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# Absence of sign problem in two-dimensional $\mathcal{N} = (2, 2)$ super Yang-Mills on lattice

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**ABSTRACT:** We show that  $\mathcal{N} = (2, 2)$   $SU(N)$  super Yang-Mills theory on lattice does not have sign problem in the continuum limit, that is, under the phase-quenched simulation phase of the determinant localizes to 1 and hence the phase-quench approximation becomes exact. Among several formulations, we study models by Cohen-Kaplan-Katz-Unsal (CKKU) and by Sugino. We confirm that the sign problem is absent in both models and that they converge to the identical continuum limit *without fine tuning*. We provide a simple explanation why previous works by other authors, which claim an existence of the sign problem, do not capture the continuum physics.

**KEYWORDS:** Supersymmetric gauge theory, Lattice Gauge Field Theories, Extended Supersymmetry

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**1 Introduction**

Supersymmetric Yang-Mills theory (SYM) attracts broad interests as a candidate of the physics beyond the standard model [1]. It is also a promising candidate for the nonperturbative formulation of the superstring theory [2–7]. Although it is important to study its nonperturbative properties for both applications, and one of the most successful nonperturbative approaches for gauge theory is the lattice simulation, however, notorious difficulty of the lattice SYM prevented it for long time. Sometime ago there appeared a breakthrough for two-dimensional theories of extended supersymmetries [8–14], (see also [15] for a recent review); the correct continuum limit is obtained without parameter fine tunings for most of these models, at least at perturbative level.<sup>1</sup> Furthermore it turned out that by combining two-dimensional lattice with matrix model technique (fuzzy sphere) [19], 4d  $\mathcal{N} = 4$  theory is constructed without relying on the parameter fine tunings to all order in perturbation theory [20, 21]. (For other very elegant formulation in the planar limit, which preserves 16 supersymmetries manifestly, see [22–24]. This method is applicable to various kind of theories for which lattice and/or fuzzy sphere technique are not applicable [25, 26].) These formulations provide robust ways to test the  $AdS_5/CFT_4$  correspondence, the gauge/gravity

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<sup>1</sup>There is a lattice formulation of 3d maximally supersymmetric Yang-Mills without fine tuning [13]. For 4d  $\mathcal{N} = 1$  pure supersymmetric Yang-Mills, the chiral symmetry guarantees supersymmetric continuum limit. Recent results are found in [16–18].

duality [6, 7] and matrix string conjecture [4, 5] at nonperturbative level. In one dimension, such a program has been pursued extensively by using a non-lattice method [27] and the gauge/gravity duality [6, 7] has been confirmed very precisely [28–32], including the stringy  $\alpha'$  corrections [30, 32]. The lattice simulations are also applicable for this system and qualitatively consistent results have been obtained [33–35]. We can expect study of two and four dimensional theories will provide even richer insights; in two dimensions simulations in this context are already ongoing [36–38] and it is urgent to establish the validity of the lattice models at nonperturbative level by detailed simulations.

However there is a possible obstacle for simulations: supersymmetric theories with eight and sixteen supersymmetries suffer from the sign problem [39].<sup>2,3</sup> On the other hand, with four supersymmetries (i.e. 4d  $\mathcal{N} = 1$  pure SYM and its dimensional reductions), there is no sign problem. It can easily be seen as follows. In Weyl notation, with an appropriate choice of the gamma matrices, the Dirac operator is written as

$$D \equiv i\sigma^\mu D_\mu, \tag{1.1}$$

where  $\sigma^0 = -i\mathbf{1}_2$  and  $\sigma^i (i = 1, 2, 3)$  are Pauli matrices. By using  $\sigma^2(i\sigma^\mu)\sigma^2 = (i\sigma^\mu)^*$  and the fact that  $D_\mu$  is real in adjoint representation, we obtain

$$\sigma^2 D \sigma^2 = D^*. \tag{1.2}$$

Therefore, if  $\varphi$  is an eigenvector corresponding to an eigenvalue  $\lambda$ ,  $\sigma^2\varphi^*$  is also an eigenvector, with eigenvalue  $\lambda^*$ . They are linearly independent and eigenvalues appear in a pair  $(\lambda, \lambda^*)$ . This assures the positivity of the determinant after removing  $\lambda = 0$  modes.

At discretized level, positivity of the determinant can be lost. In zero dimension (matrix model) [41], there is no sign problem, because there is no need for the regularization. In 1d theory and fuzzy sphere construction of three and four dimensional theories [22–24], by using the momentum cutoff prescription [27] the determinant is positive even at discretized level [42].

In lattice constructions of two-dimensional SYM, the determinant is in general complex at discretized level. In Sugino’s model [9] for two-dimensional theory, however, the sign problem disappears as one approaches to the continuum [43, 44]; that is, if one performs the phase-quenched simulation, distribution of the phase factor of the determinant<sup>4</sup> peaks at 1 in the continuum limit. Therefore the phase quench approximation becomes exact at continuum. In this case, in addition to the absence of sign, the agreements with analytic calculations in small volume region have been observed as well [36] by using techniques developed in [45]. Therefore the absence of the sign is the property of the correct continuum limit, as expected. Numerical studies of this model can also be found in [46–48].

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<sup>2</sup>In the case of maximally supersymmetric matrix quantum mechanics, agreement with the dual gravity prescription has been observed by ignoring the phase of the Pfaffian, even when the sign fluctuates violently [29, 31]. It has also been observed that the Pfaffian is almost real positive for SU(2) [32]. It would be nice to understand why it happens, but it is out of scope of the present paper.

<sup>3</sup>Even if there is a sign problem, measurement of the sign factor itself is interesting because it is related to the Witten index with a suitable normalization [40].

<sup>4</sup>Strictly speaking, in Sugino model fermions are Majorana and hence we calculate the Pfaffian.

On the other hand, for other two-dimensional lattice models which are supposed to have the same continuum limit, an *existence* of the sign problem has been reported [49, 50]. In Cohen-Kaplan-Katz-Unsal (CKKU) model, which is equivalent to a model by D’Adda et.al. [14] with a specific choice of parameters, Giedt reported that the determinant has a complex phase, and the phase fluctuates violently if one chooses random lattice configurations [49]. However the importance sampling has not been performed in [49], and hence this result has nothing to do with the continuum limit as the author remarked correctly. This model was studied later with importance sampling in [50],<sup>5</sup> where the sign problem was reported as well. However, it is not clear whether it is a property of the continuum, because they could not evaluate physical quantities because of the “sign problem” and hence could not estimate how close to the continuum limit they have reached.

In this paper, we resolve the confusion mentioned above. We show the absence of sign problem in the Sugino model and the CKKU model in the continuum limit. (In Sugino model, the absence of sign has been explicitly reported for SU(2) theory in [43, 44]. For  $N > 2$  theory, we briefly checked but have not mentioned it in [36], because our emphasis was put on other physical quantities. In this paper we show the detail for SU( $N$ ) with  $N > 2$ , together with new data for SU(2).) The action in the continuum is obtained from 4d  $\mathcal{N} = 1$  SYM through the dimensional reduction, and is given by

$$S = \frac{N}{\lambda} \int_0^{L_x} dx \int_0^{L_y} dy \operatorname{Tr} \left\{ \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (D_\mu X_i)^2 - \frac{1}{4} [X_i, X_j]^2 - \frac{1}{2} \bar{\psi} \Gamma^\mu D_\mu \psi - \frac{i}{2} \bar{\psi} \Gamma^i [X_i, \psi] \right\}, \quad (1.3)$$

where  $\mu$  and  $\nu$  run  $x$  and  $y$ ,  $i$  and  $j$  run 1 and 2, and  $\Gamma^I = (\Gamma^\mu, \Gamma^i)$  are gamma matrices in four dimensions.  $X_i$  are  $N \times N$  hermitian matrices,  $\psi_\alpha$  are  $N \times N$  fermionic matrices with a Majorana index  $\alpha$  and the covariant derivative is given by  $D_\mu = \partial_\mu - i[A_\mu, \cdot]$ . The only parameters of the model are the size of circles  $L_x$  and  $L_y$ . (Note that the coupling constant can be absorbed by redefining the fields and coordinates. Therefore we take the ’t Hooft coupling  $\lambda$  to be 1. Then the strong coupling corresponds to the large volume.) We study this system by using two lattice models (CKKU and Sugino) numerically and show the absence of the sign problem. We evaluate expectation values of some physical quantities and see that the results show nice agreements. Note that small volume behavior of the Sugino model is consistent with known analytic estimates [36].

