$, we find for the bilinear operators

$$[[r_\mu r_\nu]]^{n_1 11}_{r,H} = z \left\{ \frac{\rho}{2\lambda} \left(D + \frac{3\delta}{\lambda} \right) \delta_{\mu\nu} + \tau \left[\tau D + \frac{2\delta}{\lambda} (\tau - \rho x) \right] p_\mu p_\nu \right.$$
$$\left. + \frac{i\rho}{2\lambda} \left[\tau D + \frac{\delta}{\lambda} (3\tau - \rho x) \right] \{p, l_r\}_{\mu\nu} - \frac{\rho^2}{4\lambda^2} \left(D + \frac{4\delta}{\lambda} \right) l_{r\mu} l_{r\nu} \right\},$$

$$[[r_\mu r_\nu]]^{n_1 21}_{r,H} = y \left[\frac{\rho}{2\lambda} \delta_{\mu\nu} + \tau^2 p_\mu p_\nu + \frac{i\rho\tau}{2\lambda} \{p, l_r\}_{\mu\nu} - \frac{\rho^2}{4\lambda^2} l_{r\mu} l_{r\nu} \right],$$

$$[[r_\mu r_\nu]]^{n_1 12}_{r,H} = z \left[\frac{\rho}{2\lambda} \delta_{\mu\nu} + \tau^2 p_\mu p_\nu + \frac{i\rho\tau}{2\lambda} \{p, l_r\}_{\mu\nu} - \frac{\rho^2}{4\lambda^2} l_{r\mu} l_{r\nu} \right],$$

$$[[r_\mu r_\nu]]^{n_1 21}_{r,H} = yz\lambda \left[\frac{\rho}{2\lambda} \delta_{\mu\nu} + \tau^2 p_\mu p_\nu + \frac{i\rho\tau}{2\lambda} \{p, l_r\}_{\mu\nu} - \frac{\rho^2}{4\lambda^2} l_{r\mu} l_{r\nu} \right], \tag{4.52}$$

$$[[r_\mu s_\nu]]^{n_1 11}_{r,H} = \frac{z^2}{2\sigma} \left\{ \frac{-1}{2\lambda} \left(D + \frac{3\delta}{\lambda} \right) \delta_{\mu\nu} + \left[\tau D + \frac{\delta}{\lambda} (3\tau - \rho x) \right] \left(x p_\mu p_\nu - \frac{i}{2\lambda} p_\mu l_{r\nu} \right) \right.$$
$$\left. + \frac{\rho}{4\lambda^2} \left(D + \frac{4\delta}{\lambda} \right) (l_{r\mu} l_{r\nu} + 2ix\lambda l_{r\mu} p_\nu) \right\},$$

$$[[r_\mu s_\nu]]^{n_1 21}_{r,H} = \frac{yz}{\sigma} \left[\frac{-1}{2\lambda} \delta_{\mu\nu} + \tau x p_\mu p_\nu - \frac{i\tau}{2\lambda} p_\mu l_{r\nu} + \frac{i\rho x}{2} l_{r\mu} p_\nu + \frac{\rho}{4\lambda^2} l_{r\mu} l_{r\nu} \right],$$

$$[[r_\mu s_\nu]]^{n_1 12}_{r,H} = \frac{z^2}{\sigma} \left[\frac{-1}{2\lambda} \delta_{\mu\nu} + \tau x p_\mu p_\nu - \frac{i\tau}{2\lambda} p_\mu l_{r\nu} + \frac{i\rho x}{2} l_{r\mu} p_\nu + \frac{\rho}{4\lambda^2} l_{r\mu} l_{r\nu} \right],$$

$$[[r_\mu s_\nu]]^{n_1 22}_{r,H} = \frac{yz^2\lambda}{\sigma} \left[\frac{-1}{2\lambda} \delta_{\mu\nu} + \tau x p_\mu p_\nu - \frac{i\tau}{2\lambda} p_\mu l_{r\nu} + \frac{i\rho x}{2} l_{r\mu} p_\nu + \frac{\rho}{4\lambda^2} l_{r\mu} l_{r\nu} \right], \tag{4.53}$$

and

$$[[s_\mu s_\nu]]_{r,H}^{n_1 11} = \frac{z^3}{3\sigma^2}\left(D+\frac{4\delta}{\lambda}\right)\left(x^2 p_\mu p_\nu - \frac{ix}{2\lambda}\{p,l_r\}_{\mu\nu} - \frac{1}{4\lambda^2}l_{r\mu}l_{r\nu}\right)$$
$$+ \delta_{\mu\nu}\left\{-\frac{m_2^2 z}{2}\left(D+\frac{2\delta}{\lambda}\right) - \frac{Az^2}{4\rho\sigma}\left(D+\frac{2\delta}{\lambda}\right) - \frac{\tau z}{2\rho\lambda\sigma}\left(D+\frac{3\delta}{\lambda}\right)\right.$$
$$\left. + \frac{z^2}{\sigma^2}\left(D+\frac{4\delta}{\lambda}\right)\left[\frac{z+3y}{12}\left(\frac{ixp\cdot l_r}{\lambda}+\frac{l_r^2}{4\lambda^2}\right) - \frac{zx^2 p^2}{3}\right] - \frac{x^2 z^2 \delta^2 p^2}{2\lambda\sigma^2}\right\},$$

$$[[s_\mu s_\nu]]_{r,H}^{n_1 21} = \frac{yz^2}{\sigma^2}\left(x^2 p_\mu p_\nu - \frac{ix}{2\lambda}\{p,l_r\}_{\mu\nu} - \frac{1}{4\lambda^2}l_{r\mu}l_{r\nu}\right)$$
$$+ \delta_{\mu\nu}\left[\frac{1}{4}\left(\frac{\tau}{\rho}+z\right)D + \frac{z\delta}{\lambda} + \frac{z^3}{\rho\sigma^2\lambda}\right],$$

$$[[s_\mu s_\nu]]_{r,H}^{n_1 12} = \frac{z^3}{\sigma^2}\left(x^2 p_\mu p_\nu - \frac{ix}{2\lambda}\{p,l_r\}_{\mu\nu} - \frac{1}{4\lambda^2}l_{r\mu}l_{r\nu}\right)$$
$$+ \delta_{\mu\nu}\left[\frac{1}{4}\left(-\frac{\tau}{\rho}+z\right)D + \frac{z^2}{\sigma\lambda} - \frac{z^3}{2\rho\sigma^2\lambda}\right],$$

$$[[s_\mu s_\nu]]_{r,H}^{n_1 22} = \frac{yz^3\lambda}{\sigma^2}\left(x^2 p_\mu p_\nu - \frac{ix}{2\lambda}\{p,l_r\}_{\mu\nu} - \frac{1}{4\lambda^2}l_{r\mu}l_{r\nu}\right) + \delta_{\mu\nu}\left[\frac{\tau}{2\rho}(1-\tau)+\frac{z\tau}{2}\right]. \quad (4.54)$$

Given these expressions for $\langle\langle X\rangle\rangle_{r,H}$, we may proceed as for the one-loop integrals and choose between performing the summations in terms of theta functions, or evaluating the λ integral in terms of modified Bessel functions. The results quoted in eqs. (4.49)–(4.54) make no assumptions on the momentum p. Below, we restrict ourselves to the cms frame where $p\cdot l_r = 0$ or $p = (p,0,0,0)$. This case is the most commonly encountered, and the expressions for a moving frame can be obtained along similar lines.

4.3.1 Center-of-mass frame: Bessel functions

Here, we have performed the integration over λ in terms of the functions $\mathcal{K}_\nu(Y,Z)$ defined in appendix (A.2). We note that the summation only depends on l_r^2, such that eq. (3.8) is applicable. We have suppressed the arguments (Y,Z) in order to keep the expressions short and concise. The expressions always contain the abbreviated part

$$\oiint = \frac{1}{\Gamma(n_1)(16\pi^2)^2}\sum_{l_r}{}'\int_0^1 dx\int_0^{1-x}dy\,\frac{x^{n_1-1}}{\sigma^2}, \quad (4.55)$$

and numerical results for selected examples are given in section 5. For the simplest sunset integrals, we find

$$H^{r,H;n_1 11} = \oiint z\left(A\mathcal{K}_{n_1-1} + 2\delta\mathcal{K}_{n_1-2} - C\mathcal{K}_{n_1-3}\right),$$
$$H^{r,H;n_1 21} = \oiint y\,\mathcal{K}_{n_1-1},$$
$$H^{r,H;n_1 12} = \oiint z\,\mathcal{K}_{n_1-1},$$
$$H^{r,H;n_1 22} = \oiint zy\,\mathcal{K}_{n_1}, \quad (4.56)$$

and for $X = r_\mu, s_\mu$ we find

$$H_1^{r,H;n_1 11} = \oint z \left(\tau A \mathcal{K}_{n_1-1} + (2\tau - \rho x)\delta \mathcal{K}_{n_1-2} - \tau C \mathcal{K}_{n_1-3} \right),$$

$$H_1^{r,H;n_1 21} = \oint y\tau \, \mathcal{K}_{n_1-1},$$

$$H_1^{r,H;n_1 12} = \oint z\tau \, \mathcal{K}_{n_1-1},$$

$$H_1^{r,H;n_1 22} = \oint zy\tau \, \mathcal{K}_{n_1}, \tag{4.57}$$

$$H_2^{r,H;n_1 11} = \oint \frac{xz^2}{2\sigma} \left(A \mathcal{K}_{n_1-1} + 3\delta \mathcal{K}_{n_1-2} - C \mathcal{K}_{n_1-3} \right),$$

$$H_2^{r,H;n_1 21} = \oint \frac{xyz}{\sigma} \, \mathcal{K}_{n_1-1},$$

$$H_2^{r,H;n_1 12} = \oint \frac{xz^2}{\sigma} \, \mathcal{K}_{n_1-1},$$

$$H_2^{r,H;n_1 22} = \oint \frac{xyz^2}{\sigma} \, \mathcal{K}_{n_1}, \tag{4.58}$$

respectively. For $X = r_\mu r_\nu$, we have

$$H_{21}^{r,H;n_1 11} = \oint z\tau \left(\tau A \mathcal{K}_{n_1-1} + 2(\tau - \rho x)\delta \mathcal{K}_{n_1-2} - \tau C \mathcal{K}_{n_1-3} \right),$$

