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Consistent interactions and involution

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Abstract: Starting from the concept of involution of field equations, a universal method is proposed for constructing consistent interactions between the fields. The method equally well applies to the Lagrangian and non-Lagrangian equations and it is explicitly covariant. No auxiliary fields are introduced. The equations may have (or have no) gauge symmetry and/or second class constraints in Hamiltonian formalism, providing the theory admits a Hamiltonian description. In every case the method identifies all the consistent interactions.

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1 Introduction

In this paper, a universal method is suggested for solving the problem of consistent interactions in the general field theories. The method equally well applies to the Lagrangian or non-Lagrangian field equations, and it does not break the explicit covariance. The field equations may have (or have no) gauge symmetry, and/or have (or don't have) the second class constraints in the corresponding Hamiltonian form in the variational case - the method provides identification of all the consistent interactions, in all the instances.

The proposed method is based on the idea of involutive form of the field equations. Every regular system of field equations can be equivalently reformulated in the involutive form. The gauge algebra of the involutive equations is more rich, in general, then the algebra of the equivalent non-involutive system. In particular, the involutive system may have gauge identities even if the theory does not have any gauge invariance. The consistency is completely controlled by the stability of the gauge algebra of involutive system with respect to inclusion of the interaction.

The paper is organized as follows. In the next section, we explain the problem setting, and discuss previously known methods of its solution. In section 3, we explain the notion of the involutive form of field equations and preview the basic idea of our method. In section 4, we describe the gauge algebra associated to the involutive system of field equations. Then, we explain the procedure of perturbative inclusion of interactions in the involutive systems. In section 5, we consider the examples of interactions in massive spin 1 and 2 models, to illustrate the general method. The conclusion contains a brief discussion of the paper results. In the appendix we derive the relation (3.8) which is used in the paper for counting physical degrees of freedom.

2 The consistency problem of interactions

The most common view on the problem of consistent interactions can be roughly formulated in the following way. The starting point is a system of linear field equations, or a quadratic Lagrangian, which is supposed to be covariant with respect to certain global symmetry group (the most often examples are Poincaré, AdS or conformal groups). The free field model has a certain number of physical degrees of freedom. Switching on an interaction means inclusion of nonlinear covariant terms into the free equations. The interaction is said consistent if and only if the nonlinear theory has the same number of physical degrees of freedom as the original free model has had.

If the free field equations and the interacting ones follow from the least action principle, the Dirac-Bergmann algorithm [1] will allow one to examine the number of physical degrees of freedom and to check in this way consistency of the interaction. There is an extension of this algorithm [2] which is applicable to general regular dynamics with not necessarily Lagrangian equations. The Dirac-Bergmann technique and its extensions, however, break the explicit covariance. Because of that, these algorithms can help to examine the consistency of particular interaction a posteriori, but they are hardly able to serve as a tool for covariant derivation of the consistent interactions.

The explicitly covariant method is known for solving the interaction consistency problem for the gauge fields in Lagrangian theories (for introductory review see [3]). The basic idea of this method is that both the action and the gauge symmetry must be simultaneously deformed by interaction in such a way that the number of gauge transformations for the Lagrangian would remain the same after inclusion of the interaction as it has been in the free theory, though the symmetry can change. The most systematic form of this method, being based on the cohomological view of the problem, is known in the framework of the BV-BRST formalism [3, 4]. In its turn, the BRST approach to the consistency problem is based on the general theory of local BRST cohomology [5]. Many gauge field theories are known where this method either has delivered the complete solution to the consistency problem, or established a no-go theorem for the interactions. We mention several examples of such results: establishing all the consistent interactions for p-form fields [6], and the no-go theorem for graviton-graviton interactions [7]. There are numerous results of a similar type obtained in various models by various versions of this method during the two recent decades. Among the most recent ones we mention the work [8], where all cubic electromagnetic interactions are found by this method for the higher-spin fermionic fields in Minkowski space, and it is proven that the minimal couplings are not admissible.

The above mentioned method is so popular because of the explicit covariance and algebraic elegance. This method, however, is unable to provide a solution for the consistency interaction problem for any field theory. The matter is that gauge invariance is not the only cause that makes "nonphysical" some of degrees of freedom. For example, the number of

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physical degrees of freedom is less than the number of fields for the second class constrained models, though there is no gauge symmetry. That is why, even in gauge invariant systems, the non-violation of gauge symmetry by the interaction does not necessarily mean the consistency of interaction. With this regard, it is relevant to mention the idea of introducing Stückelberg fields to gauge the massive models, and then apply the usual method to introduce the gauge invariant interactions. Many works apply the Stückelberg gauge symmetry idea to find the consistent interactions or examine consistency for various massive fields. Among the examples of this type we can mention the series of recent works on massive gravity [9–11] (further references can be found in these articles, and also in [12]). Inclusion of the Stückelberg fields is rather an art than a science at the moment. No general prescription is known of doing that in such a way that could ensure the consistency of interaction just as a consequence of its consistency with the Stückelberg symmetry. To examine consistency of interactions in the Stückelberg gauged model, a non-covariant Dirac-Bergmann constrained analysis still remains inevitable in many cases, see for example [10, 11].

In this paper we suggest to control the consistency of interaction by exploiting the involutive form of the field equations. As we explain in the next section, any field theory model can be equivalently formulated in the involutive form, and the result of such a reformulation is termed an *involutive closure*. Normally, the involutive closure retains all of the symmetry of the original system that makes the method convenient for studying covariant field equations. The involutive closure of the Lagrangian equations is generally not Lagrangian anymore, even for such a simple model as Proca equations. There is no pairing between gauge identities and gauge symmetries[1] in the involutive closures of the Lagrangian equations in general. In particular, even if the Lagrangian does not have gauge symmetry, the involutive closure of the Lagrangian equations can possess non-trivial gauge identities. The gauge algebra of the involutive closure of the field equations, with unrelated gauge symmetries and identities, turns out to be more rich, in general, than the gauge algebra of the equivalent non-involutive system. It is the structure of the gauge algebra of the involutive closure of the field equations that controls the number of physical degrees of freedom. That is why, when the interactions are included, it is the stability of the gauge algebra of the involutive closure of the field equations that ensures the consistency of the theory.

The idea of involution is well developed in the theory of ODE's and PDE's, and it is effectively applied to a broad range of the problems in various fields, see [13]. It has been never systematically studied, however, as a tool for controlling the consistency of interactions - to the best of our knowledge. In the next section we explain some of the basic notions related to the involutive systems which are relevant in the context of consistent interaction problem. Our consideration, being at the physical rigor level, skips subtleties of the theory of involutive systems (for a rigorous review see [13]), because we focus at the issues of gauge algebra that are underdeveloped in this field at the moment.

[1]The gauge identities are often termed as the Noether ones, because the second Noether theorem states the isomorphism between the identities and symmetries. For the general non-Lagrangian system of equations, including involutive closure of Lagrangian one, there is no automatic Noether's correspondence between symmetries and identities. That is why, we do not use the term "Noether identity" to avoid the impression that it is related to any gauge symmetry.

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3 Involutive form of field equations

By the order of a field equation we understand the maximal order of the derivatives of the fields in the equation. The maximal order of the equations in the system is said to be the order of the system.

Definition. *A system of order t is said involutive if any differential consequence of the order less than or equal to t is already contained in the system.*

In the theory of involutive PDE's [13], the definition above is normally complemented by more strong and intricate requirements which are inessential here, given the physical rigor level accepted and the problem addressed, so we adopt this simple definition.

If the system of field equations is supplemented by all the differential consequences of the orders lower or equal to the order of the system, it becomes involutive. The system brought to involution in this way is said to be the *involutive closure* of the original system. Obviously, the original system is equivalent to its involutive closure in the sense that all the solutions are the same for both systems.

If the system of field equations can be brought to the involutive closure just by inclusion derivatives of some of the equations, we will also consider it involutive,[2] though it is not, according to the definition above. This simplification is convenient, and it does not lead to any contradiction as far as the problem of consistent interactions is concerned.

To illustrate the distinctions between the involutive and non-involutive form of field equations from the viewpoint of variational principle and gauge algebra, consider the example.

Example. *Irreducible massive spin 1 field in $d = 4$ Minkowski space.* The field equations of the model include the Klein-Gordon equations and the transversality condition for the vector field,

$$T_\mu \equiv (\Box - m^2)A_\mu = 0\,, \qquad ord(T_\mu) = 2 \tag{3.1}$$
$$T_\perp \equiv \partial^\mu A_\mu = 0\,, \qquad ord\,(T_\perp) = 1 \tag{3.2}$$

The order of equations is denoted by the symbol *ord*. This system is involutive and it is obviously non-Lagrangian as it contains five equations for the four component field A_μ. There is no gauge symmetry, though there exists a non-trivial gauge identity between the equations (3.1) and (3.2):

$$\partial^\mu T_\mu - (\Box - m^2)T_\perp \equiv 0\,. \tag{3.3}$$

It is useful to introduce the generator of gauge identity L, such that the identity would read

$$L^a T_a \equiv 0\,, \quad a = (\mu, \perp)\,, \qquad L^\mu = \partial^\mu\,, \quad L^\perp = -(\Box - m^2) \tag{3.4}$$

Define now a notion of the total order of gauge identity. Suppose the a-th component of the generator of the gauge identity L^a is a differential operator of order s_a, and the order

[2] In the theory of PDEs such consequences are termed trivial integrability conditions. These are inessential for the count of physical degrees of freedom.

of the equation T_a is t_a. The order of the a-th identity component l_a is defined as the sum $l_a = s_a + t_a$. The total order of the gauge identity generated by L is defined as the maximum

$$ord(L) = \max_a \{l_a\} \,. \tag{3.5}$$

In the case of the identity (3.3), the differential operator $L^\perp = -(\Box - m^2)$ has the order $s_\perp = 2$, and it acts on the first order equation $T_\perp$, so the order of the transverse component of identity $l_\perp = 2+1 = 3$. The order of the differential operator $L^\mu = \partial^\mu$ is $s^\mu = 1$, and it acts on the second order equation T_μ, so the order of this identity component is the same, $l_\mu = 1+2 = 3$. As we see, the total order of the gauge identity (3.3) is 3.

Alternatively, the same massive vector field model is described by the Proca equations

$$P_\mu \equiv (\delta^\mu_\nu \Box - \partial_\mu \partial^\nu - m^2\, \delta^\nu_\mu) A_\nu = 0 \,, \qquad \mathrm{ord}(P_\mu) = 2 \,, \tag{3.6}$$

that are Lagrangian. The Proca equations follow from (3.1), (3.2):

$$P_\mu \equiv T_\mu - \partial_\mu T_\perp \,,$$

and vice versa, the Klein-Gordon and transversality equations follow from (3.6),

$$T_\perp \equiv -m^{-2} \partial^\mu P_\mu \,, \qquad T_\mu \equiv (\delta^\mu_\nu - m^{-2} \partial_\mu \partial^\nu) P_\nu \,.$$

Notice that the Proca equations, being of the second order, have the first order differential consequence - the transversality condition (3.2). This means that the Proca system is not involutive. Obviously, (3.6) and (3.1), (3.2) are equivalent systems of equations. The involutive closure of Proca equations, that includes (3.6) and their first order consequence (3.2), is an involutive non-Lagrangian system that contains the gauge identity of the third order:

$$\partial^\mu P_\mu + m^2 T_\perp \equiv 0 \tag{3.7}$$

One can observe that the involutive form of the spin one equations, being non-Lagrangian, has a gauge identity, so its gauge algebra is non-trivial. Quite opposite, the Lagrangian Proca equations equations are not involutive and have trivial gauge algebra, without any gauge identity.

A similar conclusion as in the above example holds in general: if the Lagrangian equations are not involutive, their involutive closure, being a non-Lagrangian system, will have the gauge algebra with more independent gauge identity generators than the original Lagrangian system has had. Notice that if the Lagrangian system has a gauge symmetry, its involutive closure will obviously have the same symmetry, so the gauge algebra of the involutive closure will always have the original gauge symmetry as a subalgebra.

Running a couple steps ahead, notice that the structure of the gauge algebra of involutive system unambiguously defines the number of physical degrees of freedom $\mathcal{N}$ by the following formula:

$$\mathcal{N} = \sum_{k=0}^{\infty} k(t_k - l_k - r_k) \,. \tag{3.8}$$

Here t_k is a number of equations of order k in the involutive system, l_k is the number of gauge identities of k-th total order, and r_k is the number of gauge symmetry generators of kth order.[3] In this formula, both the gauge symmetry and gauge identity generators are supposed irreducible. The formula is derived in the appendix, where one can also find its extension to the case of reducible gauge identities and symmetries. The number of physical degrees of freedom is understood here as a number of the Cauchy data needed to define a solution modulo gauge ambiguity. In variational case it coincides with the dimension of the reduced phase space. So, if the configuration space count is done, $\mathcal{N}$ has to be divided by 2.

In the example above of the involutive equations for the massive spin-1 in $d = 4$, we have one first order equation and four second order ones, so $t_2 = 4, t_1 = 1$. There is one identity of the third order $l_3 = 1$, and no gauge symmetry. Substituting these numbers into the general formula (3.8) one obtains $\mathcal{N} = 1 \cdot 1 + 2 \cdot 4 - 3 \cdot 1 = 6$ that provides the correct answer, as the massive vector field has 3 physical polarizations, and the physical phase space is 6-dimensional.

