Quantum black holes in Type-IIA String Theory

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ABSTRACT: We study black hole solutions of Type-IIA Calabi-Yau compactifications in the presence of quantum perturbative corrections. We define a class of black holes that only exist in the presence of quantum corrections and that, consequently, can be considered as purely *quantum* black holes. The regularity conditions of the solutions impose the topological constraint $h^{1,1} > h^{2,1}$ on the Calabi-Yau manifold, defining a class of admissible compactifications, which we prove to be non-empty for $h^{1,1} = 3$ by explicitly constructing the corresponding Calabi-Yau manifolds, new in the literature.

Contents

General remarks.   Supergravity solutions have played, and continue to play, a prominent rôle in the new developments of String Theory. The body of literature about black hole solutions (and p-branes) that has been accumulated during the past thirty years is enormous, but only recently the issue of non-extremality was systematically investigated, and by now, it could be said that we have at our disposal well-established methods to deal with non-extremal solutions [2–7, 20–22] in Supergravity. However, explicit non-extremal solutions to Supergravity models with perturbative quantum corrections are yet to be constructed. These kind of solutions may be relevant in order to understand how the deformation of the scalar geometry modifies the solutions of the theory, and also in order to relate the macroscopic computation of the entropy with the microscopic calculation in a String Theory set-up, once sub-leading corrections to the prepotential are taken into account [8, 9]. These kinds of corrections differ from the higher order corrections, which, together with the corresponding microscopic String Theory computation, have been extensively studied in the literature [10] (for a very nice review about this and related topics, as well as for further references, see [11]).

In this note we are going to use the so-called *H-formalism* [20–22] in order to take a small step in the study of non-extremal black holes in Supergravity in the presence of quantum corrections. The *H-formalism*, as it has been used so far to produce new solutions, is based on a change of variables in the $\mathcal{N} = 2$, $d = 4$ ungauged Supergravity action (to new ones H^M that transform linearly under duality and become harmonic functions on $\mathbb{R}^3$ in the extremal case) plus a hyperbolic Ansatz for them, that allows to transform the system of differential equations of motion into a system of algebraic equations, easier to handle. It is, of course, also possible to make the change of variables and try to solve the resulting system of differential equations by other means, not involving any particular Ansatz for the H^M.

Through a consistent truncation, we are going to define a particular class of black holes, which is characterized by existing only when the quantum perturbative corrections are included in the action. These kinds of solutions, which we have chosen to call *quantum black holes*, display a remarkable behavior: the so called *large-volume limit* $\Im m z^i \to \infty$ is in fact not a large volume limit of the Calabi-Yau (C.Y.) manifold, whose volume remains constant and fixed by topological data. In addition, the regularity conditions of the black hole solutions impose the topological restriction $h^{1,1} > h^{2,1}$ in the compactification C.Y. For small $h^{1,1}$ the condition is particularly restrictive, and since this case is the most manageable one from the point of view of black hole solutions, we prove the existence of C.Y. manifolds obeying $h^{1,1} > h^{2,1}$ by explicit construction for the $h^{1,1} = 3$ case. These C.Y. manifolds are new in the literature.

The perturbative corrections, encoded in a single term $i\frac{c}{2}$ in the prepotential, introduce a highly non-trivial difficulty in the model, which makes almost hopeless the resolution of the equations of the theory. Surprisingly enough, we are able to find a black hole solution with non-constant scalars, similar to the $D0-D4-D4-D4$ black hole solution of the *STU* model, and which can be used as a toy model to study the microscopic description of black holes in the presence of quantum perturbative corrections and away from extremality.

1 Type-IIA String Theory on a Calabi-Yau manifold

Type-IIA String Theory compactified to four dimensions on a C.Y. manifold, with Hodge numbers $(h^{1,1}, h^{2,1})$, is described by a $\mathcal{N} = 2, d = 4$ Supergravity whose prepotential is given in terms of an infinite series around $\Im m z^i \to \infty$,[1] [15–17]

$$\mathcal{F} = -\frac{1}{3!}\kappa^0_{ijk} z^i z^j z^k + \frac{ic}{2} + \frac{i}{(2\pi)^3} \sum_{d_i} n_{d_i} Li_3 \left(e^{2\pi i d_i z^i} \right) , \qquad (1.1)$$

where z^i, $i = 1, \dots, n_v = h^{1,1}$, are the scalars in the vector multiplets,[2] $c = \frac{\chi\zeta(3)}{(2\pi)^3}$ is a model dependent number,[3] κ^0_{ijk} are the classical intersection numbers, $d_i \in \mathbb{Z}^+$ is a n_v-dimensional summation index and

$$Li_3(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^3} . \qquad (1.2)$$

The non-perturbative part of the prepotential (1.1), according to his stringy origin, is given by

$$\mathcal{F}_{\text{Non-Pert}} = \frac{i}{(2\pi)^3} \sum_{d_i} n_{d_i} Li_3 \left(e^{2\pi i d_i z^i} \right) , \qquad (1.3)$$

whereas the rest of the prepotential includes the tree level contribution and the quantum perturbative corrections

$$\mathcal{F}_{\text{Pert}} = -\frac{1}{3!}\kappa^0_{ijk} z^i z^j z^k + \frac{ic}{2} . \qquad (1.4)$$

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[1] Actually, the prepotential obtained in a Type-IIA C.Y. compactification is *symplectically equivalent* to the prepotential (1.1).