$$H_{21}^{r,H;n_1 21} = \oint y\tau^2 \, \mathcal{K}_{n_1-1},$$

$$H_{21}^{r,H;n_1 12} = \oint z\tau^2 \, \mathcal{K}_{n_1-1},$$

$$H_{21}^{r,H;n_1 22} = \oint yz\tau^2 \, \mathcal{K}_{n_1}, \tag{4.59}$$

$$H_{22}^{r,H;n_1 11} = \oint \frac{z\rho}{2} \left(A \mathcal{K}_{n_1-2} + 3\delta \mathcal{K}_{n_1-3} - C \mathcal{K}_{n_1-4} \right),$$

$$H_{22}^{r,H;n_1 21} = \oint \frac{y\rho}{2} \, \mathcal{K}_{n_1-2},$$

$$H_{22}^{r,H;n_1 12} = \oint \frac{z\rho}{2} \, \mathcal{K}_{n_1-2},$$

$$H_{22}^{r,H;n_1 22} = \oint \frac{yz\rho}{2} \, \mathcal{K}_{n_1-1}, \tag{4.60}$$

$$H_{27}^{r,H;n_1 11} = \oint \frac{-z\rho^2 l_r^2}{12} \left(A \mathcal{K}_{n_1-3} + 4\delta \mathcal{K}_{n_1-4} - C \mathcal{K}_{n_1-5} \right),$$

$$H_{27}^{r,H;n_1 21} = \oint \frac{-y\rho^2 l_r^2}{12} \, \mathcal{K}_{n_1-3},$$

$$H_{27}^{r,H;n_1 12} = \oint \frac{-z\rho^2 l_r^2}{12} \, \mathcal{K}_{n_1-3},$$

$$H_{27}^{r,H;n_1 22} = \oint \frac{-yz\rho^2 l_r^2}{12} \, \mathcal{K}_{n_1-2}, \tag{4.61}$$

for $X = r_\mu s_\nu$, we find

$$H_{23}^{r,H;n_1 11} = \oiint \frac{xz^2}{2\sigma} \left(\tau A \mathcal{K}_{n_1-1} + (3\tau - \rho x)\delta \mathcal{K}_{n_1-2} - \tau C \mathcal{K}_{n_1-3} \right),$$

$$H_{23}^{r,H;n_1 21} = \oiint \frac{xyz\tau}{\sigma} \mathcal{K}_{n_1-1},$$

$$H_{23}^{r,H;n_1 12} = \oiint \frac{xz^2\tau}{\sigma} \mathcal{K}_{n_1-1},$$

$$H_{23}^{r,H;n_1 22} = \oiint \frac{xyz^2\tau}{\sigma} \mathcal{K}_{n_1}, \tag{4.62}$$

$$H_{24}^{r,H;n_1 11} = \oiint \frac{-z^2}{4\sigma} \left(A \mathcal{K}_{n_1-2} + 3\delta \mathcal{K}_{n_1-3} - C \mathcal{K}_{n_1-4} \right),$$

$$H_{24}^{r,H;n_1 21} = \oiint \frac{-yz}{2\sigma} \mathcal{K}_{n_1-2},$$

$$H_{24}^{r,H;n_1 12} = \oiint \frac{-z^2}{2\sigma} \mathcal{K}_{n_1-2},$$

$$H_{24}^{r,H;n_1 22} = \oiint \frac{-yz^2}{2\sigma} \mathcal{K}_{n_1-1}, \tag{4.63}$$

$$H_{28}^{r,H;n_1 11} = \oiint \frac{z^2 \rho l_r^2}{24\sigma} \left(A \mathcal{K}_{n_1-3} + 4\delta \mathcal{K}_{n_1-4} - C \mathcal{K}_{n_1-5} \right),$$

$$H_{28}^{r,H;n_1 21} = \oiint \frac{yz\rho l_r^2}{12\sigma} \mathcal{K}_{n_1-3},$$

$$H_{28}^{r,H;n_1 12} = \oiint \frac{z^2 \rho l_r^2}{12\sigma} \mathcal{K}_{n_1-3},$$

$$H_{28}^{r,H;n_1 22} = \oiint \frac{yz^2 \rho l_r^2}{12\sigma} \mathcal{K}_{n_1-2}, \tag{4.64}$$

and for $X = s_\mu s_\nu$, we have

$$H_{25}^{r,H;n_1 11} = \oiint \frac{x^2 z^3}{3\sigma^2} \left(A \mathcal{K}_{n_1-1} + 4\delta \mathcal{K}_{n_1-2} - C \mathcal{K}_{n_1-3} \right),$$

$$H_{25}^{r,H;n_1 21} = \oiint \frac{x^2 yz^2}{\sigma^2} \mathcal{K}_{n_1-1},$$

$$H_{25}^{r,H;n_1 12} = \oiint \frac{x^2 z^3}{\sigma^2} \mathcal{K}_{n_1-1},$$

$$H_{25}^{r,H;n_1 22} = \oiint \frac{x^2 yz^3}{\sigma^2} \mathcal{K}_{n_1}, \tag{4.65}$$

$$\begin{aligned}
H_{26}^{r,H;n_1 11} = \oiint \Bigg\{ & \frac{-z}{12} A \left(6m_2^2 + 3\frac{z}{\rho\sigma} A + \frac{4x^2 z^2 p^2}{\sigma^2} \right) \mathcal{K}_{n_1-1} \\
& + \left[-z\delta m_2^2 + \frac{z^2}{6\rho\sigma^2}(m_3^2 - m_2^2)(5z + 3y) - \frac{z^2 A}{6\rho\sigma^2}(2z + 9y) \right] \mathcal{K}_{n_1-2} \\
& + \left[\frac{z^2 l_r^2}{24\sigma^2} \left((z + 3y)A - 2z(m_3^2 - m_2^2) \right) - \frac{3yz^2 \delta}{2\rho\sigma^2} + \frac{z}{2} m_2^2 C \right] \mathcal{K}_{n_1-3} \\
& + \frac{z^2 l_r^2 \delta}{24\sigma^2}(2z + 9y)\mathcal{K}_{n_1-4} - \frac{z^2 C l_r^2}{48\sigma^2}(z + 3y)\mathcal{K}_{n_1-5} \Bigg\},
\end{aligned}$$

$$H_{26}^{r,H;n_121} = \oiint \left[\frac{1}{4} \left(\frac{\tau}{\rho} + z \right) \left(A\mathcal{K}_{n_1-1} - \frac{2z}{\sigma}\mathcal{K}_{n_1-2} - C\mathcal{K}_{n_1-3} \right) + \tau\mathcal{K}_{n_2-2} \right],$$

$$H_{26}^{r,H;n_112} = \oiint \frac{z^2}{4\rho\sigma} \left(A\mathcal{K}_{n_1-1} + \frac{2y}{\sigma}\mathcal{K}_{n_1-2} - C\mathcal{K}_{n_1-3} \right),$$

$$H_{26}^{r,H;n_122} = \oiint \frac{\tau}{2\rho} \left(1 + \frac{z^2}{\sigma} \right) \mathcal{K}_{n_1-1}, \tag{4.66}$$

$$H_{29}^{r,H;n_111} = \oiint \frac{-z^3 l_r^2}{36\sigma^2} \left(A\mathcal{K}_{n_1-3} + 4\delta\mathcal{K}_{n_1-4} - C\mathcal{K}_{n_1-5} \right),$$

$$H_{29}^{r,H;n_121} = \oiint \frac{-yz^2 l_r^2}{12\sigma^2} \mathcal{K}_{n_1-3},$$

$$H_{29}^{r,H;n_112} = \oiint \frac{-z^3 l_r^2}{12\sigma^2} \mathcal{K}_{n_1-3},$$

$$H_{29}^{r,H;n_122} = \oiint \frac{-yz^3 l_r^2}{12\sigma^2} \mathcal{K}_{n_1-2}. \tag{4.67}$$

4.3.2 Center-of-mass frame: theta functions

Next, instead of computing the integrals over x, y and λ, we have performed the summation in terms of the theta functions, previously encountered for the one-loop and simplest sunset integrals. In the cms frame, we make use of eqs. (3.10), (3.18), and

$$\sum_{n \in \mathbb{Z}^3} (n^2)^2 q^{(n^2)} = \left(q\frac{\partial}{\partial q} \right)^2 \left(\sum_{n \in \mathbb{Z}^3} q^{(n^2)} \right) = \left(q\frac{\partial}{\partial q} \right)^2 (\theta_{30}(q)^3)$$

$$= 3\theta_{34}(q)\theta_{30}(q)^2 + 6\theta_{32}(q)^2\theta_{30}(q), \tag{4.68}$$

where we note that eq. (4.68) can immediately be used for the primed sums by setting $l_r = nL$, as the term with $n = 0$ does not contribute. We rescale λ such that the argument of all theta functions is $e^{-1/\lambda}$, which we suppress for brevity. Further, we introduce the abbreviation

$$\fint = \frac{1}{\Gamma(n_1)(16\pi^2)^2} \int_0^1 dx \int_0^{1-x} dy \int_0^\infty d\lambda \frac{(x\hat\lambda)^{n_1-1}}{\lambda\sigma^2} e^{-\hat\lambda Z}, \tag{4.69}$$

where $\hat\lambda \equiv \lambda\rho L^2/4$. For the simplest sunset integral, we have

$$H^{r,H;n_111} = \fint z \left[\left(A + \frac{2\delta}{\hat\lambda} \right) (\theta_{30}^3 - 1) - \frac{3\delta\rho}{4\hat\lambda^2} L^2 \theta_{32}\theta_{30}^2 \right],$$

$$H^{r,H;n_121} = \fint y \left(\theta_{30}^3 - 1 \right),$$

$$H^{r,H;n_112} = \fint z \left(\theta_{30}^3 - 1 \right),$$

$$H^{r,H;n_122} = \fint yz\hat\lambda \left(\theta_{30}^3 - 1 \right), \tag{4.70}$$