Consider one more example illustrating relation (3.8): a scalar field $\phi(t, x)$ in two-dimensional space subject to a pair of the second order equations

$$T_t \equiv \partial_t^2 \phi(t,x) = 0\,, \quad T_x \equiv \partial_x^2 \phi(t,x) = 0\,, \qquad ord(T_t) = ord(T_x) = 2\,. \tag{3.9}$$

There are no differential consequences of the second or lower orders, so the system is in involution. There is the fourth order gauge identity,

$$\partial_t^2 T_x - \partial_x^2 T_t \equiv 0\,. \tag{3.10}$$

With two second order equations ($t_2 = 2$) and one forth order identity ($l_4 = 1$), relation (3.8) brings zero as the number of physical degrees of freedom. Let us directly check that it is a correct count. The general solution reads

$$\phi(t,x) = Axt + Bx + Ct + D\,, \tag{3.11}$$

with A, B, C, D being arbitrary integration constants. No arbitrary function of x or t is involved in the general solution. This means that the system, being $2d$ field theory, has no local physical degrees of freedom.

Let us comment on formula (3.8). At first, we notice that the relation is valid for involutive equations. So, prior to applying this formula to a non-involutive system, one has to take an involutive closure. The involutive closure can be taken in an explicitly covariant way for covariant equations, it does not require any $(3+1)$-splitting, unlike the Dirac-Bergman algorithm. Also, it is important that the involutive closure can be found for any system, be it Lagrangian or not, and it does not require any special (e.g., first order)

[3] By the order of gauge symmetry generator we understand the highest order of the derivative of the gauge parameter involved in the gauge transformation of the fields. From this point of view, in Lagrangian theory, where the gauge identities and gauge symmetries have the same generators, the order of symmetry can be different from the total order of the identity. This is because the total order of identity is defined taking into account the order of the equations it involves, while the order of the gauge symmetry is indifferent to the order of equations.

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formulation. The orders of equations, symmetries and identities can be easily found, so formula (3.8) provides a simple tool to covariantly control the number of physical degrees of freedom. The second peculiarity of this formula is that it is somewhat counterintuitive: it involves neither the number of fields in the system, nor is it sensitive to the number of zero order gauge symmetries, identities and equations.

The problem of computing the number of the physical degrees of freedom has been thoroughly studied in the theory of involutive systems [13], though the answer has been known in a completely different terms,[4] not explicitly appealing to the total orders of equations, identities and symmetries. In the context of the problem of consistent interactions, where the structure of the gauge algebra is the principal object to study, it is important to control the degrees of freedom in terms of the gauge algebra constituents. In the appendix we deduce the formula (3.8) proceeding from the definition accepted in the theory of involutive systems and based on the concept of strength of a system of equations. This concept was introduced by Einstein when he counted degrees of freedom in General Relativity [14] and it has been further developed in many works (among which we mention [15–18]) and related to the count by Cauchy data [13].

Notice that for Lagrangian equations, whenever they are involutive from the outset (in this case, the gauge identity generators coincide with the gauge symmetry ones), the receipt (3.8) for the degree of freedom count takes a special form (A.25), which has been well known before [19]. Notice another special form of field equations where the asymmetry may occur between gauge identities and symmetries: the unfolded formulation of the higher spin fields (for review see [20]). This method utilizes the involutive form of the unfolded equations, and it also benefits from the fact that all the equations, symmetry and identity generators are of the first order. The unfolded formalism involves, however, infinite number of field equations, symmetries and identities. Formula (3.8) can not be immediately applied to the unfolded systems because all the numbers in (3.8) are supposed to be finite. In this case, the method of σ_--cohomology [21–23] provides a tool for the degree of freedom count. The σ_--cohomology method allows one to pick out a finite involutive subsystem such that the unfolded system will be its infinite jet prolongation. The degree of freedom count in the finite subsystem, being made by the formula (3.8), delivers the answer for the number of degrees of freedom in the complete unfolded theory.

Let us formulate now the key stages of the procedure we suggest for constructing consistent interactions, given the original free field equations:

1. The free system is to be brought to the involutive form.

2. All the gauge symmetries and identities are to be identified in the free involutive system.

3. The interaction vertices are perturbatively included to comply with the three basic requirements in every order of coupling constant:

[4]The analysis is made by means of the theory of Cartan's differential systems, and the answers are formulated in terms of the Hilbert polynomials [13]. In principle, this way of counting physical degrees of freedom is sufficient, though it seems inconvenient in the context of relativistic field theories because it requires cumbersome and not always explicitly covariant computations.

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(a) The field equations have to remain involutive;

(b) The gauge algebra of the involutive system can be deformed, though the number of gauge symmetry and gauge identity generators remains the same as it has been in the free theory;

(c) The number of physical degrees of freedom, being established by relation (3.8), cannot change, though some of the involved orders can.

This procedure ensures finding all the consistent interaction vertices, for any regular system of field equations.

4 Gauge algebra of involutive systems

As it has been already explained, if the Lagrangian system of field equations is not involutive, its involutive closure will be non-Lagrangian. It is the structure of the gauge algebra of the involutive closure, not the original system of equations, that controls number of physical degrees of freedom. That is why, one has to study the gauge algebra of dynamics in its non-Lagrangian involutive form even if the Lagrangian exists. The exception is the case where the Lagrangian equations are involutive from the outset. In this special case, the known methods [3] work well, being insufficient for general Lagrangians. Notice once again that the non-involutive Lagrangian equations are not exceptional - many models in physics are of this class, e.g. massive fields with spin. This leads us to consider first the gauge algebra of the non-Lagrangian dynamics.

The gauge algebra of the general (not necessarily Lagrangian) system is known in the same details as in the Lagrangian case, and the corresponding BRST complex is also well studied [24, 25] that allows one to systematically control all the compatibility conditions. Below, we provide a simplified description of the gauge algebra, without resorting to the corresponding cohomological tools and leaving aside the higher compatibility conditions, as these are less important in the context of interaction problem.

4.1 Algebra of gauge symmetries and identities in general field theory

It is common to consider general structures of gauge algebra by making use of the condensed notation, and we will follow this tradition as it is convenient for presenting the general idea of the method. In this notation, the fields are collectively denoted by ϕ^i, with i being the condensed index that includes all the discrete indices, and also the space-time points. For example, the vector field $A^\mu(x)$ is indexed by $i = (\mu, x)$. Summation over the condensed index implies integration over x.

In the condensed notation, any system of field equations reads

$$T_a(\phi) = 0\,, \qquad ord(T_a) = t_a, \tag{4.1}$$

where a is a condensed index, and T_a is understood as a function of the fields and their space-time derivatives up to some finite order t_a. The discrete part of the condensed index a labeling the equations is different, in general, from that of the condensed index

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i numbering the fields. For the Lagrangian equations, i coincides with a, though this is not true if the involutive closure is considered instead of the original equations. For example, the involutive closure of the Proca equations includes both the original Lagrangian equations (3.6) and the transversality condition (3.2), so the indices belong to the different sets: $i = (\mu, x)$, and $a = (\mu, \perp, x)$. For the regular field equations, the order t_a depends only on the discrete part of the index a, not on the space-time point.

The general field equations can enjoy gauge symmetry transformations

$$\delta_\epsilon \phi^i = \epsilon^\alpha R^i_\alpha(\phi)\,, \qquad \delta_\epsilon\, T_a(\phi)|_{T=0} = 0\,, \quad \forall \epsilon^\alpha\,, \qquad ord(R_\alpha) = r_\alpha < \infty\,, \tag{4.2}$$

where the gauge parameters ϵ^α and generators $R^i_\alpha(\phi)$ are understood in the sense of condensed notation, i.e. the summation over α implies integration over x. For example, in electrodynamics, $\delta_\epsilon A_\mu(x) = \partial_\mu \epsilon(x)$, and hence $i = (\mu, x)$, $\alpha = y$, $R_\mu(x, y) = \partial_\mu \delta(x - y)$, so that $\delta_\epsilon A_\mu(x) = \int dy R_\mu(x, y)\epsilon(y)$. Locality of the gauge symmetry implies that the gauge generators R_α are the differential operators of finite order, denoted r_α, with coefficients depending on the fields and their derivatives.

The condition (4.1) defines the on-shell invariance of the equations that off-shell reads

$$R^i_\alpha(\phi)\partial_i T_a(\phi) = U^b_{\alpha a}(\phi) T_b(\phi)\,, \tag{4.3}$$

where the derivative ∂_i is understood as variational and the structure coefficients $U^b_{\alpha a}(\phi)$ are supposed to be regular on shell.

The gauge identities can also take place for the general field equations, being not necessarily related to the gauge symmetries

$$L^a_A(\phi) T_a(\phi) \equiv 0\,, \qquad ord(L_A) = l_a\,. \tag{4.4}$$

The gauge identity generators L_A are supposed to be local differential operators. The total order of the identity $ord(L_A)$ is defined by the order of the differential operator and the order of the equation it acts on as explained in section 3 below relation (3.4).

The gauge symmetry and gauge identity generators are considered as trivial whenever they vanish on shell, that is

$$R^i_{\alpha(triv)} = \rho^{i\,a}_\alpha(\phi) T_a\,, \qquad L^a_{A(triv)} = \zeta^{ab}_A(\phi) T_b\,, \quad \zeta^{ab}_A = -\zeta^{ba}_A\,, \tag{4.5}$$

where ρ and ζ can be arbitrary local differential operators of finite order with the coefficients depending on the fields and their derivatives.

The sets $\{R^i_\alpha\}$, $\{L^a_A\}$ of the gauge symmetry and gauge identity generators are supposed to be complete. The completeness means that any other generator of gauge symmetry or identity, satisfying (4.3) or (4.4), must be a linear combination of the generators from the given set modulo the trivial ones (4.5).

Let us discuss now the equivalence relations for the systems of field equations. Two systems of field equations are considered as equivalent if they are related by a locally invertible change of fields and/or by locally invertible linear combination of the left hand sides of the equations. The admissible class of changes of fields reads

$$\phi^i \to \phi'^i = \phi'^i(\phi, \partial\phi, \partial^2\phi, \dots)\,, \tag{4.6}$$

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where the existence is implied for the inverse change belonging to the same class, i.e. the original fields ϕ have to be unambiguously determined by (4.6) as functions of the fields ϕ' and their derivatives up to some finite order.

For example, consider the system of vector and scalar field. The change $A'_\mu = A_\mu - \partial_\mu\phi$, $\phi' = \phi$ is admissible as the local inverse change exists.

Given a set of fields, the admissible class of equivalence transformations for the field equations reduces to the linear combining with invertible coefficients:

$$T_a \sim T'_a \;\Leftrightarrow\; T'_a = K_a^b(\phi)T_b\,, \qquad T_a = (K^{-1})_a^b(\phi)T'_b\,, \tag{4.7}$$

where the elements of the transformation matrices K and K^{-1} are the differential operators of finite order.

Assuming the completeness of the generators of gauge symmetries and gauge identities, one can derive the following consequences from the relations (4.3), (4.4):

$$R^j_\alpha\partial_j R^i_\beta - R^j_\beta\partial_j R^i_\alpha = U^\gamma_{\alpha\beta}R^i_\gamma + W^{i\,a}_{\alpha\beta}T_a\,; \tag{4.8}$$

$$R^j_\alpha\partial_j L^a_A = U^B_{\alpha A}L^a_B + W^{ab}_{\alpha A}T_b\,, \qquad W^{ab}_{\alpha A} = -W^{ba}_{\alpha A}\,, \tag{4.9}$$

where U, W are some structure functions. These relations have further compatibility conditions involving higher structure functions (see for details [24, 25]). The existence of all the higher structure functions and their locality have been proven in [26] under the condition that the generators L, R and the structure function U involved in (4.3) are all local. The corresponding existence theorem for Lagrangian theories has been known long before [27]. So, with the existence theorem, to ensure consistency of the field theory (4.1), it is sufficient to fulfill relations (4.3), (4.4) with some differential operators R, Z, U of finite order.

4.2 Gauge algebra and perturbative inclusion of interactions

Consider involutive system of free field equations

$$T^{(0)}_a(\phi) = 0\,, \qquad ord(T^{(0)}_a) = t^{(0)}_a\,. \tag{4.10}$$

As the free field equations are supposed to be linear, the generators of gauge symmetries and gauge identities are the differential operators with field-independent coefficients. With this regard, relations (4.3), (4.4) in the free theory should have identically vanishing on-shell terms:

$$R^{(0)i}_\alpha\partial_i T^{(0)}_a(\phi) \equiv 0\,, \qquad ord(R^{(0)}_\alpha) = r^{(0)}_\alpha\,; \tag{4.11}$$

$$L^{(0)a}_A T^{(0)}_a(\phi) \equiv 0\,, \qquad ord(L^{(0)}_A) = l^{(0)}_A\,. \tag{4.12}$$

Given the orders of the equations, gauge symmetries and identities, the number of physical degrees of freedom in the free model, $\mathcal{N}^{(0)}$, is defined by (3.8).

