Evidence for F(uzz) theory

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ABSTRACT: We show that in the decoupling limit of an F-theory compactification, the internal directions of the seven-branes must wrap a non-commutative four-cycle $\mathcal{S}$. We introduce a general method for obtaining fuzzy geometric spaces via toric geometry, and develop tools for engineering four-dimensional GUT models from this non-commutative setup. We obtain the chiral matter content and Yukawa couplings, and show that the theory has a finite Kaluza-Klein spectrum. The value of $1/\alpha_{\mathrm{GUT}}$ is predicted to be equal to the number of fuzzy points on the internal four-cycle $\mathcal{S}$. This relation puts a non-trivial restriction on the space of gauge theories that can arise as a limit of F-theory. By viewing the seven-brane as tiled by D3-branes sitting at the N fuzzy points of the geometry, we argue that this theory admits a holographic dual description in the large N limit. We also entertain the possibility of constructing string models with large fuzzy extra dimensions, but with a high scale for quantum gravity.

Contents

1 Introduction

In local string model building, the degrees of freedom that comprise the Standard Model are assumed to be localized within a small neighborhood of the compactification manifold, arranged such that one can *in principle* take a decoupling limit, in which the four-dimensional Planck scale is sent off to infinity [1–4]. The gauge and matter fields are engineered via branes wrapping small cycles of a geometric singularity, and decoupling is achieved by a combined decompactification and zero slope limit. The decoupled theory reduces to a pure four-dimensional quantum field theory (QFT), with *a priori* freely tunable coupling constants. In recent years, there has been some notable progress in local model building of supersymmetric grand unified field theories (GUTs) via F-theory [5–9]. See [10] for a recent review of F-theory GUTs. In this setting, the existence of a decoupling limit turns out to be quite restrictive: it requires that the gauge fields must live on spacetime filling seven-branes that wrap a del Pezzo surface. As argued in [8], decoupling gravity from an F-theory GUT imposes significant restrictions, and provides a way to narrow the search for potentially promising vacua. The local approach in M-theory is less well developed, but presumably gives rise to a similarly restrictive framework.

Local string model building is only the first stage of a modular bottom-up approach to string phenomenology [1–4] in which the task of embedding particle physics within string theory is made more tractable, by decomposing it into a sequence of four steps:

(i) Geometrically engineer a (semi-)realistic GUT via a local M-theory or F-theory model.

(ii) Investigate its low energy phenomenology, while assuming that couplings can be tuned.

(iii) Formulate the topological and geometric compatibility conditions that follow from the requirement that the local model can be embedded inside a global string compactification.

(iv) Establish the existence of stabilized compactifications that fulfill these requirements.

Each stage has its own set of challenges and potential for learning new lessons. In the first two steps one develops a link between the effective field theory language of GUT phenomenology and the geometric language of extra dimensions, strings and branes. This local geometric perspective is flexible yet restrictive, and when combined with minimal naturalness requirements, produces a concrete and predictive framework. Via the last two steps one imposes the requirement that the local model can be extended into a UV complete theory with gravity. This second half of the program is harder, as it involves dealing with the full complexity and apparent arbitrariness of the closed string landscape, but is equally essential, since one should expect to uncover additional non-trivial restrictions and predictions. Central model building components, such as supersymmetry breaking, may involve all four stages.

Here we concern ourselves with the first step, the map between the geometric ingredients of F-theory and four-dimensional quantum field theory. In a strict decoupling limit,

the two systems should be equivalent: all higher dimensional and stringy degrees of freedom should either be explicitly encoded in terms of four-dimensional quantum fields, or integrated out and absorbed via computable threshold corrections. This raises the obvious question:

What is the subspace of all four-dimensional QFTs that can be obtained via a decoupling limit of F-theory?

We will find that the landscape of decoupling limits of F-theory is distinctly smaller than the space of all four-dimensional QFTs. Specifically, for any F-theory GUT with a given matter content, we will establish a concrete upper bound on the gauge coupling. Conversely, for a given gauge coupling, we will find restrictions on the possible matter content of the gauge theory.

1.1 F(uzz) theory as the decoupling limit of F-theory

Suppose we set out to engineer a grand unified theory by means of a stack of seven-branes, wrapped on a contractible del Pezzo four-cycle $\mathcal{S}$ of an F-theory compactification. To gain a better understanding and control over the physical content of the model, it is helpful to isolate the four-dimensional QFT data from the string and extra dimensional degrees of freedom. However, as long as the four-cycle $\mathcal{S}$ has some finite size, the seven-brane theory still has an elaborate spectrum of Kaluza-Klein (KK) and string excitations, and interacts with the closed strings in its neighborhood. To get a decoupled four-dimensional gauge theory, we must let the four-cycle $\mathcal{S}$ shrink to zero size[1]

$$\text{Vol}_{\text{closed}}(\mathcal{S}) \ \to \ 0, \tag{1.1}$$

where $\text{Vol}_{\text{closed}}(\mathcal{S})$ denotes the volume of the four-cycle $\mathcal{S}$ evaluated in the *closed string* metric. The relevant RG scale, which typically would be equal to M_{GUT}, is assumed to be far below the string scale. This assumption amounts to taking a zero slope limit. The small volume limit pushes up the KK mass scale, and the zero slope limit decouples the massive string excitations. The combined limit decouples the closed string dynamics, and sends the four-dimensional Planck mass to infinity.[2]

Local geometric engineering of four-dimensional gauge theories thus typically requires that parts of the internal geometry become highly curved. Besides the decoupling argument, there is also a more dynamical motivation for considering the small volume limit. A general important issue is how the moduli that determine the local size and shape of the seven-brane worldvolume in fact get stabilized. One common way that volume moduli are stabilized is through instanton effects. This is potentially problematic for the GUT volume modulus, because if we are to treat the gauge coupling as a fixed parameter up to high scales (in accord with the paradigm of unification), the corresponding modulus must have

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[1]We use units with $2\pi\alpha' = 1$. The zero slope limit then corresponds to working at energy scales below the scale of massive string excitations.

[2]The assumption that gravity can be decoupled from a local F-theory model imposes conditions on possible global completions [11–13].

a very large mass. On the other hand, the naive mass-squared induced by an instanton of the GUT theory is suppressed by $e^{-2\pi/\alpha_{\rm GUT}}$. In particular, to achieve an appropriate GUT scale mass for the volume modulus, it would seem necessary to consider a regime of parameters where some volume has collapsed to zero size, so as to enhance the associated instanton induced mass scale. Related to this is the problem noted in [14] that one expects at least some of the parameters of the model to be stabilized at strong coupling or high curvature. Our approach is to turn this problem on its head. Rather than trying to avoid high curvature or strong coupling, we instead set out to make use of the fact that closed string moduli may naturally be stabilized in a small volume regime.

The commonly used approach to study string and brane dynamics in a highly curved setting is to make use of the fact that the geometric methods remain accurate for characterizing topological and holomorphic gauge theory data, which can be computed at large volume and extrapolated to the small volume limit. However, as we will now argue, there is another physical mechanism and independent reason for why the geometric perspective remains valid and useful even in the zero volume limit.

To explain our argument, let us consider the value of the four-dimensional gauge coupling in the decoupling limit. In the context of compactifications of perturbative type IIB vacua with D7-branes, the four-dimensional gauge coupling is given by

$$\frac{4\pi}{g_{\rm GUT}^2} = \frac{{\rm Vol}_{\rm open}(\mathcal{S})}{(2\pi)^2 g_s} \tag{1.2}$$

where ${\rm Vol}_{\rm open}(\mathcal{S})$ denotes the internal volume of the GUT seven-brane as measured by the *open string* metric. Since we are interested in producing four-dimensional gauge theories with a perturbative coupling, we must ensure that the open string volume remains finite, while taking the small closed string volume limit (1.1). From the DBI action, we read off that at large closed string volume the open string volume can be expressed as

$$\text{Vol}_{\rm open}(\mathcal{S}) = \text{Vol}_{\rm closed}(\mathcal{S}) + \int_{\mathcal{S}} \mathcal{B} \wedge \mathcal{B} + \cdots. \tag{1.3}$$

Here $\mathcal{B}$ is the background two-form flux on the D7-branes, given by the sum

$$\mathcal{B} = F + \hat{B} \tag{1.4}$$

of the worldvolume U(1) flux $F = dA$ and the pull-back $\hat{B}$ of the NS two-form field $B_{\rm NS}$ to the seven-brane worldvolume. In addition, the "$\cdots$" refers to various correction terms associated with the Kähler form. In the ${\rm Vol}_{\rm closed}({\rm S}) \to 0$ limit we can keep ${\rm Vol}_{\rm open}(\mathcal{S})$ finite, provided we switch on a non-zero background $\mathcal{B}$-flux along the internal directions. The decoupling limit (1.1) then produces a four-dimensional gauge theory with finite gauge coupling

$$\frac{4\pi}{g_{\rm GUT}^2} = \frac{1}{(2\pi)^2 g_s} \int_{\mathcal{S}} \mathcal{B} \wedge \mathcal{B}. \tag{1.5}$$

Note that because $\mathcal{B} \wedge \mathcal{B}$ is a topological quantity, we expect this formula to be correct in the decoupling limit we are considering.

In flat space, taking the zero slope limit at zero closed string volume, in tandem with holding the two-form field $\mathcal{B}$ and the open string metric G_{ij}^{open} fixed coincides with the Seiberg-Witten limit. This produces a non-commutative gauge theory along the internal directions of the seven-brane [15].[3] Non-commutative theories on tori [16] and in flat space [15] have been studied extensively in the literature. Some generalizations of the flat space case have been studied in the string theory literature for example in [17, 18]. See for example [19–21] for reviews of some aspects of non-commutative geometry and its possible applications to physics. Though somewhat orthogonal to the limit we consider here, related fluxes are often considered in theories of magnetized branes (see [22] for a review).

Our present application requires a non-trivial generalization of the flat space situation considered in [15], but it seems reasonable to assume that the same conclusion should carry over to our context. In other words, we learn that from the perspective of the seven-brane, the internal four-cycle should be treated as a non-commutative or "fuzzy" space $\mathcal{S}_{\text{NC}}$ with a finite volume and a non-trivial symplectic form equal to $\mathcal{B}$. At a schematic level, the commutative coordinates $(z, \bar{z})$ on $\mathcal{S}$ should be replaced by operators $(Z, Z^\dagger)$ which satisfy a commutation relation of the form[4]

$$[\, Z_i, Z_j^\dagger \,] = \theta_{ij}\,, \qquad \theta_{ij} = \mathcal{B}_{ij}^{-1}\,. \tag{1.6}$$

The four uncompactified space-time dimensions remain as ordinary commuting directions.

Motivated by this observation, we will set out to develop some basic algebraic tools that are needed to build local F-theory models with a non-commutative internal geometry $\mathcal{S}$. In this paper, we will show that all the necessary ingredients can be realized on an internal fuzzy geometry with only a finite number of points. The number of points on $\mathcal{S}$ is equal to the dimension of the associated state space on which the non-commuting coordinates Z_i and $Z_j^\dagger$ act as linear operators. The space of functions on $\mathcal{S}$ is identified with the space of all linear maps from this state space to itself, and is therefore finite-dimensional. Hence by design, the decoupled seven-brane theory has a finite Kaluza-Klein expansion, and reduces to a pure four-dimensional quantum field theory with a finite number of fields. Our goal in the following sections is to develop the basic elements of the non-commutative geometry probed by the seven-brane and give a precise algebraic prescription for computing the spectrum of fields, the Lagrangian and the gauge and Yukawa interactions of the four-dimensional QFT.

At a practical level, making the internal space non-commutative also provides us with a regulator for the seven-brane, which softens the usually problematic high energy behavior of gauge theory in extra dimensions. Ordinarily, one would appeal to a lattice formulation, which comes at the cost of destroying some of the geometric structure of the internal space. By contrast, the fuzzy prescription provides us a way to retain the holomorphic geometry used to define the compactification in the first place. Schematically, the higher-dimensional

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[3]In most discussions of D-brane engineering of non-commutative gauge theories, one only considers the NS 2-form B_{NS}. The U(1) flux $F = dA$ can be viewed as the integral part of B_{NS}. Indeed, it is clear that in the presence of a U(1) magnetic field and in a suitable low energy limit, the charged endpoints of the open strings will relax into the states of the lowest Landau level, which can be thought of as occupying unit Planck cells of a non-commutative geometry.

[4]Here, and in the following, we assume that $\mathcal{B}$ is a (1,1)-form.
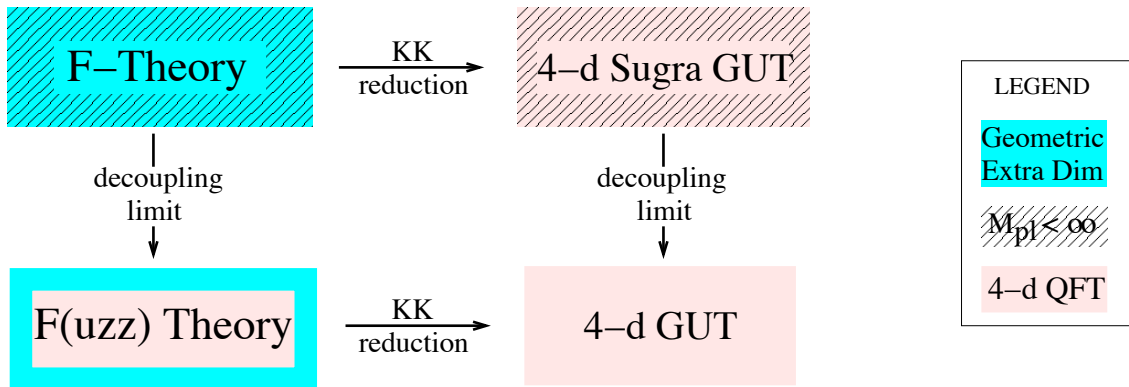
Figure 1. F(uzz) theory arises from F-theory via a decoupling limit in which M_{pl} is sent to infinity. It represents a four-dimensional QFT, but preserves the geometric higher dimensional perspective of the local F-theory construction. The extra dimensions are non-commutative, and give rise to a finite KK spectrum. The arrows in the above diagram are not surjective and the diagram does not commute: each reverse arrow represents an embedding into a more complete theory, which give rise to non-trivial consistency requirements on the possible GUT models that can arise from F-theory.

fields become operators via the replacement:

$$\phi(x_\mu, z, \overline{z}) \to \phi(x_\mu, Z^\dagger, Z) = \sum_{n,m} \phi_{n,m}(x^\mu) Z^{\dagger n} Z^m \tag{1.7}$$

where each $\phi_{n,m}(x_\mu)$ corresponds to a four-dimensional field. As we show, when the internal space $\mathcal{S}$ is compact, the oscillators Z and $Z^\dagger$ act on a finite-dimensional Hilbert space. As a consequence, our power series truncates to a polynomial, and only a finite number of the $\phi_{n,m}$ wind up being dynamical four-dimensional fields. Similar conclusions about models with fuzzy extra dimensions have been drawn from a somewhat different viewpoint for example in [23–25].

Each point on the non-commutative space $\mathcal{S}$ can be viewed as a localized quantum state, that occupies one unit Planck cell. The size of the Planck cells is governed by the $\mathcal{B}$-field, and the total number of points is given by the symplectic volume

$$\# \text{ points in } \mathcal{S} = \frac{1}{(2\pi)^2} \int_{\mathcal{S}} \mathcal{B} \wedge \mathcal{B}. \tag{1.8}$$

For small $\mathcal{B}$, there are only a few fuzzy points. To recover the commutative continuum gauge theory, one needs to take the large $\mathcal{B}$-field limit. In order to quantize $\mathcal{S}$, the periods of $\mathcal{B}$ around any two-cycle Σ_a within $\mathcal{S}$ must be an integer multiple of 2π. This in particular ensures that the right-hand side of (1.8) is an integer.

At a general point in moduli space, when $\mathcal{S}$ has finite size, the periods of $\mathcal{B}$ do not automatically take quantized values. The quantization constraint is a natural consequence of the small volume limit, both from a dynamical and a topological perspective. Like any stringy axion, the B_{NS}-periods acquire a periodic potential, generated via string instantons wrapping the associated 2-cycles. It is reasonable to expect that in the small volume limit, when the string instantons are unsuppressed, the induced potential enforces the constraint

that $\mathcal{B}$ has integer periods. Topologically, shrinking a two-cycle Σ can be thought of as going towards a geometric transition point, at which the three-chain bounded by Σ turns into a three-cycle B_3. Cutting off the non-compact geometry, B_3 becomes compact. It is then consistent to find that $\int_\Sigma \mathcal{B} \simeq \int_{B_3} H_3$ attains a quantized value at this geometric transition point. Thus, after taking the zero volume limit, the periods of $\mathcal{B}$ do not represent freely tunable moduli, but are more properly viewed as discrete parameters.

An illustrative alternative perspective on the situation is as follows. The integral part of the periods of $\mathcal{B}$ and of the symplectic volume $\int_{\mathcal{S}} \mathcal{B} \wedge \mathcal{B}$ represent, respectively, wrapped five-branes and three-branes, that are dissolved inside of the seven-brane. In the decoupling limit, the internal open string volume of the wrapped seven-brane is fully supported by the induced D3-brane charge. This suggests that one can view the seven-brane as completely built up from constituent D3-branes, each occupying one Planck cell. The fuzzy points on $\mathcal{S}$ are in one-to-one correspondence with the constituents D3-branes. We will return to this perspective later, in section 7.

$(\alpha_{\mathbf{GUT}})^{-1} \simeq$ number of fuzzy points. Our considerations uncover some useful new insights into the structure of the subspace of four-dimensional QFTs that can arise as a decoupling limit of F-theory. As a particular example, we can combine equations (1.5) and (1.8) to obtain the following intriguing relation between the four-dimensional gauge coupling and the number of fuzzy points on the internal space

$$\frac{4\pi}{g_{\mathrm{GUT}}^2} = \frac{1}{g_s} \left(\# \ \text{points in } \mathcal{S} \right) \tag{1.9}$$

A priori, it would seem that one could relax the proportionality between the GUT gauge coupling and the number of points on $\mathcal{S}$ by varying the value of the string coupling g_s over its full range. However, g_s is not an arbitrarily adjustable coupling constant.

In an F-theory compactification, the axio-dilaton $\tau = C_0 + i/g_s$ represents the shape of the elliptic fiber of the Calabi-Yau fourfold. Like any complex structure modulus, this shape can be stabilized by turning on appropriate fluxes. In general τ does not stabilize at a constant value, but varies non-trivially over the compactification manifold. In the neighborhood of seven-branes, τ behaves in a well prescribed but possibly singular way. (For example, near a D7-brane it behaves as $\tau \sim \frac{1}{2\pi i} \log z$.) The string coupling at the seven-brane thus depends on the value of the short-distance cut-off near the brane. In this paper we shall assume that τ varies adiabatically over the holomorphic divisor defined by the GUT seven-brane, and so we shall specify a cut-off value for τ compatible with its profile near an E-type Yukawa point. In principle such position dependence can also be taken into account [26], though throughout this paper we shall neglect this subtlety.