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and for the $H_1^{r,H}$ and $H_2^{r,H}$, we find

$$H_1^{r,H;n_1 11} = \oint z \left[\left(\tau A + \frac{2\tau\delta}{\hat{\lambda}} - \frac{x\delta\rho}{\hat{\lambda}} \right) \left(\theta_{30}^3 - 1 \right) - \frac{3\delta\rho\tau}{4\hat{\lambda}^2} L^2 \theta_{32}\theta_{30}^2 \right],$$

$$H_1^{r,H;n_1 21} = \oint y\tau \left(\theta_{30}^3 - 1 \right),$$

$$H_1^{r,H;n_1 12} = \oint z\tau \left(\theta_{30}^3 - 1 \right),$$

$$H_1^{r,H;n_1 22} = \oint yz\tau\hat{\lambda} \left(\theta_{30}^3 - 1 \right), \tag{4.71}$$

$$H_2^{r,H;n_1 11} = \oint \frac{xz^2}{2\sigma} \left[\left(A + \frac{3\delta}{\hat{\lambda}} \right) \left(\theta_{30}^3 - 1 \right) - \frac{3\delta\rho}{4\hat{\lambda}^2} L^2 \theta_{32}\theta_{30}^2 \right],$$

$$H_2^{r,H;n_1 21} = \oint \frac{xyz}{\sigma} \left(\theta_{30}^3 - 1 \right),$$

$$H_2^{r,H;n_1 12} = \oint \frac{xz^2}{\sigma} \left(\theta_{30}^3 - 1 \right),$$

$$H_2^{r,H;n_1 22} = \oint \frac{xyz^2\hat{\lambda}}{\sigma} \left(\theta_{30}^3 - 1 \right), \tag{4.72}$$

respectively. For the $H_{21}^{r,H}$, $H_{22}^{r,H}$, and $H_{27}^{r,H}$, we find

$$H_{21}^{r,H;n_1 11} = \oint z\tau \left[\left(\tau A + \frac{2\tau\delta}{\hat{\lambda}} - \frac{2x\delta\rho}{\hat{\lambda}} \right) \left(\theta_{30}^3 - 1 \right) - \frac{3\tau\delta\rho}{4\hat{\lambda}^2} L^2 \theta_{32}\theta_{30}^2 \right],$$

$$H_{21}^{r,H;n_1 21} = \oint y\tau^2 \left(\theta_{30}^3 - 1 \right),$$

$$H_{21}^{r,H;n_1 12} = \oint z\tau^2 \left(\theta_{30}^3 - 1 \right),$$

$$H_{21}^{r,H;n_1 22} = \oint yz\tau^2\hat{\lambda} \left(\theta_{30}^3 - 1 \right), \tag{4.73}$$

$$H_{22}^{r,H;n_1 11} = \oint \frac{z\rho}{2\hat{\lambda}} \left[\left(A + \frac{3\delta}{\hat{\lambda}} \right) \left(\theta_{30}^3 - 1 \right) - \frac{3\delta\rho}{4\hat{\lambda}^2} L^2 \theta_{32}\theta_{30}^2 \right],$$

$$H_{22}^{r,H;n_1 21} = \oint \frac{y\rho}{2\hat{\lambda}} \left(\theta_{30}^3 - 1 \right),$$

$$H_{22}^{r,H;n_1 12} = \oint \frac{z\rho}{2\hat{\lambda}} \left(\theta_{30}^3 - 1 \right),$$

$$H_{22}^{r,H;n_1 22} = \oint \frac{yz\rho}{2} \left(\theta_{30}^3 - 1 \right), \tag{4.74}$$

$$H_{27}^{r,H;n_1 11} = \oint \frac{-z\rho^2}{4\hat{\lambda}^2} \left[\left(A + \frac{4\delta}{\hat{\lambda}} \right) L^2 \theta_{32}\theta_{30}^2 - \frac{\delta\rho}{4\hat{\lambda}^2} L^4 \left(\theta_{34}\theta_{30}^2 + 2\theta_{32}^2\theta_{30} \right) \right],$$

$$H_{27}^{r,H;n_1 21} = \oint \frac{-y\rho^2}{4\hat{\lambda}^2} L^2 \theta_{32}\theta_{30}^2,$$

$$H_{27}^{r,H;n_1 12} = \oint \frac{-z\rho^2}{4\hat{\lambda}^2} L^2 \theta_{32}\theta_{30}^2,$$

$$H_{27}^{r,H;n_1 22} = \oint \frac{-yz\rho^2}{4\hat{\lambda}} L^2 \theta_{32}\theta_{30}^2, \tag{4.75}$$

respectively, and for the $H_{23}^{r,H}$, $H_{24}^{r,H}$, and $H_{28}^{r,H}$, we have

$$H_{23}^{r,H;n_1 11} = \oiint \frac{xz^2}{2\sigma} \left[\left(\tau A + \frac{3\tau\delta}{\hat{\lambda}} - \frac{x\delta\rho}{\hat{\lambda}} \right) (\theta_{30}^3 - 1) - \frac{3\tau\delta\rho}{4\hat{\lambda}^2} L^2 \theta_{32} \theta_{30}^2 \right],$$

$$H_{23}^{r,H;n_1 21} = \oiint \frac{xyz\tau}{\sigma} (\theta_{30}^3 - 1),$$

$$H_{23}^{r,H;n_1 12} = \oiint \frac{xz^2\tau}{\sigma} (\theta_{30}^3 - 1),$$

$$H_{23}^{r,H;n_1 22} = \oiint \frac{xyz^2\tau\hat{\lambda}}{\sigma} (\theta_{30}^3 - 1), \tag{4.76}$$

$$H_{24}^{r,H;n_1 11} = \oiint \frac{-z^2}{4\sigma\hat{\lambda}} \left[\left(A + \frac{3\delta}{\hat{\lambda}} \right) (\theta_{30}^3 - 1) - \frac{3\delta\rho}{4\hat{\lambda}^2} L^2 \theta_{32} \theta_{30}^2 \right],$$

$$H_{24}^{r,H;n_1 21} = \oiint \frac{-yz}{2\sigma\hat{\lambda}} (\theta_{30}^3 - 1),$$

$$H_{24}^{r,H;n_1 12} = \oiint \frac{-z^2}{2\sigma\hat{\lambda}} (\theta_{30}^3 - 1),$$

$$H_{24}^{r,H;n_1 22} = \oiint \frac{-yz^2}{2\sigma} (\theta_{30}^3 - 1), \tag{4.77}$$

$$H_{28}^{r,H;n_1 11} = \oiint \frac{z^2\rho}{8\sigma\hat{\lambda}^2} \left[\left(A + \frac{4\delta}{\hat{\lambda}} \right) L^2 \theta_{32} \theta_{30}^2 - \frac{\delta\rho}{4\hat{\lambda}^2} L^4 \left(\theta_{34}\theta_{30}^2 + 2\theta_{32}^2\theta_{30} \right) \right],$$

$$H_{28}^{r,H;n_1 21} = \oiint \frac{yz\rho}{4\sigma\hat{\lambda}^2} L^2 \theta_{32} \theta_{30}^2,$$

$$H_{28}^{r,H;n_1 12} = \oiint \frac{z^2\rho}{4\sigma\hat{\lambda}^2} L^2 \theta_{32} \theta_{30}^2,$$

$$H_{28}^{r,H;n_1 22} = \oiint \frac{yz^2\rho}{4\sigma\hat{\lambda}} L^2 \theta_{32} \theta_{30}^2, \tag{4.78}$$

respectively. Finally, for the $H_{25}^{r,H}$, $H_{26}^{r,H}$, and $H_{29}^{r,H}$, we find

$$H_{25}^{r,H;n_1 11} = \oiint \frac{x^2 z^3}{3\sigma^2} \left[\left(A + \frac{4\delta}{\hat{\lambda}} \right) (\theta_{30}^3 - 1) - \frac{3\delta\rho}{4\hat{\lambda}^2} L^2 \theta_{32}\theta_{30}^2 \right],$$

$$H_{25}^{r,H;n_1 21} = \oiint \frac{x^2 yz^2}{\sigma^2} (\theta_{30}^3 - 1),$$

$$H_{25}^{r,H;n_1 12} = \oiint \frac{x^2 z^3}{\sigma^2} (\theta_{30}^3 - 1),$$

$$H_{25}^{r,H;n_1 22} = \oiint \frac{x^2 yz^3\hat{\lambda}}{\sigma^2} (\theta_{30}^3 - 1), \tag{4.79}$$

$$H_{26}^{r,H;n_1 11} = \oiint \left\{ \left[-\frac{zm_2^2}{2} \left(A + \frac{2\delta}{\hat{\lambda}} \right) + \frac{z^2}{6\rho\sigma^2\hat{\lambda}} (5z + 3y)(m_3^2 - m_2^2) - \frac{3yz^2}{2\rho\sigma^2\hat{\lambda}} \left(A + \frac{\delta}{\hat{\lambda}} \right) \right. \right.$$
$$\left. - \frac{z^2 A}{12\rho\sigma} \left(3A + \frac{4z}{\hat{\lambda}\sigma} + \frac{4x^2 z\rho p^2}{\sigma} \right) \right] (\theta_{30}^3 - 1)$$
$$+ \left[\frac{z^2\delta}{8\sigma^2\hat{\lambda}^3} (2z + 9y) + \frac{z^2}{8\sigma^2\hat{\lambda}^2} \left((z + 3y)A - 2z(m_3^2 - m_2^2) \right) + \frac{3z\delta\rho}{8\hat{\lambda}^2} m_2^2 \right] L^2 \theta_{32}\theta_{30}^2$$
$$\left. - \frac{z^2\delta\rho}{64\sigma^2\hat{\lambda}^4} (z + 3y) L^4 \left(\theta_{34}\theta_{30}^2 + 2\theta_{32}^2\theta_{30} \right) \right\},$$

$$H_{26}^{r,H;n_1 21} = \oiint \left\{ \left[\frac{1}{4} \left(z + \frac{\tau}{\rho} \right) A + \frac{z\delta}{\hat{\lambda}} + \frac{z^3}{2\rho\sigma^2\hat{\lambda}} \right] (\theta_{30}^3 - 1) - \frac{3\delta\rho}{16\hat{\lambda}^2} \left(z + \frac{\tau}{\rho} \right) L^2 \theta_{32}\theta_{30}^2 \right\},$$

$$H_{26}^{r,H;n_1 12} = \oiint \left\{ \left[\frac{1}{4} \left(z - \frac{\tau}{\rho} \right) A + \frac{z^2}{2\sigma\hat{\lambda}} - \frac{z^3}{2\rho\sigma^2\hat{\lambda}} \right] (\theta_{30}^3 - 1) - \frac{3\delta\rho}{16\hat{\lambda}^2} \left(z - \frac{\tau}{\rho} \right) L^2 \theta_{32}\theta_{30}^2 \right\},$$

$$H_{26}^{r,H;n_1 22} = \oiint \frac{\tau}{2\rho} \left(1 + \frac{z^2}{\sigma} \right) (\theta_{30}^3 - 1), \tag{4.80}$$

and

$$H_{29}^{r,H;n_1 11} = \oiint \frac{-z^3}{12\sigma^2\hat{\lambda}^2} \left[\left(A + \frac{4\delta}{\hat{\lambda}} \right) L^2 \theta_{32}\theta_{30}^2 - \frac{\delta\rho}{4\hat{\lambda}^2} L^4 \left(\theta_{34}\theta_{30}^2 + 2\theta_{32}^2\theta_{30} \right) \right],$$

$$H_{29}^{r,H;n_1 21} = \oiint \frac{-yz^2}{4\sigma^2\hat{\lambda}^2} L^2 \theta_{32}\theta_{30}^2,$$

$$H_{29}^{r,H;n_1 12} = \oiint \frac{-z^3}{4\sigma^2\hat{\lambda}^2} L^2 \theta_{32}\theta_{30}^2,$$

$$H_{29}^{r,H;n_1 22} = \oiint \frac{-yz^3}{4\sigma^2\hat{\lambda}} L^2 \theta_{32}\theta_{30}^2. \tag{4.81}$$