[2] There are also $h^{2,1} + 1$ hypermultiplets in the theory. However, they can be consistently set to a constant value.

[3] χ is the Euler characteristic, which for C.Y. three-folds is given by $\chi = 2(h^{1,1} - h^{2,1})$.

The non-perturbative corrections (1.3) are exponentially suppressed and therefore can be safely ignored going to the large volume limit. Therefore our starting point is going to be eq. (1.4), which in homogeneous coordinates $\mathcal{X}^\Lambda$, $\Lambda = (0, i)$, can be written as

$$F(\mathcal{X}) = -\frac{1}{3!}\kappa^0_{ijk}\frac{\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0} + \frac{ic}{2}\left(\mathcal{X}^0\right)^2 . \tag{1.5}$$

The scalars z^i are given by[4]

$$z^i = \frac{\mathcal{X}^i}{\mathcal{X}^0} . \tag{1.6}$$

The scalar geometry defined by (1.5) is the so called *quantum corrected d*-SK geometry[5] [18, 19]. In this scenario, the classical case is modified and the scalar manifold, due to the correction encoded in c, is no longer homogeneous, and therefore, the geometry has been *corrected* by quantum effects.

We are interested in studying spherically symmetric, static, black hole solutions of the theory defined by eq. (1.5). In order to do so we are going to use the so-called *H-formalism*, developed in [20–22], based on the use of a new set of variables H^M, $M = (\Lambda, \Lambda)$, that transform linearly under duality and reduce to harmonic functions on the transverse space $\mathbb{R}^3$ in the supersymmetric case.[6] This is the subject of the next section.

1.1 A *quantum* class of black holes

The most general static, spherically symmetric space-time metric solution of an ungauged Supergravity is given by[7] [1, 21]

$$ds^2 = e^{2U}dt^2 - e^{-2U}\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}},$$
$$\gamma_{\underline{mn}}dx^{\underline{m}}dx^{\underline{n}} = \frac{r_0^4}{\sinh^4 r_0\tau}d\tau^2 + \frac{r_0^2}{\sinh^2 r_0\tau}d\Omega^2_{(2)} . \tag{1.7}$$

Using eq. (1.7) and following the *H-formalism*, we obtain that the equations of the theory are given by

$$\mathcal{E}_P = \frac{1}{2}\partial_P\partial_M\partial_N \log \mathsf{W}\left[\dot{H}^M\dot{H}^N - \frac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right] + \partial_P\partial_M \log \mathsf{W}\,\ddot{H}^M - \frac{d}{d\tau}\left(\frac{\partial\Lambda}{\partial\dot{H}^P}\right) + \frac{\partial\Lambda}{\partial H^P} = 0, \tag{1.8}$$

together with the *Hamiltonian constraint*

$$\mathcal{H} \equiv -\frac{1}{2}\partial_M\partial_N \log \mathsf{W}\left[\dot{H}^M\dot{H}^N - \frac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right] + \left(\frac{\dot{H}^M H_M}{\mathsf{W}}\right)^2 - \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}}\right)^2 - r_0^2 = 0, \tag{1.9}$$

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[4]This coordinate system is therefore only valid away from the locus $\mathcal{X}^0 = 0$.

[5]The attractor points of this model have been extensively studied in [12]. Related works can be found in [13, 14].

[6]It has been conjectured that this is also true in the extremal non-supersymmetric case [21, 22].

[7]The conformastatic coordinates (t, τ, θ, ϕ) cover the outer region of the event horizon when $\tau \in (-\infty, 0)$ and the inner region, between the Cauchy horizon and the physical singularity when $\tau \in (\tau_S, \infty)$, where $\tau_S \in \mathbb{R}^+$ is a model dependent number. The event horizon is located at $\tau \to -\infty$ and the Cauchy horizon at $\tau \to \infty$.

where

$$\Lambda \equiv \left(\frac{\dot{H}^M H_M}{\mathsf{W}}\right)^2 + \left(\frac{\mathcal{Q}^M H_M}{\mathsf{W}}\right)^2 , \tag{1.10}$$

and

$$\mathsf{W}(H) \equiv \mathcal{R}_M(H)H^M = e^{-2U}, \quad \mathcal{R} + i\mathcal{I} = \mathcal{V}^M/X . \tag{1.11}$$

$\mathcal{V}^M$ is the covariantly holomorphic symplectic section of $\mathcal{N} = 2$ Supergravity, and X is a complex variable with the same Kähler weight as $\mathcal{V}^M$. $\mathcal{R}^M(\mathcal{I})$ stands for the real part $(\mathcal{R}^M)$ of $\mathcal{V}^M/X$ written as a function of the imaginary part $\mathcal{I}^M$, something that can always be done by solving the so-called *stabilization equations*. $\mathsf{W}(H)$ is usually known in the literature as the *Hesse potential*.