A closely related point is that building an F-theory GUT necessarily involves non-perturbative (p, q) seven-branes, around which τ has a monodromy given by some non-trivial element of the $SL(2, \mathbb{Z})$ S-duality group. One could argue that a self-contained discussion of non-commutative seven-branes must be formulated in an $SL(2, \mathbb{Z})$ covariant way. The non-commutativity parameter of a (p, q) seven-brane gauge theory is given by the overall U(1) flux

$$\mathcal{B} = F + p\hat{B}_{\mathrm{NS}} - q\hat{B}_{\mathrm{RR}}. \tag{1.10}$$

Similarly, we need to generalize the formula (1.5) for the four-dimensional gauge coupling by replacing the prefactor $1/g_s$ by the (p, q) seven-brane tension (in string frame). Upon switching to Einstein frame, this provides an S-duality invariant characterization of the parameters of the seven-brane gauge theory. Note in particular, that the covariance under S-duality allows one to impose the restriction that g_s is always less than one (or more accurately, less than $2/\sqrt{3}$). See [27] for a discussion of the DBI action for (p, q) seven-branes.[5]

Now, in a local F-theory model, the requirement that the theory incorporates all the necessary matter and Yukawa couplings dictates that the seven-brane worldvolume must contain a locus where the singularity type is enhanced to E_6. At the E_6 singularity, the value of τ is fixed to be equal to the cubic root of unity. Combining this with our argument above, we learn that in an F-theory GUT, the string coupling is naturally stabilized at the special value $g_s = 2/\sqrt{3}$. Hence it is reasonable to fix g_s to be close to this maximal value. Setting $g_s \simeq 1$, equation (1.9) turns into a direct equality between $1/\alpha_{\text{GUT}}$ and the number of points on $\mathcal{S}$

$$\frac{1}{\alpha_{\text{GUT}}} \quad \simeq \quad \# \text{ points in } \mathcal{S} \tag{1.11}$$

Given a typical value $\alpha_{\text{GUT}} \simeq 1/25$, this prescribes that the number of points in $\mathcal{S}$ must be equal to an integer on the order of 20 to 30. Given that $\mathcal{S}$ is four-dimensional, this is a rather small number — which means that non-commutative effects can be sizable.[6]

1.2 Outline

The rest of this paper is organized as follows. In section 2 we review the main ingredients of commutative F-theory. One of our goals will be to translate the holomorphic data of the commutative theory to the non-commutative setting. In section 3 we develop some aspects of fuzzy toric geometry. A general non-commutative Kähler manifold can be written as a subspace of a toric space and we define forms and differential calculus on such subspaces. Using this construction, in section 4 we formulate the main ingredients of Fuzz theory, and develop the theory of seven-branes that wrap a non-commutative four-cycle. Sections 5 and 6 contain some applications of Fuzz theory. In section 7 we comment on the limit where the number of fuzzy points N becomes large, and on the possible closed string duals. Section 8 contains our conclusions. Additional details of fuzzy $\mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are given in appendices A and B.

2 Review of commutative F-theory

In this section we briefly review the main geometric elements of F-theory constructions [28–30].

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[5]For bound states of seven-branes of different (p, q)-type, turning on a $\mathcal{B}$-field would appear differently to the constituent branes. For example, the exceptional seven-branes with gauge group E_6, E_7 and E_8 can be realized as bound states of the form $E_n = A^{n-1}BC^2$, where A, B and C correspond to seven-branes of different (p, q) type. As the $\mathcal{B}$-flux experienced by each type of seven-brane factor is different, it is as if we had switched on a GUT breaking flux, breaking the corresponding group to $SU(n - 1) \times SU(2) \times U(1)$.

[6]Note that even if one relaxes the assumption that the local string coupling is close to one, the requirement that $g_s \lesssim 1$ (since one can always go to an S-duality frame in which this is the case) gives the same strong upper bound on the number of points.

To achieve four-dimensional $\mathcal{N} = 1$ supersymmetry, we compactify F-theory on an elliptically fibered Calabi-Yau fourfold, fibered over a complex threefold B_3. The elliptic fiber is infinitesimally small, and the base B_3 represents the actual compactification manifold of the IIB string theory. A minimal presentation of the Calabi-Yau fourfold is in terms of the Weierstrass equation

$$y^2 = x^3 + f(z_i)x + g(z_i), \tag{2.1}$$

where the f's and g's are given by sections of suitable line-bundles over the base B_3. We can represent f and g as polynomials of the complex coordinates z_i that parameterize B_3. At any point z_j on the base, equation (2.1) defines a torus with shape modulus τ, which specifies the value of the IIB axion and dilaton field via $\tau = C_0 + ie^{-\phi}$. In F-theory compactications, this axio-dilaton field varies non-trivially over the internal directions.

The location of seven-branes in the compactification is specified by the discriminant locus

$$\Delta = 4f^3 + 27g^2 = 0, \tag{2.2}$$

where the elliptic curve degenerates. The discriminant locus (2.2) in general factorizes into several irreducible components:

$$\Delta = \prod_\alpha \Delta_\alpha(z_i) = 0. \tag{2.3}$$

Each vanishing locus ($\Delta_\alpha = 0$) defines a hypersurface $\mathcal{S}_\alpha$ in B_3 where a seven-brane is located. In the direct neighborhood of a seven-brane worldvolume $\mathcal{S}_\alpha$, the Calabi-Yau fourfold can be represented as a local $K3$ fibration over $\mathcal{S}_\alpha$. The ADE degeneration of the local K3 then dictates the gauge group of the worldvolume gauge theory on the seven-brane.

Our main interest in the present context will be the study of the worldvolume theory of the "GUT seven-brane" wrapping the hypersurface

$$\mathcal{S} = \left(\Delta_{\mathrm{GUT}} = 0\right). \tag{2.4}$$

This seven-brane hosts the gauge sector of the F-theory GUT, which for concreteness we will take to have gauge group $G = \mathrm{SU}(5)$. In accord with the decoupling principle, we will restrict our attention to the case that the four-cycle $\mathcal{S}$ can shrink to zero size within the base B_3. This condition requires that $\mathcal{S}$ describes a del Pezzo surface. The detailed geometric aspects of del Pezzo geometries will not be important to us in what follows, except that it is always possible to represent them as intersections of hypersurfaces inside of a toric space.

In the vicinity of $\mathcal{S}$, we can switch to a local coordinate z such that $z = 0$ indicates the location of the GUT seven-brane. In this case, the presence of an SU(5) gauge theory on the GUT brane indicates the discriminant locally looks as:

$$\Delta = z^5 \prod_{\Delta_\alpha \neq \Delta_{\mathrm{GUT}}} \Delta_\alpha(z_j, z). \tag{2.5}$$

Here we have treated the local coordinate z somewhat differently from the other coordinates z_j parameterizing the local region of the threefold. At a typical point at $z = 0$, the equation

$\Delta = 0$ defines an A_4 singularity. Further enhancement in the singularity type indicates where chiral matter localizes. Each vanishing locus

$$\Sigma_\alpha = \big(\Delta_\alpha(z_j, z) = 0\big) \cap (z = 0) \qquad (2.6)$$

defines a complex curve inside of $\mathcal{S}$, at which the GUT seven-brane intersects with one or more other seven-branes. We will refer to these intersections Σ_α as "matter curves", since they give rise to six-dimensional matter. This matter arises from the ground states of open strings that connect the two stacks of seven-branes, and correspondingly, are charged as "bifundamentals" under the two gauge groups.

The existence of chiral four-dimensional matter can be assured by activating appropriate background fluxes on the worldvolume of the flavor seven-branes. These fluxes specify that the higher-dimensional matter transform as sections of vector bundles defined on the matter curves, and the localized chiral zero modes are then guaranteed to exist and are counted via the appropriate index theorem. The intersections between the matter curves indicate additional enhancements of the singularity type, and give rise to Yukawa couplings of the compactification. We will now briefly describe in a bit more detail how the matter curves and their intersections can be described from the point of view of the worldvolume gauge theory of the GUT seven-brane.

2.1 Seven-brane gauge theory

The worldvolume theory of the GUT seven-brane is given by a partially twisted version of the maximally supersymmetric gauge theory defined on $\mathbb{R}^{3,1} \times \mathcal{S}$. We follow conventions as in [6] (see also [5]). The four-dimensional effective theory is obtained by treating the eight-dimensional fields as a collection of four-dimensional fields parameterized by points of $\mathcal{S}$. In [6], the component form of the Lagrangian was presented in which the eight-dimensional gauge symmetry was manifest. For brevity, here we present a formulation of the higher-dimensional gauge theory which is manifestly $\mathcal{N} = 1$ supersymmetric [31, 32]. Organize the eight-dimensional fields according to four-dimensional $\mathcal{N} = 1$ supermultiplets, the gauge sector of the seven-brane theory then consists of: an adjoint-valued vector multiplet V, which represents an ordinary function on $\mathcal{S}$, a collection of adjoint-valued chiral superfields A, which transform as a $(0, 1)$-form along the internal directions, and a collection of adjoint-valued chiral superfields Φ which combine into a $(2, 0)$-form on $\mathcal{S}$.

The eight-dimensional Lagrangian $\mathcal{L}$ can then be written in four-dimensional superspace notation as

$$\mathcal{L} = \int d^4\theta\, \mathcal{K} + \int d^2\theta\, \mathcal{W} + h.c. + \text{WZW}\,, \qquad (2.7)$$

where the WZW term is a non-local term which vanishes in WZ gauge [31]. The gauge sector Kähler potential decomposes into a contribution from Φ and A:

$$\mathcal{K}_{\text{gauge}} = \mathcal{K}_\Phi + \mathcal{K}_A \qquad (2.8)$$

where the Kähler potential is fixed by the inner product $(\cdot, \cdot)$ induced by the open string Hermitian metric associated with the gauge bundles for Φ and A:

$$\mathcal{K}_\Phi = \big(\Phi^\dagger e^{-V}, \Phi\, e^V\big) \qquad (2.9)$$

$$\mathcal{K}_A = \big((\overline{\partial} + A)^\dagger e^{-V}, (\overline{\partial} + A)e^V\big) - (\overline{\partial}^\dagger e^{-V}, \overline{\partial} e^V). \qquad (2.10)$$

For example, a canonical Kähler potential for Φ is given by the pairing $(\Phi^\dagger, \Phi)_{\text{canonical}} = \Phi^\dagger \Phi$.[7] The F-terms are:

$$\mathcal{W}_{\text{gauge}} = \text{Tr}\big(W_\alpha^2\big) + \text{Tr}\big(\Phi \wedge (\overline{\partial}A + A \wedge A)\big). \qquad (2.11)$$

The fields V, A and Φ all have non-trivial profiles on the internal manifold $\mathcal{S}$, and thus decompose into a full KK tower of massive four dimensional fields. The bulk zero mode content will in general depend on the choice of background gauge field flux. The zero mode spectrum will consist of the constant modes of the vector multiplet, and possibly additional chiral superfields descending from Φ and A. In realistic applications, one typically requires the flux to be chosen so as to exclude these additional zero modes.

2.2 Matter localization

The singularity type of an elliptic fibration enhances along the intersection of seven-branes. Consider two stacks of seven-branes, one wrapping the four-cycle $\mathcal{S}$ and the other one wrapping a transverse four-cycle $\mathcal{S}_\alpha$, with respective gauge groups G and G_α. The physics near the intersection curve $\Sigma_\alpha = \mathcal{S} \cap \mathcal{S}_\alpha$ can be modelled in terms of a breaking pattern of $G_\Sigma \supset G \times G_\alpha$, where G_Σ is the gauge group that specifies the enhanced singularity type along the matter curve Σ_α [33]. In a local patch on the GUT seven-brane $\mathcal{S}$ near the matter curve we can view both stacks of seven-branes as part of one stack, and adopt a G_Σ gauge theory perspective. The breaking pattern can then be viewed as induced via a vev Φ_0 of the Higgs scalar field valued in the adjoint of the enhanced gauge group G_Σ. The Higgs field Φ_0 represents the geometric distance between the two seven-brane stacks on $\mathcal{S}$ and $\mathcal{S}_\alpha$. Hence it is non-zero everywhere except at the location of the matter curve.

The G_Σ gauge theory has the same matter content (V, Φ, A) as the G gauge theory on the GUT seven-brane, except that all fields transform as adjoints of the enlarged gauge group G_Σ. Φ_0 is a vev of the $(2,0)$ field Φ, that takes the form of a vortex configuration centered at the matter curve Σ_α. In the vicinity of the matter curve, localized fluctuation modes Ψ_α of Φ and A_α of A satisfy the equations [6]:

$$\overline{\partial}_A \Psi_\alpha + \big[\Phi_0, A_\alpha\big] = 0 \qquad (2.12)$$

$$k \wedge \partial_A A_\alpha + \big[\Phi_0^\dagger, \Psi_\alpha\big] = 0 \qquad (2.13)$$

where k is the Kähler form in the patch. The commutator with the Higgs field gives a mass to A_α and Ψ_α, and modes localize where this commutator vanishes. These modes organize as six-dimensional fields which represent the open strings that stretch between the two stacks of seven-branes.

The thickness of the vortex is governed by the slope of the Higgs field, or in geometric terms, the angle at which the two seven-brane worldvolumes $\mathcal{S}$ and $\mathcal{S}_\alpha$ intersect each other. When this angle is large, open strings that stretch between the two sets of seven-branes

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[7]To emphasize the connection with the usual Kähler potential, we deviate from the standard inner product notation, and include the explicit Hermitian conjugation.

acquire a large mass as soon as they separate from the matter curve. In the large Higgs field limit, the derivative terms drop out of the equation of motion (2.12) of the fluctuation fields

$$\big[\Phi_0, A_\alpha\big] = 0, \qquad \big[\Phi_0^\dagger, \Psi_\alpha\big] = 0. \tag{2.14}$$

The commutator equation is equivalent to the algebraic data defining the location of the holomorphic matter curves.

Working in a neighborhood of the compact curve Σ_α, it is enough to consider a configuration where the Higgs field Φ_0 takes values in the Cartan subalgebra of G_Σ. Introduce Cartan generators $\mathfrak{h}_i$ and generators $\mathfrak{t}_\alpha$ parameterized by simple roots of G_Σ. For simplicity, let us now assume that the symmetry enhancement at the matter curve $\mathcal{S}_\alpha$ amounts to intersecting with just a single seven-brane, so that $G_\alpha = \mathrm{U}(1)$. The matter curve Σ_α is then associated with some specified simple root α, and the corresponding localized matter fields can be decomposed as $\Psi = \Psi_\alpha \mathfrak{t}_\alpha$ and $A = A_\alpha \mathfrak{t}_\alpha$. Using that $[\mathfrak{h}_i, \mathfrak{t}_\alpha] = \langle \alpha, \mathfrak{h}_i \rangle \mathfrak{t}_\alpha$, the equation of motion (2.14) then assumes the form

$$\langle \alpha, \Phi_0 \rangle A_\alpha = 0, \qquad \langle \alpha, \Phi_0^\dagger \rangle \Psi_\alpha = 0. \tag{2.15}$$

The fluctuation fields must vanish everywhere except at the location of the matter curve

$$\Sigma_\alpha = \big(\langle \alpha, \Phi_0 \rangle = 0 \big). \tag{2.16}$$

The large Higgs field limit thus amounts to a delta function approximation: the matter fields Ψ_α and A_α turn into distribution valued fields on $\mathcal{S}$, transforming in irreducible representations R_α of the unbroken gauge group G, specified by the simple root α of the enlarged gauge group associated with the matter curve Σ_α.[8] It is important to note that here we are describing the representation content of a *six-dimensional* field. Dimensionally reducing, four-dimensional matter fields localized on the curve will transform in the representation R_α, as well as the conjugate representation which we denote by $R_{-\alpha}$. To emphasize this point, we shall sometimes write such fields as A_α and $\Psi_{-\alpha}$.

The above description of the matter localization in terms of a Higgs bundle of a gauge theory with an extended gauge group is in general only valid in a local patch around each matter curve. In principle, one could consider the special situation in which all matter curves admit a unified description as some appropriate Higgs bundle of some maximal rank gauge theory defined globally over $\mathcal{S}$ [11]. The prescribed choice for this maximal rank gauge group would be E_8, and the local geometry would then take the form of a partially unfolded E_8 singularity. However, the assumption that such a unified perspective exists is very restrictive, and the required global conditions are not easily satisfied. Instead, we will view the collection of the matter curves and associated matter representations as extra data, external to the seven-brane gauge theory on $\mathcal{S}$, and specified by the local geometry of the Calabi-Yau fourfold.

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[8]The representation R_α is specified via the decomposition of the adjoint representation of G_Σ into irreducible representations of $G \times G_\alpha \subset G_\Sigma$: $\mathrm{ad}(G_\Sigma) = \mathrm{ad}(G) \oplus \mathrm{ad}(G_\alpha) \oplus \oplus_\tau (R_\tau, R'_\tau)$, so that $\tau = \alpha$ corresponds to one or more of the summands. When $G_\alpha = \mathrm{U}(1)$, R_α is an irreducible representation of G.

Along the matter curve Σ_α, the equation of motion (2.12) for the localized fields reduces to

$$\overline{\partial}_{A+A'}\Psi_\alpha = 0, \qquad k \wedge \partial_{A+A'}A_\alpha = 0 \,. \qquad (2.17)$$

Here $\overline{\partial}_{A+A'}$ denotes the Dolbault operator associated with the difference of the background gauge fields defined on the two seven-brane stacks, restricted to Σ_α. The independent solutions to these equations are the charged massless chiral matter fields of the four-dimensional gauge theory. The number of zero modes depends on the choice of gauge bundle. This bundle data is dictated by the breaking patterns of the fibration singularities of the compactification, and also by the choice of background fluxes. The number of chiral zero modes on a matter curve can be adjusted by turning on an appropriate background U(1) flux on the transverse seven-brane worldvolume $\mathcal{S}_\alpha$, as well as by activating fluxes of the GUT brane.

The intersection of matter curves gives rise to Yukawa couplings in the four-dimensional theory. At the intersection, the singularity type of the ADE fibration enhances further to G_p. This situation can locally be modelled in terms of Higgsing of a G_p gauge theory. Such intersections can either involve just two matter curves meeting or three intersecting matter curves. Though the latter possibility may look fine-tuned, both possibilities are generic from the perspective of the G_p gauge theory, and in the geometry this naturally occurs once the data of the fibration is included.

Putting this all together, the matter sector is described by an $\mathcal{N} = 1$ supersymmetric Lagrangian defined on $\mathbb{R}^{3,1} \times \mathcal{S}$, specified by a Kähler and superpotential of the form [6]

$$\mathcal{K}_{\mathrm{matter}} = \left(\Psi_{-\alpha}^{\dagger}, e^{-V-V'} \Psi_{-\alpha} \right) + \left(A_\alpha^{\dagger}, e^{V+V'} A_\alpha \right) \qquad (2.18)$$

$$\mathcal{W}_{\mathrm{matter}} = \Psi_{-\alpha} \wedge \overline{\partial}_{A+A'} A_\alpha + f_{\alpha\beta\gamma} \, \Psi_\alpha \wedge A_\beta \wedge A_\gamma \,. \qquad (2.19)$$

In the above we have included possible couplings to the gauge fields of the flavor seven-branes, with their associated vector multiplet V' and internal gauge field A'. In our conventions, matter fields A_α and $\Psi_{-\alpha}$ on a curve Σ_α respectively transform in the representation R_α, and its conjugate $R_{-\alpha}$. Here $f_{\alpha\beta\gamma}$ is only non-zero if the corresponding three matter curves intersect, and if the tensor product of the three representations $R_\alpha \otimes R_\beta \otimes R_\gamma$ contains the trivial representation; in this case, $f_{\alpha\beta\gamma}$ denotes the appropriate Clebsch-Gordon coefficient. The matter fields are all viewed as distributions on $\mathcal{S}$ that are localized on the associated matter curves. Note that the Yukawa couplings involve the overlap of three such distributions. This triple overlap needs to be regularized, by locally resolving the matter curves as Higgs vortices, as described above. In section 4 we will see that in the non-commutative theory, this triple overlap is automatically well-defined, and the Yukawa couplings are immediately finite.