4.4 Sunset integrals with two quantized loop momenta

Here, we follow the treatment of section 4.1.2, and generalize to all integrals $\langle\langle X \rangle\rangle_{rs}$ with $X = 1, r_\mu, s_\mu, r_\mu r_\nu, r_\mu s_\nu$ and $s_\mu s_\nu$. All of these are not needed for completeness, but the redundant ones enable a check on our results by means of the relations given in section 4.2. We again introduce Gaussian parameterizations for the propagators using eq. (A.1), and then shift the momenta using eqs. (4.21) and (4.22). This leads to

$$\langle\langle X \rangle\rangle_{rs} = \frac{1}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(4\pi)^d} \sum_{l_r, l_s}'' \int_0^\infty d\lambda_1 d\lambda_2 d\lambda_3 \frac{\lambda_1^{n_1-1}\lambda_2^{n_2-1}\lambda_3^{n_3-1}}{\tilde{\lambda}^{d/2}} [[X]]_{rs}\, e^{-\tilde{M}^2}, \tag{4.82}$$

where $\tilde{M}^2$ is defined in eq. (4.23), and $\tilde{\lambda} \equiv \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$. For the $[[X]]_{rs}$, we find

$$[[1]]_{rs} = 1,$$

$$[[r_\mu]]_{rs} = \frac{1}{\tilde{\lambda}} \left(\lambda_2\lambda_3 p_\mu + \frac{i}{2}\lambda_2 l_{r\mu} + \frac{i}{2}\lambda_3 l_{n\mu} \right),$$

$$[[s_\mu]]_{rs} = \frac{1}{\tilde{\lambda}} \left(\lambda_1\lambda_3 p_\mu + \frac{i}{2}\lambda_1 l_{s\mu} - \frac{i}{2}\lambda_3 l_{n\mu} \right),$$

$$[[r_\mu r_\nu]]_{rs} = \frac{\lambda_2^2\lambda_3^2}{\tilde{\lambda}^2} p_\mu p_\nu + \frac{\lambda_2 + \lambda_3}{2\tilde{\lambda}}\delta_{\mu\nu} + \frac{i\lambda_2^2\lambda_3}{2\tilde{\lambda}^2}\{p, l_r\}_{\mu\nu} + \frac{i\lambda_2\lambda_3^2}{2\tilde{\lambda}^2}\{p, l_n\}_{\mu\nu}$$
$$- \frac{1}{4\tilde{\lambda}^2} \left(\lambda_2^2 l_{r\mu}l_{r\nu} + \lambda_2\lambda_3\{l_r, l_n\}_{\mu\nu} + \lambda_3^2 l_{n\mu}l_{n\nu} \right),$$

$$[[r_\mu s_\nu]]_{rs} = \frac{\lambda_1\lambda_2\lambda_3^2}{\tilde{\lambda}^2} p_\mu p_\nu - \frac{\lambda_3}{2\tilde{\lambda}}\delta_{\mu\nu} - \frac{i\lambda_2\lambda_3^2}{2\tilde{\lambda}^2} p_\mu l_{n\nu} + \frac{i\lambda_1\lambda_3^2}{2\tilde{\lambda}^2} p_\nu l_{n\mu} + \frac{i\lambda_1\lambda_2\lambda_3}{2\tilde{\lambda}^2} \left(p_\mu l_{s\nu} + p_\nu l_{r\mu} \right)$$
$$+ \frac{1}{4\tilde{\lambda}^2} \left(\lambda_3^2 l_{n\mu}l_{n\nu} + \lambda_2\lambda_3 l_{r\mu}l_{n\mu} - \lambda_1\lambda_3 l_{n\mu}l_{s\nu} - \lambda_1\lambda_2 l_{r\mu}l_{s\nu} \right),$$

$$[[s_\mu s_\nu]]_{rs} = \frac{\lambda_1^2 \lambda_3^2}{\tilde{\lambda}^2} p_\mu p_\nu + \frac{\lambda_1 + \lambda_3}{2\tilde{\lambda}} \delta_{\mu\nu} + \frac{i\lambda_1^2 \lambda_3}{2\tilde{\lambda}^2} \{p, l_s\}_{\mu\nu} - \frac{i\lambda_1 \lambda_3^2}{2\tilde{\lambda}^2} \{p, l_n\}_{\mu\nu}$$
$$- \frac{1}{4\tilde{\lambda}^2} \left(\lambda_1^2 l_{s\mu} l_{s\nu} - \lambda_1 \lambda_3 \{l_s, l_n\}_{\mu\nu} + \lambda_3^2 l_{n\mu} l_{n\nu} \right), \tag{4.83}$$

where $l_n \equiv l_r - l_s$. We may now switch integration variables to to $x, y, z \equiv 1 - x - y$ and λ as in eq. (4.25), which gives us an integral similar to eq. (4.27).

In what follows, we restrict ourselves to the cms frame with $p \cdot l_r = p \cdot l_s = 0$, which simplifies the expressions greatly. The results for a moving frame can again be obtained using the same methods. In the cms frame, the exponential factors depend only on the components of l_r and l_s via l_r^2, l_s^2 and l_n^2. This allows us to write

$$\sum_{l_r, l_s}^{''} l_{r\mu} f(l_r^2, l_s^2, l_n^2) = \sum_{l_r, l_s}^{''} l_{r\mu} f(l_r^2, l_s^2, l_n^2) = 0,$$

$$\sum_{l_r, l_s}^{''} l_{r\mu} l_{r\nu} f(l_r^2, l_s^2, l_n^2) = \frac{t_{\mu\nu}}{3} \sum_{l_r, l_s}^{''} l_r^2 f(l_r^2, l_s^2, l_n^2),$$

$$\sum_{l_r, l_s}^{''} l_{s\mu} l_{s\nu} f(l_r^2, l_s^2, l_n^2) = \frac{t_{\mu\nu}}{3} \sum_{l_r, l_s}^{''} l_s^2 f(l_r^2, l_s^2, l_n^2),$$

$$\sum_{l_r, l_s}^{''} l_{r\mu} l_{s\nu} f(l_r^2, l_s^2, l_n^2) = \frac{t_{\mu\nu}}{3} \sum_{l_r, l_s}^{''} l_r \cdot l_s f(l_r^2, l_s^2, l_n^2),$$

$$l_r \cdot l_s = \frac{1}{2} \left(l_r^2 + l_s^2 - l_n^2 \right). \tag{4.84}$$

4.4.1 Center-of-mass frame: Bessel functions

As for the sunset integrals with one quantized loop momentum, the integral over λ can again be performed in terms of the modified Bessel functions $\mathcal{K}_\nu(Y_{rs}, Z_{rs})$, where Y_{rs} and Z_{rs} are defined in eq. (4.26). These arguments will be suppressed for brevity. While the sextuple summation over the components of l_r and l_s can be reduced to a triple sum using eq. (4.29), we find that the remaining summations converge fairly slowly for moderate values of $m_i L$. In the following expressions, we set $d = 4$ since no divergences appear. Using the notation

$$\oint \equiv \frac{1}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2)^2} \sum_{l_r, l_s}^{''} \int_0^1 dx \int_0^{1-x} dy \frac{x^{n_1-1} y^{n_2-1} z^{n_3-1}}{\sigma^2}, \tag{4.85}$$

and $m \equiv n_1 + n_2 + n_3 - 4$, we obtain

$$H^{rs;n_1 n_2 n_3} = \oint \mathcal{K}_m,$$

$$H_1^{rs;n_1 n_2 n_3} = \oint \frac{yz}{\sigma} \mathcal{K}_m,$$

$$H_2^{rs;n_1 n_2 n_3} = \oint \frac{xz}{\sigma} \mathcal{K}_m, \tag{4.86}$$

for the simplest sunset integral and the scalar components of the integrals with one Lorentz index. For the components of the sunset integrals with two Lorentz indices, we find

$$H_{21}^{rs;n_1 n_2 n_3} = \oint \frac{y^2 z^2}{\sigma^2} \mathcal{K}_m,$$

$$H_{22}^{rs;n_1 n_2 n_3} = \oint \frac{y+z}{2\sigma} \mathcal{K}_{m-1},$$

$$H_{27}^{rs;n_1 n_2 n_3} = \oint \frac{1}{12\sigma^2}\Big[-y(y+z)\,l_r^2 + yz\,l_s^2 - z(y+z)\,l_n^2\Big]\mathcal{K}_{m-2}, \qquad (4.87)$$

$$H_{23}^{rs;n_1 n_2 n_3} = \oint \frac{xyz^2}{\sigma^2} \mathcal{K}_m,$$

$$H_{24}^{rs;n_1 n_2 n_3} = \oint \frac{-z}{2\sigma} \mathcal{K}_{m-1},$$

$$H_{28}^{rs;n_1 n_2 n_3} = \oint \frac{1}{24\sigma^2}\Big[(2yz-\sigma)\,l_r^2 + (2xz-\sigma)\,l_s^2 + (2z^2+\sigma)\,l_n^2\Big]\mathcal{K}_{m-2}, \qquad (4.88)$$

and

$$H_{25}^{rs;n_1 n_2 n_3} = \oint \frac{x^2 z^2}{\sigma^2} \mathcal{K}_m,$$

$$H_{26}^{rs;n_1 n_2 n_3} = \oint \frac{x+z}{2\sigma} \mathcal{K}_{m-1},$$

$$H_{29}^{rs;n_1 n_2 n_3} = \oint \frac{1}{12\sigma^2}\Big[xz\,l_r^2 - x(x+z)\,l_s^2 - z(x+z)\,l_n^2\Big]\mathcal{K}_{m-2}. \qquad (4.89)$$