The theory is now expressed in terms of $2\,(n_v+1)$ variables H^M and depends on $2\,(n_v+1)+1$ parameters: $2\,(n_v+1)$ charges $\mathcal{Q}^M$ and the non-extremality parameter r_0, from which one can reconstruct the solution in terms of the original fields of the theory (that is it, the space-time metric, scalars and vector fields).

For eq. (1.4), the general $\mathsf{W}(H)$ is an extremely involved function, and one cannot expect to solve in full generality the corresponding differential equations of motion, or even the associated algebraic equations of motion obtained by making use of the hyperbolic Ansatz for the H^M. Therefore, we are going to consider a particular truncation, which will give us the desired *quantum* black holes

$$H^0 = H_0 = H_i = 0, \quad p^0 = p_0 = q_i = 0 . \tag{1.12}$$

Eq. (1.12) implies

$$\mathsf{W}(H) = \alpha \left| \kappa_{ijk}^0 H^i H^j H^k \right|^{2/3} , \tag{1.13}$$

where $\alpha = \frac{(3!c)^{1/3}}{2}$ must be positive in order to have a non-singular metric. Hence $c > 0$ is a necessary condition in order to obtain a regular solution and a consistent truncation. The corresponding *black hole potential* reads

$$V_{\text{bh}} = \frac{\mathsf{W}(H)}{4} \partial_{ij} \log \mathsf{W}(H) \mathcal{Q}^i \mathcal{Q}^j , \tag{1.14}$$

The scalar fields, purely imaginary, are given by

$$z^i = i\,(3!c)^{1/3} \frac{H^i}{\left(\kappa_{ijk}^0 H^i H^j H^k\right)^{1/3}} , \tag{1.15}$$

and are subject to the following constraint, which ensures the regularity of the Kähler potential ($\mathcal{X}^0 = 1$ gauge)

$$\kappa_{ijk}^0 \Im m z^i \Im m z^j \Im m z^k > \frac{3c}{2} . \tag{1.16}$$

Substituting eq. (1.15) into eq. (1.16), we obtain

$$c > \frac{c}{4} , \tag{1.17}$$

which is an identity (assuming $c > 0$) and therefore imposes no constraints on the scalars. This phenomenon can be traced back to the fact that the the Kähler potential is constant when evaluated on the solution, and given by

$$e^{-\mathcal{K}} = 6c, \tag{1.18}$$

which is well defined, again, if $c > 0$. Since the volume of the C.Y. manifold is proportional to $e^{-\mathcal{K}}$, eq. (1.18) implies that such volume remains constant and, in particular, that the limit $\Im m z^i \to \infty$ does not imply a large volume limit of the compactification C.Y. manifold, a remarkable fact that can be seen as a purely quantum characteristic of our solution.[8] Notice that it is also possible to obtain the classical limit $\Im m z^i \gg 1$ taking $c \gg 1$, that is, choosing a Calabi-Yau manifold with large enough c. In this case we would have also a truly large volume limit.

We have seen that, in order to obtain a consistent truncation, a necessary condition is $c > 0$, which implies that $\mathsf{W}(H)$ is well defined. We can go even further and argue that this is a sufficient condition by studying the equations of motion $\mathcal{E}_P$:

A consistent truncation requires that the equation of motion of the truncated field is identically solved for the *truncation value* of the field. First, notice that the set of solutions of eqs. (1.8) and (1.9), taking into account (1.12), is non-empty, since there is a *model-independent* solution, given by

$$H^i = a^i - \frac{p^i}{\sqrt{2}}\tau, \quad r_0 = 0, \tag{1.19}$$

which corresponds to a supersymmetric black hole. However, the equations of motion $\mathcal{E}_P$ don't know about supersymmetry: it is system of differential equations whose solution can be written as

$$H^M = H^M(a, b), \tag{1.20}$$

where we have made explicit the dependence in $2n_v + 2$ integration constants. When the solution (1.20) is plugged into (1.9) is when we impose, through r_0, a particular condition about the extremality of the black hole. If $r_0 = 0$ the integration constants are fixed such as the solution is extremal. In general there is not a unique way of doing it, one of the possibilities being always the supersymmetric one. Therefore, given that for our particular truncation the supersymmetric solution always exists, we can expect the existence also of the corresponding solution (1.20) of the equations of motion, from which the supersymmetric solution may be obtained through a particular choice of the integration constants that make (1.20) fulfilling (1.9) for $r_0 = 0$.

We conclude, hence, that

$$\left\{ H^P = 0, \mathcal{Q}^P = 0 \right\} \quad \Rightarrow \mathcal{E}_P = 0, \tag{1.21}$$

and therefore the truncation of as many H's as we want, together with the correspondet $\mathcal{Q}$'s, is consistent as long as $\mathsf{W}(H)$ remains well defined, something that in our case is

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[8]Notice that in order to consistently discard the non-perturbative terms in eq. (1.1) we only need to take the limit $\Im m z^i \to \infty$. Therefore, the behavior of the C.Y. volume in such limit plays no role.

assured if $c > 0$. From eq. (1.1) it can be checked that the case $c = 0$, that is $h^{1,1} = h^{2,1}$, can be cured by non-perturbative effects.