Perturbative inclusion of interaction is understood as a deformation of the equations, identities and gauge symmetries by nonlinear terms,

$$T^{(0)}_a \;\rightarrow\; T_a = T^{(0)}_a + gT^{(1)}_a + g^2T^{(2)}_a + \dots\,, \tag{4.13}$$

$$R^{(0)i}_\alpha \;\rightarrow\; R^i_\alpha = R^{(0)i}_\alpha + gR^{(1)i}_\alpha + g^2R^{(2)i}_\alpha + \dots\,, \tag{4.14}$$

$$L^{(0)a}_A \;\rightarrow\; L^a_A = L^{(0)a}_A + gL^{(1)a}_A + g^2L^{(2)a}_A + \dots\,. \tag{4.15}$$

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Here g is a coupling constant considered as formal deformation parameter, generators $L_A^{(1)a}$ and $R_\alpha^{(1)i}$ are linear in fields and their derivatives; $T_a^{(1)}$, $L_A^{(2)a}$, and $R_\alpha^{(2)i}$ are bi-linear, etc. Notice that in each order of the deformation, the orders of equations, identities and symmetries can never decrease.

Now, we can give a more specific formulation of the consistency conditions for the interactions than the general explanation in the end of section 3. The consistency of the interaction is provided if the three conditions are fulfilled: (a) the system remains involutive at each order in g; (b) the deformations do not break the gauge algebra generated by relations (4.3), (4.4), though the structure functions can change in (4.3) as well as the higher relations; (c) the orders of the equations, symmetries, and identities may increase, though the overall balance established by relation (3.8) cannot change, i.e., it is required $\mathcal{N}^{(0)} = \mathcal{N}$ in every order in g. The conditions (a) and (b) provide algebraic consistency of the system with perturbatively included interactions, and (c) ensures that the interacting system has the same number of physical degrees of freedom.

Let us elaborate on the perturbative procedure of the interaction inclusion. Suppose we have taken the most general ansatz for $T_a^{(1)}$, $ord(T_a^{(0)} + gT_a^{(1)}) = t_a^{(1)}$, that does not break involutivity, so (a) is fulfilled. Substituting this ansatz into relations (4.3), (4.4) and considering that in the first order in g, we find the following relations between $T^{(1)}, R^{(1)}, L^{(1)}$:

$$R_\alpha^{(0)i}\partial_i T_a^{(1)} = U_{\alpha a}^{(1)b} T_b^{(0)} - R_\alpha^{(1)i}\partial_i T_a^{(0)}\,, \tag{4.16}$$

$$L_A^{(0)a} T_a^{(1)} + L_A^{(1)a} T_a^{(0)} = 0. \tag{4.17}$$

These relations impose nontrivial restrictions on the first order interaction.

The first relation means that the free theory gauge transformation has to leave the first order interaction on-shell invariant modulo linear combination of variations of free equations. "On-shell" hereafter means on the free equations.

The second relation means that the free gauge identity generators must leave the first order interaction on-shell invariant. Notice that even if the model has no gauge symmetry, the involutive closure of its equations can have non-trivial gauge identities. This means that the conditions (4.17) are essential for consistency of interactions even in the systems without any gauge invariance. If the equations are Lagrangian and involutive from the outset,[5] relations (4.17) are reduced to the the on-shell gauge invariance of the cubic vertices in the Lagrangian. For the involutive Lagrangian equations, the generators R and L coincide, and the relations (4.16) follow from (4.17) in this case. For general system, including the involutive closure of the Lagrangian equations, the relations (4.17) are not necessarily connected with (4.16). The relations (4.16) are always first examined in the Lagrangian case (see [3]–[8]) to check the first order consistency of the interaction in Lagrangian dynamics. As is seen from the explanations above, if the Lagrangian field equations are not involutive, the first order consistency of interaction requires to independently impose the extra conditions (4.17) on the vertices, because these are not necessarily connected to the gauge symmetry of the Lagrangian.

[5] As it has been already noticed in section 3, there are many non-involutive Lagrangian equations being of a considerable interest in physics.

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Given a free model, the solution does not necessarily exist in any theory for the first order consistency conditions (4.16), (4.17) imposed on the first order interactions $T^{(1)}$ and the corresponding first order corrections to the gauge identity and gauge symmetry generators $L^{(1)}, R^{(1)}$. If a solution exists, it can be explicitly found as the system is linear.

The solutions for interactions are considered modulo ambiguities related to the equivalence relations (4.7), (4.6). In particular, a nonlinear change of fields in the free equations is not considered as an interaction as well as a linear combination of the free equations with field-dependent coefficients.

If the order $t_a^{(1)}$ increases because of the first order interactions, the orders of gauge identity and symmetry generators should also increase in a corresponding way to have the same number of physical degrees of freedom (3.8). If a solution to (4.16), (4.17) exists with the correct $\mathcal{N}$, one can proceed to the next order.

In the second order in g, the basic relations of the gauge algebra (4.3), (4.4) read

$$R_\alpha^{(0)i}\partial_i T_a^{(2)} + R_\alpha^{(1)i}\partial_i T_a^{(1)} + R_\alpha^{(2)i}\partial_i T_a^{(0)} = U_{\alpha a}^{(1)b} T_b^{(1)} + U_{\alpha a}^{(2)b} T_b^{(0)} \,, \tag{4.18}$$

$$L_A^{(0)a} T_a^{(2)} + L_A^{(1)a} T_a^{(1)} + L_A^{(2)a} T_a^{(0)} = 0 \,. \tag{4.19}$$

In the first instance, these relations represent further compatibility conditions for the first order interaction. Let us explain that in the case of relation (4.19). On substituting into (4.19) the expressions for $L_A^{(1)a}, T_a^{(1)}$ previously derived from (4.16), (4.17), one has to get a combination of the free theory gauge identity generators $L^{(0)}$ modulo free equations. This requirement is not automatically fulfilled for any interaction derived from (4.16), (4.17). In some models it can be even possible that these relations are inconsistent. In this case, one arrives at a no-go theorem for the interaction. So, the second order relations (4.18), (4.19) provide an additional selection mechanism for the first order interactions. If this filter is passed by the first order interactions, then relations (4.18), (4.19) can be viewed as a consistent algebraic system of linear equations defining the second order contributions to the equations, gauge symmetry and gauge identity generators: $T_a^{(2)}, R_\alpha^{(2)i}, L_A^{(2)a}$. The solution for the second order interaction is to be considered modulo the equivalence relations (4.6), (4.7). In particular, the nonlinear changes of fields or combinations of the lower order equations with field dependent coefficients are not considered as interactions. If the solution for $T_a^{(2)}$ involves the field derivatives of a higher order than $T_a^{(1)}$ and $T_a^{(0)}$, then the orders of the gauge symmetries and gauge identities have to increase in the corresponding way to provide the same number of physical degrees of freedom according to relation (3.8).

On substituting the second order interactions into the third order expansion terms of the relations (4.3), (4.4) one arrives at the relations that represent the consistency conditions for $T_a^{(2)}, R_\alpha^{(2)i}, L_A^{(2)a}$. This is much like the relations (4.18), (4.19) work for the previous order equations and generators. Again, there can be either inconsistency found at this stage, or one derives the third order interaction, and the procedure repeats in the next order. Three different scenarios are possible for further development of the iterative constructing the interactions. The first is that the iterative analysis of the expansion in g of the conditions (4.3), (4.4) will terminate at certain order because of contradiction.

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This results in a no-go theorem. The second is that starting from certain order all the interactions become trivial. This results in a polynomial interaction. The third option is that the procedure results in nontrivial consistent interactions in every order. This leads to a non-polynomial interaction.

In the case of involutive Lagrangian equations, this procedure reduces to the commonly known method of inclusion interactions between gauge fields (see [3] for a review). It has been already mentioned that the involutive closure of non-involutive Lagrangian equations is not Lagrangian anymore. Because of that, the Lagrangian method does not apply to this case, whereas the method of this section still works well, as well as for any other involutive system of field equations. Our method exploits the same general idea as the Lagrangian gauge approach: to include interactions by a consistent deformation of the equations and the related gauge algebra. The main distinctions are related to the fact that the general gauge algebra of involutive system involves gauge identities (4.4) independently from gauge symmetries, and the involutive form of equations allows one to effectively control the number of physical degrees of freedom.

5 Examples: consistent self-interactions of massive fields of spin 1 and 2

In this section we illustrate the general procedure of perturbative inclusion of consistent interactions described in section 4 by the examples of self-interactions for massive fields of spin one and two. Applying this method we find all the consistent interaction contributions (without higher derivatives) to the field equations of the second order in fields. For the corresponding Lagrangians this would correspond to the cubic vertices, though we find that some of the admissible interactions do not follow from any Lagrangian.

5.1 The massive spin 1 in d=4

As it has been explained in previous section, there are equivalence relations for the involutive field equations, so one can choose various representatives from the equivalence class of the free equations. For the spin 1, this choice is not unique either, as it has been explained in section 3. We choose the Proca equations and the transversality condition as free involutive equations for the spin 1,

$$T^{(0)}_\mu = \partial^\nu F_{\nu\mu} - m^2 A_\mu, \qquad T^{(0)}_\perp = \partial^\nu A_\nu\,, \qquad ord(T^{(0)}_\mu) = 2\,,\; ord(T^{(0)}_\perp) = 1\,. \tag{5.1}$$

In this section, we adopt the following agreement for the strength tensor and its dual: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. The choice of free equations in the form (5.1) is slightly more convenient than the other equivalent options, e.g. (3.1), (3.2), because the second order equations in the system (5.1) are Lagrangian that makes it easier to check the consistency of the interactions.

Equations (5.1) admit the gauge identity whose generator reads

$$L^{(0)\mu} = \partial^\mu, \qquad L^{(0)\perp} = m^2\,, \qquad L^{(0)\mu}T^{(0)}_\mu + L^{(0)\perp}T^{(0)}_\perp \equiv 0\,, \quad ord(L^{(0)}) = 3. \tag{5.2}$$

So, the involutive form of the massive spin-1 free field equations includes four second order equations and one of the first order together with the third-order gauge identity between them. (The general definition is provided by relation (3.5) for the total order of the gauge identity.)

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The next step according to the general procedure of perturbative inclusion of interactions, as described in section 4, is to take the most general covariant ansatz for the first order correction to the field equations that does not break involutivity. Let us assume that no higher order derivatives of the fields are included in the interactions.[6] Then the most general ansatz reads

$$
\begin{aligned}
T^{(1)}_\mu = {} & A^\alpha\Big(\rho_1 F_{\mu\alpha} + \rho_3 \widetilde{F}_{\mu\alpha}\Big) + \rho_2\Big(A^\alpha\partial_\alpha A_\mu - T^{(0)}_\perp A_\mu\Big) + \partial^\alpha A^\beta\Big(\rho_4\partial_\beta\widetilde{F}_{\alpha\mu} + \\
& + \rho_5\partial_\beta F_{\alpha\mu} + \rho_9\partial_\alpha\widetilde{F}_{\beta\mu}\Big) + \partial_\mu\Big(\rho_7\partial_\alpha A_\beta\partial^\alpha A^\beta + \rho_8\partial_\beta A_\alpha\partial^\alpha A^\beta + \rho_6 F_{\alpha\beta}\tilde{F}^{\alpha\beta}\Big) + \\
& + \zeta_1 A_\mu T^{(0)}_\perp + \zeta_2\partial_\mu A^\alpha\partial_\alpha T^{(0)}_\perp + \zeta_3\partial^\alpha A_\mu\partial_\alpha T^{(0)}_\perp + \zeta_4\widetilde{F}^\alpha{}_\mu\partial_\alpha T^{(0)}_\perp + \zeta_5\partial_\mu\Big(T^{(0)}_\perp\Big)^2 + \\
& + \zeta_6 T^{(0)}_\alpha\partial^\alpha A_\mu + \zeta_7 T^{(0)}_\alpha\partial_\mu A^\alpha + \zeta_8\widetilde{F}^\alpha{}_\mu T^{(0)}_\alpha + \zeta_9 T^{(0)}_\perp T^{(0)}_\mu\,, \\
T^{(1)}_\perp = {} & \rho_{10}F^{\alpha\beta}F_{\alpha\beta} + \rho_{11}\partial^\beta A^\alpha\partial_\alpha A_\beta + \rho_{12}F^{\alpha\beta}\widetilde{F}_{\alpha\beta} + \rho_{13}m^2 A_\alpha A^\alpha + \\
& + (\zeta_{10} + \frac{\rho_2}{m^2})\Big(T^{(0)}_\perp\Big)^2.
\end{aligned}
\tag{5.3}
$$

The vertices with ten ζ-coefficients are trivial as they are reduced to a linear combination of the free equations with field-dependent coefficients. Inclusion/exclusion of these vertices gives an example of the equivalence transformation (4.7) with

$$
\begin{aligned}
&(K_\zeta)_\mu^\nu = \delta^\mu_\nu + \zeta_6\partial^\nu A_\mu + \zeta_7\partial_\mu A^\nu + \zeta_8\widetilde{F}^\nu{}_\mu + \zeta_9\delta^\nu_\mu T^{(0)}_\perp, && (K_\zeta)^\perp_\perp = 1 + \zeta_{10}T^{(0)}_\perp\,, \\
&(K_\zeta)^\perp_\mu = \zeta_1 A_\mu + \Big(\zeta_2\partial_\mu A^\alpha + \zeta_3\partial^\alpha A_\mu + \zeta^4\widetilde{F}^\alpha{}_\mu\Big)\partial_\alpha + 2\zeta_5\partial_\mu T^{(0)}_\perp, && (K_\zeta)^\nu_\perp = 0\,.
\end{aligned}
\tag{5.4}
$$

The inverse transformation has the form $(K_\zeta)^{-1} = K_{-\zeta} + O(\zeta)$. We keep these terms to simplify the control of trivial vertices in the next order of interactions. Besides the trivial terms, the most general covariant quadratic ansatz (5.3) includes a 13-parametrer set of the non-trivial contributions with the coupling constants ρ.