2.3 U(1) fluxes

In perturbative IIB string theory, it is convenient to combine the B-field and overall field strength in the direction of the trace U(1) of a U(n) gauge theory into the quantity $\mathcal{B}$. At first this may seem less straightforward in F-theory, because the gauge group on a seven-brane is really SU(n), and not U(n). Let us first note that the compactification

geometry supports non-trivial three-form fluxes, and so generically non-trivial $\mathcal{B}$-periods will be present. As explained in the Introduction, in the decoupling limit, the $\mathcal{B}$-periods take quantized values.

In gauge theory terms there is also a notion of an overall U(1) which is shared between intersecting seven-branes. For example, SU(6) contains a 5×5 U(5) block and a 1×1 U(1) block which are subject to the condition that $\det_{\mathrm{U}(5)} \times \det_{\mathrm{U}(1)} = 1$, which is often denoted as the subgroup $S(\mathrm{U}(5) \times \mathrm{U}(1)) \subset \mathrm{SU}(6)$. Thus, switching on a flux on the U(5) factor is compensated by a flux on the U(1) factor. In F-theory, intersecting seven-brane configurations can be described by such breaking patterns, so we see that there is still a notion of $\mathcal{B}$ on a seven-brane.

Activating the flux $\mathcal{B}$ induces a net D3-brane and five-brane charge on the seven-brane worldvolume. In an F-theory compactification, the D3-brane tadpole can be cancelled by the Euler character of the Calabi-Yau fourfold [34]. Returning to the gauge theory description of $\mathcal{B}$ fields on seven-branes, we see that the five-brane tadpole induced by one stack of seven-branes can be cancelled by a compensating contribution to the net five-brane charge induced by other intersecting seven-branes.

A convenient and by now standard way to break the SU(5) gauge group down to the Standard Model gauge group is to activate a background flux on the GUT brane, in the direction of the hypercharge U(1) generator [8, 9]

$$F_Y = \sum_{a=1}^{3} F_a \, \mathbb{1}_a, \tag{2.20}$$

where $a = 1, 2, 3$ labels the three SM gauge group factors. The gauge and matter fields then organize into Standard Model representations. The hypercharge flux also affects the line bundle data on the matter curves. This opens up the possibility of distinguishing the Higgs fields of the MSSM from the other chiral matter content. The Higgs fields localize on matter curves where the U(1) hypercharge flux is non-vanishing, and the other chiral matter of the MSSM localizes on curves where the net hyperflux vanishes. This gives a natural geometric solution to the doublet-triplet splitting problem [8].

The GUT symmetry breaking via hypercharge flux will in general perturb exact coupling constant unification. The geometric volume one would associate with each gauge coupling constant will in general be different, because the hyperflux contributes to the effective open string metric seen by each gauge group factor.

In the decoupling limit where the closed string volume is taken to zero, the geometric formula for the three Standard Model gauge couplings is then:

$$\frac{4\pi}{g_{\mathrm{a}}^2} = \frac{1}{(2\pi)^2 g_s} \int_{\mathcal{S}} \mathcal{B}_a \wedge \mathcal{B}_a \tag{2.21}$$

with $\mathcal{B}_a = F_a + \mathcal{B}$ where $\mathcal{B}$ denotes the ambient two-form flux turned on to support the open string volume of the cycle. To preserve coupling constant unification at this geometric level, one would need to choose the hyperflux such that the right-hand side of (2.21) takes approximately the same value for all three gauge groups. In models with a realistic spectrum, this condition is already met.

3 Fuzzy toric geometry

Our aim in the following sections will be to develop the four-dimensional field theory of a seven-brane wrapping a non-commutative four-cycle. To this end, we now develop some tools for dealing with such geometries.

The standard prescription for defining a non-commutative space is to start with the algebra of functions of a commutative space, and to then deform this structure to a more general algebra. Though this provides a general way to work with non-commutative geometries, it can also be somewhat unwieldy. In this section we present a practical quantization prescription for working with non-commutative or "fuzzy" geometries which exploits the additional structure present in toric geometries. In subsequent sections we shall use this prescription to study the low energy theory of seven-branes wrapping a fuzzy four-cycle.

In the commutative setting, one convenient way to realize a wide variety of possible compact spaces is via toric constructions. The main idea is that the compact space of interest, such as the ones that are relevant to F-theory, can be viewed as the intersection of hypersurfaces inside of a larger toric space. Our strategy in this section will be to first develop a suitable non-commutative analogue of toric spaces. This prescription can be applied in situations where all toric coordinates are non-commutative, and also in situations where some subset are non-commutative, while the others remain commutative. Subspaces of these fuzzy toric geometries will then provide us with a prescription for realizing non-commutative Kähler surfaces. The construction we present bears a formal resemblance to that given in [35], though as far as we are aware, the geometric interpretation presented here has not appeared in the literature before.[9] We shall therefore proceed somewhat systematically.

Let us first review classical toric geometry. In physics terms, toric geometries are characterized by the gauged linear sigma model (GLSM). Recall that in a GLSM, there are r chiral superfields $z_1, \ldots, z_r$, and a gauge group $\mathrm{U}(1)^s = \mathrm{U}(1)_{(1)} \times \ldots \times \mathrm{U}(1)_{(s)}$ under which these chiral superfields are charged. Hence each coordinate z_i is labeled by an s-component charge vector $\vec{q}_i$. In addition, the z_i obey s D-term constraints:

$$\sum_{i=1}^{r} \vec{q}_i \, |z_i|^2 = \vec{\zeta}. \tag{3.1}$$

The classical vacua of the $1 + 1$ dimensional theory then correspond to gauge equivalent orbits of the z_i subject to the D-term constraints of line (3.1). These vacua then define the classical geometry of the symplectic quotient

$$X(\vec{\zeta}) = \mathbb{C}^r /\!/ \mathrm{U}(1)^s \tag{3.2}$$

which is an $r - s$ complex dimensional manifold. More general spaces are obtained by considering hypersurfaces in X, specified as the vanishing locus of one or more weighted homogeneous polynomials $P_{\vec{w}}(z_1, \ldots, z_r)$. The weights $\vec{w}$ are constrained by the condition that $P_{\vec{w}}$ has a well-defined GLSM charge.

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[9]For another approach to defining non-commutative toric geometries, see [36].

This toric characterization can be straightforwardly quantized and used to construct a general class of non-commutative geometries. In principle one would need to show that this particular quantization prescription is the correct one for describing the zero slope limit of an open string theory on a wrapped seven-brane in a $\mathcal{B}$-field. We will not try to give such a derivation here, but do note that the GLSM formulation given here appears to be a particularly suitable starting point for making such an attempt.

3.1 Quantization procedure

We now provide a quantization prescription for this toric setup. In general terms, the data necessary to quantize our system consists of a symplectic form ω defined on the ambient space $\mathbb{C}^r$, which we use to define commutation relations. The D-term constraint of the classical toric geometry is then promoted to the Hamiltonian constraint of the quantized system. Imposing the Hamiltonian constraint on the state space associated with the quantized ambient space produces the non-commutative toric space X.

Let us now describe each step in more detail. We begin by quantizing $\mathbb{C}^r$, by viewing it as a symplectic manifold with a closed symplectic form ω. To keep our treatment as simple as possible, we choose ω to be a (1,1)-form. In the F-theory context, this means that we assume that the $\mathcal{B}$-flux is a $(1,1)$-form on the seven-brane worldvolume.[10] Assuming ω does not vanish, Darboux's theorem ensures that there exists a general coordinate redefinition on $\mathbb{C}^r$ such that ω takes the form

$$\omega = \sum_{i=1}^{r} i\,dz_i \wedge d\overline{z}_i \tag{3.3}$$

This defines a Poisson bracket for the z's and $\overline{z}$'s: $\{z_i, \overline{z}_j\}_{\mathrm{PB}} = \delta_{ij}$. This means that the metric on $\mathbb{C}^r$ takes the general form

$$ds^2 = \sum_{i,j=1}^{r} G_{ij}\,dz_i\,d\overline{z}_j \tag{3.4}$$

with G_{ij} some general matrix. In this section, we will mostly focus on the symplectic geometry. In particular, we shall define the toric $U(1)^s$ action on the z_i's and the GSLM D-term constraint with respect to this symplectic coordinate basis.

We now take the quantization step and replace each coordinate z_i and $\overline{z}_j$ by operators Z_i and $Z_j^\dagger$ obeying the commutation relations:

$$\left[Z_i, Z_j^\dagger\right] = \hbar_{\mathrm{NC}}\,\delta_{ij}\,, \tag{3.5}$$

$$\left[Z_i, Z_j\right] = \left[Z_i^\dagger, Z_j^\dagger\right] = 0\,, \tag{3.6}$$

for $i,j = 1,\ldots,r$. The parameter $\hbar_{\mathrm{NC}}$ is Planck's constant, which here has the dimension of an area: it sets the size of the Planck cell on the fuzzy space. For convenience, we shall mostly work in units where $\hbar_{\mathrm{NC}} = 1$.

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[10]For a discussion of local F-theory models with non-commutativity induced by a holomorphic bi-vector, see [37, 38].

To build up the quantized geometry, we introduce a vacuum state $|0\rangle$ annihilated by the Z_i. Acting with creation operators on the vacuum realizes a Fock space

$$\mathcal{F}(\mathbb{C}^r) = \text{span} \left\{ \prod_{i=1}^r \frac{(Z_i^\dagger)^{n_i}}{\sqrt{n_i!}} |0\rangle \right\}. \tag{3.7}$$

The basis elements of $\mathcal{F}(\mathbb{C}^r)$ can be thought of as the quantized points of $\mathbb{C}^r$, and the Fock space is obtained by treating these points as basis elements of a linear vector space. For brevity, we shall often refer to these vector spaces as defining the non-commutative space itself, since the distinction will be largely unimportant in what follows.

We now want to take the symplectic quotient, and define the non-commutative version of the toric space $X(\vec{\zeta})$. As we will see, the FI-parameters $\vec{\zeta}$ that determine the overall size and shape of $X(\vec{\zeta})$ will be quantized (in the sense that they can only take integer values) via the requirement that the various compact two-cycles inside X support an integer number of Planck cells. Treating the Z_i as the quantized analogues of the GLSM fields, there is an associated set of $\text{U}(1)^s$ charges for each Z_i. Concretely, the vector of D-term operators

$$\vec{D} = \sum_{i=1}^r \vec{q}_i Z_i^\dagger Z_i, \tag{3.8}$$

now assumes the natural role of the vector of conserved charges that generate the $\text{U}(1)^s$ rotations. These conserved charges will act like Hamiltonians on the large phase space $\mathbb{C}^r$. The toric space is obtained by treating these Hamiltonians as operator constraints.

The Fock space $\mathcal{F}(\mathbb{C}^r)$ admits a grading in terms of the spectrum of the $\vec{D}$ operator

$$\mathcal{F}(\mathbb{C}^r) = \bigoplus_{\vec{\zeta}} \mathcal{F}_X(\vec{\zeta}) \tag{3.9}$$

so that for all states $|\Psi\rangle$ in the subspace $\mathcal{F}_X(\vec{\zeta})$, the vector $\vec{D}$ takes the fixed value $\vec{\zeta}$:

$$\vec{D}|\Psi\rangle = \vec{\zeta}|\Psi\rangle. \tag{3.10}$$

This defines the quantized analogue of the D-term equation (3.1) in the commuting case. Geometrically, the $\vec{\zeta}$'s define quantized versions of the FI parameters of the GLSM, and thus specify the Kähler data of the non-commutative toric space $X(\vec{\zeta})$. Roughly speaking, we view $\mathcal{F}_X(\vec{\zeta})$ as the vector space of fuzzy points on $X(\vec{\zeta})$.[11] We recover the commutative geometry in the limit in which we simultaneously rescale all of the components of the vector $\vec{\zeta}$ to infinity, while also requiring that the quantization parameter $\hbar_{\text{NC}}$ tends to zero.

The sign and magnitude of the FI parameters determine the compact and non-compact directions of the geometry. Indeed, the values of the FI parameters determine the Kähler volumes for the toric space. Viewing the norms $|z_i|^2$ as coordinates of $\mathbb{R}^r_{\geq 0}$, each D-term constraint defines a hyperplane in this space, and the toric space lies at the mutual

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[11]Later when we discuss differential forms we will extend this definition to also include "fermionic points". Since the context will hopefully be clear, we shall use similar notation to refer to both the bosonic space, and its fermionic extension.
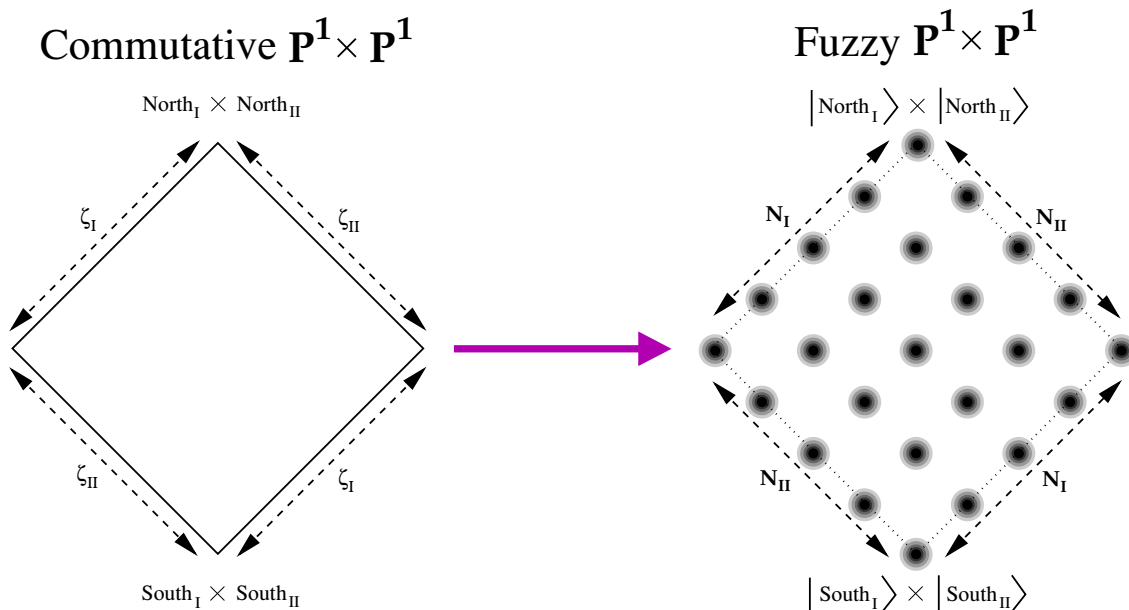
Figure 2. Toric diagram for commutative $\mathbb{P}^1 \times \mathbb{P}^1$ and its fuzzy analogue. The commutative geometry is described by the classical vacua of a $U(1) \times U(1)$ GLSM with four chiral superfields u_1, u_2, v_1, v_2 with GLSM charges $(+1, 0)$ for the u_i's and $(0, +1)$ for the v_i's. Viewing the norms $|u_i|^2$ and $|v_i|^2$ as coordinates of $\mathbb{R}^4_{\geq 0}$, the D-term constraints $|u_1|^2 + |u_2|^2 = \zeta_I$ and $|v_1|^2 + |v_2|^2 = \zeta_{II}$ define two hypersurfaces which intersect over a two-dimensional compact subspace. Here we have projected this space onto a two-dimensional plane, where it corresponds to a rectangle with sides of lengths ζ_I and ζ_{II}. Each side corresponds to a $\mathbb{P}^1$ factor of the geometry. In the non-commutative theory, the D-term becomes a quantized Hamiltonian constraint, and the state space of points is finite-dimensional. The North and South poles of the commutative $\mathbb{P}^1$'s then become highest and lowest $su(2)$ angular momentum states of the fuzzy theory. See appendix A for further discussion of fuzzy $\mathbb{P}^1$.

intersection of all the hyperplanes. In the non-commutative setting, the intersection of these hyperplanes is replaced by discretized data. In particular, this means that finite area intersections of hyperplanes translate to subspaces with an upper bound on the oscillator number. Non-compact intersections of hyperplanes correspond to subspaces with no such bound. Changing the values of the FI parameters can cause the geometry to undergo a flop transition. Such transitions correspond to changing the oscillator content of the Fock space. See section 5 for further discussion of a fuzzy flop transition.

For compact toric manifolds, the subspaces $\mathcal{F}_X(\vec{\zeta})$ are finite dimensional. The basis elements of $\mathcal{F}_X(\vec{\zeta})$ correspond to the points of the fuzzy toric space $X(\vec{\zeta})$, and hence the total number of points equals the dimension of this vector space

$$\# \text{ points in } X(\vec{\zeta}) \equiv \dim \mathcal{F}_X(\vec{\zeta}). \tag{3.11}$$

From a semi-classical viewpoint, this equals the number of Planck cells in X.

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3.2 Functions and line bundles

Having specified the formal spaces of interest, we now introduce functions and line bundles over a non-commutative toric variety $X(\vec{\zeta})$.

Fuzzy functions. In commutative geometry, a complex function on a space X is defined as a map from X to $\mathbb{C}$. In the non-commutative setting, one can similarly consider linear maps from the state space $\mathcal{F}_X(\vec{\zeta})$ to $\mathbb{C}$. The collection of all such maps is a linear space with the same dimension as the state space $\mathcal{F}_X(\vec{\zeta})$ itself. There is, however, a more substantive notion of functions on a fuzzy space, namely as the space of functions of the non-commutative coordinates on X.

Since the fuzzy coordinates do not commute among each other, we cannot view them as having some precise classical value, but rather as operators that act non-trivially on the other coordinates. Taking this into account, it is natural to define the space of non-commutative functions as the space of linear maps from the state space $\mathcal{F}_X(\vec{\zeta})$ to itself. Indeed, every linear operator from the state space $\mathcal{F}_X(\vec{\zeta})$ to itself can be presented in terms of polynomials built from the creation and annihilation operators $Z^\dagger$ and Z. More precisely, they are given by polynomials $F(Z^\dagger, Z)$, which have zero GLSM charge:

$$\Omega^0(X) \equiv \left\{ \, F(Z^\dagger, Z) \, : \, \left[\vec{D}, \, F \right] = 0 \, \right\}. \tag{3.12}$$

A matrix representative for an element of $\mathcal{F}_X(\vec{\zeta}) \otimes \mathcal{F}_X^*(\vec{\zeta})$ is then given by evaluating the operator in a specified basis of states.

On the function space we can define a natural non-commutative star product, given by operator multiplication

$$(F * G)(Z^\dagger, Z) = F(Z^\dagger, Z) \cdot G(Z^\dagger, Z) \tag{3.13}$$

Since functions are defined as normal ordered expressions, the right-hand side of (3.13) must be decomposed into a sum of normal ordered terms via Wick contractions. In the matrix representation of functions, the star product is simply given by matrix multiplication. In the following we will not explicitly write the star product $*$, as it will be automatically understood that functions are normal ordered expressions in the oscillators.

