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Anomalous charged fluids in 1+1d from equilibrium partition function

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Abstract: In this note we explore the constraints imposed by the existence of equilibrium partition on parity violating charged fluids in 1+1 dimensions at zero derivative order. We write the equilibrium partition function consistent with 1+1 dimensional CPT invariance and which reproduces the correct anomaly in the charge current. The constraints on constitutive relations obtained in this way matches precisely with those obtained using the second law of thermodynamics.

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Contents

1 Introduction

Relativistic Fluid dynamics is an effective large wavelength description (at length scales much bigger than the mean free path) of certain phases of matter which at microscopic level are described by relativistic quantum field theories. The basic equations governing dynamics in this description are the conservation laws corresponding to the global symmetries of the underlying theory. More specifically, these are the conservation equations of the stress energy tensor and charge currents. These equations are to be supplemented by constitutive relations which expresses the stress-energy tensor and charge current in terms of the basic fluid variables namely velocity, temperature and chemical potential.

Consistency with second law of thermodynamics has been used as a constraining principle on the constitutive relations in fluid dynamics (see [1–8] and references therein). This gives two kinds of relations: a) Inequality type relations on dissipative coefficients(which contributes to entropy increase). b) Equality type relations on non dissipative coefficients(which do not contribute to entropy increase). Recently in [9, 10] it was shown that the requirement of the existence of stationary equilibrium which is generated from a partition function gives all the equality type relations. One of the cases that was studied in [9] was charged fluid dynamics in 3+1 dimensions when the charge current is anomalous. In this case, the results of Son and Surowka [2] on the chiral magnetic and chiral vorticity flows, were recovered without making any reference to an entropy current.

In this note we study the anomalous charged fluid dynamics in 1+1 dimensions using the equilibrium partition function. This system has earlier been studied in [11] using the second law of thermodynamics as well as from an action point of view. In this note we write down the equilibrium partition function for this system at zero derivative order which reproduces the anomalous charge conservation and on comparison with the most general constitutive relations in fluid dynamics, gives the results obtained in [11].

2 1+1d parity violating charged fluid dynamics

Consider the parity violating charged fluids in 1+1 dimensions with background metric and gauge field

$$\begin{aligned} ds^2 &= -e^{2\sigma}(dt + a_1 dx)^2 + g_{11}dx^2 \\ \mathcal{A} &= \mathcal{A}_0 dt + \mathcal{A}_1 dx^1 \quad . \end{aligned} \tag{2.1}$$

The equations of motion are the following anomalous conservation laws

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \mathcal{F}^{\nu\lambda}\tilde{J}_\lambda \\ \nabla_\mu \tilde{J}^\mu &= C\epsilon^{\mu\nu}\mathcal{F}_{\mu\nu} \\ \nabla_\mu J^\mu &= \frac{C}{2}\epsilon^{\mu\nu}\mathcal{F}_{\mu\nu} \end{aligned} \tag{2.2}$$

here $\tilde{J}$, J are covariant and consistent currents respectively ([12], see also [9]) and C denotes the coefficient of gauge anomaly.

The most general partition function consistent with Kaluza-Klein gauge invariance,[1] diffeomorphism along the spatial direction and U(1) gauge invariance upto anomaly is

$$\begin{aligned} \mathcal{W} &= \mathcal{W}_{\rm inv} + \mathcal{W}_{\rm anom} \\ \mathcal{W}_{\rm inv} &= C_1 T_0 \int A_1 dx - C_2 T_0 \int a_1 dx \\ \mathcal{W}_{\rm anom} &= -\frac{C}{T_0}\int A_0 A_1 dx \end{aligned} \tag{2.4}$$

where C, C_1 and C_2 are constants independent of σ and A_0 and

$$A_0 = \mathcal{A}_0 + \mu_0, \quad A_i = \mathcal{A}_i - A_0 a_i. \tag{2.5}$$

Equation (2.4) is written in terms of A_i which unlike $\mathcal{A}_i$, are Kaluza-Klein gauge invariant.

Under U(1) gauge transformation $A_0 \to A_0,\ A_1 \to A_1 + \partial_1\phi$, we obtain[2]

$$\begin{aligned} \delta\mathcal{W}_{\rm inv} &= 0 \\ \delta\mathcal{W}_{\rm anom} &= \frac{C}{T_0}\int \phi\ \partial_1 A_0 dx = -\frac{C}{2}\int d^2x\sqrt{-g_2}\ \phi\ \epsilon^{\mu\nu}\mathcal{F}_{\mu\nu}. \end{aligned} \tag{2.6}$$

Table 1 lists the action of 2 dimensional C, P and T on various fields. Requiring CPT invariance sets C_1 to zero since the term with coefficient C_1 is odd under CPT.