One obstacle for the simulation is the existence of the flat direction, along which two scalar fields  $X_1$  and  $X_2$  commute. In contrary to a theory on  $\mathbb{R}^{1,3}$ , there is no superselection of the moduli parameter in this case. That is, eigenvalues of scalars are determined dynamically. Therefore, some mechanism which restrict eigenvalues to a finite distribution is necessary for the stable simulation. In addition, to obtain an interesting dynamical system, having a (small) finite region for the eigenvalues is important as well; if the eigenvalues of the scalar spread so large, the theory would run into the abelian phase, which is just a free

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<sup>5</sup>In [50], the bosonic fields are defined as  $e^z$  where  $z$  is a complex field, and thus a natural way of extracting physical quantities is different from the original CKKU model. For SU( $N$ ) gauge group as in [50], this definition is different from a general complex field originally defined in [8]. We would like to thank S. Catterall for detailed explanations on his work.

theory.<sup>6</sup> In this work, we introduce soft SUSY-breaking mass to scalar fields

$$\mu^2 N \int d^2x \sum_{i=1,2} \text{Tr} X_i^2, \tag{1.4}$$

so that the flat direction is lifted.<sup>7</sup> It is crucial to control the flat direction for various reasons. We have just mentioned two of them — stability of the simulation and interesting non-abelian phase. There is one more; in order to guarantee the correct continuum limit, the eigenvalue must be smaller than the cut off scale  $\sim 1/a$ . Especially in the CKKU model, we can decompose the bosonic field to appear scalars  $X$  as a log of positive Hermitian variables  $H$  [51],

$$H = \exp(aX), \tag{1.5}$$

where  $a$  is the lattice spacing. In order to obtain the tree level action, one has to assume  $aX \ll 1$ , expand it in powers of  $aX$  and neglect higher order terms. Therefore, unless  $aX \ll 1$ , one cannot get to the continuum limit even at tree level. Actually it turns out that one of previous works, whose conclusion contradicts with ours, does not satisfy this condition.

This paper is organized as follows. In section 2 we study the CKKU model. We introduce the model in section 2.1 and then consider a structure of the Dirac operator in section 2.2. Then we show the absence of the sign problem in section 2.3. In section 3 we show the absence of the sign problem in Sugino model. Then in section 4 we confirm that two models (CKKU and Sugino) converge to the same continuum limit. To our best knowledge, this is the first result from the CKKU model in the continuum limit and shows in fact we can take a supersymmetric continuum limit without any fine tunings. In section 5 we explain why previous works by other authors fail to capture the continuum physics.

## 2 CKKU model

In this section we study the CKKU model. In section 2.1 we introduce the model. Then in section 2.2 we discuss the structure of light modes in the model, which is crucial for the analysis of the sign problem shown in section 2.3.

### 2.1 The model

Here we consider the  $U(N)$  gauge group. Note that the  $U(1)$  part is decoupled in the continuum limit and hence the physics is the same as  $SU(N)$  theory. The action is given by

$$S = Na^2 \sum_{\vec{n}} \text{Tr} \left\{ \frac{1}{2} (\bar{x}_{\vec{n}-\hat{x}} x_{\vec{n}-\hat{x}} - x_{\vec{n}} \bar{x}_{\vec{n}} + \bar{y}_{\vec{n}-\hat{y}} y_{\vec{n}-\hat{y}} - y_{\vec{n}} \bar{y}_{\vec{n}})^2 + 2|x_{\vec{n}} y_{\vec{n}+\hat{x}} - y_{\vec{n}} x_{\vec{n}+\hat{y}}|^2 \right. \\ \left. + \sqrt{2} (\alpha_{\vec{n}} \bar{x}_{\vec{n}} \lambda_{\vec{n}} - \alpha_{\vec{n}-\hat{x}} \lambda_{\vec{n}} \bar{x}_{\vec{n}-\hat{x}}) + \sqrt{2} (\beta_{\vec{n}} \bar{y}_{\vec{n}} \lambda_{\vec{n}} - \beta_{\vec{n}-\hat{y}} \lambda_{\vec{n}} \bar{y}_{\vec{n}-\hat{y}}) \right. \\ \left. - \sqrt{2} (\alpha_{\vec{n}} y_{\vec{n}+\hat{x}} \xi_{\vec{n}} - \alpha_{\vec{n}+\hat{y}} \xi_{\vec{n}} y_{\vec{n}}) + \sqrt{2} (\beta_{\vec{n}} x_{\vec{n}+\hat{y}} \xi_{\vec{n}} - \beta_{\vec{n}+\hat{x}} \xi_{\vec{n}} x_{\vec{n}}) \right.$$

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<sup>6</sup>Which phase is preferred is in fact a dynamical question. At large- $N$ , the flat direction is lifted and the system stays non-abelian phase; see [36]. This phase is an analogue of the black 1-brane solution in type IIB supergravity.

<sup>7</sup>For 8 and 16 SUSY models, there exists SUSY-preserving mass deformation [20, 21].

$$\begin{aligned}
 & \left. + a^2 \mu^2 \left( x_{\bar{n}} \bar{x}_{\bar{n}} - \frac{1}{2a^2} \right)^2 + a^2 \mu^2 \left( y_{\bar{n}} \bar{y}_{\bar{n}} - \frac{1}{2a^2} \right)^2 \right\} \\
 & + N a^2 \sum_{\bar{n}} \left\{ a^2 \nu^2 \left| \frac{\text{Tr}(x_{\bar{n}} \bar{x}_{\bar{n}})}{N} - \frac{1}{2a^2} \right|^2 + a^2 \nu^2 \left| \frac{\text{Tr}(y_{\bar{n}} \bar{y}_{\bar{n}})}{N} - \frac{1}{2a^2} \right|^2 \right\}. \tag{2.1}
 \end{aligned}$$

Here  $x, y$  are  $N \times N$  complex matrices and  $\alpha, \beta, \lambda, \xi$  are  $N \times N$  complex Grassmanian matrices. They are related to the fields in the continuum by

$$x = \frac{1}{a\sqrt{2}} + \frac{X_1 + iA_1}{\sqrt{2}}, \quad y = \frac{1}{a\sqrt{2}} + \frac{X_2 + iA_2}{\sqrt{2}}, \tag{2.2}$$

$$\psi = \begin{pmatrix} \lambda \\ \xi \end{pmatrix}, \quad \bar{\psi} = i(\alpha, \beta). \tag{2.3}$$

The fermion is in the Weyl representation and is complex. Hence we study the determinant of the Dirac operator rather than the Pfaffian. The Dirac operator  $D$  is obtained by writing the fermion part as  $\bar{\psi}_{j\alpha x} D_{ij\alpha x, kl\beta y} \psi_{kl\beta y}$ , where suffixes  $i, j, k, l$  refer to color,  $\alpha, \beta$  to spinor and  $x, y$  to coordinate.

It is convenient to introduce (semi-)compact decomposition of the bosonic fields [51]

$$x = \frac{1}{\sqrt{2}a} U_1 H_1, \quad y = \frac{1}{\sqrt{2}a} U_2 H_2, \tag{2.4}$$

where  $U_i$  are unitary,  $H_i$  are Hermitian and positive definite, and

$$U_i = \exp(iaA_i), \quad H_i = \exp(aX_i). \tag{2.5}$$

From  $x$  and  $y$ ,  $H_i$  can be obtained as

$$H_1 = \sqrt{2a^2 x^\dagger x}, \quad H_2 = \sqrt{2a^2 y^\dagger y}. \tag{2.6}$$

A mass parameter  $\mu$  gives mass to  $U(N)$  scalar fields. The last two terms are not present in the original proposal;<sup>8</sup> it gives mass only to  $U(1)$  part of the scalar. As observed in [28, 36], at large volume and/or with periodic boundary condition for the fermion, flat direction in  $SU(N)$  sector is dynamically lifted and nonabelian phase (i.e. bound state of scalar eigenvalues) becomes meta-stable. (It becomes stabler as  $N$  increases.) On the other hand  $U(1)$  flat direction is never lifted, and in the CKKU model, it can destroy the lattice structure. But to stabilize this  $U(1)$  flat direction we do not have to turn on the  $U(N)$  mass  $\mu$ ; the  $U(1)$  mass  $\nu$  is fine enough. Given that the  $U(1)$  sector is free and decouples from the dynamics, nonzero value of  $\nu$  does not affect the supersymmetry in the  $SU(N)$  sector in the continuum limit. We explicitly confirm this statement numerically (See figures 5, 6, 7, 8). Note that this decoupling contains a delicate issue. At lattice level, the  $U(1)$  and  $SU(N)$  sectors are not completely decoupled. In order to stabilize the lattice structure, the heavy  $U(1)$  mass is suitable. However, if the  $U(1)$  mass is too large, the SUSY breaking effect in the  $U(1)$  sector becomes large and it might be mediated through the lattice artifact to the

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<sup>8</sup>We would like to thank O. Aharony for suggesting the use of  $U(1)$  mass term.

SU( $N$ ) sector. Our numerical results support the decoupling in a wide range of the U(1) mass. It would be nice if the decoupling could be explained analytically. We will show that with non-zero value of  $\nu$  the obtained values of the observables are the same as those from SU( $N$ ) Sugino model (see section 4), which justifies the treatment of the U(1) mass term. At finite volume and finite  $N$  we need nonzero  $\mu$  to completely remove the instability, but at large- $N$  we can take  $\mu = 0$  and  $\nu \neq 0$  so that the supersymmetry in the SU( $N$ ) sector is fully restored already at finite volume. It enables one to study interesting finite-volume physics like black hole/black string phase transition [52, 53].