Substituting the ansatz (5.3) into the structure relations (4.17), we obtain the following consistency conditions for $T^{(1)}_\mu, T^{(1)}_\perp$:

$$
\begin{aligned}
L^{(0)\alpha}T^{(1)}_\alpha + L^{(0)\perp}T^{(1)}_\perp \equiv {} & \partial^\alpha T^{(1)}_\alpha + m^2 T^{(1)}_\perp = \\
= {} & \Big(\frac{\rho_1}{2} + m^2(\rho_7+\rho_{10})\Big)F^{\alpha\beta}F_{\alpha\beta} + \Big(\frac{\rho_3}{2} + m^2(2\rho_6+\rho_{12})\Big)F^{\alpha\beta}\widetilde{F}_{\alpha\beta} + \\
& + \Big(\rho_2 + m^2(2\rho_8 + 2\rho_7 - \rho_5 + \rho_{11})\Big)\partial^\beta A^\alpha\partial_\alpha A_\beta + m^2\Big(\rho_1 + m^2\rho_{13}\Big)A_\alpha A^\alpha + \\
& + \Big(-\frac{\rho_9}{2} + 2\rho_6\Big)\partial_\gamma F_{\alpha\beta}\partial^\gamma\tilde{F}^{\alpha\beta} + 2\rho_7\partial_\gamma\partial_\beta A_\alpha\partial^\gamma\partial^\alpha A^\beta + 2\rho_8\partial_\gamma\partial_\alpha A_\beta\partial^\gamma\partial^\alpha A^\beta + \\
& + \rho_1 A^\alpha T^{(0)}_\alpha + \partial^\alpha A^\beta\Big(2\rho_7\partial_\alpha T^{(0)}_\beta + (2\rho_8-\rho_5)\partial_\beta T^{(0)}_\alpha + 2(\rho_7+\rho_8)\partial_\alpha\partial_\beta T^{(0)}_\perp\Big) + \\
& + 4\rho_6\widetilde{F}^{\alpha\beta}\partial_\alpha T^{(0)}_\beta + O(\zeta) = 0 \qquad (\mathrm{mod}\ \ T^{(0)})\,.
\end{aligned}
\tag{5.5}
$$

[6] This assumption does not restrict generality. One can see that the inclusion of higher derivatives would inevitably increase the number of physical degrees of freedom, as it is defined by relation (3.8). We omit the detailed proof of this fact, though the simple evidence of that can be easily seen. If, for example, the third order derivatives are included into $T^{(1)}_\mu$, there will be four equations of the third order, so the positive contribution to $\mathcal{N}$ will increase by 4. There is only one gauge identity, so its total order should increase at least by four to compensate that. To achieve such a growth of the order of the identity, one has to raise the order of $T_\perp$. As the order of the scalar equation raised, this will again increase $\mathcal{N}$ with no way to compensate the latter growth of the order.

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The ζ-terms vanish on-shell, and for this reason, the parameters ζ remain arbitrary at this stage. The consistency requirement (5.5) imposes seven conditions on thirteen interaction parameters ρ:

$$\rho_7 = 0, \qquad \rho_8 = 0, \qquad \rho_9 = 4\rho_6, \qquad \rho_{10} = -\frac{\rho_1}{2m^2}, \qquad \rho_{11} = \rho_5 - \frac{\rho_2}{m^2}\,,$$
$$\rho_{12} = -\frac{\rho_3}{2m^2} - 2\rho_6, \qquad \rho_{13} = -\frac{\rho_1}{m^2}\,. \tag{5.6}$$

Obviously, six parameters $\rho_1, \dots, \rho_6$ remain arbitrary, while the seven others are fixed by these relations. Having the consistency conditions (5.5) fulfilled, we arrive at the following six-parametrer set of vertices:

$$\begin{aligned}
T^{(1)}_\mu =& A^\alpha\Big(\rho_1 F_{\mu\alpha} + \rho_3 \widetilde{F}_{\mu\alpha}\Big) + \rho_2\Big(A^\alpha\partial_\alpha A_\mu - T^{(0)}_\perp A_\mu\Big) + \partial^\alpha A^\beta\Big(\rho_4\partial_\beta\widetilde{F}_{\alpha\mu} + \\
&+ \rho_5\partial_\beta F_{\alpha\mu}\Big) + \rho_6\Big(4\partial^\alpha A^\beta\partial_\alpha\widetilde{F}_{\beta\mu} + \partial_\mu(F^{\alpha\beta}\widetilde{F}_{\alpha\beta})\Big)\,,\\
T^{(1)}_\perp =& -\frac{\rho_1}{m^2}\Big(\frac{1}{2}F^{\alpha\beta}F_{\alpha\beta} + m^2 A_\alpha A^\alpha\Big) + \Big(\rho_5 - \frac{\rho_2}{m^2}\Big)\partial^\beta A^\alpha\partial_\alpha A_\beta - \\
&- \Big(\frac{\rho_3}{2m^2} + 2\rho_6\Big)F^{\alpha\beta}\widetilde{F}_{\alpha\beta} + \frac{\rho_2}{m^2}\Big(T^{(0)}\Big)^2 \qquad\qquad (\mathrm{mod}\ \zeta).
\end{aligned} \tag{5.7}$$

The corresponding contributions to the gauge identity generators read

$$L^{(1)\nu} = -\rho_1 A^\nu + \Big(\rho_5\partial^\nu A^\alpha - 4\rho_6\widetilde{F}^{\alpha\nu}\Big)\partial_\alpha, \qquad L^{(1)\perp} = 0\,. \tag{5.8}$$

Consider now the problem of compatibility of the first-order interactions at the next order. Following the general prescription of section 4, we have to substitute the first-order gauge identity generators and equations $L^{(1)}, T^{(1)}$ obtained above, into relations (4.19) and examine their consistency. We have

$$\begin{aligned}
L^{(1)\alpha}T^{(1)}_\alpha =& -\rho_1\rho_2\Big(\frac{1}{2}A^\alpha\partial_\alpha A^2 - A^2 T^{(0)}_\perp\Big) - \rho_1 A^\nu\partial^\alpha A^\beta\Big(\rho_4\partial_\beta\widetilde{F}_{\alpha\nu} + \rho_5\partial_\beta F_{\alpha\nu} + \\
&+ 4\rho_6\partial_\alpha\widetilde{F}_{\beta\nu}\Big) - \rho_1\rho_6 A^\nu\partial_\nu(F^{\alpha\beta}\widetilde{F}_{\alpha\beta}) + \Big(\rho_5\partial^\nu A^\alpha - 4\rho_6\widetilde{F}^{\alpha\nu}\Big)\partial_\alpha T^{(1)}_\nu
\end{aligned} \tag{5.9}$$

As is seen from (4.19) the first order interactions $L^{(1)}, T^{(1)}$, having the form (5.9) with six parameters involved, will be compatible in the second order if there exist functions $T^{(2)}_\alpha$, $T^{(2)}_\perp$ such that $ord(T^{(2)}_\alpha) \leq 2$, $ord(T^{(2)}_\perp) \leq 1$ and the following conditions are fulfilled:

$$L^{(1)\alpha}T^{(1)}_\alpha + \partial^\alpha T^{(2)}_\alpha + m^2 T^{(2)}_\perp = 0 \qquad\qquad (\mathrm{mod}\ T^{(0)}). \tag{5.10}$$

On substituting (5.9) into (5.10), one can find that no obstructions occur to the existence of $T^{(2)}_\alpha$, $T^{(2)}_\perp$ with appropriate orders of field derivatives. For example, we can always take

$$\begin{aligned}
T^{(2)}_\mu =& -\Big(\rho_5\partial^\beta A_\mu - 4\rho_6\widetilde{F}_\mu{}^\beta\Big)T^{(1)}_\beta + \rho_1 A^\beta\partial^\alpha A_\mu\Big(\rho_4\widetilde{F}_{\alpha\beta} + \rho_5 F_{\alpha\beta}\Big) + \\
&+ \rho_1\rho_6\Big(4A^\beta\partial_\mu A^\alpha\widetilde{F}_{\alpha\beta} + A_\mu F^{\alpha\beta}\widetilde{F}_{\alpha\beta}\Big)\,,\\
T^{(2)}_\perp =& \frac{\rho_1\rho_2}{m^2}\Big(\frac{1}{2}A^\alpha\partial_\alpha A^2 - A^2 T^{(0)}_\perp\Big) - \frac{\rho_1}{m^2}\partial_\beta A^\nu\partial^\alpha A^\beta\Big(\rho_4\widetilde{F}_{\alpha\nu} + \rho_5 F_{\alpha\nu}\Big)\,.
\end{aligned} \tag{5.11}$$

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This means that the six-parametrer set of the first-order interactions (5.7), being the general solution to the first-order condition (4.17), admits a consistent extension to the second order without any restriction on the parameters ρ. We will not further elaborate here on the most general interactions of the higher orders, although the method allows one to study the issue in its full generality in any order, as it can be seen from the first-order example. Instead, we will just notice some special cases, where the perturbative procedure of interaction inclusion can be consistently interrupted already at the second order level.

At first, notice that if the parameters ρ are chosen in such a way that $L_\perp^{(2)} = 0$ and $T_\mu^{(2)} = 0$, the identity will be consistent without higher order contributions, i.e., with $T^{(n)}, L^{(n)} = 0, n > 2$. The corresponding first-order vertices are called self-consistent.

Two special combinations of the parameters are possible that define inequivalent self-consistent first-order interactions:

1. $\rho_1\,, \rho_2\,, \rho_3 \neq 0, \qquad \rho_4 = \rho_5 = \rho_6 = 0$ that corresponds to the equations at most cubic in fields. A further specialized option $\rho_1\rho_2 = 0$ leads to the at most quadratic in fields interactions;

2. $\rho_2\,, \rho_3\,, \rho_4 \neq 0, \qquad \rho_1 = \rho_5 = \rho_6 = 0$ results in quadratic interaction.

In the first case, the corresponding equations read

$$
\begin{aligned}
T_\mu &= \partial^\alpha F_{\alpha\mu} - m^2 A_\mu + A^\alpha\Big(\rho_1 F_{\mu\alpha} + \rho_3 \widetilde{F}_{\mu\alpha}\Big) + \rho_2\Big(A^\alpha\partial_\alpha A_\mu - T_\perp^{(0)} A_\mu\Big),\\
T_\perp &= \partial^\alpha A_\alpha - \frac{\rho_1}{m^2}\Big(\frac{1}{2}F^{\alpha\beta}F_{\alpha\beta} + m^2 A_\alpha A^\alpha\Big) - \frac{\rho_2}{m^2}\Big(\partial^\beta A^\alpha\partial_\alpha A_\beta - \Big(T^{(0)}\Big)^2\Big) - \\
&\quad - \frac{\rho_3}{2m^2}F^{\alpha\beta}\widetilde{F}_{\alpha\beta} + \frac{\rho_1\rho_2}{m^2}\Big(\frac{1}{2}A^\alpha\partial_\alpha A^2 - A^2 T_\perp^{(0)}\Big),\\
L^\alpha &= \partial^\alpha - \rho_1 A^\alpha, \qquad L^\perp = m^2.
\end{aligned}
\tag{5.12}
$$

The second item results in a different self-consistent interaction of the first order

$$
\begin{aligned}
T_\mu &= \partial^\alpha F_{\alpha\mu} - m^2 A_\mu + \rho_3 A^\alpha \widetilde{F}_{\mu\alpha} + \rho_2\Big(A^\alpha\partial_\alpha A_\mu - T_\perp^{(0)} A_\mu\Big) + \rho_4 \partial^\alpha A^\beta \partial_\beta \widetilde{F}_{\alpha\mu},\\
T_\perp &= \partial^\alpha A_\alpha - \frac{\rho_2}{m^2}\left(\partial^\beta A^\alpha\partial_\alpha A_\beta - \Big(T^{(0)}\Big)^2\right) - \frac{\rho_3}{2m^2}F^{\alpha\beta}\widetilde{F}_{\alpha\beta},\\
L^\alpha &= \partial^\alpha, \qquad L^\perp = m^2\,.
\end{aligned}
\tag{5.13}
$$

These two different quadratic interactions, being self-consistent as such, can be complemented by the higher order interactions. The more general quadratic interactions (5.3) need cubic corrections to ensue consistency. Though such corrections exist, as we have explained above, they can be inconsistent in the next order of interaction.