An important feature of our present discussion is that when all of the GLSM charges are positive, it is not possible to write down any functions which are purely holomorphic in the Z's. This reflects the fact that on a space of positive curvature, a holomorphic function will have a pole somewhere on the surface. Holomorphic sections of non-trivial line bundles correspond to holomorphic polynomials in the Z's.

Fuzzy line bundles. We now present a prescription for defining line bundles on a fuzzy toric space. Roughly speaking, non-trivial line bundles can be thought of as non-square matrices.[12] In terms of GLSM data, we can characterize the data of a line bundle in terms of operators built from the z_i's. In particular, we can organize all operators according to

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[12]Line bundles on fuzzy $\mathbb{P}^1$ and the generalization to fuzzy $\mathbb{P}^n$ have been discussed in [39] and [40].

their GLSM charges. The space of operators with the same GLSM charge $\vec{Q}$ then define sections of a line bundle of degree $\vec{Q}$, which we call $L^{(\vec{Q})}$.

In the non-commutative setting these sections are represented by polynomials in the Z_i and $Z_i^\dagger$ oscillators with GLSM charge $\vec{Q}$. In other words, the space of sections valued in the line bundle $L^{(\vec{Q})}$ is defined as the space of polynomials $G(Z^\dagger, Z)$ that satisfy[13]

$$L^{(\vec{Q})} = \left\{ G(Z^\dagger, Z) \; : \; \left[\vec{D}, G \right] = -\vec{Q}\, G \right\}. \tag{3.14}$$

Expressions given purely in terms of the Z's with no $Z^\dagger$'s correspond to holomorphic sections of non-trivial line bundles.

Fixing the Kähler class of the fuzzy space, we can present sections of the line bundle $L^{(\vec{Q})}$ in terms of a collection of rectangular matrices valued in $\mathcal{F}_X(\vec{\zeta} - \vec{Q}) \otimes \mathcal{F}_X^*(\vec{\zeta})$.

3.3 Fuzzy p-forms

We now construct differential forms on a non-commutative toric space X. Our strategy will be to first construct the requisite geometry on $\mathbb{C}^r$, and to then perform the symplectic quotient to construct differential forms on X.

Let us begin with differential forms on $\mathbb{C}^r$. To this end, introduce a set of fermionic oscillators obeying the anti-commutation relations:

$$\{C_i, C_j^\dagger\} = \hbar_{\mathrm{NC}}\, \delta_{ij}, \tag{3.15}$$

$$\{C_i, C_j\} = \{C_i^\dagger, C_j^\dagger\} = 0, \tag{3.16}$$

for $i, j = 1, \ldots, r$. In what follows we continue to set $\hbar_{\mathrm{NC}} = 1$. The C_i's are the quantized versions of the differentials dz_i, and the $C_i^\dagger$'s are the quantized analogues of the $d\bar{z}_i$'s. Note that with this prescription the non-commutative analogue of the identity $dz \wedge d\bar{z} = -d\bar{z} \wedge dz$ is violated due to Wick contraction between C and $C^\dagger$. This additional non-commutative structure provides a covariant extension of functions to differentials. We can view the oscillators $(Z_i, Z_j^\dagger, C_i, C_j^\dagger)$ as coordinates on the quantized cotangent bundle $T^*\mathbb{C}^r$.

Introducing a vacuum state $|0\rangle$ annihilated by the Z_i and the C_i, the Fock space of states spanned by acting with $Z_i^\dagger$ and $C_i^\dagger$ then provides a natural extension of the fuzzy points of $\mathbb{C}^r$:

$$\mathcal{F}(T^*_{\mathrm{hol}}\mathbb{C}^r) = \mathrm{span}\left\{ \prod_{i=1}^r \frac{(Z_i^\dagger)^{n_i}}{\sqrt{n_i!}} (C_i^\dagger)^{\sigma_i} |0\rangle \right\}. \tag{3.17}$$

Geometrically, the subspace of states with fermion number zero corresponds to the space of fuzzy points on $\mathbb{C}^r$, and the subspace of states with fermion number zero or one corresponds to the fuzzy points of the holomorphic cotangent bundle $T^*_{\mathrm{hol}}\mathbb{C}^r$. Here, we are viewing the cotangent bundle as a phase space in which the bosonic points correspond to the "position coordinates" and the fermion number one points are the "momentum coordinates". Similarly, the n-fermion subspace is the nth exterior power $\Lambda^n T^*_{\mathrm{hol}}\mathbb{C}^r$. The

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[13]Our sign conventions are chosen to maximally adhere to the sign conventions in the commutative theory. For example, a positive degree line bundle on $\mathbb{P}^1$ has holomorphic sections.

Fock space corresponds to exterior powers of the holomorphic cotangent bundle. We view the Hermitian conjugate space of states as the fuzzy points of the anti-holomorphic cotangent bundle of $\mathbb{C}^r$ and its exterior powers:

$$\mathcal{F}(T^*_{\text{hol}}\mathbb{C}^r) = \mathcal{F}(T^*_{\text{hol}}\mathbb{C}^r)^*. \tag{3.18}$$

Differential forms of $\mathbb{C}^r$ correspond to maps which send one fuzzy point of $\mathcal{F}(T^*_{\text{hol}}\mathbb{C}^r)$ to another. The most general operator of this type can be expanded as a power series in the oscillators which is normal ordered in the Z's, and anti-normal ordered in the C's. An operator with p C oscillators and q $C^\dagger$ oscillators is then given by the (p,q)-form:

$$\omega_{(p,q)} = \sum_{I,J} f_{I,J}(Z^\dagger, Z)\, C_{i_1} \wedge \ldots \wedge C_{i_p} \wedge C^\dagger_{j_1} \wedge \ldots \wedge C^\dagger_{j_q} \tag{3.19}$$

where $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ are multi-indices running over i and j holomorphic and anti-holomorphic indices respectively.

The space of all (p,q)-forms admits a $\mathbb{Z}_2$ grading according to whether there are an even or odd number of fermionic oscillators. The wedge product of differential forms is given by operator multiplication

$$\left(\omega_1 \wedge_* \omega_2\right)(Z^\dagger, Z, C, C^\dagger) = \omega_1(Z^\dagger, Z, C, C^\dagger) \cdot \omega_2(Z^\dagger, Z, C, C^\dagger) \tag{3.20}$$

This non-commutative wedge product preserves the $\mathbb{Z}_2$ grading. Note, however, that the $\wedge_*$-product involves non-trivial Wick contractions between the fermionic oscillators. This means that, as opposed to the commutative theory, the wedge product of a (p,q)- and (p',q')-form will generally be of mixed (p,q)-type. This feature should be viewed as additional non-commutative structure along the cotangent space direction of the fuzzy space.

3.3.1 Dolbeault operators

On the space of differential forms on $\mathbb{C}^r$, we can define the operators $\overline{\partial}$ and ∂ which act on a form ω as:

$$\overline{\partial}\omega = \sum_{i=1}^{r} \left[Z_i, \omega\right] C^\dagger_i, \qquad \partial\omega = \sum_{i=1}^{r} C_i \left[Z^\dagger_i, \omega\right]. \tag{3.21}$$

By virtue of the non-trivial commutator between Z and $Z^\dagger$, this acts as a differential operator. As opposed to the commutative theory, ∂ and $\overline{\partial}$ do not commute. Nevertheless, we can still define Dolbeault cohomology. Just as in the commutative theory, the operators $\overline{\partial}$ and ∂ map (p,q) forms to $(p, q+1)$ and $(p+1, q)$-forms, and are both nilpotent:

$$\overline{\partial}^2 = 0 \qquad \text{and} \qquad \partial^2 = 0. \tag{3.22}$$

Note that when $\overline{\partial}$ acts on $(0,q)$-forms, the definition given in equation (3.21) can also be written as a super-commutator with the operator $\overline{\partial} = \sum_i Z_i C^\dagger_i$, which manifestly satisfies $\overline{\partial}^2 = 0$. Thus, we can also speak of a Dolbeault complex in this case as well. This latter version of $\overline{\partial}$ can then be interpreted as a BRST operator. As a matter of notation, we will sometimes write

$$\overline{\partial}\omega_{(0,q)} = \left[\overline{\partial}, \omega_{(0,q)}\right\}, \qquad \partial\omega_{(p,0)} = \left[\partial, \omega_{(p,0)}\right\}. \tag{3.23}$$

3.3.2 p-forms on toric spaces

We now turn to differential forms on the symplectic quotient $X = \mathbb{C}^r /\!/ U(1)^s$. To frame our discussion, first recall how differentials are constructed in the symplectic quotient of the commutative geometry. There, the restriction to the cotangent bundle T^*X requires us to impose the projection onto the subspace of the holomorphic cotangent bundle orthogonal to the Hamiltonian vector fields associated with the $U(1)^s$ action, and to the gradients $d\vec{D}$ of the D-term constraint. Concretely, this means that we need to project the forms onto those components that are orthogonal to the holomorphic and anti-holomorphic gradients $\partial\vec{D}$ and $\overline{\partial}\vec{D}$ of the D-terms.

We now extend our Hamiltonian constraints to the extended phase space. In the GLSM, we can consider operators made from bosons, but also operators involving fermions. These bosons and fermions combine into supermultiplets of the two-dimensional gauge theory. Thus, we see that the fermions also possess a GLSM charge, and our Hamiltonian constraint must be modified to reflect this fact. The natural extension of the original Hamiltonian constraint is then given by extending the definition of the D-term constraint operators $\vec{D}$ to act on the fermionic coordinates:

$$\vec{D} = \sum_{i=1}^{r} \left(\vec{q}_i \, Z_i^\dagger Z_i + \vec{q}_i C_i^\dagger C_i \right) \tag{3.24}$$

This operator $\vec{D}$ acts on functions and forms via the commutator $\mathrm{ad}(\vec{D})\,\omega \equiv [\vec{D}, \omega]$. The constraint on the rest of the "phase space" for the holomorphic cotangent bundle of X is enforced by the operator:

$$\partial\vec{D} = \sum_i \vec{q}_i Z_i^\dagger C_i. \tag{3.25}$$

The ket states $|\Psi\rangle$ that define the holomorphic cotangent bundle and its exterior powers $\Lambda^\bullet T_{\mathrm{hol}}^* X$ are then given by the subspace of $\mathcal{F}(T_{\mathrm{hol}}^* \mathbb{C}^r)$ which satisfy the D-term constraint and are annihilated by $\partial\vec{D}$:

$$\vec{D}\,|\Psi\rangle = \vec{\zeta}\,|\Psi\rangle\,, \qquad \text{and} \qquad \partial\vec{D}\,|\Psi\rangle = 0. \tag{3.26}$$

Importantly, all of the original bosonic fuzzy points are retained by this extension. Indeed, what has been added is the additional structure of the holomorphic cotangent bundle. Further note that precisely because this is a holomorphic object, it is appropriate to only enforce a vanishing condition from $\partial\vec{D}$, and *not* its Hermitian conjugate:

$$\overline{\partial}\vec{D} = \sum_i \vec{q}_i Z_i C_i^\dagger. \tag{3.27}$$

To avoid cluttering notation, we shall refer to the entire state space defined in this way as $\mathcal{F}_X(\vec{\zeta})$.

The bra states $\langle\Psi|$ which are elements of the anti-holomorphic cotangent bundle and its exterior powers $\Lambda^\bullet T_{\overline{\mathrm{hol}}}^* X$ similarly satisfy

$$\langle\Psi|\vec{D} = \langle\Psi|\vec{\zeta}, \qquad \text{and} \qquad \langle\Psi|\overline{\partial}\vec{D} = 0\,. \tag{3.28}$$

Differential forms on X are given by operators which send a state of $\mathcal{F}_X(\vec{\zeta})$ to another state of $\mathcal{F}_X(\vec{\zeta})$. Compatibility with the Hamiltonian constraints (3.26) imposed by $\vec{D}$ and $\partial\vec{D}$ require that a (p,q)-form $\omega_{(p,q)}$ on X satisfies:

$$\left[\vec{D}, \omega_{(p,q)}\right] = 0 \quad \text{and} \quad \left[\partial\vec{D}, \omega_{(p,q)}\right] = 0. \tag{3.29}$$

In other words, $\omega_{(p,q)}$ must have zero GLSM charge and preserve the structure of $T^*_{\text{hol}}X$.

Note that in (3.29) we do not impose that $\omega_{(p,q)}$ commutes with $\bar{\partial}\vec{D}$. So when acting to the left, the form does not need to preserve the hermitian conjugate GLSM constraints (3.28). Our definition of differential forms on X thus looks asymmetric under complex conjugation on X. However, the following observation restores the symmetry. Physical quantities are given by the matrix element of (products of) differential forms between bra and ket states, that satisfy (3.26) and (3.28). Consider such a matrix element

$$\langle\mu|\,\omega_{(p,q)}|\nu\rangle. \tag{3.30}$$

The bra states span the dual space to $\mathcal{F}_X(\vec{\zeta})$, in the sense that *any* bra state in the big Fock space $\mathcal{F}(T^*_{\text{hol}}\mathbb{C}^r)$ can be decomposed into an element of $\mathcal{F}^*_X(\vec{\zeta})$ — that is, a state that satisfies the conditions (3.28) — plus a state in the orthocomplement of $\mathcal{F}^*_X(\vec{\zeta})$. So in equation (3.30), the inner product with the ket state $|\nu\rangle$ automatically projects the state $\langle\mu|\,\omega_{(p,q)}$ along the component in $\mathcal{F}^*_X(\vec{\zeta})$, that satisfies (3.28). By explicitly implementing this projection, we can map a fuzzy differential form $\omega_{(p,q)}$ on X in the 'holomorphic representation' (3.29) into an equivalent differential form $\overline{\omega}_{(p,q)}$ on X in the 'anti-holomorphic representation', satisfying

$$\left[\vec{D}, \overline{\omega}_{(p,q)}\right] = 0 \quad \text{and} \quad \left[\bar{\partial}\vec{D}, \overline{\omega}_{(p,q)}\right] = 0. \tag{3.31}$$

Next consider the Dolbeault operator on X. The adjoint action of the D-term constraint $\vec{D}$ and the Dolbeault operators on functions and forms are compatible, in the sense that

$$\left[\text{ad}(\vec{D}), \bar{\partial}\right] = \left[\text{ad}(\vec{D}), \partial\right] = 0 \tag{3.32}$$

This property guarantees that the conditions of line (3.29) are preserved by the action of the Dolbeault operators. In other words, ∂ and $\bar{\partial}$ map differential forms on X to differential forms on X, and thus properly project to Dolbeault operators on X. Just as for the commutative theory, the explicit form of the Dolbeault operator on X will in general not be as simple as its presentation on $\mathbb{C}^r$.

This provides us with a prescription for constructing differential forms on X. In general, given a differential form of the commutative geometry, there is a corresponding differential form on X, given as a polynomial expression in the Z and C oscillators. Since Dolbeault cohomology is also retained in the fuzzy theory, this means that in practice, there is a direct translation of all of the necessary algebraic data of the commutative geometry.

At a practical level, all of the differential forms on X can be viewed as descending from (p,q)-forms on $\mathbb{C}^r$. Viewing X as embedded in $\mathbb{C}^r$, there is a corresponding projection

operator given by summing over an orthonormal basis of states for $\mathcal{F}_X(\vec{\zeta})$:

$$\pi_X(\vec{\zeta}) = \sum_{|\mu\rangle \in \mathcal{F}_X(\vec{\zeta})} |\mu\rangle \langle \mu| . \tag{3.33}$$

Given a (p,q)-form $\omega_{(p,q)}$ on $\mathbb{C}^r$, the restriction to $X(\vec{\zeta})$ is given by sandwiching between projection operators:

$$\omega_{(p,q)}|_{X(\vec{\zeta})} = \pi_X(\vec{\zeta}) \cdot \omega_{(p,q)} \cdot \pi_X(\vec{\zeta}). \tag{3.34}$$

For actual computations, the explicit basis of differential forms for a toric space will depend on details of the geometry. Indeed, there is a certain amount of redundancy built into our description because there are more Z and $Z^\dagger$ coordinates than the dimension of the fuzzy toric space. For explicit computations, it is often convenient to eliminate this redundancy. To do this, we choose a collection of holomorphic and anti-holomorphic monomials in the Z's and $Z^\dagger$'s which we refer to as E_I and $\overline{E}_J$. The E_k's should correspond to monomials in the Z's such that for each monomial, it is not possible for all of the Z's to simultaneously vanish (while satisfying the D-term constraint). The collection of such E's determines the Stanley-Reisner ideal of the toric variety X [41].[14] We then introduce a collection of differentials: $dE_I = [\partial, E_I]$ and $d\overline{E}_J = [\overline{\partial}, \overline{E}_J]$. A priori there could be additional non-vanishing differentials on X, which we can add to the collection of dE's and $d\overline{E}$'s. This is really an issue in the commutative geometry, so we shall not dwell on this subtlety here. A general differential form on X can then be expanded in terms of this basis as

$$\omega_{(p,q)} = \sum_{I,J} : f_{I,J}(Z^\dagger, Z) \otimes \wedge_I dE_I \wedge_J d\overline{E}_J : . \tag{3.35}$$

In the following, we will also consider (p,q)-forms on X that take values in a non-trivial line bundle $L^{(\vec{Q})}$. These are characterized as forms with non-zero $U(1)^s$ GLSM charge:

$$[\vec{D}, \omega_{(p,q)}] = -\vec{Q}\omega_{(p,q)}. \tag{3.36}$$

Note that varying with respect to the Z's, the phase space constraint on differential forms is the same as before

$$[\partial \vec{D}, \omega_{(p,q)}] = 0. \tag{3.37}$$

Indeed, a (p,q)-form valued in the line bundle $L^{(\vec{Q})}$ can be viewed as a linear map from $\mathcal{F}_X(\vec{\zeta})$ to $\mathcal{F}_X(\vec{\zeta} - \vec{Q})$. The Dolbeault operator $\overline{\partial}_A$ acting on such line-bundle valued (p,q) forms requires a non-trivial connection, which is defined using the projection operators introduced in equation (3.33) as

$$\overline{\partial}_A \omega_{(p,q)} = \pi_X(\vec{\zeta} - \vec{Q}) \cdot \overline{\partial}\omega_{(p,q)} \cdot \pi_X(\vec{\zeta}) \tag{3.38}$$

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[14]See [42] for a recent discussion of Stanley-Reisner ideals in the physics literature.

3.4 Subspaces

Consider a subspace P of the ambient toric space X defined as the vanishing locus of a weighted homogeneous polynomial which we denote (by abuse of notation) by $P_{\vec{w}}(z_1, \ldots, z_r)$. To specify the embedding of $(P = 0)$ in X, we can also introduce a local basis of differential forms corresponding to directions normal to the divisor labelled as dP, and directions parallel to the divisor, which we label as $dz_{\parallel}$.

In the quantized version of this construction, the polynomial P becomes an operator $P(Z_1, \ldots, Z_r)$ with fixed GLSM charges:

$$\left[\vec{D}, P \right] = \vec{w} P. \tag{3.39}$$

Moreover, the formal differential dP is now given by

$$dP = \left[\partial, P \right], \tag{3.40}$$

with $\partial = \sum_i C_i Z_i^{\dagger}$. The formal differential dP can be viewed as the local holomorphic direction normal to the fuzzy subspace.