## 2.2 Structure of light modes

At the classical vacuum of the U( $N$ ) theory, the Dirac operator has  $2N^2$  fermion zero-modes, which correspond to zero-momentum. Apart from the classical vacuum,  $2(N^2 - 1)$  of them are lifted and there remain two zero-modes which correspond to the U(1) part. At a discretized level, because of the special property of the CKKU model, only one of them is exactly zero [49]. The other approaches to zero in the continuum limit. Let us call it the *pseudo zero-mode*.

In our simulation, the phase quenched ensemble with  $\det \sqrt{MM^\dagger}$  is used, where  $M = iD$ . To avoid the exact zero-mode, we add a regulator term to  $MM^\dagger$  in the simulation,

$$MM^\dagger \rightarrow MM^\dagger + \epsilon \mathbf{1}. \tag{2.7}$$

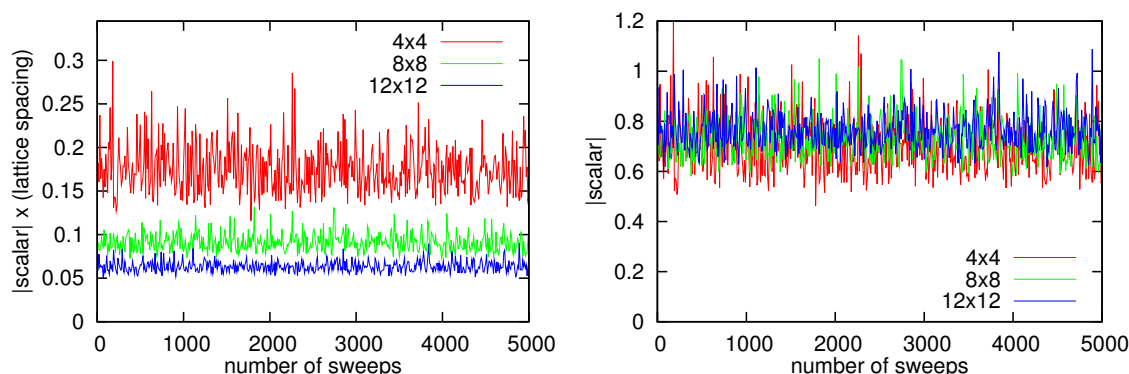
In practice  $\epsilon$  is fixed to be a small enough value ( $\epsilon \sim 10^{-6}$ ) compared to all nonzero eigenvalues of  $MM^\dagger$ .

When we calculate the determinant, we remove the exact zero-mode and the pseudo zero-mode by hand. (In practice we remove one exact zero eigenvalue and smallest nonzero eigenvalue of the Dirac operator.) As we will see, removal of the pseudo zero-mode is crucial to establishing the positivity of the determinant. The reason is simple — in the continuum, the Dirac operator has positive determinant because its eigenvalues form pairs  $(\lambda, \lambda^*)$  *after* removing zero-modes. The reason why we *can* remove this pseudo zero mode is clear; it will decouple from the dynamics in any case. The corresponding zero-modes which should be removed in the lattice simulation are the exact zero-mode and the pseudo zero-mode. If the lattice artifact to zero-modes had a pair structure which keeps the positivity of the determinant we would not need to remove these two modes. However, phase of the pseudo zero-mode fluctuates violently and it dominates the fluctuation of the phase of the determinant because pseudo zero-mode does not appear in a pair; without removing it the phase of the determinant becomes completely random, just because the phase of the pseudo zero-mode is random.

We numerically calculated the eigenvalues and the determinant of  $iD$ . We observed that at very small lattice spacing, the other  $2(N^2 - 1)$  light modes have a pair structure  $(\lambda, -\lambda^*)$ , which is exactly expected from the continuum argument.

## 2.3 Absence of the sign problem

As we have explained, it is important to control the flat direction in order to study continuum physics. In figure 1 we show histories of the extent of the scalar fields in the lattice



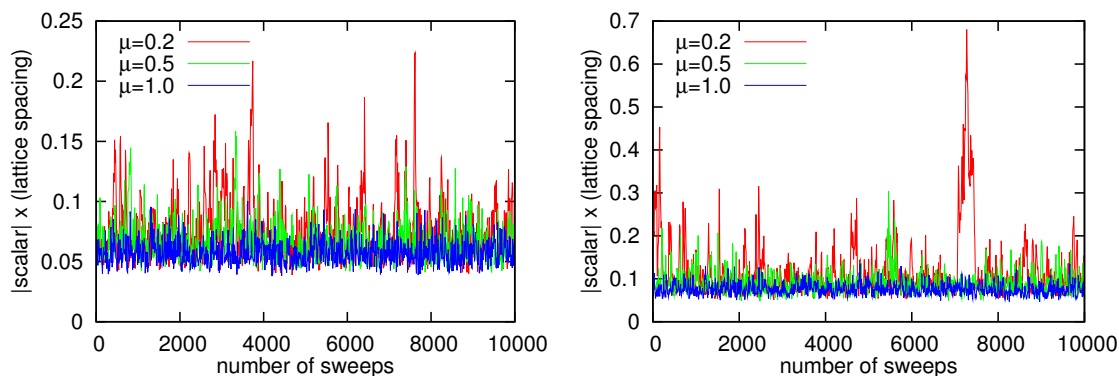
**Figure 1.** [CKKU] History of the average scalar extent, in the lattice unit (left) and in the physical unit (right). At each sweep the average over  $x$  and  $i$  of  $\sqrt{\text{Tr}(aX_i(x))^2/N}$  (a contraction w.r.t.  $i$  is *not* taken), where  $aX_i \equiv \log H_i$ , is plotted. Clear convergence as  $\sim a$  in the left panel, which corresponds to fixed physical extent of the scalar in the right panel, can be seen. The physical volume is  $L = 1.0$ , the  $U(N)$  mass is  $\mu = 1.0$  and the  $U(1)$  mass is  $\nu = 5.0$ . The gauge group is  $U(2)$ .

unit (left) and in the physical unit (right). When the extent is close to the cutoff scale, 1 in the lattice unit, the simulation is not reliable. As we can see from the left panel of figure 1, at the physical volume  $L = 1.0$ , the  $U(N)$  mass  $\mu = 1.0$  and the  $U(1)$  mass  $\nu = 5.0$ , scalars take sufficiently small values, which shows that the flat direction is well under control to guarantee the correct continuum limit. Thus there is no instability caused by a lattice artifact. Moreover the scalar fields stay finite in physical unit (the right panel). That is, this lattice model describes a system without the flat direction in the continuum limit. The obtained continuum system has no instability along the flat direction. It is expected of course, because we added a mass term. Note that if the scalar converged in the lattice unit but diverged in physical unit we would obtain a continuum limit with flat direction, which is just a free theory in the Abelian phase. As the mass is decreased, the flat direction emerges gradually. In figure 2 we show the histories at  $L = 0.5$  with a few values of  $\mu$ . With  $8 \times 8$  ( $a = 0.0625$ ) lattice the fluctuation is not violent even at  $\mu = 0.2$  (the left panel). As one can easily imagine, with smaller lattice the instability — caused by the lattice artifact — appears more easily; see the right panel. This plot uses the same physical parameters as the left panel, but a smaller lattice. Note that for the smaller lattice (i.e. larger lattice spacing) more spikes appear at  $\mu = 0.2$ , which is a signal of the instability. The reason is obvious — with smaller  $\mu$  the extent of the scalar in physical unit is larger, and to make it to be small in lattice unit lattice spacing must be smaller. By extrapolating to  $\mu = 0$  by using data from stable region, we obtain a finite extent of the scalar (figure 3; here we have assumed a simple linear extrapolation, based on the obtained plot). Therefore we can expect the phase we are looking at is smoothly connected to the meta-stable non-abelian phase [27, 28, 36].

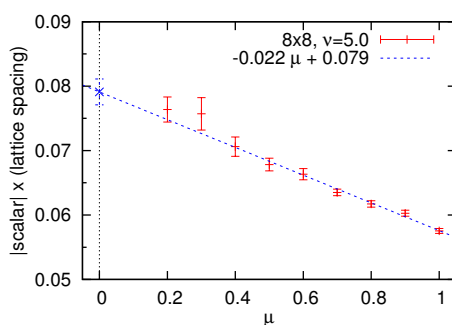
In the following, for all data we show, we have confirmed the simulation does not run away to the flat direction.

Now we are ready to study the sign problem. As discussed in detail in appendix A, the determinant of  $iD$  has a sign  $(-1)^{N-1}$ , at least for the constant configurations. Therefore,





**Figure 2.** [CKKU] History of the average scalar extent in the U(2) theory at  $L = 0.5$ ,  $\nu = 5.0$  and  $\mu = 1.0, 0.5$  and  $0.2$ . At each sweep the average of  $\sqrt{\text{Tr}(aX_i(x))^2/N}$  is plotted. (left)  $8 \times 8$  lattice, even at  $\mu = 0.2$  the flat direction is under control. (right)  $6 \times 6$  lattice, the flat direction appears around  $\mu = 0.2$ . Note the difference of the scales between the panels.