4.4.2 Center-of-mass frame: theta functions

In the cms frame, the double summation can be performed in terms of the theta functions, as encountered in the treatment of the simplest sunset integral. If we define

$$\bar{\lambda} \equiv \frac{4\sigma}{L^2}\lambda, \qquad l_r \equiv n_r L, \qquad l_s \equiv n_s L, \qquad n_n \equiv n_r - n_s, \qquad (4.90)$$

we find

$$\sideset{}{''}\sum_{l_r, l_s} e^{-\frac{y l_r^2}{4\sigma\lambda} - \frac{x l_s^2}{4\sigma\lambda} - \frac{z l_n^2}{4\sigma\lambda}} = \sideset{}{''}\sum_{n_r, n_s} e^{-\frac{y}{\bar{\lambda}} n_r^2 - \frac{x}{\bar{\lambda}} n_s^2 - \frac{z}{\bar{\lambda}} n_n^2}$$

$$= \sum_{n_r, n_s} e^{-\frac{y}{\bar{\lambda}} n_r^2 - \frac{x}{\bar{\lambda}} n_s^2 - \frac{z}{\bar{\lambda}} n_n^2} - \sum_{n_r} e^{-\frac{y+z}{\bar{\lambda}} n_r^2} - \sum_{n_s} e^{-\frac{x+z}{\bar{\lambda}} n_s^2} - \sum_{n_r} e^{-\frac{x+y}{\bar{\lambda}} n_r^2} + 2$$

$$= \theta_0^{(2)}\left(\frac{y}{\bar{\lambda}}, \frac{x}{\bar{\lambda}}, \frac{z}{\bar{\lambda}}\right)^3 - \theta_{30}\left(e^{-\frac{y+z}{\bar{\lambda}}}\right)^3 - \theta_{30}\left(e^{-\frac{x+z}{\bar{\lambda}}}\right)^3 - \theta_{30}\left(e^{-\frac{x+y}{\bar{\lambda}}}\right)^3 + 2$$

$$\equiv \Theta_0\left(\frac{y}{\bar{\lambda}}, \frac{x}{\bar{\lambda}}, \frac{z}{\bar{\lambda}}\right), \qquad (4.91)$$

which was already used in eq. (4.30). Here, the terms involving θ_{30} subtract the contributions with $(n_s = 0, n_n = n_r)$, $(n_r = 0, n_n = -n_s)$, and $(n_n = 0, n_r = n_s)$. The constant term corrects for the case when $(n_r = n_s = 0)$ is subtracted to often. By taking derivatives

w.r.t. x, y, z, we also find

$$\sum_{l_r, l_s}'' l_r^2 \, e^{-\frac{y l_r^2}{4\sigma\lambda} - \frac{x l_s^2}{4\sigma\lambda} - \frac{z l_n^2}{4\sigma\lambda}} = 3L^2 \, \theta_{02}^{(2)}\left(\frac{y}{\lambda}, \frac{x}{\lambda}, \frac{z}{\lambda}\right) \theta_0^{(2)}\left(\frac{y}{\lambda}, \frac{x}{\lambda}, \frac{z}{\lambda}\right)^2$$

$$- 3L^2 \, \theta_{32}\left(e^{-\frac{y+z}{\lambda}}\right) \theta_{30}\left(e^{-\frac{y+z}{\lambda}}\right)^2 - 3L^2 \, \theta_{32}\left(e^{-\frac{x+y}{\lambda}}\right) \theta_{30}\left(e^{-\frac{x+y}{\lambda}}\right)^2$$

$$\equiv 3L^2 \, \Theta_{02}\left(\frac{y}{\lambda}, \frac{x}{\lambda}, \frac{z}{\lambda}\right). \tag{4.92}$$

If we introduce the abbreviation

$$\oint \equiv \frac{1}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)(16\pi^2)^2} \int_0^1 dx \int_0^{1-x} dy \int_0^\infty d\lambda$$

$$\times \frac{x^{n_1-1} y^{n_2-1} z^{n_3-1} \lambda^{n_1+n_2+n_3-5}}{\sigma^2} \, e^{-\lambda Z_{rs}}, \tag{4.93}$$

we can express the scalar components as

$$H^{rs;n_1 n_2 n_3} = \oint \Theta_0\left(\frac{y}{\lambda}, \frac{x}{\lambda}, \frac{z}{\lambda}\right),$$

$$H_1^{rs;n_1 n_2 n_3} = \oint \frac{yz}{\sigma} \, \Theta_0\left(\frac{y}{\lambda}, \frac{x}{\lambda}, \frac{z}{\lambda}\right),$$

$$H_2^{rs;n_1 n_2 n_3} = \oint \frac{xz}{\sigma} \, \Theta_0\left(\frac{y}{\lambda}, \frac{x}{\lambda}, \frac{z}{\lambda}\right), \tag{4.94}$$

$$H_{21}^{rs;n_1 n_2 n_3} = \oint \frac{y^2 z^2}{\sigma^2} \, \Theta_0\left(\frac{y}{\lambda}, \frac{x}{\lambda}, \frac{z}{\lambda}\right),$$

$$H_{22}^{rs;n_1 n_2 n_3} = \oint \frac{y+z}{2\lambda\sigma} \, \Theta_0\left(\frac{y}{\lambda}, \frac{x}{\lambda}, \frac{z}{\lambda}\right),$$

$$H_{27}^{rs;n_1 n_2 n_3} = \oint \frac{L^2}{4\sigma^2\lambda^2} \left[-y(y+z)\, \Theta_{02}\left(\frac{y}{\lambda}, \frac{x}{\lambda}, \frac{z}{\lambda}\right) + yz\, \Theta_{02}\left(\frac{x}{\lambda}, \frac{y}{\lambda}, \frac{z}{\lambda}\right) \right.$$

$$\left. - z(y+z)\, \Theta_{02}\left(\frac{z}{\lambda}, \frac{x}{\lambda}, \frac{y}{\lambda}\right) \right], \tag{4.95}$$

$$H_{23}^{rs;n_1 n_2 n_3} = \oint \frac{xyz^2}{\sigma^2} \, \Theta_0\left(\frac{y}{\lambda}, \frac{x}{\lambda}, \frac{z}{\lambda}\right),$$

$$H_{24}^{rs;n_1 n_2 n_3} = \oint \frac{-z}{2\lambda\sigma} \, \Theta_0\left(\frac{y}{\lambda}, \frac{x}{\lambda}, \frac{z}{\lambda}\right),$$

$$H_{28}^{rs;n_1 n_2 n_3} = \oint \frac{L^2}{8\sigma^2\lambda^2} \left[(2yz-\sigma)\, \Theta_{02}\left(\frac{y}{\lambda}, \frac{x}{\lambda}, \frac{z}{\lambda}\right) + (2xz-\sigma)\, \Theta_{02}\left(\frac{x}{\lambda}, \frac{y}{\lambda}, \frac{z}{\lambda}\right) \right.$$

$$\left. + (2z^2+\sigma)\, \Theta_{02}\left(\frac{z}{\lambda}, \frac{x}{\lambda}, \frac{y}{\lambda}\right) \right], \tag{4.96}$$

and

$$H_{25}^{rs;n_1 n_2 n_3} = \oint \frac{x^2 z^2}{\sigma^2} \, \Theta_0\left(\frac{y}{\lambda}, \frac{x}{\lambda}, \frac{z}{\lambda}\right),$$

$$H_{26}^{rs;n_1 n_2 n_3} = \oint \frac{x+z}{2\lambda\sigma} \, \Theta_0\left(\frac{y}{\lambda}, \frac{x}{\lambda}, \frac{z}{\lambda}\right),$$

$$H_{29}^{rs;n_1 n_2 n_3} = \oint \frac{L^2}{4\sigma^2\lambda^2} \left[xz\, \Theta_{02}\left(\frac{y}{\lambda}, \frac{x}{\lambda}, \frac{z}{\lambda}\right) - x(x+z)\, \Theta_{02}\left(\frac{x}{\lambda}, \frac{y}{\lambda}, \frac{z}{\lambda}\right) \right.$$

$$\left. - z(x+z)\, \Theta_{02}\left(\frac{z}{\lambda}, \frac{x}{\lambda}, \frac{y}{\lambda}\right) \right]. \tag{4.97}$$

	n	$L = 3\,\text{fm}$	$L = 4\,\text{fm}$	$L = \infty$
A^V	1	$2.99758 \cdot 10^{-4}$	$7.79162 \cdot 10^{-5}$	$-4.21046 \cdot 10^{-4}$
A^V	2	$1.85663 \cdot 10^{-2}$	$5.98396 \cdot 10^{-3}$	$1.53036 \cdot 10^{-2}$
A_{22}^V	1	$3.81017 \cdot 10^{-6}$	$7.16805 \cdot 10^{-7}$	$2.34818 \cdot 10^{-6}$
A_{22}^V	2	$1.49879 \cdot 10^{-4}$	$3.89581 \cdot 10^{-5}$	$-2.10523 \cdot 10^{-4}$
A_{23}^V	1	$-7.02467 \cdot 10^{-6}$	$-1.46116 \cdot 10^{-6}$	–
A_{23}^V	2	$-2.20354 \cdot 10^{-4}$	$-6.47885 \cdot 10^{-5}$	–

Table 1. Numerical results for the one-propagator "tadpole" integrals, for $m = 0.1395\,\text{GeV}$, which corresponds to $mL \approx 2.12$ ($L = 3\,\text{fm}$) and $mL \approx 2.83$ ($L = 4\,\text{fm}$). The corresponding continuum integrals are shown in the column labeled $L = \infty$. The continuum results employ the $\overline{\text{MS}}$ subtraction scheme with $\mu = 0.77\,\text{GeV}$. Note that the "23" case has no continuum counterpart. All results are given in units of the appropriate powers of GeV, and the pole configurations n of the propagators are given in appendix D.

5　Numerical results

As a numerical check of the results presented here, we have evaluated all integrals in terms of modified Bessel functions as well as theta functions, and checked these for agreement with each other. We have also verified the expected integral relations by numerical differentiation w.r.t. m_1^2, m_2^2 and m_3^2. Furthermore, we have checked that the expected symmetries under interchange of masses are satisfied. For the sunset integrals, this can be non-trivial as the permutation symmetries are not explicitly conserved by the analytical methods employed here. We have also verified that the one-loop results satisfy the integral relations in eq. (3.35) and (3.36). For reference, we present numerical results with 6 digits of precision. Implementations of the full set of sunset integrals are available from the authors in `C++` and `Mathematica`.