The subspace of states in $\mathcal{F}_X(\vec{\zeta})$ annihilated by both P and dP then defines the vector space of fuzzy points associated with P:

$$\mathcal{F}_P(\vec{\zeta}) = \ker P \cap \ker dP. \tag{3.41}$$

Provided the polynomial P has positive GLSM charge with respect to each $U(1)_i$ factor, the space of states annihilated by P will always be non-trivial.[15] Hence, there is always a sense in which we can speak of the collection of fuzzy points associated with this divisor. We can also consider more general intersections of hypersurfaces $(P_1 = 0) \cap \ldots \cap (P_M = 0)$. The associated vector space generated by the collection of points of this subspace are then given by the overlap: $\mathcal{F}_{P_1 \cap \ldots \cap P_M}(\vec{\zeta}) = \mathcal{F}_{P_1}(\vec{\zeta}) \cap \ldots \cap \mathcal{F}_{P_M}(\vec{\zeta})$.

Functions and differential forms on a subspace P are defined by a similar procedure to that used for X. Viewing $\mathcal{F}_P(\vec{\zeta})$ as the exterior powers of the holomorphic cotangent bundle of P, a differential form corresponds to a map from $\mathcal{F}_P(\vec{\zeta})$ back to itself. Let us now describe this procedure in practical terms.

One way to obtain the function space on P is to introduce projection operators π_P given in terms of an orthonormal basis $\{|\mu\rangle\}$ spanning $\mathcal{F}_P(\vec{\zeta})$ as:

$$\pi_P = \sum_{|\mu\rangle \in \mathcal{F}_P(\vec{\zeta})} |\mu\rangle \langle \mu|. \tag{3.42}$$

The projection operator π_P can be viewed as the characteristic function localized on the subspace $\mathcal{F}_P(\vec{\zeta})$. A general function G_P on P is then given by sandwiching a polynomial $G(Z^{\dagger}, Z)$ in the oscillators $Z^{\dagger}$ and Z between two projection factors

$$G_P = \pi_P \cdot G \cdot \pi_P. \tag{3.43}$$

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[15] To prove this, we view P as a linear map from $\mathcal{F}_X(\vec{\zeta})$ to $\mathcal{F}_X(\vec{\zeta} - \vec{w})$, with $\vec{w}$ the weight of the polynomial P. It follows from the rank nullity theorem that $\dim \ker P + \dim \operatorname{im} P = \dim \mathcal{F}_X(\vec{\zeta})$. Since all components in the vector $\vec{w}$ are positive this implies: $\dim \ker P = \dim \mathcal{F}_X(\vec{\zeta}) - \dim \operatorname{im} P \geq \dim \mathcal{F}_X(\vec{\zeta}) - \mathcal{F}_X(\vec{\zeta} - \vec{w}) > 0$, where the final inequality follows from the fact that the degrees of the polynomial in P are all positive.

Again by evaluating specific matrix elements in a basis for $\mathcal{F}_P(\vec{\zeta}) \otimes \mathcal{F}_P^*(\vec{\zeta})$, we can present such zero forms as explicit matrices. Similar considerations hold for further restrictions onto intersections of subspaces $P_1, \ldots, P_n$.

Differential forms of a subspace P of X can be obtained in the same way, by restriction of (p, q) forms on X to P. In the commutative theory, we can decompose a general (p, q) form as $\omega_{(p,q)} = \omega_\parallel + \omega_\perp$ where $\omega_\perp$ has formal differentials which include dP or $\overline{dP}$, and $\omega_\parallel$ is defined as the remainder. Formally setting to zero $\omega_\perp$ then defines the restriction of the form. In the fuzzy theory, the analogous prescription is given by projecting out dP, and performing an appropriate restriction on the remaining components. All of this is captured by sandwiching a general fuzzy (p, q) form between two projection operators:

$$\omega_{(p,q)|P} = \pi_P \cdot \omega_{(p,q)} \cdot \pi_P. \tag{3.44}$$

In the commutative theory, it is often convenient to adopt differentials intrinsic to a subspace such as P. In most of our applications, it will be enough to work in terms of the ambient space differentials defined on the larger toric space X.

3.5 Integration

Finally, we need to provide a suitable prescription for integrating functions and forms.

Consider some non-commutative subspace P of dimension d inside of the toric space $X = \mathbb{C}^r//\mathrm{U}(1)^s$. We wish to define integration over P. To motivate our definition, we first characterize a natural prescription for integrating forms over the underlying commutative space P. This space P is a symplectic manifold with a symplectic form ω_P that gets inherited from the standard symplectic form ω (given in (3.3)) defined on $\mathbb{C}^r$, by applying the appropriate projection operator. As is clear from the construction of P, the symplectic form ω_P is a (1,1)-form on P. Hence we can use it define the integration of general (p, p) forms $W_{(p,p)}$ over P, by taking the wedge product with the required number of ω_P's to produce a volume form which is then integrated over P

$$\int_P W = \int_P \omega_P \wedge \ldots \wedge \omega_P \wedge W. \tag{3.45}$$

Equivalently, we can first use the inverse of the symplectic form ω_P to contract all indices of $W_{(p,p)}$, and define a function $W^{(0)}$ via

$$W^{(0)} = W_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_p} \, \omega_P^{i_1 \bar{j}_1} \ldots \omega_P^{i_p \bar{j}_p} \tag{3.46}$$

We can then multiply this function with the standard symplectic volume element and integrate

$$\int_P W = \int_P \omega_P \wedge \ldots \wedge \omega_P \cdot W^{(0)}. \tag{3.47}$$

The latter prescription is most directly generalized to the non-commutative context.

Given a function $W^{(0)}(Z^\dagger, Z)$ defined on the space P, the analogue of the integral (3.47) is to take the trace over the bosonic state space $\mathcal{F}_P(\vec{\zeta})$

$$\sum_{|\mu\rangle \in \mathcal{F}_P(\vec{\zeta})} \langle \mu | \, W^{(0)}(Z^\dagger, Z) \, | \mu \rangle \tag{3.48}$$

Indeed, the states in $\mathcal{F}_P(\vec{\zeta})$ represent the points on P, and the trace amounts to summing the values of the function $W^{(0)}(Z^\dagger, Z)$ evaluated at all the points of the geometry.

The contraction (3.46) from a differential form W to a function $W^{(0)}$ also has a direct non-commutative realization. The fuzzy (p, p)-form represents an operator $W(Z^\dagger, Z, C, C^\dagger)$ on the extended state space, that includes the anti-commuting oscillators. Let $|0\rangle_C$ denote the fermionic vacuum state annihilated by the C oscillators. We then define

$$W^{(0)}(Z^\dagger, Z) = {}_C\langle\, 0\, |W(Z^\dagger, Z, C, C^\dagger)|\, 0\, \rangle_C \tag{3.49}$$

Note that we defined the (p, q)-differentials $W(Z^\dagger, Z, C, C^\dagger)$ to be anti-normal ordered in the fermionic oscillators. The above vacuum expectation value thus generates Wick contractions between the $C^\dagger$ and C oscillators. These Wick contractions contract the form indices with the inverse symplectic form. So the two steps (3.48) and (3.49) provide a direct fuzzy realization of the commutative integration prescription as given above.

4 F(uzz) theory

After collecting the necessary geometrical tools, we now return to our original task of describing the worldvolume gauge theory of a stack of seven-branes wrapping a fuzzy internal del Pezzo four-cycle $\mathcal{S}$, supported by a non-zero $\mathcal{B}$-flux, within an F-theory compactification. Our approach will be as follows.

We can locally specify the elliptically fibered Calabi-Yau four-fold in the neighborhood of the seven-branes via a Weierstrass equation

$$y^2 = x^3 + f(z; Z_j)x + g(z; Z_j). \tag{4.1}$$

Here z denotes an ordinary commuting local coordinate, chosen such that the stack of seven-branes is located at $z = 0$. The other coordinates Z_j will be used to represent non-commutative coordinates that parametrize the fuzzy four-cycle $\mathcal{S}$.

Following the general treatment presented in section 3, we start by defining the Z_i as coordinates on some ambient toric variety $X = \mathbb{C}^r//\mathrm{U}(1)^s$, which we can view as the classical vacuum of a GLSM. The four-cycle $\mathcal{S}$ wrapped by the GUT seven-brane is defined as an intersection of hypersurfaces within X. Concretely, X has dimension $r - s$ so the Kähler surface $\mathcal{S}$ is defined as the intersection of $r - s - 2$ hypersurfaces $P_1, \ldots, P_{r-s-2}$, each of complex codimension one, inside of X.

We can now apply the quantization procedure for a general toric space X given in section 3, and promote the coordinates Z_i and $Z_i^\dagger$ to annihilation and creation operators. This provides us with a non-commutative deformation of the toric space X, and an associated finite-dimensional state space $\mathcal{F}_X(\vec{\zeta})$, specified by a set of discrete FI-parameters $\vec{\zeta}$. From this ambient toric space, we obtain the fuzzy four-cycle $\mathcal{S}$ and the associated state space $\mathcal{F}_\mathcal{S}(\vec{\zeta})$ by applying the quantized version of the projections onto the hypersurfaces $P_1, \ldots, P_{r-s-2}$, following the procedure outlined in subsection 3.4.

Our main interest will be in formulating the gauge theory living on the seven-brane worldvolume $\mathbb{R}^{3,1} \times \mathcal{S}$, and the spectrum and interactions of chiral matter localized on fuzzy

divisors inside of this space. Applying our discussion of non-commutative spaces provided above, we introduce a Fock space $\mathcal{F}_{\mathcal{S}}(\vec{\zeta})$ associated with the vector space of points of $\mathcal{S}$. The vector space of points of the matter curves then correspond to subspaces of this larger Fock space. We can expand the functions $f(z; Z_i)$ and $g(z; Z_j)$ that appear in the Weierstrass equation (4.1) in powers of z. The coefficients are then defined as sections of the appropriate line bundles on the ambient toric space X, restricted to the hypersurface $\mathcal{S}$. The discriminant $\Delta = 4f^3 + 27g^2$ then locally looks as

$$\Delta = z^5 \prod_{\Delta_\alpha \neq \Delta_{\text{GUT}}} \Delta_\alpha(Z_j, z). \tag{4.2}$$

The discriminant locus $\Delta = 0$ decomposes into the set of fuzzy divisors $\Delta_\alpha(Z_j, z) = 0$. Along the locus $z = 0$, the component $\Delta_\alpha(Z_j, z = 0)$ defines the equation of a matter curve in $\mathcal{S}$. These divisors need to be treated as operator equations that restrict the state and function space of the matter curves, on which the chiral matter lives.

4.1 Fuzzy seven-branes

In this section we will describe how to construct the eight-dimensional Lagrangian $\mathcal{L}$ of a fuzzy seven-brane. This Lagrangian defines an operator $\mathcal{L}(Z^\dagger, Z, C, C^\dagger)$ built from the bosonic and fermionic oscillators. The four-dimensional Lagrangian L is given by integrating $\mathcal{L}$ over the non-commutative internal space $\mathcal{S}$ using the prescription outlined in subsection 3.5:

$$L = \sum_{|\mu\rangle \in \mathcal{F}_{\mathcal{S}}(\vec{\zeta})} \langle \mu | \mathcal{L}^{(0)}(Z^\dagger, Z) | \mu \rangle \tag{4.3}$$

where in the above, the sum runs over an orthonormal basis of bosonic states of $\mathcal{F}_{\mathcal{S}}(\vec{\zeta})$. Here, $\mathcal{L}^{(0)}$ is the bosonic projection of $\mathcal{L}$:

$$\mathcal{L}^{(0)}(Z^\dagger, Z) = {}_C\langle 0 | \mathcal{L}(Z^\dagger, Z, C, C^\dagger) | 0 \rangle_C \tag{4.4}$$

Since $\mathcal{S}$ is compact and its state space is finite-dimensional, the sum in equation (4.3) truncates to a finite number of terms. The effective action then involves only a finite number of four-dimensional fields.

Gauge sector. The worldvolume theory of the seven-brane describes an eight-dimensional gauge theory of $\mathbb{R}^{3,1} \times \mathcal{S}$ with gauge group G, as well as chiral modes which localize on matter curves Σ_α.

As before, we shall find it convenient to organize the fields in terms of four-dimensional $\mathcal{N} = 1$ supersymmetry. The field content of the gauge sector includes an adjoint-valued vector multiplet V, a collection of adjoint-valued chiral superfields A and Φ, which transform respectively as an ordinary function, a $(0,1)$-form and a $(2,0)$-form on $\mathcal{S}$. So the dependence on the internal fuzzy coordinates of these fields is of the form

$$\begin{aligned} V &= V(Z^\dagger, Z) \\ A &= A_i(Z^\dagger, Z) \, C_i^\dagger \end{aligned} \tag{4.5}$$

$$\Phi = \Phi_{ij}(Z^\dagger, Z)\, C_i \wedge C_J$$

The presentation of the Lagrangian is similar to the commutative case, though the interpretation of the fields is quite different. Returning to our discussion around equations (2.7)–(2.11), the Lagrangian density $\mathcal{L}$ takes the form:

$$\mathcal{L} \;=\; \int d^4\theta\, \mathcal{K} \;+\; \int d^2\theta\, \mathcal{W} \;+\; h.c. \;+\; \text{WZW}\,. \tag{4.6}$$

where the WZW term is a non-local term which vanishes in WZ gauge [31]. Repeating the equations already presented in the commutative case, first consider the eight-dimensional gauge sector of the seven-brane action. In terms of four-dimensional $\mathcal{N} = 1$ superspace, the D-terms are:

$$\mathcal{K}_{\text{gauge}} = \mathcal{K}_\Phi + \mathcal{K}_A \tag{4.7}$$

where the Kähler terms for Φ and A are:

$$\mathcal{K}_\Phi \;=\; \big(\Phi^\dagger e^{-V},\, \Phi\, e^V\big) \tag{4.8}$$

$$\mathcal{K}_A \;=\; \big((\bar{\partial} + A)^\dagger e^{-V},\, (\bar{\partial} + A) e^V\big) - (\bar{\partial}^\dagger e^{-V},\, \bar{\partial} e^V). \tag{4.9}$$

The F-terms are:

$$\mathcal{W}_{\text{gauge}} = \text{Tr}\big(W_\alpha^2\big) \;+\; \text{Tr}\big(\Phi \wedge (\bar{\partial} A + A \wedge A)\big). \tag{4.10}$$

Here we chose the gauge kinetic term to be quadratic. Since all fields are now normal ordered functions of the oscillators, the multiplication between fields proceeds via a star product. In addition the pairing $(\cdot, \cdot)$ present in the above definition represents the fact that in general, the pairing of a field with its conjugate need not be the same as the inner product associated with the Fock space. Each field consists of a zero mode component, as well as an infinite sum of Kaluza-Klein excitations. Note, however, that the finite size of the fuzzy space causes this infinite sum to truncate to a finite number of dynamical degrees of freedom.

Fuzzy matter. In the non-commutative theory, matter localization proceeds via the same mechanism as in the commutative case. A matter curve

$$\Sigma_\alpha = \big(\Delta_\alpha(Z_j, z) = 0\big) \cap (z = 0) \tag{4.11}$$

is the intersection of the GUT seven-brane at $z = 0$ with another component of the discriminant locus of equation (4.2), and indicates a local enhancement of the singularity type of the elliptic fibration. We denote by $\Delta_\alpha(Z_j)$ the restriction of $\Delta_\alpha(Z_j, z)$ to $(z = 0)$.

The degrees of freedom in a local patch near Σ_α can then be captured by means of a gauge theory with an enlarged gauge group G_Σ, in the presence of a vortex shaped Higgs profile Φ_0 that induces a breaking pattern $G_\Sigma \supset G \times G_\alpha$. The matter curve Σ_α then corresponds to the locus where the component of the Higgs field along the simple root vanishes $\langle \alpha, \Phi_0 \rangle = 0$. All these data can be straightforwardly reproduced within the non-commutative setting, by using the technology developed in section 3.

Here we will follow a more practical approach, and directly take the limit in which the Higgs field generating the breaking pattern is very large. This limit is particularly natural in the decoupling limit, in which the closed string volume of the four-cycle wrapped by the seven-brane is taken to zero. Then following the discussion in subsection 2.2, we can introduce the chiral matter content by associating to Σ_α two chiral superfields

$$A_\alpha = A_{\alpha,i}(Z^\dagger, Z) \, C_i^\dagger \tag{4.12}$$
$$\Psi_\alpha = \Psi_{\alpha,ij}(Z^\dagger, Z) \, C_i \wedge C_j$$

transforming in the representation R_α of the unbroken gauge group G. These fields are localized on the matter curve. Just as in the commutative theory, the six-dimensional mode content is chiral, and therefore the theory of an isolated matter curve is anomalous. These anomalies are cancelled through the presence of Chern-Simons terms of the seven-brane, which induce anomaly inflow (and outflow) between the curves and closed string modes. In practice, the cancellation of all such anomalies is guaranteed by compactifying on an elliptic Calabi-Yau fourfold.

To describe the localization of matter fields in the non-commutative geometry, we first construct the state space $\mathcal{F}_{\Sigma_\alpha}(\vec{\zeta})$ associated to the matter curve. Following the prescription outlined in section 3.4, we introduce the operator $\Delta_\alpha(Z)$. We then define $\mathcal{F}_{\Sigma_\alpha}(\vec{\zeta})$ as the subspace of the total state space $\mathcal{F}_\mathcal{S}(\vec{\zeta})$ annihilated by $\Delta_\alpha(Z)$

$$\mathcal{F}_{\Sigma_\alpha}(\vec{\zeta}) = \ker(\Delta_\alpha). \tag{4.13}$$

The fact that A_α and $\Psi_{-\alpha}$ localize on Σ_α means that they are given by differential forms on $\mathcal{S}$, though the bosonic content of this form is localized on a subspace. Introduce a bosonic projection operator $\pi_\Sigma^{(0)}$ onto the subspace $\mathcal{F}_{\Sigma_\alpha}$ which acts as the identity on the fermionic oscillators. Given a bulk form Ψ on $\mathcal{S}$, one way to localize its bosonic content is to sandwich it in between such bosonic projections via $\pi_\Sigma^{(0)} \cdot \Psi \cdot \pi_\Sigma^{(0)}$. Let us stress that we view these matter fields as forms on $\mathcal{S}$.[16] This is distinct from the notion of a differential form defined exclusively on a subspace, as in subsection 3.4. In addition, in the commutative theory, there is the phenomenon of seven-brane monodromy [11, 43–46]. Again, the main point here is that algebraic expressions of the commutative theory have direct analogues in the fuzzy theory, so the explicit presentation of the higher-dimensional modes in such cases is really an issue in the commutative theory. Once we have this commutative data, the non-commutative translation follows. Accordingly, we can write an eight-dimensional matter sector Lagrangian, specified by a Kähler potential and superpotential of the same form as equation (2.18):

$$\mathcal{K}_\text{matter} = \left(\Psi_{-\alpha}^\dagger, e^V \, \Psi_{-\alpha} \right) + \left(A_\alpha^\dagger, e^V A_\alpha \right) \tag{4.14}$$
$$\mathcal{W}_\text{matter} = \Psi_{-\alpha} \wedge \overline{\partial}_A A_\alpha + f_{\alpha\beta\gamma} \, \Psi_\alpha \wedge A_\beta \wedge A_\gamma. \tag{4.15}$$

---

[16]This is essentially the same procedure one follows in the spectral cover construction [11, 43] in which an auxiliary local Calabi-Yau threefold $O(K_\mathcal{S}) \to \mathcal{S}$ is erected over the GUT seven-brane. Zero modes are identified with elements of appropriate Ext groups on the threefold as $Ext^p(\mathcal{V}, \mathcal{V}')$ for sheaves $\mathcal{V}$ and $\mathcal{V}'$ with support on subspaces.