**Figure 3.** [CKKU] Mass dependence of  $\left\langle \sqrt{\frac{1}{N} \text{Tr} X_i^2(x)} \right\rangle \times (\text{lattice spacing})$  in the U(2) CKKU model. As in figures 1 and 2, a contraction w.r.t.  $i$  is not taken and an average over  $i$  and  $x$  is taken. The lattice size is  $8 \times 8$ , the physical volume is  $0.50 \times 0.50$  and the U(1) mass is  $\nu = 5.0$ .

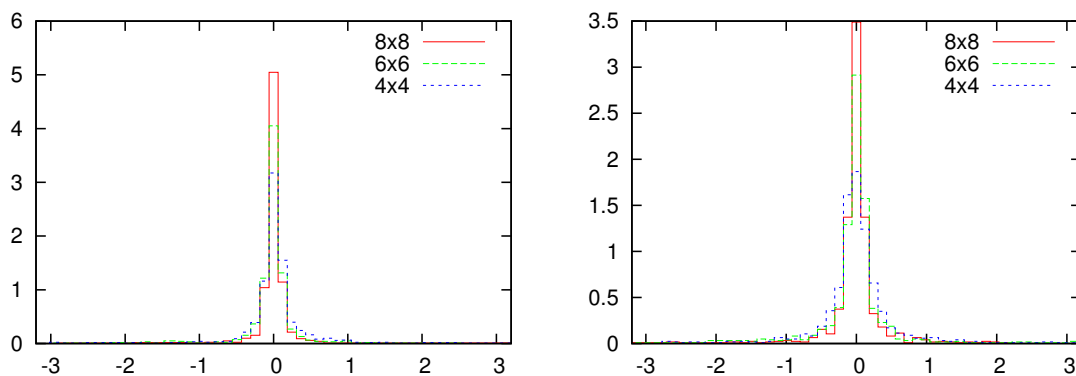
throughout this section, we multiply  $(-1)^{N-1}$  to the phase factor of the determinant so that it localizes around  $+1$ .

In short, what happens both in the CKKU and Sugino is

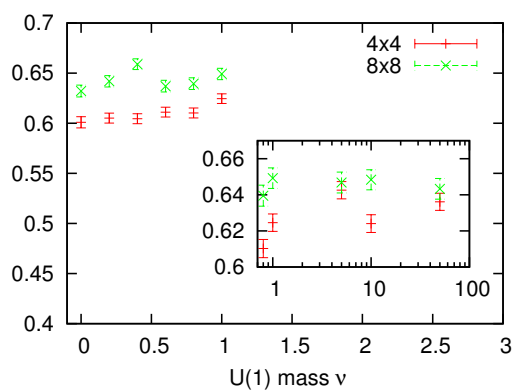
- For fixed lattice spacing, the phase fluctuation becomes smaller at smaller volume and/or smaller  $N$ .
- For fixed  $N$  and fixed volume, the phase disappears in the continuum limit (small lattice spacing).

Let us start with the U(2) theory. In figure 4 we have shown how the distributions of the argument of the determinant peaks to 0. In each panel, the physical volume is fixed and the number of sites is changed. There is a clear tendency that the peak becomes sharper as one goes closer to the continuum.

In order to justify our treatment of the U(1) mass, we have checked the U(1) mass dependence of the Wilson loop (figure 5), the norm of the SU( $N$ ) part of scalar (figure 6),



**Figure 4.** [CKKU] Distribution of the argument of the determinant in U(2) theory. The U(N) mass is  $\mu = 1.0$  and the U(1) mass is  $\nu = 5.0$ . The physical volume is  $L = 0.5$  (left) and  $L = 0.75$  (right).



**Figure 5.** [CKKU] The U(1) mass dependence of the Wilson loop in the U(2) CKKU model. The U(N) mass is fixed to  $\mu = 1.0$  and the physical volume is fixed to  $1.0 \times 1.0$ .

the distribution of the argument of the determinant (figure 7) and its cosine (figure 8). One can see that there is almost no U(1) mass dependence.

In figure 9 we have plotted the phase distribution at various values of the U(N) mass  $\mu$ , while other parameters are fixed. It turns out that the  $\mu$ -dependence is small.

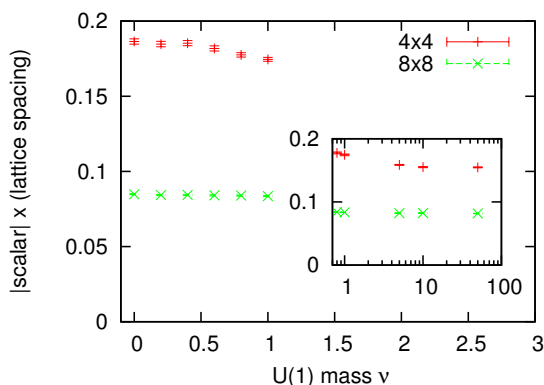
In figure 10 we plot the phase distribution in the U(3) theory. The distribution is broader compared to the U(2) case, but peaks around 0 in the continuum limit.

In figure 11 we plot the real part of the phase. It is clearly seen that it approaches to 1 as lattice spacing becomes small for each physical volume. The scalar mass dependence of the real part is plotted in figure 12, which shows almost no dependence. The detailed values are listed in tables 1, 2 and 3.

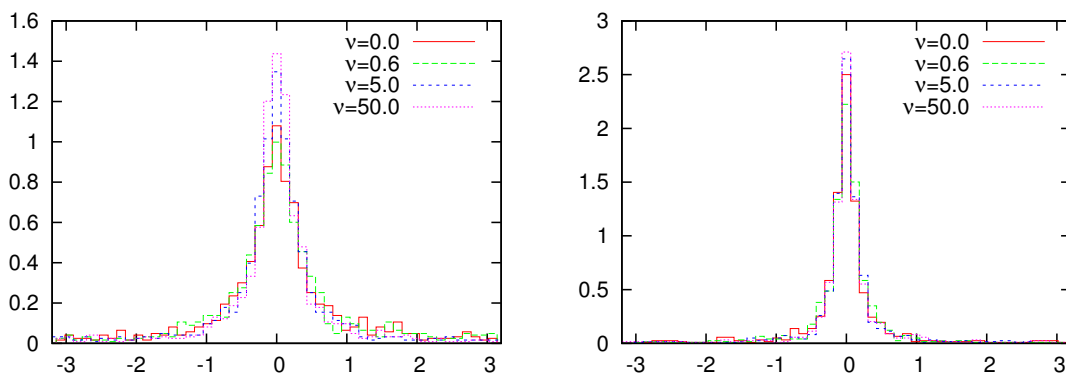
### 3 Absence of the sign problem in Sugino model

In this section, we study the Sugino model<sup>9</sup> and observe the argument of the Pfaffian of the Dirac operator. As before, we use the scalar mass term to regularize the flat direction

<sup>9</sup>See appendix B for the details of the model.



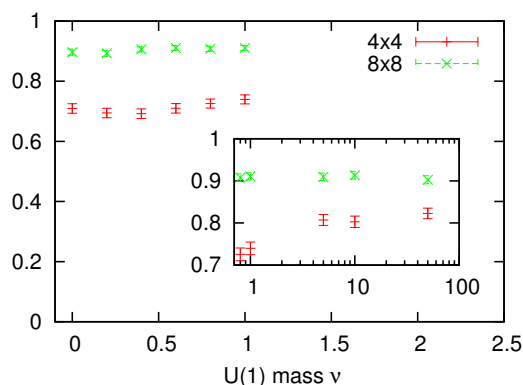
**Figure 6.** [CKKU] The U(1) mass dependence of  $\left\langle \sqrt{\frac{1}{N} \text{Tr} X_i^2(x)} \right\rangle \times (\text{lattice spacing})$  in the U(2) CKKU model. As in figures 1 and others, a contraction w.r.t.  $i$  is not taken and an average over  $i$  and  $x$  is taken. The U( $N$ ) mass is fixed to  $\mu = 1.0$  and the physical volume is fixed to  $1.0 \times 1.0$ .



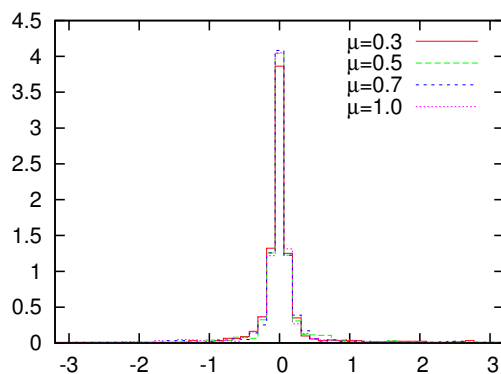
**Figure 7.** [CKKU] The distribution of the argument of the determinant in the U(2) CKKU model, with various values of the U(1) mass  $\nu$ . The left panel is for  $4 \times 4$  lattice and the right is for  $8 \times 8$  lattice. The U( $N$ ) mass is fixed to  $\mu = 1.0$  and the physical volume is fixed to  $1.0 \times 1.0$ . A factor  $(-1)^{N-1}$  is multiplied.

of the potential. As shown in [36], we have checked that the scalar eigenvalues remain close enough to the origin. Therefore we are observing the non-abelian phase, and at the same time, can avoid an unphysical vacuum with large scalar eigenvalues of the cutoff scale. The configurations for  $N = 2$  is taken in this work, while those for  $N = 3, 4, 5$  are taken from the previous work [36].