It is easy to see that the truncation is not consistent in the classical limit, and therefore, we can conclude that the corresponding solutions are *genuinely* quantum solutions, which only exist when perturbative quantum effects are incorporated into the action.

Hence, we can conclude that if we require our theory to contain regular *quantum* black holes there is a topological restriction on the Calabi-Yau manifolds that we can choose to compactify Type-IIA String Theory. The condition can be expressed as

$$c > 0 \quad \Rightarrow \quad h^{11} > h^{21} \,. \tag{1.22}$$

Eq. (1.22) is a stringent condition on the compactification C.Y. manifolds, in particular for small h^{11}. In fact, for small enough h^{11} it could be even possible that no Calabi-Yau manifold existed such eq. (1.22) is fulfilled. We will investigate this issue for $h^{11} = 3$, explicitly constructing the corresponding C.Y. manifolds and finding also particular *quantum* black hole solutions, in the next section.[9]

2   New Calabi-Yau manifolds

In this section we will present the construction of new Calabi-Yau manifolds which satisfy $h^{1,1} = 3$ and $h^{2,1} < 3$, as required for the truncation presented in the previous section.

Calabi-Yau threefolds with both Hodge numbers small are relatively rare; two large and useful databases are the complete intersections in products of projective spaces (CICY's) [23], and hypersurfaces in toric fourfolds [24, 25], but the manifolds in these lists all satisfy the inequality $h^{1,1} + h^{2,1} > 21$. Smaller Hodge numbers can be found by taking quotients by groups which have a free holomorphic action on one of these manifolds (see, e.g., [26–28] and references therein), but none of the known spaces constructed this way satisfies our requirements.

Our technique here will be to begin with known manifolds with $h^{1,1} < 3$, $h^{2,1} < 4$, and a non-trivial fundamental group, and find hyperconifold transitions [29, 30] to new manifolds with the required Hodge numbers. Briefly, these transitions occur because a generically-free group action on a Calabi-Yau will develop fixed points on certain codimension-one loci in the moduli space. The fixed points are necessarily singular, and typically nodes [29], so the quotient space develops a point-like singularity which is a quotient of the conifold — a *hyperconifold*. These singularities can be resolved to give a new smooth Calabi-Yau. If the subgroup which develops a fixed point is $\mathbb{Z}_N$, then the change in Hodge numbers for one of these transitions is $\delta(h^{1,1}, h^{2,1}) = (N - 1, -1)$.

Interestingly, there are examples which one might naïvely believe would lead to manifolds with $(h^{1,1}, h^{2,1}) = (3, 0)$ and $(h^{1,1}, h^{2,1}) = (3, 2)$, but none of these work out;[10]

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[9]It is known in the literature the existence of the so called *rigid* C.Y. manifolds [33–35], which obey $h^{11} > 0, h^{21} = 0$, being therefore admissible compactification spaces. However, in order to have a tractable theory, we need a small enough h^{11}, yet not too small to yield a trivial theory. The choice $h^{11} = 3$ fulfills both conditions.

[10]In each case, the spaces have unavoidable symmetries which make it impossible to create just a single hyperconifold singularity. Resolving the extra singularities pushes $h^{1,1}$ higher.

instead, we have two examples with $(h^{1,1}, h^{2,1}) = (3,1)$, but with different intersection forms and Kähler cones.

2.1 $(h^{1,1}, h^{2,1}) = (3,1)$ and diagonal intersection form

For the first example, we start with a manifold $X^{1,3}$, where the superscripts are the Hodge numbers $(h^{1,1}, h^{2,1})$, and fundamental group $\mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_{10} \times \mathbb{Z}_2$. It was first discovered in [26], and we briefly review the construction here. The manifold is obtained as a free quotient of a CICY $X^{5,45}$ that is given by the vanishing of two multilinear polynomials in a product of five $\mathbb{P}^1$'s; the configuration matrix [23] is

$$
\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array}
\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}
$$

Let us call the two polynomials p_1, p_2, and take homogeneous coordinates $t_{i,a}$ on the ambient space, where $i = 0, 1, 2, 3, 4$ is understood mod 5, and $a = 0, 1$ is understood mod 2. Then the action of the quotient group is generated by

$$
\begin{aligned}
g_{10} : \quad & t_{i,a} \to t_{i+1,a+1} ; & p_1 \leftrightarrow p_2 , \\
g_2 : \quad & t_{i,a} \to (-1)^a t_{i,a} ; & p_1 \to p_1 , \ p_2 \to -p_2 .
\end{aligned}
$$

Note that these commute only up to projective equivalence, but this is sufficient. To define polynomials which transform appropriately, we start with the following quantities:

$$
m_{abcde} = \sum_{i=0}^{4} t_{i,a} t_{i+1,b} t_{i+2,c} t_{i+3,d} t_{i+4,e} .
$$

Then it is easily checked that the following are the most general polynomials which transform correctly:

$$
\begin{aligned}
p_1 &= \frac{A_0}{5} m_{00000} + A_1 m_{00011} + A_2 m_{00101} + A_3 m_{01111} , \\
p_2 &= \frac{A_0}{5} m_{11111} + A_1 m_{11100} + A_2 m_{11010} + A_3 m_{10000} ,
\end{aligned}
$$

where the A_α are arbitrary complex constants. For generic values of the coefficients, these polynomials define a smooth manifold on which the group $\mathbb{Z}_{10} \times \mathbb{Z}_2$ acts freely; in this way we find a smooth quotient family $X^{1,3} = X^{5,45} / \mathbb{Z}_{10} \times \mathbb{Z}_2$.