Now let us look at the most general constitutive relations allowed by symmetry in the parity violating case at zero derivative order. At this order, there are no gauge invariant parity odd scalar or tensor. But one can construct a gauge invariant vector[3]

$$\tilde{u}^\mu = \epsilon^{\mu\nu}u_\nu. \tag{2.8}$$

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[1]

$$V_i' = V_i - \partial_i\phi V_0, \quad (V')^0 = V^0 + \partial_i\phi V^i. \tag{2.3}$$

[2]Since we are interested in time independent background fields, we consider only time independent gauge transformations.

[3]In components the parity odd vector is

$$\tilde{u}_0 = 0, \quad \tilde{u}^1 = \epsilon^{10}u_0 = \epsilon^1 \tag{2.7}$$

where $\epsilon^1 = e^\sigma\epsilon^{01} = \frac{1}{\sqrt{g_{11}}}$.

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Field	C	P	T	CPT
σ	$+$	$+$	$+$	$+$
a_1	$+$	$-$	$-$	$+$
g_{11}	$+$	$+$	$+$	$+$
A_0	$-$	$+$	$+$	$-$
A_1	$-$	$-$	$-$	$-$

Table 1. Action of CPT.

The most general allowed constitutive relations allowed by symmetry in Landau frame thus take the form

$$\begin{aligned} T^{\mu\nu} &= (\epsilon + p)u^\mu u^\nu + p g^{\mu\nu} \\ \tilde{J}^\mu &= q u^\mu + \xi_j \tilde{u}^\mu . \end{aligned} \tag{2.9}$$

2.1 Equilibrium from partition function

In this subsection we will use the equilibrium partition function (2.4) to obtain the stress tensor and charge current at zero derivative order. Setting C_1 to zero in (2.4) we have

$$\mathcal{W} = -\frac{C}{T_0}\int A_0 A_1 dx - C_2 T_0 \int a_1 dx \tag{2.10}$$

Using the partition function (2.10) it is straightforward to compute the stress tensor and charge current[4] in equilibrium to be

$$\begin{aligned} &T_{00} = 0, \qquad T^{11} = 0, \qquad T_0^1 = e^{-\sigma}\epsilon^1\left(-T_0^2 C_2 + C A_0^2\right), \\ &J_0 = C\epsilon^1 A_1 e^{\sigma}, \quad J^1 = -C\epsilon^1 e^{-\sigma} A_0 . \end{aligned} \tag{2.12}$$

The covariant current ($\tilde{J}^\mu$) can be obtained from the consistent current (J^μ) by an appropriate shift as follows

$$\tilde{J}^\mu = J^\mu + J^\mu_{\rm sh}, \quad J^\mu_{\rm sh} = C\epsilon^{\mu\nu}A_\nu . \tag{2.13}$$

In components the covariant current is then

$$\tilde{J}_0 = 0, \quad \tilde{J}^1 = -2Ce^{-\sigma}\epsilon^1 A_0 . \tag{2.14}$$

2.2 Equilibrium from hydrodynamics

We are interested in the stationary equilibrium solutions to conservation equations corresponding to the constitutive relations (2.9). The equilibrium solution in the parity even

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$$\begin{aligned} &T_{00} = -\frac{T_0 e^{2\sigma}}{\sqrt{-g_{(p+1)}}}\frac{\delta W}{\delta\sigma}, \qquad T_0^i = \frac{T_0}{\sqrt{-g_{(p+1)}}}\left(\frac{\delta W}{\delta a_i} - A_0 \frac{\delta W}{\delta A_i}\right), \\ &T^{ij} = -\frac{2T_0}{\sqrt{-g_{(p+1)}}} g^{il} g^{jm}\frac{\delta W}{\delta g^{lm}}, \quad J_0 = -\frac{e^{2\sigma}T_0}{\sqrt{-g_{(p+1)}}}\frac{\delta W}{\delta A_0}, \quad J^i = \frac{T_0}{\sqrt{-g_{(p+1)}}}\frac{\delta W}{\delta A_i}. \end{aligned} \tag{2.11}$$

where, for instance, the derivative w.r.t. A_0 is taken at constant σ, a_i, A_i, g^{ij}, T_0 and μ_0. See [9] for details.

Type	Data	Evaluated at equilibrium $T = T_0 e^{-\sigma},\ \mu = e^{-\sigma} A_0,\ u^\mu = u^\mu_K$
Scalars	None	None
Vectors	u^μ	$\delta^\mu_0 e^{-\sigma}$
Pseudo-Vectors	$\epsilon_{\mu\nu} u^\nu$	ϵ_1
Tensors	None	None

Table 2. Zero derivative fluid data.