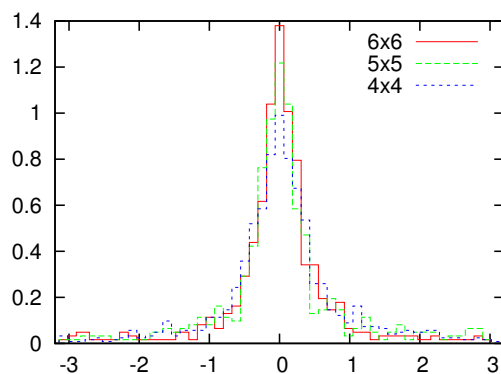
Let us start with SU(2). In figure 13, we plot the distribution of the argument of the Pfaffian for fixed physical volume on the left panel. The peak around 0 becomes sharper as we go close to the continuum. On the right panel, the scalar mass dependence is plotted. Heavier mass gives slightly sharper peak around 0, but the mass-dependence is small. The average of the real part of the Pfaffian phase factor is plotted in figure 14, which shows clear convergence to 1 as the lattice spacing becomes small. (Note that SU(2) case in the plot corresponds to the continuum limit, which uses the fixed volume.) See also figure 12 for the scalar mass dependence.



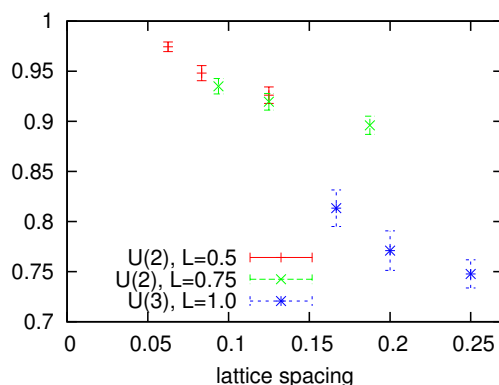
**Figure 8.** [CKKU] The expectation value of real part of the determinant phase factor vs. the U(1) mass  $\nu$ , in the U(2) CKKU model. The U( $N$ ) mass is fixed to  $\mu = 1.0$  and the physical volume is fixed to  $1.0 \times 1.0$ . A factor  $(-1)^{N-1}$  is multiplied.



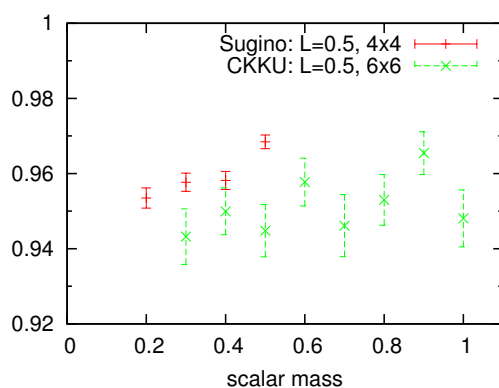
**Figure 9.** [CKKU] The scalar mass dependence of the argument of the determinant in the U(2) theory. The lattice size is  $6 \times 6$ , the physical volume is fixed to  $0.5 \times 0.5$ . The U( $N$ ) mass is varied while the U(1) mass is fixed to be  $\nu = 5.0$ .



**Figure 10.** [CKKU] The argument of the determinant in the U(3) theory. The physical volume is the  $L = 1.00$ , the U( $N$ ) mass is  $\mu = 1.0$  and the U(1) mass is  $\nu = 5.0$ .



**Figure 11.** [CKKU] The expectation value of the real part of the determinant phase factor vs. the lattice spacing. A factor  $(-)^{N-1}$  is multiplied.  $\mu = 1.0$ ,  $\nu = 5.0$ .



**Figure 12.** [CKKU and Sugino] The scalar mass  $\mu$  dependences of the real part of the phase factor in the SU(2) theory (Sugino model,  $4 \times 4$  lattice) and the U(2) theory (CKKU model,  $6 \times 6$  lattice and  $\nu = 5.0$ ). The physical volume is  $0.5 \times 0.5$  for both cases.

In figure 15, we plot the lattice spacing dependence for the SU(3) theory on the left panel. The peak becomes sharper as we go closer to the continuum. On the right panel, we show that the phase distribution with fixed lattice size  $4 \times 4$ . The peak becomes sharper at smaller volume (or equivalently at smaller lattice spacing).

The dependence on  $N$  is plotted in figure 16. As  $N$  becomes large, the distribution spreads. This can be also seen in figure 14, where the lattice spacing dependence of the real part of the phase factor is plotted for  $N = 2, 3$  and 5. The smaller the lattice spacing is, the closer the real part to 1. And larger  $N$  shows slower approach to 1.

The results are listed in terms of the real part of the phase factor in tables 4 and 5.

#### 4 Comparison of the CKKU model and the Sugino model

In order to confirm that our simulation captures the continuum physics, we compare the CKKU model and the Sugino model. In the latter, detailed studies have been performed; it correctly reproduces analytic results in continuum [36] and also restoration of the full su-

$N$	$L$	$N_x = N_y$	Real part of the phase factor
2	0.50	4	0.926(8)
	0.50	6	0.948(8)
	0.50	8	0.974(5)
	0.75	4	0.896(9)
	0.75	6	0.919(8)
	0.75	8	0.935(8)
	1.00	4	0.807(13)
	1.00	6	0.908(8)
	1.00	8	0.907(9)
3	1.00	4	0.748(14)
	1.00	5	0.771(20)
	1.00	6	0.813(18)

**Table 1.** [CKKU] The real part of the phase factor in the  $U(N)$  theory. The  $U(N)$  mass and the  $U(1)$  mass are fixed to  $\mu = 1.0$  and  $\nu = 0.5$ , respectively.

$N$	$\mu$	Real part of the phase factor
2	0.3	0.943(7)
	0.4	0.950(6)
	0.5	0.945(7)
	0.6	0.958(6)
	0.7	0.946(8)
	0.8	0.953(7)
	0.9	0.965(6)
	1.0	0.948(8)

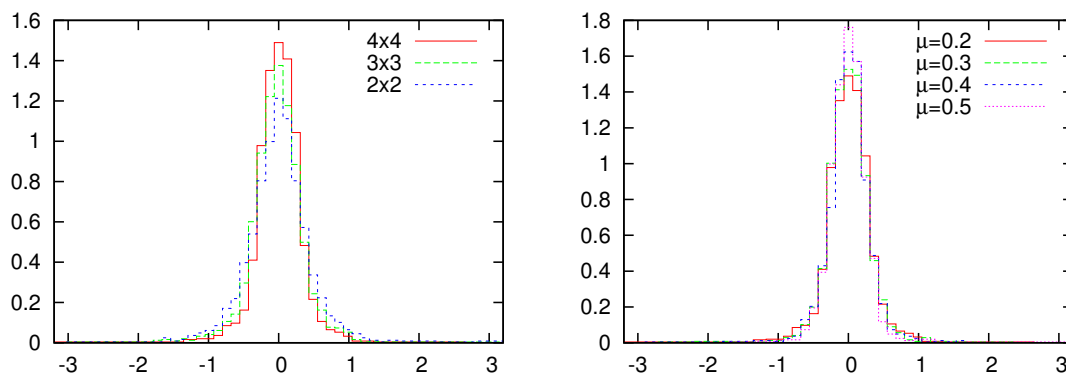
**Table 2.** [CKKU] The real part of the phase factor in the  $U(2)$  theory. The physical volume is fixed to be  $0.5 \times 0.5$  and the lattice size is  $6 \times 6$ . The  $U(1)$  mass is fixed to  $\nu = 5.0$ .

persymmetry has been confirmed [54]. Here we compare the simulation result of the CKKU model with the one of the Sugino model with periodic boundary condition for fermions.

Because the simulation of Sugino model was performed with  $SU(N)$  gauge group while for the CKKU model gauge group was chosen to be  $U(N)$ , we compare the absolute value of the Wilson loop  $W = \frac{1}{N} \text{Tr} e^{i \oint dx A_x}$ , from which the  $U(1)$  part decouples. As can be seen from figure 17, two models give the same result in the continuum limit. We also compare the size of the  $SU(N)$  part of the scalar fields (figure 18). Two models agree reasonably well with each other. Therefore we conclude that the both models converge to the same continuum limit as expected.

$N$	$\nu$	Real part of the phase factor	
		$N_x = N_y = 4$	$N_x = N_y = 8$
2	0.0	0.709(16)	0.897(10)
	0.2	0.694(16)	0.893(10)
	0.4	0.692(16)	0.906(09)
	0.6	0.710(16)	0.911(08)
	0.8	0.726(15)	0.908(09)
	1.0	0.740(15)	0.911(08)
	5.0	0.807(13)	0.910(09)
	10.0	0.803(13)	0.913(09)
	50.0	0.823(13)	0.902(10)

**Table 3.** [CKKU] The real part of the phase factor in the U(2) theory, with various values of U(1) mass  $\nu$ . The physical volume is fixed to be  $0.5 \times 0.5$  and the lattice size is  $4 \times 4$  and  $8 \times 8$ . The U( $N$ ) mass is fixed to  $\mu = 1.0$ .