We now need to specialise to a sub-family of $X^{5,45}$ which *does* have fixed points of the group generator g_2. Specifically, consider the point given by

$$
t_{0,1} = t_{1,1} = t_{2,1} = t_{3,0} = t_{4,0} = 0,
$$

which is fixed by the action of g_2. Substituting the above into the polynomials gives the values $p_1 = A_1$, $p_2 = 0$, so if we set $A_1 = 0$, $X^{5,45}$ will contain this point. The argument

of [29] guarantees that it will be a singularity, and one can check that for general values of the other coefficients, it is a node, so the quotient space $X^{1,3}$ develops a $\mathbb{Z}_2$-hyperconifold singularity. In fact, there are nine other points related to the above by the action of the other group generator g_{10}, so the covering space $X^{5,45}$ actually has ten nodes. Since these are all identified by the group action, $X^{1,3}$ develops only a single $\mathbb{Z}_2$-hyperconifold. This can be resolved by a single blow-up, and we obtain a new manifold with Hodge numbers $(h^{1,1}, h^{2,1}) = (2, 2)$, and fundamental group $\mathbb{Z}_{10}$.

To get all the way to $X^{3,1}$, we need to go through another $\mathbb{Z}_2$-hyperconifold transition. If we also set $A_2 = 0$, then $X^{5,45}$ also passes through another fixed point of g_2, given by

$$t_{0,1} = t_{1,1} = t_{2,0} = t_{3,1} = t_{4,0} = 0 \,,$$

as well as the nine points related to this by the action of g_{10}.

It can be checked that when $A_1 = A_2 = 0$, $X^{5,45}$ has exactly twenty nodes, at the points described above, and is smooth elsewhere. Therefore $X^{1,3}$ has precisely two $\mathbb{Z}_2$-hyperconifold singularities, which we can resolve independently to obtain a new smooth Calabi-Yau manifold $X^{3,1}$.

2.1.1 The intersection form and Kähler cone

To find the Supergravity theory coming from compactification on $X^{3,1}$, we need to calculate its triple intersection form, and for this we need a basis for $H^2(X^{3,1}, \mathbb{Z})$ (throughout, we will implicitly talk about only the torsion-free part of the cohomology). There is a natural basis which consists of one divisor class inherited from $X^{1,3}$, and the two exceptional divisor classes coming from the two blow-ups.

First, let us find an integral generator of $H^2(X^{1,3}, \mathbb{Z})$. On the covering space $X^{5,45}$, let H_i be the divisor class given by the pullback of the hyperplane class from the ith $\mathbb{P}^1$. Then the invariant divisor classes are multiples of $H \equiv H_0 + H_1 + H_2 + H_3 + H_4$. However, H itself, although an invariant *class*, does not have an invariant *representative*. The class $2H$ does, however; an example of an invariant divisor in $2H$ is the surface given by the vanishing of

$$f = (m_{00011})^2 + (m_{11100})^2 + (m_{00101})^2 + (m_{11010})^2 \,.$$

This is a particularly convenient choice, as it gives a smooth divisor even on the singular family of threefolds given by $A_1 = A_2 = 0$, which misses the singular points.

Let D_1 be the divisor given by setting $f = 0$ and then taking the quotient, and let D_2 and D_3 be the two exceptional divisors. Then, since $f \neq 0$ on the fixed points of the group action, we immediately see that D_1, D_2, and D_3 are all disjoint, and the only intersection numbers which might be non-zero are D_1^3, D_2^3, and D_3^3.

For D_2^3 and D_3^3, we can use an easy general argument. For any smooth surface S in a Calabi-Yau threefold, the adjunction formula gives $S\big|_S \sim K_S$, where K_S is the canonical divisor class. The triple intersection number S^3 is therefore equal to K_S^2. Each of D_2 and D_3 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, so we find $D_2^3 = D_3^3 = 8$.

To calculate D_1^3, we note that D_1 descends from the divisor class $2H$ on $X^{5,45}$. Since this is embedded in a product of projective spaces, we can calculate intersection numbers

purely from degrees; it is easy to check that on $X^{5,45}$, $(2H)^3 = 960$. We divide by a freely-acting group of order twenty, so on the quotient space we find $D_1^3 = \frac{960}{20} = 48$.

To summarise, the non-vanishing triple intersection numbers of $X^{3,1}$, in the basis D_1, D_2, D_3, are

$$\kappa_{111}^0 = 48\,, \quad \kappa_{222}^0 = \kappa_{333}^0 = 8\,.$$

We can also say something about the Kähler cone. Certainly D_1 is positive everywhere except on the exceptional divisors, where it is trivial. On the other hand, each exceptional divisor D contains curves C for which $D \cdot C = -1$. From this information, we can glean that the Kähler cone is some sub-cone of $t_1 > 0, t_2 < 0, t_3 < 0$, and certainly includes the region where t_1 is much larger than $|t_2|$ and $|t_3|$.