Scalars	None
Vectors	none , none
Pseudo-Vectors	$\epsilon^1 f(\sigma, A_0)$
Tensors	None

Table 3. Zero derivative background data.

sector in background (2.1) at zero derivative order is

$$u^\mu = u^\mu_{(0)} = e^{-\sigma}(1,0), \quad T = T_0 e^{-\sigma}, \quad \mu = A_0 e^{-\sigma}. \tag{2.15}$$

Since there are no gauge invariant parity odd scalars in table 3, temperature and chemical potential do not receive any correction. However, the fluid velocity in equilibrium receives correction as

$$u^\mu = u^\mu_{(0)} + b\epsilon^{\mu\nu} u^{(0)}_\nu. \tag{2.16}$$

From (2.9), (2.15) and (2.16) we get the parity odd correction to the equilibrium stress tensor and charge current, which receive contribution from correction to the constitutive relations as well as from correction to the equilibrium fluid velocity, to be

$$\begin{aligned} \delta T_{00} &= \delta J_0 = \delta T^{ij} = 0, \\ \delta T^1_0 &= -e^\sigma(\epsilon + P) b \epsilon^1, \\ \delta \tilde{J}^1 &= (qb + \xi_j)\epsilon^1. \end{aligned} \tag{2.17}$$

2.3 Constraints on hydrodynamics

Comparing the non trivial components of the equilibrium stress tensor and charge current of (2.12) and (2.17) we find that the coefficient of velocity correction (2.16) is

$$b = -\frac{T^2}{\epsilon + p}\left(-C_2 + C\nu^2\right) \tag{2.18}$$

and the coefficient in correction to charge current (2.9) is

$$\xi_j = C\left(\frac{q\mu^2}{\epsilon + p} - 2\mu\right) - C_2 \frac{qT^2}{\epsilon + p}. \tag{2.19}$$

where $\nu = \frac{\mu}{T} = \frac{A_0}{T_0}$.

The expressions (2.19) agree exactly with the results of [11] based on the requirement of positivity of the entropy current and effective action.

2.4 The entropy current

The equilibrium entropy can be obtained from the partition function using

$$\begin{aligned} S &= \frac{\partial}{\partial T_0}(T_0 \log Z) \\ &= -2C_2 T_0 \int \sqrt{g_{11}}\epsilon^1 a_1 dx \ . \end{aligned} \tag{2.20}$$

In this subsection we determine the constraints on the hydrodynamical entropy current J_S^μ from the requirement that (2.20) agree with the local integral

$$S = \int dx \sqrt{-g_2} J_S^0 \quad . \tag{2.21}$$

The most general form of the entropy current allowed by symmetry,[5] at zero derivative order is

$$J_S^\mu = s u^\mu + \xi_s \tilde{u}^\mu + h \epsilon^{\mu\nu} \mathcal{A}_\nu \,, \tag{2.22}$$

where h is a constant.

The parity odd correction to the entropy current in equilibrium, which receives contributions both from correction to the hydrodynamical entropy current and equilibrium velocity, is given by

$$J_S^0|_{\text{correction}} = s\delta u^0 + \xi_s \tilde{u}^0 + h\epsilon^{01}\mathcal{A}_1. \tag{2.23}$$

Now using

$$\nu = \frac{A_0}{T_0}, \quad \delta u^0 = -a_1 \delta u^1 = -b\epsilon^1 a_1, \quad \tilde{u}^0 = -\epsilon^1 a_1$$

the correction to the hydrodynamical entropy in equilibrium is given by

$$\int dx \sqrt{-g_2} J_s^0|_{\text{correction}} = \int dx \ e^\sigma \left((-sb - \xi_s)\epsilon^1 a_1 + h\epsilon^1 (A_1 + A_0 a_1) \right) . \tag{2.24}$$

Comparing this expression with (2.20) and using (2.19) we find

$$\xi_s = C\frac{s\mu^2}{\epsilon + p} + C_2 T \left(1 + \frac{\rho\mu}{\epsilon + p} \right) , \qquad h = 0. \tag{2.25}$$

This result is in precise agreement with those of [11].

3 Conclusion

To conclude, for $1+1$d parity violating charged fluid in a time independent background with anomaly one can write down a local equilibrium partition function and the constraints obtained on the constitutive relations by demanding consistency with this partition function are in agreement with those obtained from a local form of entropy increase principle. In [9], by demanding the existence of a partition function it was noted that, for first order

[5] Let us note that the entropy current need not be gauge invariant, see [9] for more details.

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$2 + 1$d parity violating charged fluid and second order $3 + 1$d uncharged fluid, one obtains weaker constraints on the non dissipative part of the entropy current as compared to that obtained by demanding entropy increase. However, for the case of first order $3 + 1$d charged fluid with anomaly, the entropy current obtained in both ways agree so is also for $1 + 1$d anomalous case, as shown in this note. It would be interesting to check this for an anomalous fluid in arbitrary dimensions.

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