Mode decomposition. In the commutative setting the matter fields of the higher-dimensional theory transform as forms of appropriate line bundles. The zero mode content is computed by the Dolbeault cohomology, and the KK spectrum is given by the higher harmonics of the Laplacian operator.

All of these statements have analogues in the non-commutative setting. Consider first the spectrum of zero modes. Because we still have a notion of a Dolbeault operator, we can define Dolbeault cohomology for fuzzy line bundles, following the same algebraic prescription in the commmutative theory. Similar considerations hold for the cohomology theory of line bundles on subspaces. Our discussion also holds for non-toric spaces as well. Indeed, in the commutative theory, there is a well-known prescription for determining cohomology groups based on the Koszul complex. The main idea in Koszul complex computations is that once we have specified a divisor D of the toric space X, the cohomology theory of a line bundle defined over D can be computed in terms of the cohomology theory on X [47].[17] Roughly speaking, the elements of the cohomology groups should be viewed as operators which are "holomorphic" in the sense that they are annihilated by the Dolbeault operator of the fuzzy subspace.

Sections and forms of a line bundle not annihilated by the Dolbeault operator then determine the spectrum of KK modes. To illustrate the general treatment of chiral matter fields, consider the modes associated with a higher-dimensional field $\Psi^{(\vec{Q})}$ corresponding to a section of a degree $\vec{Q}$ fuzzy line bundle $L^{(\vec{Q})}$. This field is given by a formal sum of normal ordered polynomials in the oscillators Z and $Z^\dagger$ of weighted degree $\vec{Q}$:

$$\Psi^{(\vec{Q})} = \sum_{I,\,\vec{v}-\vec{u}=\vec{Q}} \psi_{I(\vec{u},\vec{v})} \overline{\sigma}_{I(\vec{u})}(Z^\dagger)\, \sigma_{I(\vec{v})}(Z) \tag{4.16}$$

where in the above, the σ's are a basis of polynomials of a given weight

$$\left[\vec{D}, \overline{\sigma}_{I(\vec{u})}\right] = +\vec{u}\,\overline{\sigma}_{I(\vec{u})} \quad \left[\vec{D}, \sigma_{I(\vec{v})}\right] = -\vec{v}\,\sigma_{I(\vec{v})}. \tag{4.17}$$

The overall weight of $\Psi^{(\vec{Q})}$ is fixed to be equal to $\vec{v}-\vec{u}=\vec{Q}$. The indices I label the linearly independent weighted homogeneous polynomials, and the coefficients $\psi_{I(\vec{u},\vec{v})}$ correspond to four-dimensional chiral superfields. Note that the sum in $\Psi^{(\vec{Q})}$ runs over an infinite set of terms. The terms with $\vec{u} = 0$ are to be viewed as the massless modes (corresponding to holomorphic sections), and the terms with $\vec{u} \neq 0$ define massive Kaluza-Klein excitations.

The presence of an infinite number of terms might at first appear contrary to the intuition that making the geometry non-commutative truncates the Kaluza-Klein spectrum. Here this occurs once we fix an overall size for the fuzzy space on which $\Psi^{(\vec{Q})}$ localizes. Indeed, tracing over the state space of bosonic points, the kinetic term is

$$\sum_{|\mu\rangle \in \mathcal{F}_S(\vec{\zeta})} \int d^4\theta \, \langle\mu| \left(\Psi^{(\vec{Q})\dagger}, \Psi^{(\vec{Q})}\right) |\mu\rangle. \tag{4.18}$$

---

[17]See for example [48] for a recent discussion in the physics literature for how this algorithm is implemented in practice.

Writing the explicit bundle indices in the open string metric, the pairing is given by inserting the appropriate Hermitian metric in between the two factors, so that

$$\left(\Psi^{(\vec{Q})\dagger}, \Psi^{(\vec{Q})} \right) = \Psi^{(\vec{Q})\dagger} \cdot G_{\text{open}} \cdot \Psi^{(\vec{Q})}. \tag{4.19}$$

Thus, once the magnitude of the vector $\vec{v}$ given in equation (4.16) becomes sufficiently large, all states of $\mathcal{F}_{\mathcal{S}}(\vec{\zeta})$ will be annihilated. Hence the number of dynamical four-dimensional fields is finite. Similar considerations hold for the massive modes descending from differential-forms valued in the line bundle $L^{(\vec{Q})}$.

The field content of the seven-brane theory is then given by sums over operators defined as in equation (4.16). These fields correspond to chiral superfields in the case of the modes Φ and A, which propagate in the bulk, and their analogues Ψ_α and A_α which localize on a curve. To treat all such modes on a uniform footing, we shall simply write all of the modes as Ψ_i and A_i, where $i = \mathcal{S}$ denotes the modes propagating in the bulk, and $i = \Sigma$ denotes a mode localized on the subspace Σ.

One of the virtues of the above description is that it provides us with an explicit list of the Kaluza-Klein modes. The actual mass matrix for the KK modes will depend on the details of the open string metric. Nevertheless, the fact that we can now catalogue this spectrum means that we can treat this theory in four-dimensional terms.

GUT breaking by a $U(1)_Y$ hypercharge flux works just as in the commutative theory [8, 9]. In four dimensions, the unbroken gauge group is the commutant of $U(1)_Y$ inside of $SU(5)_{\text{GUT}}$, and the fuzzy modes organize into representations of $SU(3)_C \times SU(2)_L \times U(1)_Y$. The hypercharge flux (2.20) can either be viewed as a component of the non-commutativity parameter of each MSSM gauge group factor, or as a background flux of the $SU(5)$ gauge theory, with non-commutativity parameter set by the overall $U(1)$ flux $\mathcal{B}$. From our discussion above we see that these two ways of representing the hypercharge flux are completely equivalent.

Fluxes from the GUT seven-brane as well as flavor seven-branes affect the line bundle data of the gauge and matter fields. After activating a hyperflux, the massless mode content in the gauge sector corresponds to the Standard Model gauge group, as well as massive vector multiplets which also transform in the adjoint representation of the Standard Model gauge group. In addition, there will also be Kaluza-Klein modes corresponding to the off-diagonal components of the GUT group.

Matter delocalization. Fuzz theory retains much of the structure of the commutative local F-theory model, though there are also differences. An example of a new phenomenon is that matter localization on curves may break down in the non-commutative setting. To illustrate the general issue, consider the weighted homogeneous polynomial $P(Z_1, \ldots, Z_r)$ in the Z's. Our definition of matter localized along the vanishing locus of P is given by states $|\psi\rangle$ in $\mathcal{F}_X(\vec{\zeta})$ subject to the condition:

$$P(Z_1, \ldots, Z_r) |\psi\rangle = 0. \tag{4.20}$$

Note, however, that when the FI parameter $\vec{\zeta}$ is sufficiently small and the weighted degree of the operator $P(Z_1, \ldots, Z_r)$ is sufficiently high, all of the states of $\mathcal{F}_X(\vec{\zeta})$ will be

annihilated. In other words, in this case, there is no sense in which the matter remains localized. Rather, the matter field spreads out over the entire space. In the commutative theory, the degree of P determines the genus of the curve. Here, we see that high genus matter curves do not really correspond to curves at all.

This in turn raises the question as to how to count the number of such delocalized matter fields. Ordinarily in the commutative theory, the main task is to specify a line bundle over an ambient toric space. Once this is accomplished, we can perform an appropriate restriction of this line bundle to subspaces, and count the number of associated zero modes. Here we face the issue that when the space is sufficiently small, the notion of a subspace itself breaks down, and thus the restriction of line bundles onto a given subspace also becomes ill-defined.

Our definition of holomorphic sections of a line bundle, however, remains well-defined at arbitrary values of the radii because it is determined by purely algebraic data. Indeed, at an abstract level, we have specified the zero modes localized on a matter curve simply as appropriate polynomials in the Z and C oscillators sandwiched between projection factors π_P. When the projection π_P acts as the identity, the matter field is no longer localized.

One consequence of this delocalization is that the number of points sampled by the Yukawa coupling will now increase. This then causes a jump in the rank of the Yukawa coupling matrix. Assuming a suitable hierarchy in the presentation of the original holomorphic sections, matter delocalization then induces novel textures for Yukawas.

4.2   A toy example: Yukawa couplings on fuzzy $\mathbb{P}^1 \times \mathbb{P}^1$

In this section we illustrate the general ideas developed earlier, and compute the Yukawa couplings in a toy model based on fuzzy $\mathbb{P}^1 \times \mathbb{P}^1$. Even though the general discussion we have presented applies equally well to cases where the GUT surface $\mathcal{S}$ is non-toric, note that because $\mathbb{P}^1 \times \mathbb{P}^1$ is itself toric, we can bypass some of this general discussion and simply treat the GLSM for $\mathbb{P}^1 \times \mathbb{P}^1$. For simplicity, we focus on the case where all matter fields remain localized on well-defined subspaces. We find that just as in the commutative theory, only one overlap of wave functions is non-zero. However, background fluxes can distort this structure, producing flavor hierarchies [37, 49].

Let us first describe the geometry of fuzzy $\mathbb{P}^1 \times \mathbb{P}^1$. The classical GLSM for $\mathbb{P}^1 \times \mathbb{P}^1$ is a $\mathrm{U}(1)^2 = \mathrm{U}(1)_I \times \mathrm{U}(1)_{II}$ gauge theory with chiral superfields u_i and v_i for $i = 1, 2$. Under the two $\mathrm{U}(1)$ factors, the charges of u_i and v_i are $(+1, 0)$ and $(0, +1)$. The fields are subject to the D-term constraints:

$$
\begin{aligned}
|u_1|^2 + |u_2|^2 &= \zeta_I \\
|v_1|^2 + |v_2|^2 &= \zeta_{II}.
\end{aligned}
\tag{4.21}
$$

The symplectic quotient $\mathbb{C}^4 // \mathrm{U}(1)^2$ yields $\mathbb{P}^1 \times \mathbb{P}^1$. The u_i and v_i correspond to projective coordinates for the two $\mathbb{P}^1$ factors.

We now turn to the non-commutative geometry. Here we focus on the bosonic content, treating all differential forms more implicitly. Introduce bosonic oscillators for each $\mathbb{P}^1$

factor, U_i for $\mathbb{P}^1_I$ and V_i for $\mathbb{P}^1_{II}$. These are subject to the commutation relations:

$$\left[U_i, U_j^\dagger \right] = \left[V_i, V_j^\dagger \right] = \delta_{ij} \tag{4.22}$$

with all other commutators vanishing.

Next, introduce a vacuum $|0\rangle$ annihilated by the U's and V's. The Fock space $\mathcal{F}(\mathbb{C}^4)$ is given by acting on the vacuum state with the creation operators. The basis elements of $\mathcal{F}(\mathbb{C}^4)$ are

$$|n_1, n_2; m_1, m_2\rangle = \frac{(U_1^{n_1} U_2^{n_2} V_1^{m_1} V_2^{m_2})^\dagger}{\sqrt{n_1! n_2! m_1! m_2!}} |0\rangle. \tag{4.23}$$

We introduce $\mathcal{F}_X(N_I, N_{II})$ as the subspace of $\mathcal{F}(\mathbb{C}^4)$ subject to the oscillator number constraint

$$\begin{aligned} n_1 + n_2 &= N_I, \\ m_1 + m_2 &= N_{II}. \end{aligned} \tag{4.24}$$

This corresponds to the linear space of bosonic points for a $\mathbb{P}^1_I \times \mathbb{P}^1_{II}$ with FI parameters N_I and N_{II}.

We now compute the Yukawa coupling associated with a triple intersection of matter curves. In the commutative geometry, we consider a configuration of matter fields in the local patch where u_2 and v_2 are both non-zero. Local coordinates for this patch are $u = u_1/u_2$ and $v = v_1/v_2$. The configuration of matter curves we consider are described by Higgsing of U(3) gauge theory down to U(1)^3 according to the background vev:

$$\Phi_0 = \begin{bmatrix} v & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u \end{bmatrix}. \tag{4.25}$$

In this patch, the three matter curves of this configuration correspond to enhancements in the singularity type at $u = 0$, $v = 0$ and $u = v$. In terms of the original u_i and v_i variables, the three matter curves are:

$$\begin{aligned} \mathbb{P}^1_I &= \mathbb{P}^1_{(12)} = (v_1 = 0) \\ \mathbb{P}^1_{II} &= \mathbb{P}^1_{(23)} = (u_1 = 0) \\ \mathbb{P}^1_{\text{diag}} &= \mathbb{P}^1_{(13)} = (u_1 v_2 = v_1 u_2), \end{aligned} \tag{4.26}$$

where we have indicated the color space assignment associated with each curve.

We now discuss matter localization. We are interested in states of $\mathcal{F}_X(N_I, N_{II})$ which are respectively annihilated by V_1, U_1 or $U_1 V_2 - U_2 V_1$. The set of states annihilated by V_1 and U_1 are:

$$\begin{aligned} \ker V_1 &= \text{span}\left\{ |n_1, n_2; 0, N_{II}\rangle : n_1 + n_2 = N_I \right\} \\ \ker U_1 &= \text{span}\left\{ |0, N_I; m_1, m_2\rangle : m_1 + m_2 = N_{II} \right\}. \end{aligned} \tag{4.27}$$

Geometrically, $\ker V_1$ corresponds to $\mathbb{P}^1_I \times |\text{South}\rangle_{II}$ and $\ker U_1$ corresponds to $|\text{South}\rangle_I \times \mathbb{P}^1_{II}$. See appendix A for further discussion of fuzzy $\mathbb{P}^1$.

Next consider the set of states annihilated by $U_1 V_2 - U_2 V_1$. One state which is annihilated by this operator is the South pole state of both $\mathbb{P}^1$ factors:

$$|\text{South}\rangle_I \otimes |\text{South}\rangle_{II} = |0, N_I; 0, N_{II}\rangle \qquad (4.28)$$

To generate the remaining states of the diagonal $\mathbb{P}^1$, note that $U_1 V_2 - U_2 V_1$ commutes with the diagonal raising operator

$$J_+^{\text{diag}} = J_+^{(I)} + J_+^{(II)} = U_1^\dagger U_2 + V_1^\dagger V_2 \qquad (4.29)$$

In other words, the remaining states of the diagonal $\mathbb{P}^1$ are obtained by acting by successive powers of J_+^{diag}:

$$\ker(U_1 V_2 - U_2 V_1) = \text{span}\left\{ \left(J_+^{\text{diag}}\right)^m |0, N_I; 0, N_{II}\rangle : m \geq 0 \right\}. \qquad (4.30)$$

We now present a computation of Yukawa couplings for this configuration of matter curves. To keep the computation as symmetric as possible, we assume that the modes on $\mathbb{P}^1_I$ and $\mathbb{P}^1_{II}$ correspond to $(0,1)$-forms with localized bosonic content, while the modes of $\mathbb{P}^1_{\text{diag}}$ correspond to $(2,0)$-forms with localized bosonic content.

The number of zero modes is controlled by the line bundle assignments on each matter curve. For simplicity, we view the line bundles as descending from bulk line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$. The localized matter fields are then given by a projection on the bosonic content. Actually, the line bundle assignments need only be defined over the matter curves, so we merely view this as a useful device for deducing the operator content of the localized modes. Stated in terms of the commutative geometry, the configuration of line bundles we take for the three matter curves are:

$$\begin{aligned} L_I &= O_{\mathbb{P}^1_I \times \mathbb{P}^1_{II}}(d\,\sigma_2), \\ L_{II} &= O_{\mathbb{P}^1_I \times \mathbb{P}^1_{II}}(d\,\sigma_1), \qquad\qquad d = g - 1, \\ L_{\text{diag}} &= O_{\mathbb{P}^1_I \times \mathbb{P}^1_{II}}(-d\,(\sigma_1 + \sigma_2)), \end{aligned} \qquad (4.31)$$

where σ_1 and σ_2 denote the divisor classes for $\mathbb{P}^1_I$ and $\mathbb{P}^1_{II}$. In the fuzzy setting, differential forms of these bundles correspond to operators of fixed degree. Note that the tensor product $L_I \otimes L_{II} \otimes L_{\text{diag}}$ is indeed trivial, so the product $A_I \wedge_* A_{II} \wedge_* \Psi_{\text{diag}}$ can indeed form a Yukawa. Further, the restriction of each line bundle onto the appropriate matter curve yields a degree d line bundle on both $\mathbb{P}^1_I$ and $\mathbb{P}^1_{II}$, while on $\mathbb{P}^1_{\text{diag}}$ we find a degree $-2d$ line bundle.

To compute the Yukawa couplings for this configuration, we first specify the profile of the zero mode wave-functions. First consider the zero modes localized on $\mathbb{P}^1_I$. Such modes correspond to zero-forms on $\mathbb{P}^1_I$, and have the form content of a one-form on $\mathbb{P}^1_{II}$. Since line bundles on $\mathbb{P}^1$ are fully classified by their degree, holomorphic sections of this line bundle can be presented as holomorphic polynomials in U_1 and U_2 of homogeneous degree d. The $g = d + 1$ zero mode wave functions are then:

$$A_{(12)} = \pi_{\mathbb{P}^1_{(12)}} \cdot \sum_{i=0}^{d} a_i^{(12)} U_1^i U_2^{d-i} \cdot \pi_{\mathbb{P}^1_{(12)}} \qquad (4.32)$$

here we are suppressing the one-form content to avoid cluttering the presentation. The π's denote bosonic projection factors to the matter curves and the a's correspond to four-dimensional chiral superfields, and define the coefficients of the degree d polynomials. Similar considerations hold for the g zero modes localized on $\mathbb{P}^1_{II}$ so that:

$$A_{(23)} = \pi_{\mathbb{P}^1_{(23)}} \cdot \sum_{i=0}^{d} a_i^{(23)} V_1^i V_2^{d-i} \cdot \pi_{\mathbb{P}^1_{(23)}}. \tag{4.33}$$

Next consider the zero modes associated with Ψ_{diag}. Since we have already demanded the bulk line bundle assignments tensor to the trivial bundle, we deduce that the GLSM charge is $(-d, -d) + (2, 2)$ for the zero modes of Ψ_{diag}. Here we have split up the contributions to the GLSM charge from the $(2, 0)$ form, and its localized coefficient. Suppressing the form content, such zero modes can all be written as:

$$\Psi_{(31)} = \pi_{\mathbb{P}^1_{(31)}} \cdot \sum_i a_i^{(31)} O_i(U^\dagger, V^\dagger) \cdot \pi_{\mathbb{P}^1_{(31)}}. \tag{4.34}$$

To be a section of the appropriate line bundle, an operator O_i must commute with $U_1 V_2 - U_2 V_1$. In other words, acting on the vacuum by O_i can be viewed as generating a state of a $\mathbb{P}^1_{\text{diag}}$ with $2d + 1$ points:

$$\mathcal{F}_{\mathbb{P}^1_{\text{diag}}}(2d + 1) = \text{span}\left\{ \left(J_+^{\text{diag}}\right)^m \left(U_2^\dagger\right)^d \left(V_2^\dagger\right)^d |0\rangle : 0 \le m \le 2d \right\}. \tag{4.35}$$

There are $2d + 1$ linearly independent O_i's of this form. We order the modes in accord with the basis of states given above so that for example $O_0 = \left(U_2^\dagger\right)^d \left(V_2^\dagger\right)^d$ and $O_{2d} = \left(U_1^\dagger\right)^d \left(V_1^\dagger\right)^d$. The zero modes localized on the diagonal $\mathbb{P}^1_{\text{diag}}$ are then given as:

$$\Psi_{(31)} = \pi_{\mathbb{P}^1_{(31)}} \cdot \sum_{i=0}^{2d} a_i^{(31)} O_i(U^\dagger, V^\dagger) \cdot \pi_{\mathbb{P}^1_{(31)}} \tag{4.36}$$

where in the above expression we have suppressed the form content of the zero mode.