2.2 $(h^{1,1}, h^{2,1}) = (3, 1)$ and non-diagonal intersection form

For our second example, we will again start with a free quotient of a CICY manifold, with configuration matrix

$$
\begin{array}{c}
\mathbb{P}^1 \\
\mathbb{P}^1 \\
\mathbb{P}^1 \\
\mathbb{P}^1 \\
\mathbb{P}^1 \\
\mathbb{P}^1 \\
\mathbb{P}^1 \\
\end{array}
\left[
\begin{array}{cccc}
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
\end{array}
\right]
$$
$$ p \quad q \quad r_1 \ r_2$$

where the labels on the columns denote the respective polynomials. This manifold has Euler number zero, and a series of splittings and contractions (explained in [23, 26, 31]) establishes that it is in fact isomorphic to the 'split bicubic' or Schoen manifold, with Hodge numbers $(h^{1,1}, h^{2,1}) = (19, 19)$.

Let us take homogeneous coordinates σ_a on the first $\mathbb{P}^1$, $s_{i,a}$ on the next three, and $t_{i,a}$ on the last three, where $i = 0, 1, 2$, and $a = 0, 1$ are understood mod 3 and mod 2 respectively. The quotient group of interest is the dicyclic group $\mathrm{Dic}_3 \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$, which is the only non-trivial semi-direct product of $\mathbb{Z}_3$ and $\mathbb{Z}_4$. It is generated by two elements g_3 and g_4, of orders given by their subscripts, with the relation $g_4 g_3 g_4^{-1} = g_3^2$, and acts on the ambient space and polynomials as follows:

$$g_3 : \quad \sigma_a \to \sigma_a\,, \qquad s_{i,a} \to s_{i+1,a}\,, \qquad t_{i,a} \to t_{i+1,a}\,; \ \text{all polynomials invariant}\,,$$

$$g_4 : \quad \sigma_a \to (-1)^a \sigma_a\,, \quad s_{i,a} \to (-1)^{a+1} t_{-i,a}\,, \quad t_{i,a} \to s_{-i,a}\,; \ p \to -q\,, \ q \to p\,, \ r_1 \leftrightarrow r_2\,.$$

In order to write down polynomials which transform appropriately, let us first define the g_3-invariant quantities

$$m_{abc} = \sum_i s_{i,a} s_{i+1,b} s_{i+2,c}\,, \quad n_{abc} = \sum_i t_{i,a} t_{i+1,b} t_{i+2,c}\,.$$

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Then by choice of coordinates (consistent with the above action), we can take the polynomials to be

$$p = \frac{1}{3} m_{000} + m_{011} \,, \qquad q = \frac{1}{3} n_{000} + n_{011} \,,$$

$$r_1 = \left(a_0 m_{001} + \frac{1}{3} a_1 m_{111} \right) \sigma_0 + \left(a_0 m_{001} + \frac{1}{3} a_2 m_{111} \right) \sigma_1 \,,$$

$$r_2 = \left(a_0 n_{001} + \frac{1}{3} a_1 n_{111} \right) \sigma_0 - \left(a_0 n_{001} + \frac{1}{3} a_2 n_{111} \right) \sigma_1 \,,$$

where a_0, a_1, a_2 are arbitrary complex coefficients, defined only up to overall scale. It can be checked that for generic values of these coefficients, the corresponding manifold is smooth, and the group acts on it without fixed points. We therefore obtain a smooth quotient manifold $X^{2,2}$, where the value $h^{2,1} = 2$ corresponds to the two free coefficients (once we factor out overall scale) in the above polynomials.[11]

We will now show that there is a $\mathbb{Z}_2$-hyperconifold transition from $X^{2,2}$ to a manifold with $(h^{1,1}, h^{2,1}) = (3, 1)$. To do this, we need to arrange for the unique order-two element, g_4^2, to develop a fixed point. Consider the point in the ambient space given by

$$\frac{\sigma_1}{\sigma_0} = -1 \,, \ s_{0,1} = s_{1,1} = s_{2,0} = t_{0,0} = t_{1,0} = t_{2,0} = 0 \,.$$

This is fixed by g_4^2, but the other elements of the group permute this and five other g_4^2-fixed points. If we evaluate the polynomials at the point above, we find

$$p = q = r_1 = 0 \,, \ r_2 = a_1 + a_2 \,,$$

and their values at the other five fixed points are related by the group action to the ones above. So if $a_1 + a_2 = 0$, the Calabi-Yau will intersect these fixed points. By expanding the polynomials around any one of these points, we find that it has a node at each of them, so on the quotient space, we obtain a single $\mathbb{Z}_2$-hyperconifold singularity. Resolving this takes us to a new smooth manifold $Y^{3,1}$ (we use the letter Y to distinguish this from the other $(3, 1)$ manifold we constructed). Its fundamental group is $\mathrm{Dic}_3 / \langle g_4^2 \rangle \cong S_3$, the symmetric group on three letters (the behaviour of fundamental groups under hyperconifold transitions such as this one is described in [28]).