Notice that the three-parameter sets of self-consistent interactions (5.12), (5.13), being the most general in this class, are not necessarily compatible with variational principle, though the free theory admits Lagrangian formulation. One can see that only the one-parameter family of the vertices (5.12), (5.13) is variational. It is the case of $\rho_2 = -\rho_1 = g$ and the other ρ's and ζ's vanishing. The corresponding Lagrangian reads

$$
\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} - \frac{m^2}{2}A^\alpha A_\alpha - \frac{g}{2}\partial^\alpha A_\alpha A^\beta A_\beta\,.
$$

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All other vertices of (5.12), (5.13) do not follow from variational principle. This demonstrates that the general class of the consistent interactions can be much broader than that of Lagrangian ones. This fact can also mean that some of the no-go theorems for the interactions, known in the Lagrangian framework, may be bypassed if the requirement is relaxed for the vertices to be variational.

5.2 The massive spin 2 in d=4

The irreducible spin-2 massive field theory can be described by a traceless, symmetric, rank-2 tensor field $h_{\mu\nu}$ subject to the Klein-Gordon equations and the transversality condition

$$T^{(0)}_{\mu\nu} \equiv (\Box - m^2)h_{\mu\nu} = 0, \quad T^{(0)}_{\nu} \equiv \partial^\nu h_{\mu\nu} = 0\,, \qquad ord(T^{(0)}_{\mu\nu}) = 2, \quad ord(T^{(0)}_{\mu}) = 1\,. \tag{5.14}$$

These equations are involutive as there are no low order differential consequences. Unlike spin 1, the equations are inequivalent to any Lagrangian system formulated in terms of the original irreducible field. The Fierz-Pauli Lagrangian [28] that involves auxiliary scalar field and traceless tensor $h_{\mu\nu}$ leads to the equations that are equivalent to (5.14). The Fierz-Pauli equations (FPE) are not involutive. Their involutive closure is not Lagrangian and it has a more complex structure than the equations (5.14) formulated without any auxiliary field. A similar picture is observed for all the higher-spin massive fields. The Lagrangian formulation due to Singh and Hagen [29] needs auxiliary fields, that makes the system non-involutive. The involutive closure of the Sing-Hagen equations is not Lagrangian anymore, and is more complex than the system of Klein-Gordon equations and the transversality condition for the traceless tensors. The aim of this subsection is to demonstrate by the example of the spin-2 field that the minimal formulation of the irreducible field equations, involving just the mass shell and transversality conditions, is sufficient for iterative construction of consistent interactions. Though this formulation is not Lagrangian, it admits quantization and can enjoy all the other advantages of Lagrangian formalism, including Noether's correspondence between symmetries and conserved currents. The matter is that the model admits a Lagrange anchor. As is known, the Lagrange anchor [25], being identified for not necessarily Lagrangian field equations, allows one to path-integral quantize the theory [25, 30, 31], and also to connect symmetries with conservation laws [26, 32].

Prior to seeking for consistent interactions, we have to identify the gauge identity and gauge symmetry generators for the free field equations (5.14). The model has no gauge symmetry and there exists four third-order gauge identities. The generators are given by

$$L^{(0)\mu\nu}_{\alpha} = \frac{1}{2}\left(\delta^{\mu}_{\alpha}\partial^{\nu} + \delta^{\nu}_{\alpha}\partial^{\mu}\right)\,, \qquad L^{(0)\nu}{}_{\alpha} = -(\Box - m^2)\delta^{\mu}_{\alpha}, \tag{5.15}$$

$$L^{(0)\mu\nu}_{\alpha}T^{(0)}_{\mu\nu} + L^{(0)\nu}{}_{\alpha}T^{(0)}_{\nu} \equiv 0\,, \qquad ord(L^{(0)})_{\alpha} = 3\,. \tag{5.16}$$

Following the general procedure of section 4, to switch on the first order interactions, one has to find the quadratic vertices $T^{(1)}_{\mu\nu}, T^{(1)}_{\nu}$ such that the identities (4.17) hold with $T^{(0)}_{\mu\nu}, T^{(0)}_{\mu}, L^{(0)\mu\nu}_{\alpha} L^{(0)\mu}_{\alpha}$ defined by (5.14), (5.15).

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We do not study the most general case, restricting the quadratic vertices by the ansatz with at most two derivatives in every term:[7]

$$
\begin{aligned}
T^{(1)}_{\mu\nu} = & \rho_8\left(\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}-\frac{1}{4}\eta_{\mu\nu}\left(\partial h\right)^2\right)+\rho_3\left(\partial_\alpha h_{\beta\mu}\partial^\alpha h^\beta{}_\nu-\frac{1}{4}\eta_{\mu\nu}\left(\partial h\right)^2\right)+\\
&+\rho_9\left(\partial_\mu h_{\alpha\beta}\partial^\alpha h^\beta{}_\nu+\partial_\nu h_{\alpha\beta}\partial^\alpha h^\beta{}_\mu-\frac{1}{2}\eta_{\mu\nu}\left(\widetilde{\partial h}\right)^2\right)+\rho_6 h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}+\\
&+\rho_4\left(\partial_\alpha h_{\beta\mu}\partial^\beta h^\alpha{}_\nu-\frac{1}{4}\eta_{\mu\nu}\left(\widetilde{\partial h}\right)^2\right)+\rho_7\left(h^{\alpha\beta}\partial_\nu\partial_\mu h_{\alpha\beta}-\frac{1}{4}\eta_{\mu\nu}h^{\alpha\beta}\Box h_{\alpha\beta}\right)+\\
&+m^2\rho_5\left(h_{\alpha\mu}h^\alpha{}_\nu-\frac{1}{4}\eta_{\mu\nu}h^2\right)+\rho_1 h^{\alpha\beta}\left(\partial_\nu\partial_\alpha h_{\beta\mu}+\partial_\mu\partial_\alpha h_{\beta\nu}-\frac{1}{2}\eta_{\mu\nu}\partial_\alpha T^{(0)}_\beta\right),\\
T^{(1)}_\nu = & \partial^\mu\left(\rho_2 h_{\alpha\mu}h^\alpha{}_\nu+\rho_{10}\eta_{\mu\nu}h^2\right),
\end{aligned}
\tag{5.17}
$$

where we used the following abbreviations:

$$
h^2=h_{\alpha\beta}h^{\alpha\beta},\qquad \left(\partial h\right)^2=\partial_\nu h_{\alpha\beta}\partial^\nu h^{\alpha\beta},\qquad \left(\widetilde{\partial h}\right)^2=\partial_\nu h_{\alpha\beta}\partial^\alpha h^{\nu\beta}.
$$

Notice that the trivial (on-shell vanishing) terms are omitted in (5.17). All the vertices are identically traceless. Following the general procedure of section 4, we substitute the ansatz (5.17) into the relations (4.17) and examine consistency,

$$
\begin{aligned}
L^{(0)\alpha\beta}_\nu T^{(1)}_{\alpha\beta}+L^{(0)\alpha}_\nu T^{(1)}_\alpha \equiv & \partial^\nu T^{(1)}_{\mu\nu}-(\Box-m^2)T_\nu = \qquad (5.18)\\
= & \partial^\mu Q_{\mu\nu}+\rho_8\partial_\nu h^{\alpha\beta}T^{(0)}_{\alpha\beta}+\rho_9\partial^\alpha h^\beta{}_\nu T^{(0)}_{\alpha\beta}+\rho_9\partial_\nu h^{\alpha\beta}\partial_\alpha T^{(0)}_\beta-\\
&-(\rho_9+\rho_1)\partial^\alpha h^\beta{}_\nu\partial_\alpha T^{(0)}_\beta+\rho_6 h^{\alpha\beta}\partial_\alpha\partial_\beta T^{(0)}_\nu-\rho_6\partial^\beta h^\alpha{}_\nu\partial_\alpha T^{(0)}_\beta+\rho_7 h^{\alpha\beta}\partial_\nu T^{(0)}_{\alpha\beta})+\\
&+\rho_1 h^{\alpha\beta}\partial_\nu\partial_\alpha T^{(0)}_\beta+\rho_1 h^{\alpha\beta}\partial_\alpha T^{(0)}_{\beta\nu}-(\rho_9+\rho_1)m^2 h^\beta{}_\nu T^{(0)}_\beta-\\
&-\partial^\mu\left(\rho_1\eta_{\mu\nu}\frac{1}{2}h^{\alpha\beta}\partial_\alpha T^{(0)}_\beta+\rho_2\left(h^\alpha{}_\mu T^{(0)}_{\alpha\nu}+h^\alpha{}_\nu T^{(0)}_{\alpha\mu}\right)+\left(2\rho_{10}+\frac{\rho_7}{4}\right)\eta_{\mu\nu}h^{\alpha\beta}T^{(0)}_{\alpha\beta}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
Q_{\mu\nu} = & \partial^\alpha h^\beta{}_\mu\partial_\alpha h_{\beta\nu}\Big(\rho_9+\rho_3+\rho_1-2\rho_2\Big)+\partial^\beta h^\alpha{}_\mu\partial_\alpha h_{\beta\nu}\Big(\rho_4+\rho_6\Big)+\\
&+\frac{1}{4}\eta_{\mu\nu}\left(\partial h\right)^2\Big(\rho_8-\rho_3+2\rho_7-8\rho_{10}\Big)+\frac{1}{4}\eta_{\mu\nu}\left(\widetilde{\partial h}\right)^2\Big(2\rho_1-\rho_4\Big)+\\
&+m^2 h^\alpha{}_\mu h_{\alpha\nu}\Big(\rho_9+\rho_5+\rho_1-\rho_2\Big)+\frac{1}{4}m^2\eta_{\mu\nu}h^2\Big(2\rho_8-\rho_5+\rho_7-4\rho_{10}\Big).
\end{aligned}
\tag{5.19}
$$

At first order, the consistency requires the expression (5.18) to vanish modulo $T^{(0)}$. Each term in (5.19) being independent (modulo a conserved tensor), so we arrive at the system of equations restricting the interaction parameters

$$
\begin{aligned}
&\rho_9+\rho_3+\rho_1-2\rho_2=0, \qquad \rho_8-\rho_3+2\rho_7-8\rho_{10}=0, \qquad \rho_4+\rho_6=0,\\
&\rho_9+\rho_5+\rho_1-\rho_2=0, \qquad 2\rho_8-\rho_5+\rho_7-4\rho_{10}=0, \qquad 2\rho_1-\rho_4=0.
\end{aligned}
\tag{5.20}
$$

[7]In the most general case, the terms can appear with two second-order derivatives and four derivatives in total. The interactions of this type are usually considered abnormal, and we do not study them here.

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The solution to this system reads

$$\rho_8 = -\frac{\gamma}{3}, \qquad \rho_9 = \gamma - \rho_1, \qquad \rho_3 = 2\rho_2 - \gamma, \qquad \rho_4 = 2\rho_1,$$
$$\rho_5 = \rho_2 - \gamma\,, \qquad \rho_6 = -2\rho_1, \qquad \rho_7 = 4\rho_0 - \frac{\gamma}{3}, \qquad \rho_{10} = \rho_0 - \frac{\rho_2}{4}\,, \tag{5.21}$$

where $\gamma, \rho_0, \rho_1, \rho_2$ are arbitrary constants. The term with the coefficient ρ_2 can be considered as trivial, because it is generated by the diffeomorphism in the space of fields:

$$h_{\mu\nu} \mapsto h'_{\nu\nu} = h_{\mu\nu} + \rho_2\Big(h_{\alpha\mu}h^{\alpha}{}_{\nu} - \frac{1}{4}\eta_{\mu\nu}h^2\Big)\,.$$

Finally, we conclude that the set of non-trivial vertices for the massive spin-2 equations may involve at most 3 parameters. The consistent quadratic vertices are given by

$$\begin{aligned}
T^{(1)}_{\mu\nu} = \gamma\Big[& -\frac{1}{3}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} - \partial_\alpha h_{\beta\mu}\partial^\alpha h^{\beta}{}_{\nu} + \frac{1}{3}\eta_{\mu\nu}\,(\partial h)^2 + \partial_\mu h_{\alpha\beta}\partial^\alpha h^{\beta}{}_{\nu} + \\
& + \partial_\nu h_{\alpha\beta}\partial^\alpha h^{\beta}{}_{\mu} - \frac{1}{2}\eta_{\mu\nu}\left(\widetilde{\partial h}\right)^2 - \frac{1}{3}h^{\alpha\beta}\partial_\nu\partial_\mu h_{\alpha\beta} + \frac{1}{12}\eta_{\mu\nu}h^{\alpha\beta}\Box h_{\alpha\beta} - \\
& - m^2 h_{\alpha\mu}h^{\alpha}{}_{\nu} + \frac{m^2}{4}\eta_{\mu\nu}h^2\Big] + \rho_1\Big[h^{\alpha\beta}\left(\partial_\nu\partial_\alpha h_{\beta\mu} + \partial_\mu\partial_\alpha h_{\beta\nu} - \frac{1}{2}\eta_{\mu\nu}\partial_\alpha T^{(0)}_{\beta}\right) - \\
& - \partial_\mu h_{\alpha\beta}\partial^\alpha h^{\beta}{}_{\nu} - \partial_\nu h_{\alpha\beta}\partial^\alpha h^{\beta}{}_{\mu} - 2h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + 2\partial_\alpha h_{\beta\mu}\partial^\beta h^{\alpha}{}_{\nu}\Big] + \\
& + \rho_2\Big[2\partial_\alpha h_{\beta\mu}\partial^\alpha h^{\beta}{}_{\nu} - \frac{1}{2}\eta_{\mu\nu}\,(\partial h)^2 + m^2 h^{\alpha}{}_{\mu}h_{\alpha\nu} - \frac{m^2}{4}\eta_{\mu\nu}h^2\Big] + \\
& + \rho_0\Big[4h^{\alpha\beta}\partial_\nu\partial_\mu h_{\alpha\beta} - \eta_{\mu\nu}h^{\alpha\beta}\Box h_{\alpha\beta}\Big]\,, \\
T^{(1)}_{\mu} = \partial^\mu & \left(\rho_2 h_{\alpha\mu}h^{\alpha}{}_{\nu} + \Big(\rho_0 - \frac{\rho_2}{4}\Big)\eta_{\mu\nu}h^2\right)\,.
\end{aligned} \tag{5.22}$$

Notice that two parameters γ and ρ_1 are associated with the conserved currents that do not contribute to the transversality condition $T^{(1)}_{\mu}$.