Having specified the zero mode content of the theory, we now compute the Yukawa couplings for this toy model. We introduce a convenient basis of states for the zero mode wave functions:

$$
\begin{aligned}
\Psi_{(31)}^{(i)} &= \pi_{\mathbb{P}^1_{(31)}} \cdot a_i^{(31)} O_i(U^\dagger, V^\dagger) \cdot \pi_{\mathbb{P}^1_{(31)}}. \\
A_{(12)}^{(j)} &= \pi_{\mathbb{P}^1_{(12)}} \cdot a_j^{(12)} U_1^j U_2^{d-j} \cdot \pi_{\mathbb{P}^1_{(12)}} \\
A_{(23)}^{(k)} &= \pi_{\mathbb{P}^1_{(23)}} \cdot a_k^{(23)} V_1^k V_2^{d-k} \cdot \pi_{\mathbb{P}^1_{(23)}}
\end{aligned}
\tag{4.37}
$$

Working at fixed discretized FI parameters (N_I, N_{II}) for the two $\mathbb{P}^1$ factors, we must evaluate the superpotential:

$$W_{ijk} = \sum_{|\mu\rangle \in \mathcal{F}_X(N_I, N_{II})} \langle \mu | \Psi_i \wedge_* A_j \wedge_* A_k |\mu\rangle. \tag{4.38}$$

By inspection, the presence of the projection factors causes most of the Yukawas to vanish. Indeed, working at fixed Kähler volumes, the only state common to all three fuzzy divisors is the mutual South pole (4.28). Thus, we can replace the general projection operators presented by the projection onto the South pole state, which we denote by π_{South}. As a geometric operation, the projection makes sense regardless of the FI parameters for our toric space. The relevant projection is then given by a direct sum over the projection operators we would obtain by working at a fixed value of the FI parameters:

$$\pi_{\text{South}} = \sum_{M_I, M_{II}} \left| 0, M_I; 0, M_{II} \right\rangle \left\langle 0, M_I; 0, M_{II} \right|. \tag{4.39}$$

The superpotential can therefore be written as:

$$W_{ijk} = \left\langle 0, N_I; 0, N_{II} \right| a_i^{(31)} O_i(U^\dagger, V^\dagger) \cdot \pi_{\text{South}} \cdot a_j^{(12)} U_1^j U_2^{d-j} \cdot \pi_{\text{South}} \cdot a_k^{(23)} V_1^k V_2^{d-k} \left| 0, N_I; 0, N_{II} \right\rangle.$$

Since we have fixed the overall Kähler class of X, we now substitute the projection operator π_{South}:

$$\begin{aligned}
W_{ijk} = a_i^{(31)} a_j^{(12)} a_k^{(23)} \times & \left\langle 0, N_I; 0, N_{II} \right| O_i(U^\dagger, V^\dagger) \left| 0, N_I - d; 0, N_{II} - d \right\rangle \times \\
\times & \left\langle 0, N_I - d; 0, N_{II} - d \right| U_1^j U_2^{d-j} \left| 0, N_I; 0, N_{II} - d \right\rangle \times \\
\times & \left\langle 0, N_I; 0, N_{II} - d \right| V_1^k V_2^{d-k} \left| 0, N_I; 0, N_{II} \right\rangle. \tag{4.40}
\end{aligned}$$

To obtain a non-zero result, it is necessary to exclude all dependence on the oscillators U_1, V_1, $U_1^\dagger$ and $V_1^\dagger$. In other words, the only non-zero Yukawa coupling occurs for $i = j = k = 0$. A straightforward calculation then gives

$$W_{ijk} = \frac{N_I! N_{II}!}{(N_I - d)!(N_{II} - d)!} \times \delta_{i0} \delta_{j0} \delta_{k0} \times a_i^{(31)} a_j^{(12)} a_k^{(23)}. \tag{4.41}$$

Thus, as expected, we obtain only a single non-vanishing Yukawa coupling.

For commutative seven-branes, small changes to the rank of the superpotential can be phrased in terms of non-commutative deformations of the holomorphic structure in a patch of the Yukawa point [37]. This corresponds to a holomorphic deformation of the oscillator algebra.

5 Geometric applications

In this section we study some other potential applications of the formulation of non-commutative toric geometry developed in section 3.

5.1 A fuzzy flop

A well known phenomenon in commutative toric geometry are flop transitions. These correspond to transitions in the large volume regime where changing the values of the FI parameters of the GLSM causes some two-cycles to collapse, and new ones to take their place. In this section we study the corresponding fuzzy flop.

To keep our discussion concrete, we focus on the flop transition associated with the small resolution of the conifold geometry $O(-1) \oplus O(-1) \to \mathbb{P}^1$. The GLSM data of the conifold is given in terms of a single U(1) gauge theory and four chiral superfields u_1, u_2, v_1 and v_2 so that the u's have GLSM charge $+1$ and the v's have GLSM charge -1. The D-term constraint is:

$$|u_1|^2 + |u_2|^2 - |v_1|^2 - |v_2|^2 = \zeta. \tag{5.1}$$

When $\zeta > 0$, the v's should be viewed as normal coordinates to a $\mathbb{P}^1$ with Kähler class proportional to $\zeta + |v_1|^2 + |v_2|^2$, so that the u's define coordinates for the $\mathbb{P}^1$. When $\zeta < 0$, the geometry undergoes a flop transition and the u and v coordinates exchange roles.

We now describe the fuzzy flop transition. Introduce operators U_i and V_i subject to the commutation relations:

$$[U_i, U_j^\dagger] = [V_i, V_j^\dagger] = \delta_{ij} \tag{5.2}$$

with all other commutators zero. Introducing the D-term operator:

$$D = U_1^\dagger U_1 + U_2^\dagger U_2 - V_1^\dagger V_1 - V_2^\dagger V_2, \tag{5.3}$$

the Fock space $\mathcal{F}_X(\zeta)$ of bosonic states for the conifold with resolution parameter ζ is spanned by

$$|n_1, n_2; , m_1, m_2\rangle = \prod_{i,j=1}^{2} \frac{(U_i^\dagger)^{n_i}(V_i^\dagger)^{m_j}}{\sqrt{n_i!}\sqrt{m_j!}} |0\rangle \quad \text{with} \quad n_1 + n_2 - m_1 - m_2 = \zeta. \tag{5.4}$$

The $\mathbb{P}^1$ of minimal size is the subspace of states in $\mathcal{F}_X(\zeta)$ with either trivial U oscillator number, or trivial V oscillator number. In particular, we see that for $\zeta > 0$, there is a $\mathbb{P}^1$ generated by just the U creation operators

$$\mathcal{F}_{\mathbb{P}^1}(\zeta > 0) = \left\{ |n_1, n_2; 0, 0\rangle \; ; \; n_1 + n_2 = \zeta \right\}, \tag{5.5}$$

while for $\zeta < 0$, there is a $\mathbb{P}^1$ generated by just the V oscillators:

$$\mathcal{F}_{\mathbb{P}^1}(\zeta < 0) = \left\{ |0, 0; m_1, m_2\rangle \; ; \; m_1 + m_2 = \zeta \right\}. \tag{5.6}$$

5.2 Fuzzy Landau-Ginzburg

One of the remarkable features of the GLSM description of a target space is that it provides a common language for describing both the geometric and Landau-Ginzburg phases of a string compactification [50]. In this paper we have been primarily focussed on the toric phase of the GLSM. It is tempting, therefore, to see whether we can also extend this discussion to provide a fuzzy analogue of Landau-Ginzburg vacua.

Rather than be overly general, we follow the main example presented in [50] and focus on the GLSM defined by a single U(1) with r chiral superfields z_i of charges $q_i > 0$, and another chiral superfield p with charge $n < 0$. A compact hypersurface is specified by including a superpotential:

$$W = p \cdot G(z_1, \ldots, z_r). \tag{5.7}$$

Anomaly freedom of the GLSM is equivalent to the Calabi-Yau condition that $q_1 + \cdots + q_r = n$. Further note that the superpotential W is neutral with respect to the U(1) gauge symmetry. The D-term and F-term equations of motion are respectively:

$$\sum_i q_i |z_i|^2 - n|p|^2 = \zeta$$

$$G = 0 \quad \text{and} \quad p\partial_i G = 0. \tag{5.8}$$

In the geometric phase, $\zeta \gg 0$ and $G = p = 0$. In the Landau-Ginzburg phase, we instead have $\zeta \ll 0$, $p = \sqrt{-\zeta/n}$, and the vacua are described in terms of the Landau-Ginzburg superpotential $\widetilde{W} = \sqrt{-\zeta/n}\, G(z_1, \ldots, z_r)$ with vacua described by the critical points of $\widetilde{W}$.

Again, the data of the GLSM provides a canonical quantization prescription for the fuzzy target. We restrict our attention to the bosonic states. Introduce operators Z_i and P subject to the commutation relations:

$$[Z_i, Z_j^\dagger] = \delta_{ij}, \qquad [P, P^\dagger] = 1,$$

with all other commutators vanishing. Next introduce a vacuum $|0\rangle$ annihilated by the Z_i's and P. We then construct a Fock space $\mathcal{F}(\mathbb{C}^{r+1})$ by acting on the vacuum state with the $Z_i^\dagger$'s and $P^\dagger$.

The geometric phase (G) and Landau-Ginzburg phase (LG) of the target are then given in terms of the Hamiltonian constraint operators:

$$D_G = \sum_i q_i Z_i^\dagger Z_i - n P^\dagger P \tag{5.9}$$

$$D_{\text{LG}} = -n P^\dagger P \tag{5.10}$$

The hypersurface constraint in the geometric phase is that G and P both annihilate a state. The hypersurface constraint in the Landau-Ginzburg phase is that $\partial_i G$ annihilates a state for all $i = 1, \ldots, r$.

Let us consider now the vector space of points for each phase. In the geometric phase, since P must annihilate a state, we see that the space of states is entirely described by acting on the vacuum with the $Z_i^\dagger$'s. This then reproduces the toric prescription we have been using throughout this paper.

The Landau-Ginzburg phase is qualitatively different. At radius $N = -kn$, the vector space of points is given by:

$$\mathcal{F}_{\text{LG}} = \left\{ \,|\psi\rangle \in \mathcal{F}(\mathbb{C}^{r+1}) \,:\, \partial_i G\, |\psi\rangle = 0 \right\} \tag{5.11}$$

where $|\psi\rangle$ is an element of the Fock space $\mathcal{F}(\mathbb{C}^{r+1})$ spanned by states of the form

$$|n_i; k\rangle = \prod_{i=1}^r \frac{(Z_i^\dagger)^{n_i}}{\sqrt{n_i!}} \frac{(P^\dagger)^k}{\sqrt{k!}} |0\rangle \tag{5.12}$$

A candidate state is annihilated by r operators. This significantly limits the number of points, at any choice of the fuzzy radius. For example, for the quintic $G = Z_1^5 + \cdots + Z_5^5$, we have that for each i, $\partial_i G = 5Z_i^4$ annihilates the set of bosonic states. This means:

$$\mathcal{F}_{\text{LG}_{\text{quintic}}} = \text{span}\Big\{ |n_i; k\rangle \,:\, 0 \le n_i \le 3 \Big\} \tag{5.13}$$

Similar considerations hold for more generic choices of G.

One aspect of the commutative description which is missing from this discussion is the fuzzy analogue of renormalization group flow to the Landau-Ginzburg phase. It would be interesting to develop this discussion further. Let us also note that the approach to local mirror symmetry in [51] (see also [52]) relies heavily on the interplay between the toric phase of one geometry, and the Landau-Ginzburg phase of its mirror.

More generally, mirror symmetry constitutes a deep feature of stringy geometry. It is quite tempting to speculate that since toric geometry provides a natural setting for setting up mirror pairs [53], the Batyrev construction [54] may have a non-commutative analogue.

6 Physical applications

In this section we consider some other aspects and possible applications of Fuzz theory.

6.1 Discretuum of gauge couplings

One application already mentioned in the Introduction pertains to moduli stabilization. There is a natural limit where the volumes of four-cycles collapse to zero size. In this paper we have focussed on a limit where we expect these volumes to be at zero size, but in which we nevertheless retain a perturbative gauge coupling constant, which is effectively quantized in units of $1/g_s$. The basic relation we have found is that the number of points N determines the Yang-Mills coupling via:

$$\frac{g_{\mathrm{YM}}^2}{4\pi} = \frac{g_s}{N} \tag{6.1}$$

where N is in turn fixed by the amount of $\mathcal{B}$-field flux passing through the seven-brane.

Different values of the discrete parameter N define different vacua of the theory. These vacua are connected by domain walls in $\mathbb{R}^{3,1}$ which change the amount of $\mathcal{B}$-flux threading the seven-brane. A domain wall which can change the amount of $\mathcal{B}$-flux must be charged under the B-field, so we expect (in a perturbative IIB frame) the domain wall is given by an NS5-brane wrapping a three-chain Γ_3 with boundary on the two-cycle Poincaré dual to the cohomology class of $\mathcal{B}$.

To motivate this answer, consider again the case of the conifold transition with N_c D5-branes wrapping the S^2 of the resolved conifold. After passing through the geometric transition, the amount of $\mathcal{B}$-flux through the S^2 becomes the amount of NS three-form flux threading the non-compact three-cycle of the deformed conifold:

$$\int_{S^2} B_{\mathrm{NS}} = \int_{B_3} H_{\mathrm{NS}}. \tag{6.2}$$

In a non-compact setting, H_{NS} becomes quantized once we impose a suitable boundary condition for the non-compact directions of the geometry. In this case, we write $H_{\mathrm{NS}} = k \cdot H_{\mathrm{NS}}^{\min}$ where $H_{\mathrm{NS}}^{\min}$ integrates over B_3 to one unit of three-form flux. Changing the amount of three-form flux threading B_3 amounts to wrapping an NS5-brane on the three-cycle Poincaré dual to $H_{\mathrm{NS}}^{\min}$. In the present case, this is just the three-cycle B_3. Thus, the

tension of the domain wall connecting vacua is $|\Delta k| \operatorname{Vol}(B_3)/g_s^2$, where $|\Delta k|$ denotes the change in H_{NS} across the domain wall. Passing back to the original small resolution, this NS5-brane wraps a non-compact three-chain with boundary on the S^2.

One expects something similar to happen in the case of the fuzzy seven-brane theory. Here, the Poincaré dual of the flux $\mathcal{B}$ defines a two-cycle $[\mathcal{B}]$ inside of the surface wrapped by $\mathcal{S}$. An NS5-brane wrapping a non-compact three-chain with boundary on $[\mathcal{B}]$ connects vacua with different values of the $\mathcal{B}$-flux. Similar considerations hold in other duality frames.

Each value of N defines an effective tension for the seven-brane. Coupling this system to gravity and assuming all other contributions to the vacuum energy density remain fixed, this suggests that vacua with different values of N have different energy densities, which are connected by domain walls with tension $\operatorname{Vol}(\Gamma_3)$, where Γ_3 is the three-chain wrapped by the five-brane. Here we drop factors of g_s since it is an order one quantity in F-theory. In the thin-wall approximation, the tunnelling rate is then given by:

$$\Gamma_{N \to N-k} \sim \exp\left(-\frac{27\pi^2}{2} \cdot k \operatorname{Vol}(\Gamma_3)^4\right). \tag{6.3}$$

6.2 Threshold corrections

One of the limitations of higher-dimensional gauge theories is that at energy scales above the Kaluza-Klein scale, an entire tower of KK modes enter the spectrum limiting the range of validity for the effective field theory. It is then necessary to embed the theory in an ultraviolet completion of the field theory, such as string theory. In the case of fuzzy extra dimensions, the number of dynamical KK modes truncates at finite order, regulating the higher-dimensional theory.

At a practical level, making the internal directions non-commutative still retains much of the geometric structure of the original commutative space. Truncating the KK spectrum has another benefit, which is that it makes threshold corrections both easier to catalog and to compute. One loop perturbation theory in a gauge theory with N_{deg} degrees of freedom is valid provided:

$$\frac{g_{\mathrm{YM}}^2 N_{\mathrm{deg}}}{16\pi^2} \ll 1. \tag{6.4}$$

For a fuzzy seven-brane theory, we have the basic relation $\alpha_{\mathrm{GUT}} = g_s/N$. In tandem with the expectation that $N_{\mathrm{deg}} \sim N^2$, this suggests $g_{\mathrm{YM}}^2 N_{\mathrm{deg}}/16\pi^2 = g_s N/4\pi$. For $N \sim O(30)$, this might at first suggest that one loop perturbation theory breaks down, once all of the KK modes are included.

This counting is a bit naive, however, because supersymmetry will induce some additional suppression in the size of threshold corrections. Consider the contribution of the bulk KK modes of the seven-brane to the running of the four-dimensional gauge coupling. Up to subtleties connected with line bundle data (which will make some KK modes fill out rectangular matrices), the KK modes transform as $N \times N$ matrices, where N is the number of points. The mode content can be organized as N^2 $\mathcal{N} = 1$ adjoint-valued massive vector multiplets, and N^2 $\mathcal{N} = 2$ adjoint-valued hypermultiplets. Since each massive vector multiplet can be viewed as a massless vector multiplet and an adjoint-valued chiral superfield,

we see that the beta function vanishes at leading order in N^2. One does expect subleading corrections of order N, for example, from matter fields localized on curves. In this case, we obtain the more reliable estimate that the effective number of degrees of freedom entering a threshold correction is more on the order of $N_{\text{deg}} \sim N$. Thus, we obtain

$$\frac{g_{\text{YM}}^2 N_{\text{deg}}}{16\pi^2} \sim \frac{g_s}{4\pi} \ll 1 \tag{6.5}$$

so that perturbation theory remains meaningful. Detailed studies of threshold effects in commutative seven-branes in F-theory GUTs have appeared in [9, 55, 56]. It would be interesting to revisit these analyses in the present context.

6.3 Low scale fuzz

In theories with a low scale for quantum gravity, there is a potential worry that processes otherwise protected by making the Planck scale very high could now be violated more significantly. In some sense, this is just a reflection of the fact that a typical string compactification can look quite complicated at the string or Planck scale.