Numerical results for the one-propagator or "tadpole" integrals, defined in eq. (3.32), are given in table 1. We note that there is no infinite-volume counterpart of the A_{23}^V integral. In figure 1, we show the ratio of the finite-volume correction to the infinite-volume result as a function of mL. For the two-propagator or "bubble" integrals, defined in eq. (3.33), results for one set of input parameters are given in table 2. We only quote the results for $n_1 = n_2 = 1$. As evident from eq. (3.43), the necessary modifications for the remaining cases are minor. Figure 2 shows the ratio of the finite volume corrections to the corresponding infinite-volume integrals as a function of $m_1 L$.

We now turn to the main objective of this study, which is an exhaustive evaluation of the sunset integrals at finite volume. The full expressions for the sunset integrals are defined in eq. (4.37), where each one is decomposed according to eq. (4.4). The components labeled $\langle\langle X \rangle\rangle_r$ are further decomposed into a non-locally divergent part and the functions $\langle\langle X \rangle\rangle_{r,G}$ of eq. (4.43) and $\langle\langle X \rangle\rangle_{r,H}$ of section 4.3.1 or 4.3.2. The equivalent expressions for $\langle\langle X \rangle\rangle_s$ and $\langle\langle X \rangle\rangle_t$ can be obtained from the set of relations given in eqs. (4.34) and (4.35).

	$L = 3\,\text{fm}$	$L = 4\,\text{fm}$	$L = \infty$
B^V	$1.23828 \cdot 10^{-3}$	$3.21648 \cdot 10^{-4}$	$4.02489 \cdot 10^{-3}$
B_1^V	$1.28452 \cdot 10^{-4}$	$2.47609 \cdot 10^{-5}$	$4.97497 \cdot 10^{-2}$
B_{21}^V	$3.57770 \cdot 10^{-5}$	$5.14256 \cdot 10^{-6}$	$4.57124 \cdot 10^{-1}$
B_{22}^V	$1.57142 \cdot 10^{-5}$	$2.96746 \cdot 10^{-6}$	$2.11523 \cdot 10^{-3}$
B_{23}^V	$-2.87678 \cdot 10^{-5}$	$-6.05375 \cdot 10^{-6}$	$-$
B_{31}^V	$1.65184 \cdot 10^{-5}$	$1.90690 \cdot 10^{-6}$	$1.47521 \cdot 10^{-4}$
B_{32}^V	$2.36759 \cdot 10^{-6}$	$3.13466 \cdot 10^{-7}$	$3.23347 \cdot 10^{-4}$
B_{33}^V	$-5.22655 \cdot 10^{-6}$	$-7.77244 \cdot 10^{-7}$	$-$

Table 2. Numerical results for the two-propagator "bubble" integrals, for $m_1 = 0.1395\,\text{GeV}$, $m_2 = 0.495\,\text{GeV}$, and $p^2 = m_1^2$, which corresponds to $m_1 L \approx 2.12$ ($L = 3\,\text{fm}$) and $m_1 L \approx 2.83$ ($L = 4\,\text{fm}$). The corresponding continuum integrals are shown in the column labeled $L = \infty$. The continuum results employ the $\overline{\text{MS}}$ subtraction scheme with $\mu = 0.77\,\text{GeV}$. Note that the "23" and "33" cases have no continuum counterpart. All results are given in units of the appropriate powers of GeV. Only the case of $n_1 = n_2 = 1$ is given.
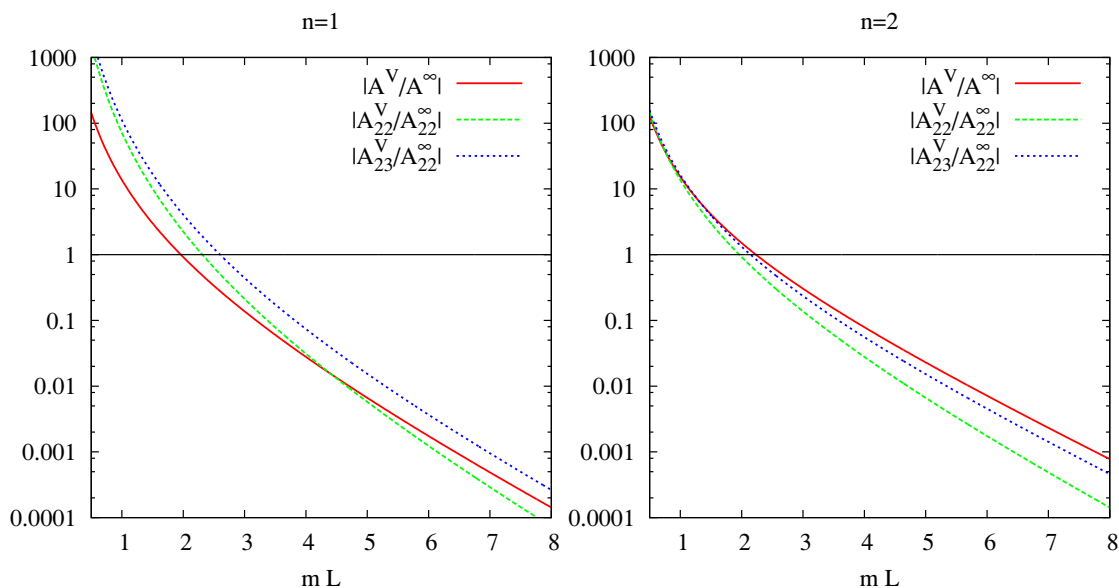

Figure 1. Ratio of finite-volume corrections to infinite-volume results for the "tadpole" integrals, for $m = 0.1395\,\text{GeV}$. The continuum results employ the $\overline{\text{MS}}$ subtraction scheme with $\mu = 0.77\,\text{GeV}$. We compare the "23" case to the "22" case at infinite volume, as the former has no infinite-volume counterpart. The left panel shows the results for $n = 1$, the right panel for $n = 2$, see appendix D for the pole configurations of the propagators. All results are in units of the appropriate powers of GeV.

Finally, the components labeled $\langle\langle X \rangle\rangle_{rs}$ are given in section 4.4.1 and 4.4.2. In order to illustrate the various components of the sunset integrals, we show $\langle\langle 1 \rangle\rangle_{r,G}$, $\langle\langle 1 \rangle\rangle_{r,H}$, $\langle\langle s \rangle\rangle_{rs}$ and the full result $\langle\langle 1 \rangle\rangle$, for two sets of input parameter values in figure 3, relative to the
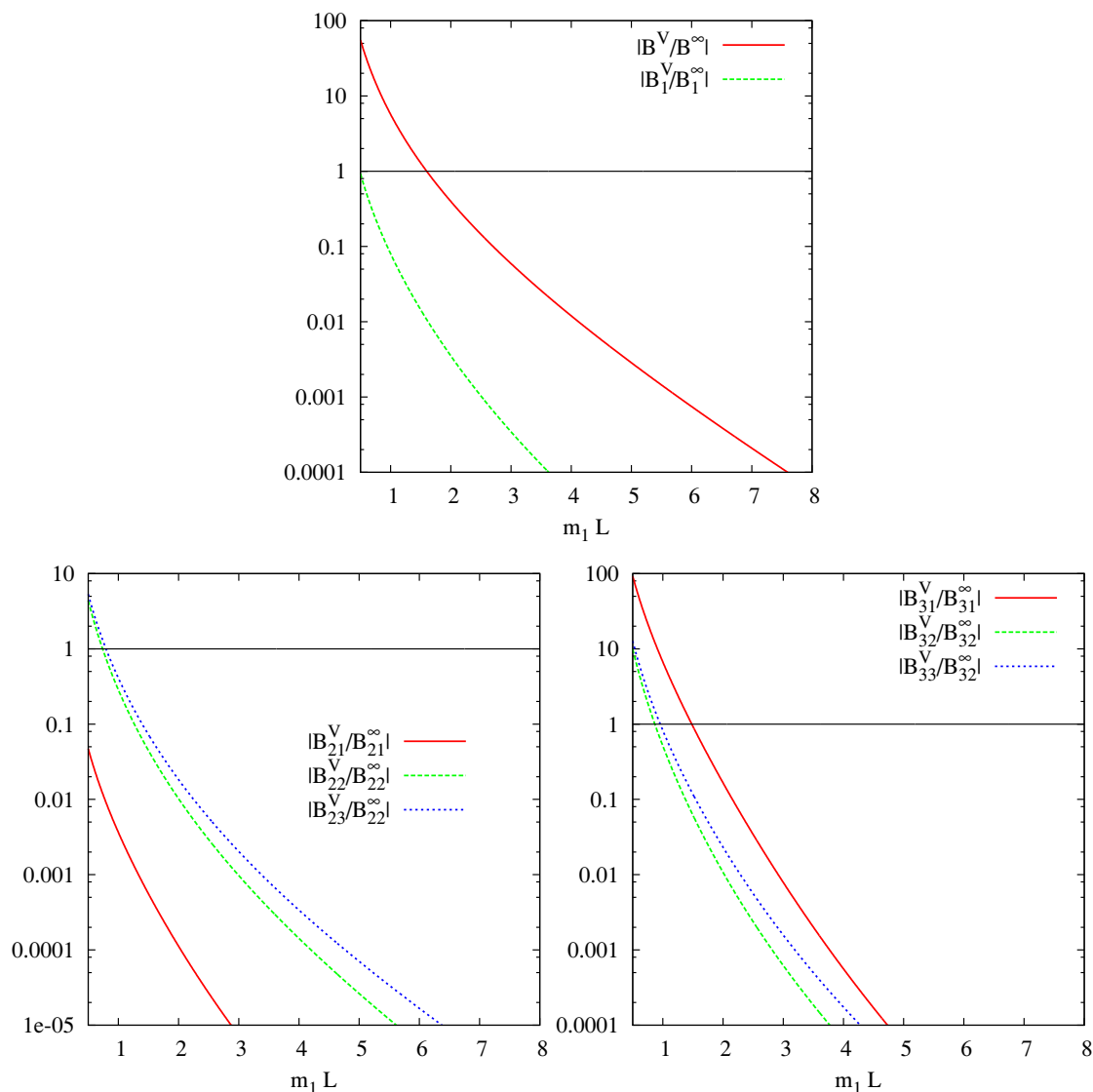
Figure 2. Ratio of finite-volume corrections to infinite-volume results for the "bubble" integrals, for $m_1 = 0.1395\,\mathrm{GeV}$, $m_2 = 0.495\,\mathrm{GeV}$, and $p^2 = m_1^2$. The continuum results employ the $\overline{\mathrm{MS}}$ subtraction scheme with $\mu = 0.77\,\mathrm{GeV}$. We compare the "23" case to the "22" case and the "33" case to the "32" case at infinite volume, as the former have no infinite-volume counterparts. The top panel shows B and B_1, the bottom left panel shows B_{21}, B_{22} and B_{23}, and the bottom right panel shows B_{31}, B_{32} and B_{33}. All results are in units of the appropriate powers of GeV. Only the case of $n_1 = n_2 = 1$ is given.