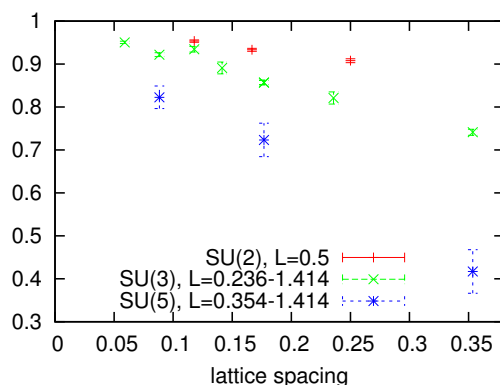


**Figure 13.** [Sugino] The argument of the Pfaffian for the SU(2) case. The physical volume is fixed to  $0.5 \times 0.5$ . The left panel is with various lattice spacings and the right is with various scalar masses  $\mu$ .

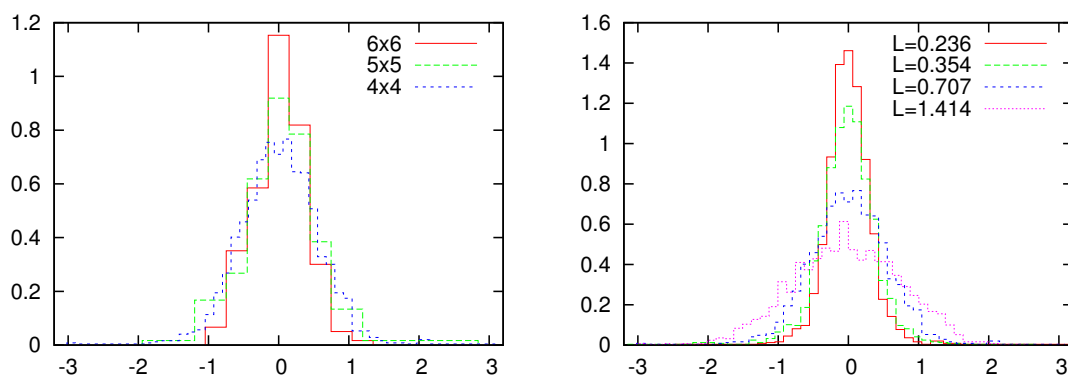
Before concluding this section, let us comment on the flat direction. In [36] it has been shown that the flat direction is lifted dynamically at large- $N$ , both in the continuum theory and in the Sugino model, and the Sugino model converges to the correct supersymmetric continuum limit. In this section we have seen the CKKU model converges to the same limit as well, and hence we can expect the same uplift of the flat direction in the CKKU model.

### 5 Why was a “sign problem” observed in previous works?

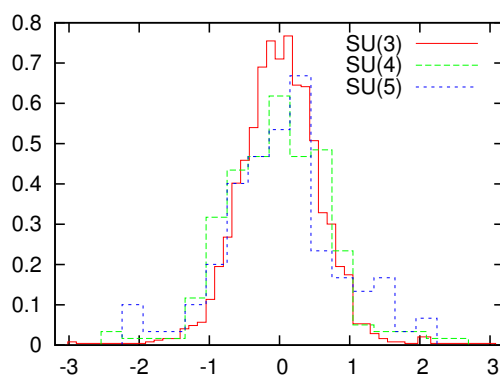
In [49] it has been pointed out that the sign of the fermion determinant of  $\mathcal{N} = (2, 2)$  CKKU model fluctuates violently if one chooses randomly generated lattice field configurations. However, as correctly argued in [49], it does not mean a problem in the continuum



**Figure 14.** [Sugino] The real part of the phase factor in the SU(2), SU(3) and SU(5) theories, with the physical volume  $0.236 \times 0.236-1.414 \times 1.414$ .



**Figure 15.** [Sugino] The argument of the Pfaffian in the SU(3) theory. The scalar mass is  $\mu = 0.20$ . The left panel is for a fixed volume  $0.707 \times 0.707$  and thus different lattice spacings. The right panel is for a fixed  $4 \times 4$  lattice with various physical volumes (thus various lattice spacings).



**Figure 16.** [Sugino] The argument of the Pfaffian in the SU(3), SU(4) and SU(5) theories. The lattice size is  $4 \times 4$  and the physical volume is  $0.707 \times 0.707$ .

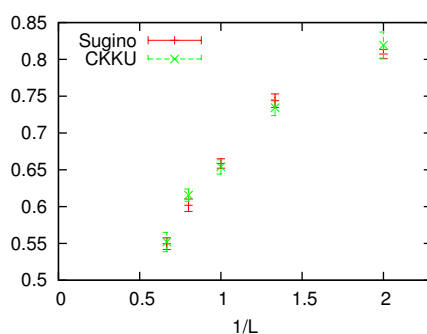


$N$	$N_x = N_y$	scalar mass $\mu$	Real part of the phase factor
2	2	0.2	0.907(4)
	3	0.2	0.933(3)
	4	0.2	0.953(2)
	4	0.3	0.958(2)
	4	0.4	0.958(2)
	4	0.5	0.968(2)

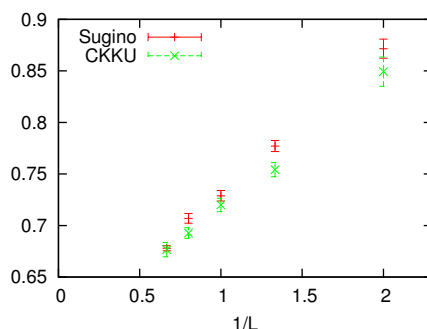
**Table 4.** [Sugino] The real part of the phase factor in the SU(2) theory. The physical volume is fixed to be  $0.5 \times 0.5$ .

$N$	$L$	$N_x = N_y$	Real part of the phase factor
3	0.236	4	0.950(3)
	0.354	4	0.922(4)
	0.707	4	0.857(5)
	0.707	5	0.891(14)
	0.707	6	0.935(7)
	1.414	4	0.742(7)
	1.414	6	0.822(14)
4	0.707	4	0.788(21)
5	0.354	4	0.82(3)
	0.707	4	0.72(4)
	1.414	4	0.42(5)

**Table 5.** [Sugino] The real part of the phase factor in the SU(3), SU(4) and SU(5) theories. The scalar mass is  $\mu = 0.2$ .



**Figure 17.** [CKKU and Sugino] The expectation value of the Wilson loop  $\langle |W| \rangle$  at  $\mu = 1.0$  and  $\nu = 5.0$ . The extrapolation to the continuum limit has been performed. The gauge group is U(2) and SU(2), respectively.



**Figure 18.** [CKKU and Sugino] The extent of the scalar  $\sqrt{\text{Tr}(X_i(x))^2/N}$  (contraction w.r.t  $i$  is not taken). The masses are  $\mu = 1.0$  (Sugino and CKKU) and  $\nu = 5.0$  (CKKU). The extrapolation to the continuum limit has been performed. For CKKU, only the  $SU(N)$  part is plotted. The gauge group is  $U(2)$  and  $SU(2)$ , respectively.

limit — randomly generated configurations are usually measure zero in the path integral, and hence it is necessary to study the distribution of the phase in the phase quenched simulation. If the distribution peaks around one (i.e. determinant is real positive) in the continuum limit, the sign problem does not exist. More crucial thing is the treatment of the pseudo zero-mode. As already mentioned in [49], removing only the exact zero-mode leads to the fluctuation of the phase factor. The pseudo zero-mode, which should give zero eigenvalue and should be decoupled in the continuum limit, gives non-zero eigenvalue due to the lattice artifact.<sup>10</sup> Since what we want to extract from the lattice simulation is the continuum limit, we must remove a contribution from the pseudo zero-mode when we calculate the determinant. Then we obtain the correct positive determinant.

Next let us consider a result from an importance sampling in  $\mathcal{N} = (2, 2)$  theory [15, 50]. According to the plot in the paper, the distribution of the scalar eigenvalues is large; the tail of the distribution reaches to (lattice spacing)<sup>-1</sup>. However, in order for the lattice considered there to converge to the continuum limit *at tree level*, the scalar eigenvalues must be of order (lattice spacing)<sup>0</sup>. Therefore it is plausible that the simulation does not capture the continuum physics.<sup>11</sup> In a simulation of the maximally supersymmetric theory reported in [50] the distribution of the scalar eigenvalues is narrower compared to  $\mathcal{N} = (2, 2)$  theory, but it is still wide (larger than the size of fluctuation in  $4 \times 4$  lattice in figure 1). Therefore it is difficult to obtain robust statements for the continuum limit unless studying smaller lattice spacing using bigger lattice. Another subtlety is that in [50] the system is projected to  $SU(N)$  from  $U(N)$ . Although it is reported that there is no effect to the supersymmetry in this projection, it might have affected to the sign of determinant.

One interesting observation in [50] for the maximally supersymmetric theory is that

<sup>10</sup>As discussed in [49], one can regard this artifact is caused by the orbifold projection, which does not commute with manipulation needed to prove the positivity of the determinant.