The first-order deformation of the Noether identity generators is given by

$$\begin{aligned}
L^{(1)\alpha\beta}_{\nu} &= \partial^\mu\Big(\mathcal{L}^{\alpha\beta}_{\mu\nu}\,\cdot\Big) + \Big(\rho_1 - \gamma\Big)\partial^{(\alpha}h^{\beta)}{}_{\nu} + 4\rho_0\partial_\nu h^{\alpha\beta} + \rho_1\delta^{(\alpha}_{\nu}T^{(0)\beta)}\,, \\
L^{(1)\alpha}_{\nu} &= \partial^\mu\Big(\mathcal{L}^{\alpha}_{\mu\nu}\,\cdot\Big) + \widetilde{\mathcal{L}}^{\alpha}_{\nu}\,,
\end{aligned} \tag{5.23}$$

where the round brackets mean symmetrization in corresponding indices and the following notation is used:

$$\begin{aligned}
\mathcal{L}^{\alpha\beta}_{\mu\nu} &= \Big(\frac{\gamma}{4} - \frac{\rho_2}{2} - \rho_0\Big)\eta_{\mu\nu}h^{\alpha\beta} + \Big(\rho_2 - \rho_1\Big)h^{(\alpha}{}_{\mu}\delta^{\beta)}{}_{\nu} + \rho_2 h^{(\alpha}{}_{\nu}\delta^{\beta)}{}_{\mu}\,, \qquad (5.24)\\
\widetilde{\mathcal{L}}^{\alpha}_{\nu} &= \gamma\Big(\partial_\nu T^{(0)\alpha} - T^{(0)\alpha}{}_{\nu}\Big) + \rho_1\Big(2\partial^\alpha T^{(0)}_{\nu} + T^{(0)\alpha}\partial_\nu - \partial_\nu T^{(0)}_{\alpha} - 2\delta^{\alpha}{}_{\nu}T^{(0)}_{\beta}\partial^\beta\Big)\,, \\
\mathcal{L}^{\alpha}_{\mu\nu} &= \rho_1\Big(\partial_\nu h^{\alpha}{}_{\mu} + 2\delta^{\alpha}{}_{\nu}h^{\beta}{}_{\mu}\partial_\beta - 2\partial^\alpha h_{\mu\nu} - h^{\alpha}{}_{\mu}\partial_\nu + \frac{1}{2}\eta_{\mu\nu}h^{\beta\alpha}\partial_\beta\Big) + \gamma\Big(\partial_\mu h^{\alpha}{}_{\nu} - \partial_\nu h^{\alpha}{}_{\mu}\Big)\,.
\end{aligned}$$

As it follows from the general requirement (4.19), the next order consistency of the vertices (5.22) is only possible under the following condition:

$$L^{(1)\mu\nu}_{\alpha}T^{(1)}_{\mu\nu} + L^{(1)\mu}_{\alpha}T^{(1)}_{\mu} = \partial^\mu\mathcal{Q}_{\alpha\mu} - m^2 T^{(2)}_{\alpha} \qquad (\mathrm{mod}\ \ T^{(0)})\,, \tag{5.25}$$

where

$$\mathcal{Q}_{\alpha\mu} = \partial_\mu T^{(2)}_\alpha - T^{(2)}_{\alpha\mu}\,. \tag{5.26}$$

Relation (5.19) is the compatibility condition for the second-order vertices. We will seek for the solutions to these equations that obey conditions

$$ord(\mathcal{Q}_{\mu\nu}) = 2, \qquad ord(T^{(2)}_\mu) = 1\,.$$

These are obviously consistent with the number of physical degrees of freedom. The admissible choice is

$$\mathcal{Q}_{\nu\mu} = \mathcal{L}^{\alpha\beta}_{\mu\nu} T^{(1)}_{\alpha\beta} + \mathcal{L}^{\alpha}_{\mu\nu} T^{(1)}_{\alpha} + \Big(\rho_1 - \gamma\Big)\Big[h^\beta{}_\nu\Big(T^{(1)}_{\beta\mu} - \partial_\mu T^{(1)}_\beta\Big) + \partial_\mu h^\beta{}_\nu T^{(1)}_\beta\Big]\,.$$

To get $\mathcal{Q}$ we integrate by parts and take into account the identity (5.18):

$$h^\beta{}_\nu \partial^\alpha T^{(1)}_{\alpha\beta} = h^\beta{}_\nu\Big(\Box - m^2\Big)T^{(1)}_\beta = \partial^\mu\Big[h^\beta{}_\nu\Big(T^{(1)}_{\beta\mu} - \partial_\mu T^{(1)}_\beta\Big) + \partial_\mu h^\beta{}_\nu T^{(1)}_\beta\Big] \qquad (\mathrm{mod}\;\; T^{(0)})\,.$$

The on-shell vanishing gauge identity generators $\widetilde{\mathcal{L}}^\alpha_\nu$ and $\delta^{(\alpha}_\nu T^{(0)\beta)}$ can not affect the relation (5.25), so the question left reads: is it possible to represent the remaining term $h^{\alpha\beta}\partial_\nu T^{(1)}_{\alpha\beta}$ in the form (5.25) for some $\mathcal{Q}$? If the answer is affirmative, one should try to express $\mathcal{Q}$ in the form (5.26) with on-shell traceless and symmetric tensor $T^{(2)}_{\alpha\beta}$. If the answer is negative, an obstruction for the second order vertex appears. We have

$$\begin{aligned} 4\rho_0\partial_\nu h^{\alpha\beta}T^{(1)}_{\alpha\beta} = \partial^\mu\Big[4\rho_0\Big(4\rho_0 - \frac{\gamma}{3}\Big)\partial_\nu h^\beta{}_\mu h^{\sigma\tau}\partial_\beta h_{\sigma\tau} + 8\rho_0\rho_1\partial_\nu h^\beta{}_\mu h^{\sigma\tau}\partial_\tau h_{\sigma\beta}\Big] - \\ - 8\rho_0\rho_1\partial_\nu h^{\alpha\beta}h^{\sigma\tau}\partial_\sigma\partial_\tau h_{\alpha\beta} + \ldots \qquad (\mathrm{mod}\;\; T^{(0)})\,, \end{aligned} \tag{5.27}$$

where the dots denote the terms of order 1 that may be included into $T^{(2)}_\nu$. There is a term of order 2 in the r.h.s. of (5.27)

$$-8\rho_0\rho_1\partial_\nu h^{\alpha\beta}h^{\sigma\tau}\partial_\sigma\partial_\tau h_{\alpha\beta}$$

that does not reduce on shell to a total divergence. It cannot also be absorbed by deformation of the transversality condition $T^{(2)}_\mu$ because of the order restriction. This means that the vertices are inconsistent in the class with $ord(T^{(2)}_\mu) = 1$ unless $\rho_0\rho_1 = 0$. Therefore there can be at most two 2-parameter families of consistent interactions in the considered class. This seems matching well the fact that the massive gravity admits a 2-parameter family of Lagrangians [33] that are consistent from the viewpoint of Hamiltonian constrained analysis [34]. The detailed comparison, however, with complete nonlinear equations of massive gravity [33] is not straightforward as the vertices for the field equations (5.14) are deduced for the traceless tensor $h_{\mu\nu}$ without any auxiliary field involved. The massive gravity equations involve the tracefull tensor. In the free limit the massive gravity reduces to the FP equations. In the FP theory, the trace vanishes on shell, leading to the equations (5.14) for traceless tensor. For the nonlinear equations of massive gravity, the explicit on-shell exclusion of the auxiliary field is unknown at the moment, though one should expect it is still possible (if it was impossible, the theory would have a different number of degrees of

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freedom). Unless the auxiliary field remains involved in the equations, the corresponding vertices can not be immediately compared with the ones in the equations formulated without auxiliary fields from outset. The goal of this section, however, is not to perturbatively re-derive the massive gravity vertices, but to exemplify the involutive technique of finding interactions without a direct use of Lagrangian, making use of a minimal set of fields.

6 Concluding remarks

Let us briefly discuss the results. In this paper, we propose a new method that allows one to examine the consistency of interactions for the general field theory models, be they Lagrangian or non-Lagrangian. The method also provides a technique for perturbative identification of all the admissible interactions, given a free field model. The method requires first to bring the field equations to the involutive form. Notice that the involutive closure is always non-variational for variational non-involutive equations. As far as the field equations are brought to the involutive form, the gauge algebra is to be identified for the equations. The consistency of the gauge algebra is examined by tools similar to those based on the BV formalism for Lagrangian systems [3, 4] with three major generalizations. The first generalization is that the consistency of gauge algebra is examined for the involutive closure of the system of field equations, not for the action functional which might be even non-existent. The second is that the gauge identity generators are involved in the gauge algebra of the involutive closure independently from the gauge symmetries. The identity generators impose their own consistency conditions that are not identified by previously known method [3] even in the Lagrangian case. The third is that the gauge algebra of the involutive system provides a convenient receipt (3.8), (A.23) for counting physical degrees of freedom. This formula is applied to the involutive equations in a covariant form, and it uniformly covers all conceivable instances. Let us also mention that the formula (A.23) has been derived in the appendix in a more general setting than it is actually utilized in the paper, as it also applies to the case of reducible gauge symmetries and gauge identities. Let us finally notice, that the involutive closure of field equations admits a BRST embedding along the lines of [25]. From the viewpoint of the corresponding local BRST complex [26], the formula for the degree of freedom count (3.8), (A.23) can get a natural cohomological interpretation. This issue will be addressed elsewhere.

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A Physical degree of freedom count

In this appendix, we explain the origin of formula (3.8) for counting of the physical degrees of freedom.

The starting point for deriving the formula (3.8) is the notion of the strength of differential equations introduced by Einstein [14]. Roughly, the strength is a number that measures the size of the solution space. The "stronger" is the system of differential equations, the smaller is its solution space. It turns out that the numerical value of strength can be immediately related with the number of Cauchy data needed to define the general solution modulo gauge freedom, i.e., with the number of physical degrees of freedom. The original Einstein's argumentation was not mathematically rigor, its justification and explanation within the modern theory of formal integrability can be found in book [13]. For earlier discussions of the concept of strength of equation as well as numerous applications to the analysis of relativistic field equations we refer the reader to [15–18].

Let us first explain the Einstein's concept of the strength of field equations. Consider a set of fields ϕ^i on d-dimensional space-time with coordinates x^μ. Assuming the fields to be analytical functions, we can expand them in Taylor series about some point x_0:

$$\phi^i(x) = \sum_{p=0}^{\infty} \frac{1}{p!} \varphi^i_{\mu_1\cdots\mu_p} (x-x_0)^{\mu_1}(x-x_0)^{\mu_2}\cdots(x-x_0)^{\mu_p} \,. \tag{A.1}$$

Let N_p denote the total number of terms of pth order in the expansion above. (The explicit expression for N_p is given below.) As far as the fields ϕ^i obey a system of PDEs, not all the Taylor coefficients $\varphi^i_{\mu_1\cdots\mu_p}$ can remain arbitrary. Denote N'_p the number of monomials of order p that are left free in the general solution of the field equations. Obviously, $N'_p < N_p$. On the other hand, if the field equations enjoy a gauge symmetry, not all the solutions are physically relevant; some of the monomials in the general solution come from Taylor series for the gauge parameters. Modding out by the gauge freedom, one can define the number N''_p of gauge inequivalent monomials of order p entering the general solution. Now, the number of physical degrees of freedom *per point*[8] is given by

$$\mathcal{N} = \lim_{p\to\infty} \frac{p}{d-1}\frac{N''_p}{N_p} \,. \tag{A.2}$$

This formula, that dates dates back to Einstein, defines the number of physical degrees of freedom as the growth of the number of "physical monomials" compared to the unconstrained ones.