2.2.1 The intersection form and Kähler cone

To calculate the intersection form of $Y^{3,1}$, we start with $X^{2,2}$ and its covering space $X^{19,19}$. Part of $H^{1,1}(X^{19,19}, \mathbb{Z})$ is generated by the pullbacks of the hyperplane classes of the $\mathbb{P}^1$ spaces. We will denote these by $H_0, H_1, \ldots, H_6$. Looking at the group action, we can see that there are exactly two invariant divisor classes constructed from these: H_0 and $H_1 + H_2 + \ldots + H_6$. In contrast to the last example, each of these actually contains an invariant representative, and we get a basis $\{D_1, D_2\}$ for $H^{1,1}(X^{2,2}, \mathbb{Z})$ by simply taking the two invariant classes above and quotienting.

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[11]Counting independent coefficients does not always give the value of $h^{2,1}$, but in this case it does; perhaps the most direct way to obtain this is to notice that the manifold is obtained via a conifold transition on a codimension two locus in the moduli space of a manifold $X^{1,4}$, which was described at length in [32].

On the covering space, we can calculate intersection numbers simply by counting degrees, and we find that

$$H_0(H_1 + H_2 + \ldots + H_6)^2 = 72 \,, \ (H_1 + H_2 + \ldots + H_6)^3 = 216 \,,$$

and all others vanish. Dividing by the order of the group, we see that on the quotient space

$$D_1 D_2^2 = 6 \,, \ D_2^3 = 18 \,,$$

and the other triple intersections are zero.

Finally, we perform the transition to $Y^{3,1}$; denote the class of the exceptional divisor by D_3. It is easy enough to check that D_1 and D_2 have representatives which miss the singularity, so their pullbacks to $Y^{3,1}$ are disjoint from the exceptional divisor, and we get $D_1 \cdot D_3 = D_2 \cdot D_3 = 0$. Once again, the exceptional divisor is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, so by the argument of the last section, $D_3^3 = 8$.

Summarising, the non-zero intersection numbers on $Y^{3,1}$ are

$$\kappa_{122}^0 = 6 \,, \ \kappa_{222}^0 = 18 \,, \ \kappa_{333}^0 = 8 \,.$$

By similar reasoning to the last case, we can say that the Kähler cone is some sub-cone of $t_1 > 0, t_2 > 0, t_3 < 0$, and includes the region where $|t_3|$ is sufficiently small compared to t_1 and t_2.

3 Quantum black hole solutions with $h^{11} = 3$

In section 1 we have presented a particular truncation of the equations of motion of $\mathcal{N} = 2, d = 4$ ungauged Supergravity in a static, spherically symmetric background, which turned out to be consistent only for positive values of the quantum perturbative coefficient c (1.22). In the next two sections we are going to explicitly construct regular non-extremal (and therefore non-supersymmetric) black hole solutions to the truncated theory. In particular, we will start studying the cases where the C.Y. manifold is of the type constructed in section 2, to wit:

$$
\begin{aligned}
X^{3,1} &\Rightarrow \quad \kappa_{111}^0 = 48 \,, \quad \kappa_{222}^0 = \kappa_{333}^0 = 8 \,, \\
Y^{3,1} &\Rightarrow \quad \kappa_{122}^0 = 6 \,, \quad \kappa_{222}^0 = 18 \,, \qquad \kappa_{333}^0 = 8 \,.
\end{aligned}
$$

For these two sets of intersection numbers, eq. (1.13) becomes, respectively

$$\mathsf{W}(H) = \alpha \left| 48 \left(H^1 \right)^3 + 8 \left[\left(H^2 \right)^3 + \left(H^3 \right)^3 \right] \right|^{2/3} , \tag{3.1}$$

$$\mathsf{W}(H) = \alpha \left| 18 \left(H^2 \right)^2 \left[H^1 + H^2 \right] + 8 \left(H^3 \right)^3 \right|^{2/3} . \tag{3.2}$$

For simplicity we take $H^1 = s_2 H^2 = s_3 H^3 \equiv H$ ($s_{2,3} = \pm 1$), $p^1 = p^2 = p^3 \equiv p$. For this particular configuration we find a non-extremal solution for each set of intersection numbers given by

$$H = a \cosh(r_0 \tau) + \frac{b}{r_0} \sinh(r_0 \tau), \quad b = s_b \sqrt{r_0^2 a^2 + \frac{p^2}{2}} \,, \tag{3.3}$$

where, now and henceforth, $s_b = \pm 1$. The scalars, which turn out to be constant, read

$$z^1 = i(3!c)^{1/3}\lambda^{-1/3} = s_{2,3}z^{2,3}\,, \tag{3.4}$$

where

$$\begin{aligned}
\lambda &= [48 + 8(s_2 + s_3)] && \text{for } X^{3,1}, \\
\lambda &= [18 + 18s_2 + 8s_3] && \text{for } Y^{3,1}.
\end{aligned} \tag{3.5}$$