infinite-volume results[3] from ref. [20], which are

$$
\begin{aligned}
H(m_\pi^2, m_\pi^2, m_\pi^2, -m_\pi^2, \mu^2) &\approx -3.73840 \cdot 10^{-5}\ \mathrm{GeV}^{-2}, \\
H(m_\pi^2, m_\pi^2, m_K^2, -m_K^2, \mu^2) &\approx -6.74071 \cdot 10^{-5}\ \mathrm{GeV}^{-2}.
\end{aligned}
\tag{5.1}
$$

---

[3]These include the finite parts of the terms containing a non-local divergence.

i	$H_i^{r,G}$	$H_i^{r,H}$	H_i^{rs}	H_i^V
0	$-2.20831 \cdot 10^{-7}$	$2.02141 \cdot 10^{-6}$	$5.94236 \cdot 10^{-7}$	$4.05528 \cdot 10^{-6}$
1	–	$1.01508 \cdot 10^{-7}$	$6.66810 \cdot 10^{-8}$	$6.04122 \cdot 10^{-7}$
2	$-1.10415 \cdot 10^{-7}$	$5.90020 \cdot 10^{-7}$	$7.22532 \cdot 10^{-8}$	–
21	–	$9.16777 \cdot 10^{-9}$	$1.58703 \cdot 10^{-8}$	$1.97612 \cdot 10^{-7}$
22	$-2.80694 \cdot 10^{-9}$	$2.54254 \cdot 10^{-8}$	$6.99086 \cdot 10^{-9}$	$-9.22444 \cdot 10^{-8}$
27	$5.17506 \cdot 10^{-9}$	$-4.65135 \cdot 10^{-8}$	$-1.22274 \cdot 10^{-8}$	$-6.10707 \cdot 10^{-8}$
23	–	$3.56590 \cdot 10^{-8}$	$9.04928 \cdot 10^{-9}$	$8.30916 \cdot 10^{-8}$
24	$1.40347 \cdot 10^{-9}$	$-7.90209 \cdot 10^{-9}$	$-9.62049 \cdot 10^{-10}$	$-1.38446 \cdot 10^{-8}$
28	$-2.58753 \cdot 10^{-9}$	$1.44459 \cdot 10^{-8}$	$1.73731 \cdot 10^{-9}$	$2.31182 \cdot 10^{-8}$
25	–	$2.63673 \cdot 10^{-7}$	$1.81386 \cdot 10^{-8}$	–
26	$-8.80371 \cdot 10^{-9}$	$-6.26120 \cdot 10^{-8}$	$7.46258 \cdot 10^{-9}$	–
29	$1.72502 \cdot 10^{-9}$	$-6.94178 \cdot 10^{-9}$	$-1.33169 \cdot 10^{-8}$	–

Table 3. Numerical results for a subset of scalar components of the sunset integrals with $n_1 = n_2 = n_3 = 1$. The contributions $H_i^{r,G}$ are defined in terms of eq. (4.43), using the decomposition into scalar components given by eq. (4.37). The expressions for the $H_i^{r,H}$ are given in section 4.3.1 and 4.3.2, and those for H_i^{rs} can be found in section 4.4.1 and 4.4.2. The full results for each scalar component in the decomposition of eq. (4.37) is given in the column labeled H_i^V (except for cases that involve a trivial exchange of m_1 and m_2). As an example, for the simplest sunset integral ($i = 0$) we have $H^V = H^{r,G} + H^{r,H} + H^{s,G} + H^{s,H} + H^{t,G} + H^{t,H} + H^{rs}$. All results are for $L = 3$ fm, $m_1 = 0.1395$ GeV, $m_2 = 0.15$ GeV, $m_3 = 0.495$ GeV, $p^2 = -0.16$ GeV^2 and $\mu = 0.77$ GeV, given in units of the appropriate powers of GeV.

For reference, we also provide the numerical values of the full sunset integrals as well as the G and H components in table 3 for a box size of $L = 3$ fm.

We cannot compare directly with the work of ref. [11] since they only quote numerical results for the correction to the mass, not for the separate integrals.

6 Conclusions

In conclusion, we have presented a complete treatment of the two-loop sunset integrals at finite volume. We have also discussed in detail the required one-loop integrals and shown how to expand these to higher order in $d - 4$ when necessary. As the main result of our work, we have provided complete expressions for the sunset integrals which are suitable for numerical evaluation. Implementations of the full set of sunset integrals are also available from the authors in `C++` and `Mathematica`. The numerical evaluation has been performed both in terms of modified Bessel functions and theta functions, which have been shown to be numerically equivalent. Depending on the desired quantity and precision, one of these methods is typically preferable. For moderate $m_i L$, the sunset integrals with two quantized loop momenta are better evaluated in terms of theta functions, as the number of terms needed in the triple summation over l_r^2, l_s^2 and l_n^2 in order to obtain acceptable precision is
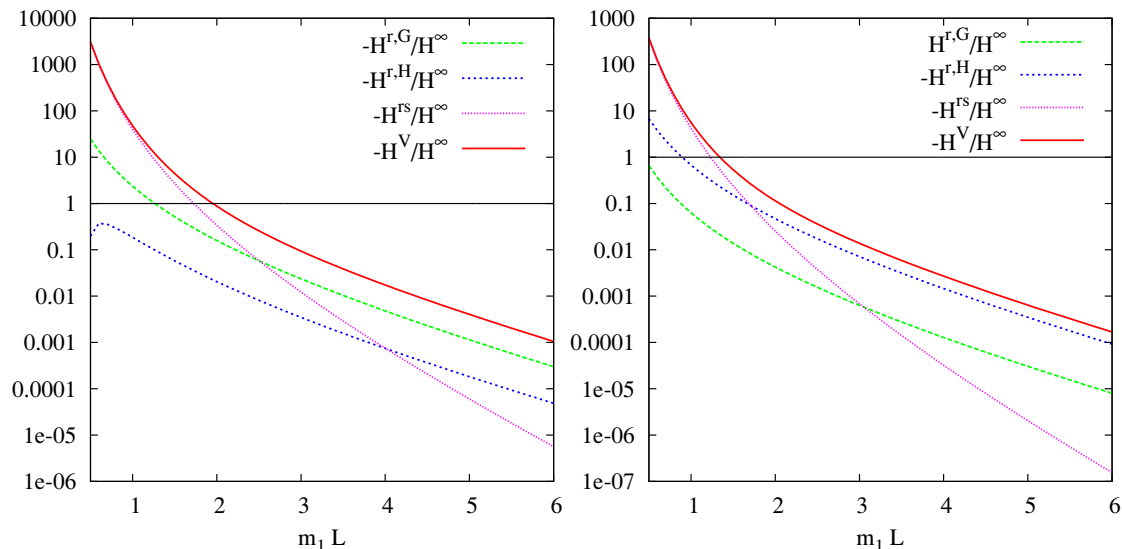
Figure 3. Ratio of finite-volume corrections to infinite-volume results for the simplest sunset integrals. The notation is according to table 3. In the left panel $m_1 = m_2 = m_3 = 0.1395\,\text{GeV}$, and in the right panel $m_1 = m_2 = 0.1395\,\text{GeV}$ with $m_3 = 0.495\,\text{GeV}$. In both cases $p^2 = -m_3^2$. All results employ the $\overline{\text{MS}}$ scheme with $\mu = 0.77\,\text{GeV}$, and are given in units of the appropriate powers of GeV. Only the case of $n_1 = n_2 = n_3 = 1$ is shown.

quite large. For small $m_i L$, the theta-function method is clearly superior in all cases. For large $m_i L$, the numerical evaluation in terms of modified Bessel functions is usually faster.

So far, we have not shown any results on the NNLO calculations at finite volume. In the extant NNLO calculations at infinite volume, many integral relations have been used which are no longer valid at finite volume. Therefore, these NNLO expressions need to first be recomputed using the more general set of finite-volume sunset integrals presented here. Work in this direction is in progress [26].

Acknowledgments

A   Modified Bessel functions

Many of the loop integrals encountered at finite volume can be expressed in terms of the modified Bessel functions $K_\nu(z)$, and we summarize here the most significant recurring results used in the main text. If the integral in question is finite, the propagator factors in

the denominator can be conveniently rewritten using the Gaussian parameterization

$$\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int_0^\infty d\lambda \, \lambda^{n-1} e^{-a\lambda}, \tag{A.1}$$

upon which the relevant integrals can be brought into the form

$$\mathcal{K}_\nu(Y, Z) = \int_0^\infty d\lambda \, \lambda^{\nu-1} e^{-Z\lambda - Y/\lambda} = 2 \left(\frac{Y}{Z}\right)^{\frac{\nu}{2}} K_\nu\left(2\sqrt{YZ}\right). \tag{A.2}$$