<sup>11</sup>It has been remarked that large phase fluctuation arise when scalars take large expectation values [55], and that this “fluctuation of sign” suggests the SUSY breaking because it can make the Witten index vanish. However, if such configuration corresponded to continuum theory, it must be an abelian phase, which does not have any dynamics, and hence the supersymmetry cannot be broken.

the phase is close to one when the gauge group is  $SU(2)$ . The same behavior is observed also in one-dimensional theory [32]. In this case there is no apparent kinematic reason like the pair structure of eigenvalues, and it is not clear if the absence of sign persists in the large volume and/or in the continuum limits. However, if it survives to some extent, it may allow detailed study of the  $SU(2)$  theory by phase-quench or reweighting.

In  $\mathcal{N} = (4, 4)$  theory, it has been reported that a lattice model a la CKKU suffers from the sign problem [56]. This 8 supercharge system is known to have the sign problem in general so the result itself is reasonable. It is interesting to see whether the result changes when pseudo zero-mode is removed, although we do not expect the pair structure of eigenvalues in this theory. In addition, it is not clear whether the flat direction was under control in [56], which is an important point to be studied.

## 6 Conclusion

In this paper we established the absence of the sign problem in 2d  $\mathcal{N} = (2, 2)$  super Yang-Mills theory on lattice. We studied two different lattice models: The Sugino model and the Cohen-Kaplan-Katz-Unsal (CKKU) model. We have clarified the structure of the light modes in the CKKU model and pointed out the importance of the removal of the pseudo zero mode. We also confirmed that the both lattice models provide the same continuum physics as expected.

As we pointed out in section 5, and also discussed in [36], in order to obtain correct continuum limit it is crucial to control the scalar flat direction. From this point of view, it is possible that some of the past simulations for two-dimensional super Yang-Mills theories failed to capture the continuum physics. It is urgent to check whether the scalar flat direction was under control.

## Acknowledgments

We would like to thank O. Aharony, L. Mannelli and M. Ünsal for stimulating discussions and comments. A part of simulation codes was developed from the ones used in collaborations of I.K. with H. Suzuki. The computations were carried out partly on PC clusters at Yukawa Institute and RIKEN RSCC. The work of I. K. was in part supported by the Nishina Memorial Foundation. The work of M. H. was in part supported by JSPS Postdoctoral Fellowship for Research Abroad. M. H. would like to appreciate wonderful hospitality of Israeli people during three years of happy time at Weizmann Institute.

## A An overall sign of the determinant of Dirac operator in the CKKU model

Let us introduce the following notation,

$$V_1^{ab}(\vec{n}) = \sqrt{2} a \operatorname{tr}(T_a x_{\vec{n}} T_b), \quad (\text{A.1})$$

$$\bar{V}_1^{ab}(\vec{n}) = \sqrt{2} a \operatorname{tr}(T_a \bar{x}_{\vec{n}} T_b), \quad (\text{A.2})$$

$$V_2^{ab}(\vec{n}) = \sqrt{2} a \operatorname{tr}(T_a y_{\vec{n}} T_b), \quad (\text{A.3})$$

$$\bar{V}_2^{ab}(\vec{n}) = \sqrt{2} a \operatorname{tr}(T_a \bar{y}_{\vec{n}} T_b), \quad (\text{A.4})$$

where  $T_a$  is a Hermitian gauge generator normalized as  $\operatorname{tr}(T_a T_b) = \delta_{ab}$ .  $V_\mu$  and  $\bar{V}_\mu$  are related by

$$\bar{V}_\mu = (V_\mu^*)^T = V_\mu^\dagger. \quad (\text{A.5})$$

The fermionic part of the action is expressed as<sup>12</sup>

$$S_F = i(\alpha_{\vec{n}}^a, \beta_{\vec{n}}^a) \begin{pmatrix} D_{\vec{n}\vec{m}}^{ab} |_{\alpha\lambda} & D_{\vec{n}\vec{m}}^{ab} |_{\alpha\xi} \\ D_{\vec{n}\vec{m}}^{ab} |_{\beta\lambda} & D_{\vec{n}\vec{m}}^{ab} |_{\beta\xi} \end{pmatrix} \begin{pmatrix} \lambda_{\vec{m}}^b \\ \xi_{\vec{m}}^b \end{pmatrix} = i(\alpha_{\vec{n}}^a, \beta_{\vec{n}}^a) D_{\vec{n}\vec{m}}^{ab} \begin{pmatrix} \lambda_{\vec{m}}^b \\ \xi_{\vec{m}}^b \end{pmatrix} \quad (\text{A.6})$$

with

$$D_{\vec{n}\vec{m}}^{ab} |_{\alpha,\lambda} = -i \left( \delta_{\vec{n},\vec{m}} \bar{V}_1(\vec{n}) - \delta_{\vec{m},\vec{n}+\hat{1}} V_1^*(\vec{n}) \right)^{ab}, \quad (\text{A.7})$$

$$D_{\vec{n}\vec{m}}^{ab} |_{\beta,\lambda} = -i \left( \delta_{\vec{n},\vec{m}} \bar{V}_2(\vec{n}) - \delta_{\vec{m},\vec{n}+\hat{2}} V_2^*(\vec{n}) \right)^{ab}, \quad (\text{A.8})$$

$$D_{\vec{n}\vec{m}}^{ab} |_{\alpha,\xi} = -i \left( -\delta_{\vec{n},\vec{m}} V_2(\vec{n} + \hat{1}) + \delta_{\vec{m},\vec{n}-\hat{2}} \bar{V}_2^*(\vec{n} - \hat{2}) \right)^{ab}, \quad (\text{A.9})$$

$$D_{\vec{n}\vec{m}}^{ab} |_{\beta,\xi} = -i \left( \delta_{\vec{n},\vec{m}} V_1(\vec{n} + \hat{2}) - \delta_{\vec{m},\vec{n}-\hat{1}} \bar{V}_1^*(\vec{n} - \hat{1}) \right)^{ab}, \quad (\text{A.10})$$

where each of the fermion components is defined as

$$\alpha_{\vec{n}} = T_a \alpha_{\vec{n}}^a, \quad \text{etc.} \quad (\text{A.11})$$

Assuming the bosonic fields are constants we obtain the momentum representation as

$$D(p) = -i \begin{pmatrix} \bar{V}_1 - e^{iap_1} V_1^* & -V_2 + e^{-iap_2} \bar{V}_2^* \\ \bar{V}_2 - e^{iap_2} V_2^* & V_1 - e^{-iap_1} \bar{V}_1^* \end{pmatrix}. \quad (\text{A.12})$$

We further decompose it into

$$D(p) = \begin{pmatrix} e^{-\frac{iap_1}{2}} & 0 \\ 0 & e^{-\frac{iap_2}{2}} \end{pmatrix} D'(p) \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{iap_1}{2} + \frac{iap_2}{2}} \end{pmatrix}, \quad (\text{A.13})$$

where  $D'$  has the same determinant as  $D$ . The explicit form of  $D'$  is

$$D'(p) = -i \begin{pmatrix} V_1 e^{-\frac{iap_1}{2}} - V_1^* e^{\frac{iap_1}{2}} & -V_2 e^{\frac{iap_2}{2}} + V_2^* e^{-\frac{iap_2}{2}} \\ V_2 e^{-\frac{iap_2}{2}} - V_2^* e^{\frac{iap_2}{2}} & V_1 e^{\frac{iap_1}{2}} - V_1^* e^{-\frac{iap_1}{2}} \end{pmatrix}. \quad (\text{A.14})$$

Note that because of a factor  $a/2$  in the exponents,  $D'(p)$  is not periodic w.r.t.  $2\pi/a$  so that we have to be careful about the treatment of the the boundary of the Brillouin zone. We use a region  $-\pi/a < p_i \leq \pi/a$ , where  $-\pi/a$  is not included. It is easy to see that it satisfies

$$\sigma_2 D'(p) \sigma_2 = D'^*(-p). \quad (\text{A.15})$$

<sup>12</sup>A factor  $N/a$  in front of  $D$ , which is irrelevant to the argument here, is omitted.

Therefore, for  $p$  which has  $-p$  in our Brillouin zone, we have the following quartet of the eigenvalues.<sup>13</sup>

$$\lambda(p), \lambda(-p), \lambda^*(p), \lambda^*(-p) \quad (\text{A.16})$$

and the contribution to the determinant is always positive. For  $p = 0$ , we have a pair  $\lambda(0), \lambda^*(0)$  which has a positive contribution as well. Note that for  $p = 0$ ,  $D(p) = D'(p)$  thus the eigenvalues of  $D(p = 0)$  make a pair  $(\lambda, \lambda^*)$  as well.

If the lattice is (odd)  $\times$  (odd), we do not have  $p_i = \pi/a$  modes so the determinant of  $D'$  and thus that of  $D$  is positive.