Since the scalars are constant and don't depend on the charges, we cannot perform the $\Im m z^i \to \infty$ limit that fully suppresses the non-perturbative corrections. Still, the exponent in eq. (1.3) is, in both cases, of order

$$2\pi i d_i z^i \sim -\frac{1}{3}\sum_{i=1}^{3} d_i, \quad d_i \geq 1\,, \tag{3.6}$$

and therefore we find small non-perturbative corrections, in particular one order smaller than the perturbative part of the prepotential $\mathcal{F}_{\text{Pert}} \sim 10 \cdot \mathcal{F}_{\text{Non-Pert}}$. The solution lies inside the Kähler cone when

$$\begin{aligned}
s_2 = s_3 = -1, &\quad \text{for } X^{3,1}, \tag{3.7} \\
s_2 = -s_3 = 1, &\quad \text{for } Y^{3,1}. \tag{3.8}
\end{aligned}$$

This can be verified by explicitly checking the positive-definiteness of the Kähler metric

$$\mathcal{G}_{ij^*} = \partial_i \partial_{j^*} \mathcal{K} \tag{3.9}$$

evaluated on the solution. It turns out that the only sets of $\{s_2, s_3\}$ which give rise to positive-definite Kähler metrics (and, as a consequence, to solutions lying inside the Kähler cone) are the ones shown above. These conditions on the signs of the scalar fields are in full agreement with those obtained in subsections 2.2.1 and 2.1.1, since $\Im m z^i = t^i$ [16].

Imposing asymptotic flatness, the constant a gets fixed to

$$a = -s_b \frac{\Im m z^1}{\sqrt{3c}}\,. \tag{3.10}$$

It is now easy to compute the mass and the entropy of the outer/inner horizon

$$M = r_0 \sqrt{1 + \frac{3cp^2}{2r_0^2(\Im m z^1)^2}}\,, \tag{3.11}$$

$$S_{\pm} = r_0^2 \pi \left(\sqrt{1 + \frac{3cp^2}{2r_0^2(\Im m z^1)^2}} \pm 1\right)^2. \tag{3.12}$$

This implies that the product of both entropies only depends on the charge

$$S_+ S_- = \frac{\pi^2 \alpha^2}{4} p^4 \lambda^{4/3}\,. \tag{3.13}$$

It is worth stressing that the Ansatz $H^i = a^i + b^i\tau$ in the extremal ($r_0 = 0$) case was successfully used to obtain solutions with constant scalars but different critical points, in some cases particularly involved. However, presumably due to the complexity of the calculations, we have not been able to find a solution with non-constant scalars for any of the two models analyzed in this section. This may suggest also a more stabilized behavior for the scalars in the presence of perturbative quantum corrections.

4   Quantum corrected *STU* model

In this section we consider a very special case, the so-called *STU* model, in the presence of perturbative quantum corrections, obtaining the first non-extremal solution with non-constant scalars. In order to do so, we set $n_v = 3$, $\kappa^0_{123} = 1$. From (1.13) we obtain[12]

$$\mathsf{W}(H) = \alpha \left| H^1 H^2 H^3 \right|^{2/3} , \tag{4.1}$$

where $\alpha = 3c^{1/3}$. The scalar fields are given by

$$z^i = ic^{1/3} \frac{H^i}{(H^1 H^2 H^3)^{1/3}} , \tag{4.2}$$

The τ-dependence of the H^M can be found by solving eqs. (1.8) and (1.9), and the solution is given by

$$H^i = a^i \cosh(r_0\tau) + \frac{b^i}{r_0} \sinh(r_0\tau) , \quad b^i = s^i_b \sqrt{r_0^2 (a^i)^2 + \frac{(p^i)^2}{2}} . \tag{4.3}$$

The three constants a^i can be fixed relating them to the value of the scalars at infinity and imposing asymptotic flatness. We have, hence, four conditions for three parameters and therefore one would expect a relation among the $\Im m z^i_\infty$, leaving c undetermined. However, the explicit calculation shows that the fourth relation is compatible with the others, and therefore no extra constraint is necessary. The a^i are given by

$$a^i = -s^i_b \frac{\Im m z^i_\infty}{\sqrt{3c}} . \tag{4.4}$$

The mass and the entropy, in turn, read

$$M = \frac{r_0}{3} \sum_i \sqrt{1 + \frac{3c(p^i)^2}{2r_0^2 (\Im m z^i_\infty)^2}} , \tag{4.5}$$

$$S_\pm = r_0^2 \pi \prod_i \left(\sqrt{1 + \frac{3c(p^i)^2}{2r_0^2 (\Im m z^i_\infty)^2}} \pm 1 \right)^{2/3} , \tag{4.6}$$

and therefore the product of the inner and outer entropy only depends on the charges

$$S_+ S_- = \frac{\pi^2 \alpha^2}{4} \prod_i (p^i)^{4/3} , \tag{4.7}$$

In the extremal limit we obtain the supersymmetric as well as the non-supersymmetric extremal solutions, depending on the sign chosen for the charges.

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[12]We have to stress that we haven't been able to construct an explicit C.Y. manifold with $\kappa^0_{123} = 1$ and $h^{21} < 3$.

Acknowledgments

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