Also, the expansion of the finite-volume integrals to $\mathcal{O}(\varepsilon)$ around $d = 4$ generates the related functions

$$\tilde{\mathcal{K}}_\nu(Y, Z) \equiv \frac{1}{2} \ln\left(\frac{Y}{Z}\right) \mathcal{K}_\nu(Y, Z) + 2 \left(\frac{Y}{Z}\right)^{\frac{\nu}{2}} \tilde{K}_\nu\left(2\sqrt{YZ}\right), \tag{A.3}$$

where $\tilde{K}_\nu(z) \equiv \partial K_\nu(z)/\partial\nu$ denotes the derivative of the modified Bessel functions w.r.t. the order ν. Further, differentiation of $\mathcal{K}_\nu(Y, Z)$ w.r.t. p^2 involves the functions $\mathcal{K}'_\nu(Y, Z)$, given by

$$\mathcal{K}'_\nu(Y, Z) \equiv \frac{\partial \mathcal{K}_\nu(Y, Z)}{\partial p^2} = \frac{\partial Z(p^2)}{\partial p^2} \left(\frac{Y}{Z(p^2)}\right)^{\frac{\nu}{2}}$$
$$\times \left[\left(\frac{Y}{Z(p^2)}\right)^{\frac{1}{2}} \tilde{K}'_\nu\left(2\sqrt{YZ(p^2)}\right) - \frac{\nu}{2Z(p^2)} \tilde{K}_\nu\left(2\sqrt{YZ(p^2)}\right)\right], \tag{A.4}$$

where $K'_\nu(z) \equiv dK_\nu(z)/dz$. For clarity, the dependence on p^2 has been made explicit in eq. (A.4). The modified Bessel functions satisfy $K_{-\nu}(z) = K_\nu(z)$, as well as the recursion relation

$$K_{\nu+1}(z) = \frac{2\nu}{z} K_\nu(z) + K_{\nu-1}(z). \tag{A.5}$$

The derivatives are given by

$$K'_\nu(z) \equiv \frac{d}{dz} K_\nu(z) = -K_{\nu-1}(z) - \frac{\nu}{z} K_\nu(z), \tag{A.6}$$

which are also directly provided by standard computer libraries for the Bessel functions. The $\tilde{K}_\nu(z) \equiv \partial K_\nu(z)/\partial\nu$ can be expressed in terms of the K_ν themselves via

$$\begin{aligned}
\tilde{K}_0(z) &= 0, \\
\tilde{K}_1(z) &= \frac{1}{z} K_0(z), \\
\tilde{K}_2(z) &= \frac{2}{z} K_1(z) + \frac{2}{z^2} K_0(z), \\
\tilde{K}_3(z) &= \frac{3}{z} K_2(z) + \frac{6}{z^2} K_1(z) + \frac{8}{z^3} K_0(z), \\
\tilde{K}_n(z) &= \frac{n!}{2} \sum_{k=0}^{n-1} \left(\frac{z}{2}\right)^{k-n} \frac{K_k(z)}{(n-k)k!},
\end{aligned} \tag{A.7}$$

where higher orders than those given explicitly are not needed for the present considerations. Finally, for large values of z, the modified Bessel functions behave as

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z} + \mathcal{O}\left(\frac{e^{-z}}{z^{3/2}}\right),\tag{A.8}$$

which leads to an exponential fall-off for large values of the argument.

B Theta functions

In the main text, we make use of a variety of theta functions. For the one-loop integrals, the third Jacobi theta function

$$\theta_3(u|\tau) \equiv \sum_n e^{\pi i(\tau n^2 + 2nu)},\tag{B.1}$$

is needed, for which an alternative definition is

$$\theta_3(u, q) \equiv \sum_n q^{(n^2)} e^{\pi i 2nu} = 1 + 2\sum_{n>0} q^{(n^2)} \cos(2\pi nu),\tag{B.2}$$

where $\tau \equiv -\frac{i}{\pi}\log q$. In the literature, the arguments q and τ are often suppressed, and the factor of π in the argument of the cosine may also be absent. The Jacobi theta function is defined for $\operatorname{Im}\tau > 0$ or $|q| < 1$, such that the series converges absolutely. An important property of θ_3 is the "modulus symmetry"

$$\theta_3(u + 1|\tau) = \theta_3(u|\tau), \qquad \theta_3(u|\tau) = \frac{1}{\sqrt{-i\tau}}\, e^{-\pi i \frac{u^2}{\tau}}\, \theta_3\left(\frac{u}{\tau}\middle|\frac{-1}{\tau}\right),\tag{B.3}$$

which is also known as Jacobi's imaginary transformation. For small q, the summation can be evaluated directly, and for larger q the second relation in eq. (B.3) may be used to obtain rapid convergence.

We also need the Riemann theta function in g dimensions, defined by

$$\theta^{(g)}(z|\tau) \equiv \sum_{n\in\mathbb{Z}^g} e^{2\pi i\left(\frac{1}{2}n^T \tau n + n^T z\right)},\tag{B.4}$$

where n denotes a g-dimensional column vector with integer components, z is a complex, g-dimensional column vector and τ is a complex, symmetric matrix with a positive-definite imaginary part. The latter requirement ensures that the summation over n converges absolutely. We note that the most commonly encountered notation is simply θ. The Riemann theta function also satisfies a modular symmetry, generated by the transformations

$$\theta^{(g)}(z + y|\tau) = \theta^{(g)}(z|\tau),$$
$$\theta^{(g)}(z|\tau) = \theta^{(g)}(az|a\tau a^T),$$
$$\theta^{(g)}(z|\tau + b) = \theta^{(g)}\left(z + \frac{1}{2}\mathrm{diag}(b)\Big|\tau\right),$$
$$\theta^{(g)}(\tau^{-1}z| - \tau^{-1}) = \sqrt{\det(-i\tau)}\, e^{\pi i z^T \tau^{-1} z}\, \theta^{(g)}(z|\tau),\tag{B.5}$$

where y denotes a column vector with integer components, a and a^{-1} are both $g \times g$ matrices with integer elements, and b is a symmetric $g \times g$ matrix with integer elements as well. The use of these transformations for the efficient evaluation of the Riemann theta function is explained in ref. [27]. The instances of the Jacobi and Riemann theta functions used in the main text are

$$\theta_{30}(q) \equiv \sum_n q^{(n^2)} = \theta_3(u = 0, q),$$

$$\theta_{32}(q) \equiv \sum_n n^2 q^{(n^2)} e^{-xn^2} = q\frac{\partial}{\partial q}\,\theta_3(u = 0, q),$$

$$\theta_{34}(q) \equiv \sum_n n^4 q^{(n^2)} = \left(q\frac{\partial}{\partial q}\right)^2 \theta_3(u = 0, q),$$

$$\theta_0^{(2)}(\alpha, \beta, \gamma) \equiv \sum_{n_1, n_2} e^{-\alpha n_1^2 - \beta n_2^2 - \gamma(n_1 - n_2)^2},$$

$$\theta_{02}^{(2)}(\alpha, \beta, \gamma) \equiv \sum_{n_1, n_2} n_1^2\, e^{-\alpha n_1^2 - \beta n_2^2 - \gamma(n_1 - n_2)^2}, \tag{B.6}$$

where it should be noted that $\theta_0^{(2)}(\alpha, \beta, \gamma)$ is fully symmetric in the arguments, and that $\theta_{02}^{(2)} = -(\partial/\partial\alpha)\,\theta^{(2)}$.

C Integrals in arbitrary dimensions

When the finite-volume integrals contain a non-local divergence, the expressions

$$\int \frac{d^d r}{(2\pi)^d} \frac{1}{(r^2 + \Delta)^n} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \Delta^{\frac{d}{2} - n}, \tag{C.1}$$

$$\int \frac{d^d r}{(2\pi)^d} \frac{r_\mu r_\nu}{(r^2 + \Delta)^n} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(n - \frac{d}{2} - 1\right)}{\Gamma(n)} \Delta^{\frac{d}{2} - n + 1} \frac{\delta_{\mu\nu}}{2}, \tag{C.2}$$

are used in Euclidean space for arbitrary dimensions $d \equiv 4 - 2\varepsilon$. As detailed in the main text, the expansion of the above results around $\varepsilon = 0$ allows for the non-local divergences to be isolated. We also recall some further results for arbitrary d,

$$\int d^d r = r^{d-1} dr\, d\Omega_d = \frac{2}{\Gamma(\frac{d}{2})} \pi^{\frac{d}{2}} r^{d-1} dr,$$

$$\int \frac{d^d \tilde{r}}{(2\pi)^d}\, e^{-\tilde{r}^2} = \frac{1}{(4\pi)^{\frac{d}{2}}},$$

$$\int \frac{d^d \tilde{r}}{(2\pi)^d}\, \tilde{r}^2 e^{-\tilde{r}^2} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{d}{2},$$

$$\int \frac{d^d \tilde{r}}{(2\pi)^d}\, \tilde{r}^4 e^{-\tilde{r}^2} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{d}{2}\left(\frac{d}{2} + 1\right), \tag{C.3}$$

which are used throughout the main text.

n	n_1	n_2	n_3
1	1	1	1
2	2	1	1
3	1	2	1
4	1	1	2
5	2	2	1
6	2	1	2
7	1	2	2
8	2	2	2

Table 4. Table of "pole configurations", i.e. the relationship between the collective index n and the exponents n_1, n_2 and n_3 of the propagator factors $(p^2 + m_i^2)^{n_i}$ in the sunset integrals.

D Notation for double poles

In the main text, the notation $A(n, m^2)$ and $B(n_1, n_2, m_1^2, m_2^2, p^2)$ has been used for the one-loop integrals with one and two propagators, respectively. However, we wish to remind the reader that the established notation in the literature reserves the symbol A for $A(1, m^2)$ and the symbol B for $B(1, 1, m_1^2, m_2^2, p^2)$. Along these lines, integrals with three and four propagators are usually denoted C and D, respectively.

For the sunset integrals in PQχPT, some or all of the propagators can appear doubled. This gives eight possible configurations of single and double poles. In earlier NNLO work on PQχPT, a collective index n was introduced to specify the pole configuration [7–9], as a short-hand notation for the triplet (n_1, n_2, n_3). The correspondence is shown in table 4. It should be noted that the cases of $n = 4$ and $n = 6$ are superfluous due to integral relations, and the case of $n = 8$ appears only in calculations of the flavour-neutral meson properties in PQχPT.

E Translation to Minkowski conventions

While we have used the Euclidean formalism throughout, it is also of interest to recall how the expressions for the one-loop and sunset integrals can be translated to Minkowski conventions. The required substitutions are

$$
\begin{aligned}
\int \frac{d^q r}{(2\pi)^d} &\longrightarrow \frac{1}{i} \int \frac{d^q r}{(2\pi)^d}, \\
\delta_{\mu\nu} &\longrightarrow -g_{\mu\nu} \\
p \cdot q, \ p^2 &\longrightarrow -p \cdot q, \ -p^2 \\
t_{\mu\nu} &\longrightarrow -t_{\mu\nu} \\
\frac{1}{p^2 + m^2} &\longrightarrow -\frac{1}{p^2 - m^2},
\end{aligned}
\tag{E.1}
$$

where $t_{\mu\nu}$ corresponds to the spatial part of the metric.

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