Consider a system of PDEs

$$T_a(\phi^i, \partial_\mu\phi^i, \ldots, \partial_{\mu_1}\cdots\partial_{\mu_m}\phi^i) = 0 \,, \qquad a = 1, \ldots, t, \tag{A.3}$$

governing the dynamics of fields ϕ^i, $i = 1, 2 \ldots, f$. The order of these equations equals to m. Substituting the expansion (A.1) into the field equations (A.3) and evaluating the result at $x = x_0$, we get the system of algebraic equations

$$T_a(\varphi^i, \varphi^i_\mu, \ldots, \varphi^i_{\mu_1\cdots\mu_m}) = 0 \,. \tag{A.4}$$

[8] Accordingly, the number of physical *polarizations* of the field ϕ^i is the half of $\mathcal{N}$.

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These equations follow from (A.3) by simply replacing the partial derivatives of fields with the corresponding Taylor coefficients. In general, the solution space for these equations can be a very complicated algebraic variety containing strata of different dimensions. So, it might be problematic to choose the independent coefficients and compute their totality. But the task is considerably simplified if one considers the conditions on the higher order coefficients. After all, the lower order monomials do not matter for evaluating the limit (A.2). Differentiating (A.3) k times by x's and setting $x = x_0$, we obtain a set of algebraic equations of the form

$$\mathcal{J}^{\nu_1\cdots\nu_{m+k}}_{ai\mu_1\cdots\mu_k}\varphi^i_{\nu_1\cdots\nu_{m+k}} + \mathcal{I}_{a\mu_1\cdots\mu_k} = 0\,, \tag{A.5}$$

where the functions $\mathcal{J}$'s and $\mathcal{I}$'s depend on φ's of order less than $m+k$. Thus, for each given k, we get a system of linear inhomogeneous equations for the coefficients $\varphi^i_{\nu_1\cdots\nu_{m+k}}$. The matrix $\mathcal{J} = \mathcal{J}_k$, defining the system, is called the *symbol matrix* of order k; it has the following structure:

$$\mathcal{J}^{\nu_1\cdots\nu_{m+k}}_{ai\mu_1\cdots\mu_k} = \mathcal{J}^{(\nu_1\cdots\nu_m}_{ai}\delta^{\nu_{m+1}}_{\mu_1}\cdots\delta^{\nu_{m+k})}_{\mu_k}\,, \qquad \mathcal{J}^{\nu_1\cdots\nu_m}_{ai} = \frac{\partial T_a(\varphi)}{\partial\varphi^i_{\nu_1\cdots\nu_m}}\,. \tag{A.6}$$

Here the round brackets mean symmetrization of the indices enclosed and the functions $T_a(\varphi)$ are given by the left hand side of equation (A.4). As is seen, the symbol matrix $\mathcal{J}_k$ of order k is expressed in a very specific way trough the symbol matrix of order 0. The latter may be called the symbol matrix of the field equations (A.3). For linear differential equations the symbol matrix $\mathcal{J}_0$ is just the highest-order or principal part of the system.

We thus see that whatever the original system of field equations may be, there is an integer m such that the space of monomials of order $p > m$ is determined by a finite system of linear inhomogeneous equations with coefficients depending on φ's of order $\leq p$. The echelon form of the algebraic equations (A.5) suggests to solve them one after another, so that at each step one deals with a *finite* system of linear inhomogeneous equations. This makes possible applying the usual theorems of linear algebra to evaluate the solution space.

First of all, the number of linearly independent solutions to equations (A.5) crucially depends on the rank of the symbol matrix $\mathcal{J}_l$. The symbol matrix in its turn is the function of the Taylor coefficients $\{\varphi^i_{\mu_1\cdots\mu_j}\}^m_{j=0}$ constrained by the algebraic equations (A.4), so that the rank of $\mathcal{J}_k$ can suddenly change. To avoid this complication we will restrict ourselves to those solutions of (A.4) for which the rank of the symbol matrix $\mathcal{J}_k(\varphi)$ is maximal. This means that we consider only the general (opposite to singular) solutions to the field equations. Following [13], we will call $\{\varphi^i_{\mu_1\cdots\mu_j}\}^m_{j=0}$ the *principal coefficients*, referring to the other Taylor coefficients as *parametric*.

By Kronecker-Capelli's criterion, the system of linear inhomogeneous equations (A.5) is compatible iff each left null-vector $\mathcal{K}$ of the symbol matrix $\mathcal{J}_k$ annihilates also the inhomogeneous term $\mathcal{I}_k$. Clearly, the null-vectors of the symbol matrix, if any, can always be chosen to be functions of the principal coefficients alone. A crucial point is that the compatibility criterion is automatically satisfied for differential equations in involution. The reason is very simple: vanishing of the function $\mathcal{K}^{a\mu_1\cdots\mu_k}\mathcal{I}_{a\mu_1\cdots\mu_k}$ would otherwise give a nontrivial constraint on the lower order coefficients that, in turn, would be manifestation

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of a hidden integrability condition. On the other hand, if the vector $\mathcal{K}$ annihilates both the symbol matrix $\mathcal{J}_k$ and the inhomogeneous term $\mathcal{I}_k$, then it defines an identity for the linear equations (A.5) and this identity must follow from a gauge identity for the original differential equations (A.3).

The general solution to an inhomogeneous linear system is given by its partial solution plus the general solution to the corresponding homogeneous system. In our case, the latter is completely determined by the symbol matrix. Thus, to evaluate the size of the solution space, we focus on the solutions to the homogeneous system. These form a linear space, whose dimension is given by the number of unknowns minus the rank of the symbol matrix. The number of unknowns $\{\varphi^i_{\nu_1\cdots\nu_{m+k}}\}$ in (A.5) coincide with the number N_p of lineally independent monomials of order $p = m + k$. It is easy to find that

$$N_p = f\binom{p+d-1}{p} = f\frac{(p+d-1)!}{p!(d-1)!}\,. \tag{A.7}$$

The rank of the symbol matrix $\mathcal{J}_k$ can be computed as the difference between the number of equations (A.5) and the number of left null-vectors of the matrix $\mathcal{J}_k$. The former is expressed through the binomial coefficients as

$$t\binom{k+d-1}{l} = t\binom{p-m+d-1}{p-m}\,. \tag{A.8}$$

As was explained above all left null-vectors for the symbol matrix of involutive equations come from gauge identities. Each gauge identity

$$\hat{L}^a T_a \equiv 0 \tag{A.9}$$

is defined by differential operators

$$\hat{L}^a = \sum_{n=0}^{q'} \mathcal{L}^{a\nu_1\cdots\nu_n}\partial_{\nu_1}\cdots\partial_{\nu_n}\,,$$

with coefficients depending on fields and their derivative up to some finite order j. If the highest coefficients $\{\mathcal{L}^{a\mu_1\cdots\mu_q}\}$ are not all equal to zero identically, then the number q' is called the *order of the gauge identity* (4.4). Differentiating (A.9) s times by x's and setting $x = x_0$, we find

$$\mathcal{L}^{a\mu_1\cdots\mu_{q'}}\mathcal{J}^{\nu_1\cdots\nu_{m+s+q'}}_{ai\mu_1\cdots\mu_{s+q'}}\varphi^i_{\nu_1\cdots\nu_{m+s+q'}} + \cdots \equiv 0\,, \tag{A.10}$$

where dots stand for the terms involving φ's of order less than $m + s + q'$. If s is large enough such that $m + s + q' > j$, then we are lead to conclude that the coefficients at $\varphi^i_{\nu_1\cdots\nu_{m+s+q'}}$ in (A.10) must be zero in order for the identity to hold. This implies that the symbol matrix $\mathcal{J}_{s+q'}$ admits the set of null-vectors $\{\mathcal{L}_{\lambda_1\cdots\lambda_s}\}$ of the form

$$\mathcal{L}^{a\mu_1\cdots\mu_{s+q'}}_{\lambda_1\cdots\lambda_s} = \mathcal{L}^{a(\mu_1\cdots\mu_{q'}}\delta^{\mu_{q'+1}}_{\lambda_1}\cdots\delta^{\mu_{p+s})}_{\lambda_s}\,, \qquad \mathcal{L}^{a\mu_1\cdots\mu_{s+q'}}_{\lambda_1\cdots\lambda_s}\mathcal{J}^{\nu_1\cdots\nu_{m+s+q'}}_{ai\mu_1\cdots\mu_{s+q'}} = 0\,.$$

All these null-vectors are linearly independent and their number is given by

$$\binom{s+d-1}{s}\,. \tag{A.11}$$

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In other words, each gauge identity of order q' for the involutive equations of order m gives

$$\binom{l-q'+d-1}{l-q} = \binom{p-q+d-1}{p-q} \tag{A.12}$$

left null-vectors for the corresponding symbol matrix $\mathcal{J}_k$ provided that k is large enough. Here $p = k + m$ and the number $q = q' + m$ is called the *total order* of gauge identity.

Now, suppose that the system (A.3) involves equations of different orders: t_0 equations of order zero (algebraic equations), t_1 equations of the first order and so on. Let us also assume that the system contains no hidden integrability conditions and becomes involutive upon adjoining trivial differential consequences. Then according to (A.8) all these differential equations give rise to

$$\sum_{n=0}^{\infty} t_n \binom{p-n+d-1}{p-n} \tag{A.13}$$

linear equations for the parametric coefficients $\varphi^i_{\mu_1\cdots\mu_p}$ with large p. Of course, only a finitely many terms are different from zero in the above sum. Let us further suppose that the field equations enjoy l_n gauge identities of *total* orders $n = 0, 1, 2, \dots$. If all these identities are independent (irreducible), then the linear equations (A.5) for the parametric coefficients of order p are possessed of exactly

$$\sum_{n=0}^{\infty} l_n \binom{p-n+d-1}{p-n} \tag{A.14}$$

dependencies (left null-vectors) provided that p is large enough. The difference between (A.13) and (A.14) is the number of independent equations for unknowns $\varphi^i_{\mu_1\cdots\mu_p}$. Subtracting this difference from (A.7), we get the dimension of the solution space, that is, the number of independent monomials of order p:

$$N'_p = f\binom{p+d-1}{p} - \sum_{n=0}^{\infty}(t_n - l_n)\binom{p-n+d-1}{p-n}\,. \tag{A.15}$$

Now, we should take into account gauge freedom. Each gauge transformation has the form

$$\delta_\epsilon \phi^i = \sum_{n=0}^{q} \mathcal{R}^{i\mu_1\cdots\mu_n}\partial_{\mu_1}\cdots\partial_{\mu_n}\epsilon\,. \tag{A.16}$$

In this expression, the coefficients $\mathcal{R}$'s are functions of the fields and their derivatives up to some finite order and the infinitesimal gauge parameter ϵ is an arbitrary function of x's. The number q is the order of the gauge transformation. The gauge invariance of the field equations (A.3) implies that

$$\delta_\epsilon T_a = \hat{U}^b_a T_b \tag{A.17}$$

for some matrix differential operator $\hat{U}$. Let us expand the gauge parameter in Taylor series

$$\epsilon(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\varepsilon_{\mu_1\cdots\mu_n}(x-x_0)^{\mu_1}\cdots(x-x_0)^{\mu_n}\,.$$

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Then s-fold differentiation of the equality (A.17) at x_0 yields the identity

$$\mathcal{J}_{ai}^{\mu_1\cdots\mu_m}\mathcal{R}^{i\mu_{m+1}\cdots\mu_{m+q}}\varepsilon_{\mu_1\cdots\mu_{m+q+s}}+\cdots=0\,. \tag{A.18}$$

Here the dots stand for ε's of order less than $m+q+s$ and all φ's are assumed to define a solution to the field equations. Since all ε's are arbitrary, we conclude that the leading term in (A.18) must vanish separately. This results in the set $\{\mathcal{R}^{\nu_1\cdots\nu_{q+m+s}}\}$ of right null-vectors for the symbol matrix $\mathcal{J}_s$:

$$\mathcal{R}^{i\nu_1\cdots\nu_{m+s+q}}_{\mu_1\cdots\mu_{m+l}}=\mathcal{R}^{i(\nu_1\cdots\nu_q}\delta^{\nu_{q+1}}_{\mu_1}\cdots\delta^{\nu_{q+m+s})}_{\mu_{m+s}}\,,\qquad \mathcal{J}^{\mu_1\cdots\mu_{m+s}}_{ai\lambda_1\cdots\lambda_l}\mathcal{R}^{i\nu_1\cdots\nu_{m+s+q}}_{\mu_1\cdots\mu_{m+s}}=0\,.$$

These null-vectors span the space of dimension

$$\binom{m+s+q+d-1}{m+s+q}=\binom{p+q+d-1}{p+q}.$$

In general, the system (A.3) may enjoy several gauge symmetry transformations. We let r_n denote the number of the gauge transformations of order n. If all these gauge symmetries are independent (irreducible), then they make

$$\sum_{n=0}^{\infty}r_n\binom{p+n+d-1}{q+n}.$$

coefficients of $\{\varphi_{\mu_1\cdots\mu_q}\}$ unphysical. Subtracting this number from (A.15), we get the number of "physically distinguishable" parametric coefficients of pth order,

$$N''_p=f\binom{p+d-1}{q}-\sum_{n=0}^{\infty}\left\{(t_n-l_n)\binom{p-n+d-1}{p-n}+r_n\binom{p+n+d-1}{p+n}\right\}. \tag{A.19}$$