If the lattice is (even)  $\times$  (even), since we have  $p_i = \pi/a$  modes and  $p_i = -\pi/a$  is not in the Brillouin zone, the sign becomes non-trivial. For  $p_1 = \pi/a$ , we have

$$\sigma_2 D'(\pi/a, p_2) \sigma_2 = D'(\pi/a, -p_2)^* \quad (\text{A.17})$$

for  $p_2 \neq \pi/2$ . Therefore we have a quartet  $(\lambda(p_2), \lambda(-p_2), \lambda(p_2)^*, \lambda^*(-p_2))$  for  $p_2 \neq 0$  and a doublet  $(\lambda, \lambda^*)$  for  $p_2 = 0$ , both of them give positive contributions to the determinant. The situation is the same when  $p_2 = \pi/a$ . The remaining combination is  $p = (\pi/a, \pi/a)$ . This time we have

$$D'(p_1 = \pi/a, p_2 = \pi/a) = - \begin{pmatrix} \bar{V}_1 + V_1^* & V_2 + \bar{V}_2^* \\ \bar{V}_2 + V_2^* & -V_1 - \bar{V}_1^* \end{pmatrix} \quad (\text{A.18})$$

which satisfies  $\sigma_2 D' \sigma_2 = -D'^*$ . The eigenvalues make a pair  $(\lambda, -\lambda^*)$ . The sign contribution to the determinant from this sector is  $(-1)^{N^2} = (-1)^N$ .

For  $iD$ , an extra factor  $i^{2N^2 \times (\text{num. of lattice sites})}$  appears, and hence the phase is  $(-1)^N$  for both odd  $\times$  odd and even  $\times$  even lattices. Since we remove the two lightest modes — one exact zero mode and one pseudo zero mode — the determinant of  $iD$  picks up an extra factor  $i^{-2}$  and thus the determinant of  $iD$  has a sign  $(-1)^{N-1}$ .

## B The action of the Sugino model

Sugino's lattice action [9] is given by<sup>14</sup>

$$S_{\text{lattice}} = a_x a_y \sum_{\vec{x}} \left\{ \sum_{i=1}^3 \mathcal{L}_{B_i}(\vec{x}) + \sum_{i=1}^6 \mathcal{L}_{F_i}(\vec{x}) \right\} + (\text{auxiliary field}), \quad (\text{B.1})$$

where

$$\mathcal{L}_{B1}(\vec{x}) = \frac{N}{8a_x^2 a_y^2} \text{Tr}[\phi(\vec{x}), \bar{\phi}(\vec{x})]^2, \quad (\text{B.2})$$

$$\mathcal{L}_{B2}(\vec{x}) = \frac{N}{8a_x^2 a_y^2} \text{Tr} \hat{\Phi}_{\text{TL}}(\vec{x})^2, \quad (\text{B.3})$$

<sup>13</sup>Do not confuse eigenvalue  $\lambda$  with one of the fermion or 't Hooft coupling.

<sup>14</sup>Here we follow the notation in [43] with a slightly different normalization. Although group theoretical normalizations are different in [43] and [8], one can absorb them by rescaling the 't Hooft couplings as  $2\lambda_{\text{CKKU}} = \lambda_{\text{Sugino}}$ . We set  $\lambda = \lambda_{\text{CKKU}} = 1$ .

$$\begin{aligned}
 \mathcal{L}_{B3}(\vec{x}) &= \frac{N}{2a_x^3 a_y} \text{Tr} \left\{ (\phi(\vec{x}) - U_x(\vec{x})\phi(\vec{x} + a_x \hat{x})U_x(\vec{x})^{-1}) \right. \\
 &\quad \left. \times (\bar{\phi}(\vec{x}) - U_x(\vec{x})\bar{\phi}(\vec{x} + a_x \hat{x})U_x(\vec{x})^{-1}) \right\} \\
 &\quad + \frac{N}{2a_x a_y^3} \text{Tr} \left\{ (\phi(\vec{x}) - U_y(\vec{x})\phi(\vec{x} + a_y \hat{y})U_y(\vec{x})^{-1}) \right. \\
 &\quad \left. \times (\bar{\phi}(\vec{x}) - U_y(\vec{x})\bar{\phi}(\vec{x} + a_y \hat{y})U_y(\vec{x})^{-1}) \right\} \quad (\text{B.4})
 \end{aligned}$$

and

$$\mathcal{L}_{F1}(\vec{x}) = -\frac{N}{8a_x^2 a_y^2} \text{Tr} (\eta(\vec{x})[\phi(\vec{x}), \eta(\vec{x})]), \quad (\text{B.5})$$

$$\mathcal{L}_{F2}(\vec{x}) = -\frac{N}{2a_x^2 a_y^2} \text{Tr} (\chi(\vec{x})[\phi(\vec{x}), \chi(\vec{x})]), \quad (\text{B.6})$$

$$\mathcal{L}_{F3}(\vec{x}) = -\frac{N}{2a_x^3 a_y} \text{Tr} \left\{ \psi_0(\vec{x})\psi_0(\vec{x}) (\bar{\phi}(\vec{x}) + U_x(\vec{x})\bar{\phi}(\vec{x} + a_x \hat{x})U_x(\vec{x})^{-1}) \right\}, \quad (\text{B.7})$$

$$\mathcal{L}_{F4}(\vec{x}) = -\frac{N}{2a_x a_y^3} \text{Tr} \left\{ \psi_1(\vec{x})\psi_1(\vec{x}) (\bar{\phi}(\vec{x}) + U_y(\vec{x})\bar{\phi}(\vec{x} + a_y \hat{y})U_y(\vec{x})^{-1}) \right\}, \quad (\text{B.8})$$

$$\mathcal{L}_{F5}(\vec{x}) = i\frac{N}{2a_x^2 a_y^2} \text{Tr} \left( \chi(\vec{x}) \cdot Q\hat{\Phi}(\vec{x}) \right), \quad (\text{B.9})$$

$$\begin{aligned}
 \mathcal{L}_{F6}(\vec{x}) &= -i\frac{N}{2a_x^3 a_y} \text{Tr} \left\{ \psi_0 (\eta(\vec{x}) - U_x(\vec{x})\eta(\vec{x} + a_x \hat{x})U_x(\vec{x})^{-1}) \right\} \\
 &\quad -i\frac{N}{a_x a_y^3} \text{Tr} \left\{ \psi_1 (\eta(\vec{x}) - U_y(\vec{x})\eta(\vec{x} + a_y \hat{y})U_y(\vec{x})^{-1}) \right\}, \quad (\text{B.10})
 \end{aligned}$$

where  $U(\vec{x}, \mu)$  are gauge link variables,  $\phi(\vec{x})$  is a complex scalar,  $\eta(\vec{x})$ ,  $\chi(\vec{x})$  and  $\psi_\mu(\vec{x})$  are fermion field,  $a_x$  and  $a_y$  are lattice spacings,<sup>15</sup>  $\epsilon$  is a real parameter which must be chosen appropriately for each  $N$ ,

$$\hat{\Phi}(\vec{x}) = \frac{-i(P(\vec{x}) - P(\vec{x})^{-1})}{1 - |1 - P(\vec{x})|^2/\epsilon^2}, \quad \hat{\Phi}_{\text{TL}}(\vec{x}) = \hat{\Phi}(\vec{x}) - \frac{1}{N} \left( \text{Tr} \hat{\Phi}(\vec{x}) \right) \cdot \mathbf{1}, \quad (\text{B.11})$$

where  $P(\vec{x}) = U_x(\vec{x})U_y(\vec{x} + \hat{x})U_x^\dagger(\vec{x} + \hat{y})U_y^\dagger(\vec{x})$  is the plaquette variable, and  $Q$  generates one of the four super transformations,

$$QU_\mu(\vec{x}) = i\psi_\mu(\vec{x})U_\mu(\vec{x}), \quad (\text{B.12})$$

$$Q\psi_\mu(\vec{x}) = i\psi_\mu(\vec{x})\psi_\mu(\vec{x}) - i(\phi(\vec{x}) - U_\mu(\vec{x})\phi(\vec{x} + a_\mu \hat{\mu})U_\mu(\vec{x})^{-1}), \quad (\text{B.13})$$

$$Q\phi(\vec{x}) = 0, \quad (\text{B.14})$$

$$Q\chi(\vec{x}) = H(\vec{x}), \quad (\text{B.15})$$

$$QH(\vec{x}) = [\phi(\vec{x}), \chi(\vec{x})], \quad (\text{B.16})$$

$$Q\bar{\phi}(\vec{x}) = \eta(\vec{x}), \quad (\text{B.17})$$

$$Q\eta(\vec{x}) = [\phi(\vec{x}), \bar{\phi}(\vec{x})]. \quad (\text{B.18})$$

Sugino's action  $S_{\text{lattice}}$  is invariant under the supersymmetry generated by  $Q$ , because  $Q$  is nilpotent up to commutators and  $S$  can be written in a  $Q$ -exact form.

<sup>15</sup>In the actual simulation we have used the isotropic lattice,  $a_x = a_y$ .

In [9], using super-renormalizability and symmetry argument, it was shown that other three supersymmetries, which is broken by a lattice artifact at the discretized level, is restored in the continuum limit. Furthermore, in [54], this restoration has been confirmed explicitly by the Monte-Carlo simulation. Absence of operator mixing/renormalization is has been shown perturbatively in [57].

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