Having computed N_p and N''_p we are ready to evaluating the limit (A.2). Making use of the asymptotic expansion for the binomial coefficients [15, 18]

$$\binom{p\pm n+d-1}{p\pm n}=\binom{p+d-1}{p}\left\{1\pm\frac{n}{p}(d-1)+O\left(\frac{1}{p^2}\right)\right\},\qquad p\to\infty\,,$$

we find

$$N''_p/N_p=(f-t+l-r)+\frac{(d-1)}{p}\sum_{n=0}^{\infty}n(t_n-l_n-r_n)+O\left(\frac{1}{p^2}\right). \tag{A.20}$$

The numbers

$$t=\sum_{n=0}^{\infty}t_n\,,\qquad l=\sum_{n=0}^{\infty}l_n\,,\qquad r=\sum_{n=0}^{\infty}r_n$$

coincide, respectively, with the total number of equations, gauge identities, and gauge symmetries. The leading term of the expansion

$$\Delta=f-t+l-r \tag{A.21}$$

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is called the *compatibility coefficient*. Let us assume that the system (A.3) is *absolutely compatible*, that means $\Delta = 0$. Then comparing (A.20) with (A.2), we finally arrive at the desired formula for the physical degrees of freedom

$$\mathcal{N} = \sum_{n=0}^{\infty} n(t_n - l_n - r_n)\,. \tag{A.22}$$

Vanishing of the compatibility coefficient Δ can be easily established under the assumption that each right null-vector of the symbol matrix $\mathcal{J} = \mathcal{J}_0$ originates from some gauge symmetry. To do this, we introduce the $n \times t$-matrix

$$J_{ai}(p) = \mathcal{J}_{ai}^{\mu_1\cdots\mu_m} p_{\mu_1} \cdots p_{\mu_m}\,,$$

whose entries are polynomials in formal variables p_μ, $\mu = 1, \ldots, d$. It then follows from (A.10) that each gauge identity provides the left null-vector

$$L^a(p) = \mathcal{L}^{a\mu_1\cdots\mu_{q'}} p_{\mu_1} \cdots p_{\mu_{q'}}$$

for the polynomial matrix $J(p)$, so that $L^a(p)J_{ai}(p) = 0$. Similarly, each gauge symmetry transformation (A.18) gives rise to the polynomial vector

$$R^i(p) = \mathcal{R}^{i\mu_1\cdots\mu_q} p_{\mu_1} \cdots p_{\mu_q}$$

annihilating the matrix $J(p)$ on the right, that is, $J_{ai}(p)R^i(p) = 0$. The vanishing condition for the compatibility coefficient (A.21) can be written as

$$t - l = f - r\,.$$

It means that the rank of the rectangular matrix $J(p)$, being computed by the number of left null-vectors, coincides with its rank defined in terms of right null-vectors. Clearly, this equality takes place for any matrix over an algebraic field, say $\mathbb{R}$ or $\mathbb{C}$. It turns out that the same statement holds true for the matrices over the ring of polynomials in p's provided that the null-vectors L's and R's are linearly independent (over the ring of polynomials in p's) and span the right and left kernel spaces of the matrix J (see, e.g. [35]).

There is also another, more direct, interpretation of the absolute compatibility condition. It can be shown [13] that the value $f - t + l$ defines the number of arbitrary functions of d variables entering the general solution to the field equations (A.3). The equality $\Delta = f - t + l - r = 0$ then implies that *all* these functional parameters owe their existence to the gauge symmetries. To the best of our knowledge, example of field equations has been yet unknown that would not be absolutely compatible. Moreover, the results of [2] suggest that any system of ODEs is absolutely compatible and the same is true for two-dimensional field theories [36]. So, it is a very plausible hypotheses that any reasonable field theory is absolutely compatible.

The above consideration can be extended to the field equations with reducible gauge symmetries and/or identities. Without further ado we just present the final formula for

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the physical degrees of freedom, which might be deduced by appropriate adjustment of the derivation in the irreducible case:

$$\mathcal{N} = \sum_{m,n=0} n\big(t_n - (-1)^m (l_n^m + r_n^m)\big)\,. \tag{A.23}$$

Here t_n is the number of the equations of order n; l_n^m is the number of gauge identities of the total order n, and the reducibility order m; and r_n^m is the number of gauge symmetry transformations of the total order n and reducibility order m. The total order of the generator of gauge symmetry/identity is defined inductively to be the sum of its order as a differential operator and the total order of a generator it annihilates. It is also assumed that the total order of the original gauge symmetry generators coincides with their order as differential operators, whereas the total order of gauge identities for the field equations is given by the order of the corresponding generators plus the order of equations they act on. For the field theories with irreducible gauge symmetries of the first order, equality (A.23) was first derived in [18].

It is curious to note that the final formula (A.23), being independent on d, holds true for the one-dimensional systems as well, whereas the original definition (A.2) becomes meaningless. The proof of (A.23) for $d = 1$ requires a different method, which is beyond the scope of this paper.

Let us concretize the formula (A.23) for the special case of involutive Lagrangian second order equations as this case has a common interest in field theory. For the Lagrangian equations, the gauge symmetries and identities are generated by the same operators. The total order of the identities, however, is shifted by the order of the equations involved in, so the careful adjustment of the general relation (A.23) for the second order involutive Lagrangian equations leads to the following count of physical degrees of freedom:

$$\mathcal{N} = 2\left(t_2 + \sum_{n,m=0}(-1)^{m+1}(n+1)r_n^{(m)}\right)\,. \tag{A.24}$$

Here t_2 is a number of the Lagrangian equations, $r_n^{(m)}$ is the number of the gauge symmetry generators of the total order n and reducibility order m. In the irreducible case, ($m = 0$) this brings the well known relation for the degrees of freedom for the second order Lagrangian equations [19]:

$$\frac{\mathcal{N}}{2} = t_2 - \sum_{n=0}(n+1)r_n\,. \tag{A.25}$$

$\mathcal{N}/2$ has the meaning of number of physical polarizations.

References

[1] P.A.M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva University, New York U.S.A. (1964).

[2] S. Lyakhovich and A. Sharapov, *Normal forms and gauge symmetries of local dynamics*, *J. Math. Phys.* **50** (2009) 083510 [arXiv:0812.4914] [INSPIRE].

JHEP01(2013)097

[3] M. Henneaux, *Consistent interactions between gauge fields: the Cohomological approach*, *Contemp. Math.* **219** (1998) 93.

[4] G. Barnich and M. Henneaux, *Consistent couplings between fields with a gauge freedom and deformations of the master equation*, *Phys. Lett.* **B 311** (1993) 123 [hep-th/9304057] [INSPIRE].

[5] G. Barnich, F. Brandt and M. Henneaux, *Local BRST cohomology in gauge theories*, *Phys. Rept.* **338** (2000) 439 [hep-th/0002245] [INSPIRE].

[6] M. Henneaux and B. Knaepen, *All consistent interactions for exterior form gauge fields*, *Phys. Rev.* **D 56** (1997) 6076 [hep-th/9706119] [INSPIRE].

[7] N. Boulanger, T. Damour, L. Gualtieri and M. Henneaux, *Inconsistency of interacting, multigraviton theories*, *Nucl. Phys.* **B 597** (2001) 127 [hep-th/0007220] [INSPIRE].

[8] M. Henneaux, G. Lucena Gomez and R. Rahman, *Higher-Spin Fermionic Gauge Fields and Their Electromagnetic Coupling*, *JHEP* **08** (2012) 093 [arXiv:1206.1048] [INSPIRE].

[9] C. de Rham, G. Gabadadze and A.J. Tolley, *Ghost free Massive Gravity in the Stúckelberg language*, *Phys. Lett.* **B 711** (2012) 190 [arXiv:1107.3820] [INSPIRE].

[10] J. Kluson, *Comments About Hamiltonian Formulation of Non-Linear Massive Gravity with Stuckelberg Fields*, *JHEP* **06** (2012) 170 [arXiv:1112.5267] [INSPIRE].

[11] S. Hassan, A. Schmidt-May and M. von Strauss, *Proof of Consistency of Nonlinear Massive Gravity in the Stúckelberg Formulation*, *Phys. Lett.* **B 715** (2012) 335 [arXiv:1203.5283] [INSPIRE].

[12] Y. Zinoviev, *On massive spin 2 interactions*, *Nucl. Phys.* **B 770** (2007) 83 [hep-th/0609170] [INSPIRE].

[13] W.M. Seiler, *Involution: The Formal Theory of Differential Equations and its Applications in Computer Algebra*, Algorithms and computations in Mathematics, Volume 24, Springer-Verlag, Berlin Heidelberg (2010).

[14] A. Einstein, *The Meaning of the Relativity*, 5th. edition, Princeton University Press, Princeton U.S.A. (1955).

[15] K.H. Mariwalla, *Application of the concept of strength of a system of partial differential equations*, *J. Math. Phys.* **15** (1974) 468.

[16] B.F. Schutz, *On the strenght of a system of partial differential equations*, *J. Math. Phys.* **16** (1975) 855.

[17] N.F.J. Matthews, *On the strenght of Maxwell's equations*, *J. Math. Phys.* **28** (1987) 810.

[18] M. Sué, *Involutive systems of differential equations: Einstein's strenght versus Cartan's degré d'arbitraire*, *J. Math. Phys.* **32** (1991) 392.

[19] M. Henneaux, C. Teitelboim and J. Zanelli, *Gauge invariance and degree of freedom count*, *Nucl. Phys.* **B 332** (1990) 169 [INSPIRE].

[20] X. Bekaert, S. Cnockaert, C. Iazeolla and M.A. Vasiliev, *Nonlinear higher spin theories in various dimensions*, IHES-P-04-47, ULB-TH-04-26, ROM2F-04-29, FIAN-TD-17-04, hep-th/0503128 [INSPIRE].

[21] V. Lopatin and M.A. Vasiliev, *Free massless bosonic fields of arbitrary spin in d-dimensional de Sitter space*, *Mod. Phys. Lett.* **A 3** (1988) 257 [INSPIRE].

JHEP01(2013)097

[22] O. Shaynkman and M.A. Vasiliev, *Scalar field in any dimension from the higher spin gauge theory perspective*, *Theor. Math. Phys.* **123** (2000) 683 [hep-th/0003123] [INSPIRE].

[23] E. Skvortsov, *Gauge fields in (A)dS(d) within the unfolded approach: algebraic aspects*, *JHEP* **01** (2010) 106 [arXiv:0910.3334] [INSPIRE].

[24] S. Lyakhovich and A. Sharapov, *BRST theory without Hamiltonian and Lagrangian*, *JHEP* **03** (2005) 011 [hep-th/0411247] [INSPIRE].

[25] P. Kazinski, S. Lyakhovich and A. Sharapov, *Lagrange structure and quantization*, *JHEP* **07** (2005) 076 [hep-th/0506093] [INSPIRE].

[26] D. Kaparulin, S. Lyakhovich and A. Sharapov, *Local BRST cohomology in (non-)Lagrangian field theory*, *JHEP* **09** (2011) 006 [arXiv:1106.4252] [INSPIRE].

[27] M. Henneaux, *Space-time locality of the BRST formalism*, *Commun. Math. Phys.* **140** (1991) 1 [INSPIRE].

[28] M. Fierz and W. Pauli, *On Relativistic Wave Equations for Particles of Arbitrary Spin in an Electromagnetic Field*, *Proc. R. Soc. London* **A 173** (1939) 211.

[29] L. Singh and C. Hagen, *Lagrangian formulation for arbitrary spin. 1. The boson case*, *Phys. Rev.* **D 9** (1974) 898 [INSPIRE].

[30] S. Lyakhovich and A. Sharapov, *Schwinger-Dyson equation for non-Lagrangian field theory*, *JHEP* **02** (2006) 007 [hep-th/0512119] [INSPIRE].

[31] S. Lyakhovich and A. Sharapov, *Quantizing non-Lagrangian gauge theories: an augmentation method*, *JHEP* **01** (2007) 047 [hep-th/0612086] [INSPIRE].

[32] D. Kaparulin, S. Lyakhovich and A. Sharapov, *Rigid Symmetries and Conservation Laws in Non-Lagrangian Field Theory*, *J. Math. Phys.* **51** (2010) 082902 [arXiv:1001.0091] [INSPIRE].

[33] S. Hassan and R.A. Rosen, *On Non-Linear Actions for Massive Gravity*, *JHEP* **07** (2011) 009 [arXiv:1103.6055] [INSPIRE].

[34] S. Hassan and R.A. Rosen, *Confirmation of the Secondary Constraint and Absence of Ghost in Massive Gravity and Bimetric Gravity*, *JHEP* **04** (2012) 123 [arXiv:1111.2070] [INSPIRE].

[35] D. Eisenbud, *The Geometry of Syzygies: A Second Course in Algebraic Geometry and Commutative Algebra*, Graduate Texts in Mathematics, Volume 229, Springer-Verlag New York Inc., NY U.S.A. (2005).

[36] S.L. Lyakhovich and A.A. Sharapov, work in progress.