As we have emphasized in this paper, there are (at least) two Kaluza-Klein scales associated with an extra-dimensional theory in string theory: One for open string KK modes $M_{\text{open}}^{\text{KK}}$, and one for the closed string KK modes $M_{\text{closed}}^{\text{KK}}$. Taking this distinction seriously, we can in principle lower the scale of open string KK excitations all the way to the TeV scale:

$$M_{\text{open}}^{\text{KK}} \sim \text{TeV} \tag{6.6}$$

while still keeping the closed string modes and quantum gravity effects decoupled.[18]

This type of scenario certainly shares features similar to those of TeV scale extra dimensions [57], those of large extra dimensions [58, 59] and those with warped extra dimensions [60, 61].[19] Let us stress, however, that our limit is qualitatively different from these scenarios, because we have explicitly decoupled the closed string modes from the description. Indeed, since $M_{\text{closed}}^{\text{KK}} \gg M_{\text{open}}^{\text{KK}}$, non-commutative extra dimensions allow KK gluons and other KK open string excitations with no accompanying KK gravitons. This leads to some differences in the low energy phenomenology. For example, a common collider signature of extra-dimensional scenarios involves production of KK gravitons. With fuzzy extra dimensions, this signature would be absent.

There are clearly many possible variants of extra dimensional ideas and many mechanisms previously discussed in such frameworks likely possess fuzzy analogues. Though it is in principle possible to still maintain some aspects of GUTs in TeV to intermediate scale extra-dimensional models, (see for example [63–67]), putting $M_{\text{open}}^{\text{KK}}$ near the TeV scale instead suggests widening our scope beyond the GUT paradigm.

Indeed, one might contemplate models in which some or all of the Standard Model fields have non-commutative counterparts. In extra-dimensional models, it is common to regulate

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[18]In a scenario with TeV scale non-commutativity, but with $M_{\text{closed}}^{\text{KK}} = M_{pl}$ *finite*, the string scale M_{string} is equal to the geometric mean $M_{\text{string}} \sim \sqrt{M_{\text{open}}^{\text{KK}} \cdot M_{\text{closed}}^{\text{KK}}} \sim 10^{11}$ GeV.

[19]See for example [62] for a collider study of some aspects of non-commutative extra dimensions.

the extra dimensions with a lattice formulation. At the very least, working with a fuzzy prescription would provide a way to maintain more of the smooth structure of the internal geometry, and may also open up new avenues for model building with extra dimensions.

7 Towards a holographic dual

The seven-brane on a non-commutative four-cycle with N points can be viewed as tiled by N D3-branes that each occupy one Planck cell. In the limit where the number of fuzzy points becomes large, the D3-branes form a dense mesh, and the non-commutative geometry converges to the commutative description. This structure hints at a possible holographic dual description.[20] Here we make some qualitative remarks about the form of this holographic dual. Though the application is different, the appearance of fuzzy geometry in Klebanov-Strassler type solutions has been studied for example in [68]. Our situation also shares some similarities with the description given in [69] of extremal black holes in IIA string theory in terms of D0-branes tiling the horizon (see also [70]).

Consider a seven-brane with gauge group $G = \mathrm{SU}(N_c)$ wrapped on a non-commutative four-cycle with a large number of fuzzy points, N. The qualitative behavior of this theory depends on the energy scale. At energies below the scale of Kaluza-Klein excitations, the low energy theory consists of a single four-dimensional $\mathrm{SU}(N_c)$ gauge theory. The number of seven-branes N_c cannot be too large without overclosing the compactification [28]. As we proceed up in energy, additional light modes begin to enter the spectrum. For instance, the vector multiplet of the non-commutative theory is given by a $(0, 0)$-form which can be viewed as an $N \times N$ matrix:

$$V^{i\bar{i}} \in Mat(N \times N). \tag{7.1}$$

The other bulk KK modes roughly fill out $N \times N$ matrices with shifts $N \to N + q_i$ set by line bundle data. Matter on curves crudely correspond to $N^{1/2} \times N^{1/2}$ matrices, though the precise exponent of the N scaling depends on details of the matter curve.

Once we pass the threshold for the open string KK states, we find additional degrees of freedom. For example, the N^2 massive vector multiplets can be viewed as N^2 massless vector multiplets transforming in N^2 copies of $\mathrm{SU}(N_c)$, and N^2 adjoint-valued chiral superfields which give mass to these vector multiplets through their vevs.

The appearance of so many additional vector multiplets suggests that we have tiled the seven-brane with a collection of D3-branes. Indeed, turning on a non-trivial $\mathcal{B}$-flux through the worldvolume of the seven-brane induces a net D3-brane charge equal to the number of points N. These D3-branes have finite extent because they experience a collective Myers effect induced by the background fluxes [71], and so rather than sitting at commutative points, the D3-branes puff up and sit at fuzzy points of the non-commutative geometry. Indeed, though it is tempting to speak of the classical position of N D3-branes located at various positions on the seven-brane, in the non-commutative setting we cannot simultaneously

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[20]In practical applications, $\alpha_{\mathrm{GUT}} = g_s/N$ so that $N \sim O(30)$ when g_s is an order one parameter (as in F-theory). Insofar as this is a large number, this expansion is also potentially relevant in more phenomenological applications.

diagonalize these positions. The configuration of D3-branes is therefore better viewed as filling out an $N \times N$ matrix. The gauge group of this configuration exactly matches what we expect in the fuzzy theory. The bifundamentals correspond to $3-3'$ strings stretched between the various D3-brane factors, and set the mass scale of open string Kaluza-Klein excitations.

At even higher energies, we pass back to an unHiggsed theory in which the $3-3'$ strings no longer condense. In this limit, the N D3-branes sit on top of each other at the tip of the singularity, and organize according to an $\mathrm{SU}(N_c \times N)$ gauge theory, with some appropriate matter content and scalar potential that reflects the presence of the local del Pezzo geometry. Viewed as a collection of $N_c \times N$ D3-branes sitting at a single point of the geometry, the $\mathrm{SU}(N_c \times N)$ gauge theory has Yang Mills coupling $g_{\mathrm{HI}}^2 = 4\pi g_s$. Taking the large N limit suggests organizing our perturbative expansion according to the 't Hooft parameter:

$$\lambda_{\mathrm{HI}} = g_{\mathrm{HI}}^2 N_c \times N = 4\pi g_s N_c \times N. \tag{7.2}$$

At large λ_{HI}, we expect to obtain a holographic dual to the stack of $N_c \times N$ D3-branes, supported by the five-form flux F_5 [72]. The closed string modes in this dual geometry correspond to glueball like excitations of the large N gauge theory. Since the number of chiral matter fields is assumed to be much smaller than $N_c \times N$, the flavor seven-branes can to first order be treated in a probe approximation.

We could ask how one would recover the low energy fuzzy description from this holographic dual perspective. Descending from the high energy theory, one enters a regime where the wave functions of the D3-branes spread out over the seven-brane. Viewed from a classical perspective, we are performing a breaking pattern of the form $\mathrm{SU}(N_c \times N) \supset \mathrm{SU}(N_c)^N$, where we rearrange the three-brane configuration into N stacks of N_c D3-branes. The gauge coupling of the parent gauge theory descends directly to each stack of D3-branes so that $g_{\mathrm{HI}}^2 = g_{\mathrm{MID}}^2$. Note, however, that in the vicinity of each single stack, the gauge group is only $\mathrm{SU}(N_c)$. The radius of curvature of a putative holographic dual to this smaller stack would then be set by the 't Hooft coupling $\lambda_{\mathrm{MID}} = 4\pi g_s N_c$. Insofar as g_s is at most order one, and since we cannot arbitrarily increase the rank of a seven-brane gauge group, we see that the holographic dual of this configuration will have a string scale curvature.

Finally, we descend back down to the Higgs branch of the D3-brane configuration, in which only a single $\mathrm{SU}(N_c)$ gauge group factor remains. This can be viewed as embedded in the diagonal subgroup of $\mathrm{SU}(N_c)^N$. This breaking pattern affects the value of the gauge coupling constant of the gauge theory on the Higgs branch so that:

$$g_{\mathrm{LO}}^2 = \frac{g_{\mathrm{HI}}^2}{N} = \frac{4\pi g_s}{N}. \tag{7.3}$$

Thus we recover the by now familiar relation $g_{\mathrm{YM}}^2 = g_{\mathrm{LO}}^2 = 4\pi g_s/N$.

Even though in the IR, the gauge symmetry has been broken to $\mathrm{SU}(N_c)$, the UV presence of the large gauge group $\mathrm{SU}(N \times N_c)$ is still reflected in the spectrum of KK excitations of the low energy theory. This allows one to organize the perturbation theory of the fuzzy seven-brane in terms of a large N expansion [73]. For example, the vector multiplets organize as an $N \times N$ matrix $V^{i\bar{i}}$. The Feynman rules require that in
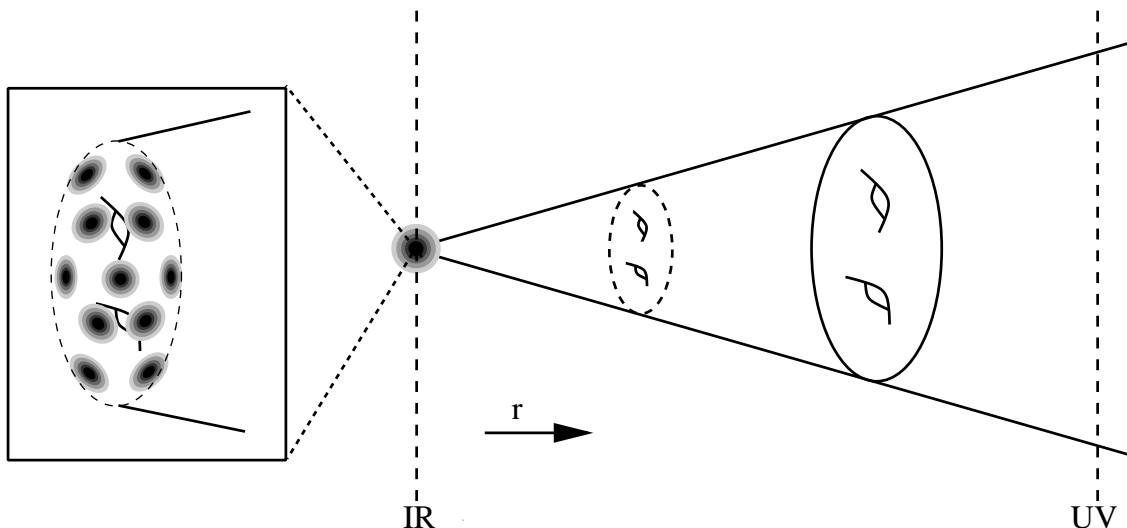
Figure 3. Depiction of the holographic dual to a seven-brane wrapping a non-commutative four-cycle. In the open string picture, this configuration can alternatively be viewed as a collection of N D3-branes sitting at the fuzzy points of the geometry. At higher holographic energy scales r, the D3-branes can clump together, and we recover a corresponding weakly curved holographic dual.

interactions involving the vector multiplets $V^{i\bar{i}}$, an i index must link up with an $\bar{i}$ index. This means that we have an analogue of the double line notation. In this $1/N$ expansion, the corresponding 't Hooft coupling would now be:

$$\lambda_{\text{LO}} \equiv g_{\text{YM}}^2 N_c \times N = 4\pi g_s N_c. \tag{7.4}$$

The resulting large N expansion for amplitudes can then be interpreted as the loop expansion of a closed string theory which lives at the highly curved tip of the closed string geometry. Indeed, for small N_c the 't Hooft coupling is order one indicating that the holographic dual geometry has string scale curvature. See figure 3 for a depiction.

8 Conclusions

Local model building provides a general strategy for embedding the Standard Model in string theory by focussing first on the open string degrees of freedom, keeping the closed string degrees of freedom decoupled. For seven-branes wrapping a compact four-cycle, fully decoupling the closed string sector requires working in the zero slope limit, and simultaneously demanding that this four-cycle collapses to zero closed string volume. The open string volume is still non-zero, and is supported by a two-form flux $\mathcal{B}$. This is a quite natural regime for high scale stabilization of Kähler moduli, and further, leads to significant simplifications in the dynamics of the higher-dimensional gauge theory defined by the seven-brane. In this limit, the internal geometry becomes non-commutative, and the spectrum of KK excitations for the seven-brane theory truncates. We have developed a general effective action for fuzzy seven-branes, and have explored some of the consequences of non-commutative extra dimensions. We now turn to some further avenues of investigation.

Our approach to non-commutative toric spaces has been based on effectively quantizing the data of a classical gauged linear sigma model. This appears to be a physically reasonable starting point in particular because $\mathcal{B}$-fields threading the worldvolume of D-branes on $\mathbb{C}^r$ are well-understood, and the symplectic quotient provides a natural formulation for treating more general toric spaces. Even so, the quantization procedure is still at the level of a motivated ansatz. *Deriving* this structure from the gauged linear sigma model would likely provide further insight into stringy realizations of fuzzy toric spaces. Along these lines, we have seen that there are analogues of many by now standard features of GLSM constructions such as flop transitions and Landau-Ginzburg phases. It is far from clear that this exhausts the full list of possibilities, and it may be that there are transitions which do not possess commutative analogues.

We have seen that the dynamics of seven-branes admits a $1/N$ expansion in the number of points N of the non-commutative geometry. This suggests an intuitive picture based on viewing the seven-brane as a collection of N puffed up D3-branes tiling the interior. Developing gravity duals of such D3-brane configurations would likely provide further insight into strongly coupled gauge theories built from such seven-branes, and would open up the possibility of finding a holographic dual of weakly coupled low rank gauge theories, like the Standard Model. This picture also suggests an intriguing interpolation between a single unified SU(5) GUT model on a fuzzy seven-brane, and the collective dynamics of $5 \times N$ D3-branes. Though this is similar to what is expected in deconstructed extra-dimensional theories [74], the construction we have presented retains additional geometric structure which is typically absent in lattice regularization of higher-dimensional theories.

Acknowledgments

A Fuzzy $\mathbb{P}^1$

In this appendix we present some additional features of fuzzy $\mathbb{P}^1$. We begin by making contact with an alternative formulation of this space, which is equivalent to what we have discused.

We start with commutative coordinates x_i of $\mathbb{R}^3$ subject to the condition:

$$x_1^2 + x_2^2 + x_3^2 = R^2. \tag{A.1}$$

The ordinary non-commutative construction then quantizes this information by promoting the coordinates x_i to operators X_i which satisfy the commutation relations of the $su(2)$ algebra:

$$[X_i, X_j] = i\theta\varepsilon_{ijk}X_k. \tag{A.2}$$

Here θ is a formal quantization parameter. The overall size of the sphere is then fixed by specifying a spin j representation of the angular momentum algebra:

$$X_1^2 + X_2^2 + X_3^2 = \theta^2 j(j+1). \tag{A.3}$$

Rescaling the X's can be absorbed into a change of the quantization parameter θ. One common convention is $\theta^2 = 1/j$, so that the volume of the sphere scales as $j+1$.

To make contact between this description and the construction we have been using in this paper, we first review the commutative toric description of $\mathbb{P}^1$. Recall that in the commutative theory, $\mathbb{P}^1$ can be defined in terms of the GLSM of a single U(1) with two chiral superfields u_1, u_2 of charge $+1$. The volume of the $\mathbb{P}^1$ is then fixed by the D-term constraint:

$$|u_1|^2 + |u_2|^2 = \zeta. \tag{A.4}$$

We now quantize this data. Denote the analogue of the u_i by the quantized variables U_i. Let $|0\rangle$ denote the vacuum state annihilated by the U's. The Fock space $\mathcal{F}(\mathbb{C}^2)$ is given by all states obtained by acting on $|0\rangle$ with the $U^\dagger$'s. The spectrum of the D-term operator $D = U_1^\dagger U_1 + U_2^\dagger U_2$ then defines a grading of $\mathcal{F}(\mathbb{C}^2)$. We denote by $\mathcal{F}_X(N)$ the bosonic points of $X = \mathbb{P}^1$ with quantized FI parameter N. The space $\mathcal{F}_X(N)$ is spanned by the states:

$$|\, n_1, n_2 \rangle = \frac{(U_1^\dagger)^{n_1}(U_2^\dagger)^{n_2}}{\sqrt{n_1! n_2!}}|0\rangle \tag{A.5}$$

with $n_1 + n_2 = N$.

Associated with the U's there is a canonical $su(2)$ algebra given as:

$$J_i = \frac{1}{2}U^\dagger \sigma_i U \tag{A.6}$$

where $U^\dagger$ is a two-component vector, and the σ's are the Pauli matrices. Introducing $J_+ = J_1 + iJ_2$ and $J_- = J_1 - iJ_2$, the angular momentum generators are:

$$\begin{aligned} J_+ &= U_1^\dagger U_2 \\ J_- &= U_2^\dagger U_1 \\ J_3 &= \frac{1}{2}(U_1^\dagger U_1 - U_2^\dagger U_2). \end{aligned} \tag{A.7}$$

The South pole state of $\mathbb{P}^1$ corresponds to:

$$|\text{South}\rangle = |\, 0, N \,\rangle. \tag{A.8}$$

Indeed, acting by J_- annihilates this state, and acting by J_+ $N+1$ times yields, up to a constant of proportionality, the North pole state:

$$|\text{North}\rangle = |\, N, 0 \,\rangle. \tag{A.9}$$

Line bundle cohomology on fuzzy $\mathbb{P}^1$. We now discuss line bundle cohomology for fuzzy $\mathbb{P}^1$. Our general strategy is to map the algebraic data of the commutative theory to the non-commutative geometry.

A general zero form of the degree Q line bundle $\mathcal{O}_{\mathbb{P}^1}(Q)$ can be presented as a polynomial in the u's and $\overline{u}$'s with net GLSM charge Q. Global sections of $\mathcal{O}_{\mathbb{P}^1}(Q)$ correspond to degree Q polynomials in the u's:

$$H^0(\mathbb{P}^1, \mathcal{O}(Q)) = \left\{ G(u_1, u_2) | \deg G = Q \right\} \tag{A.10}$$

This implies that there are global sections for $Q \geq 0$, but for $Q < 0$, $\mathcal{O}_{\mathbb{P}^1}(Q)$ does not possess any global sections. In particular, for $Q \geq 0$, the number of global sections is $Q + 1$.

The case of $(0,1)$-forms is similar. The canonical line bundle of $\mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}(-2)$, so a natural basis element for holomorphic one-forms will be $de_{\mathbb{P}^1}$, where $e_{\mathbb{P}^1}$ has GLSM charge $+2$. The condition that this monomial remains well-defined over all of $\mathbb{P}^1$ effectively fixes $e_{\mathbb{P}^1}$ to be $u_1 u_2$, since the differentials $d(u_1^2)$ and $d(u_2^2)$ both vanish on $\mathbb{P}^1$. Note that $d(u_1 u_2) = u_1 du_2 + u_2 du_1$, so that since u_1 and u_2 cannot both simultaneously vanish, this differential is the unique choice.[21]

To fix actual representatives, it is actually enough to work in terms of the Serre dual description of the cohomology group. Using the isomorphism:

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(Q)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-Q) \otimes \mathcal{O}_{\mathbb{P}^1}(-2))^*, \tag{A.11}$$

we conclude that the elements of $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(Q))$ can be viewed as living in the dual space to $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-Q-2))$. In other words, a convenient basis for this cohomology group can be presented as:

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(Q)) \simeq \{G(u_1^*, u_2^*) \otimes d(\overline{u_1 u_2}) : -\deg G - 2 = Q\} \tag{A.12}$$

where the u^*'s are to be viewed as dual coordinates such that $u^*(u) = 1$. This convention requires the dual coordinates u^* to have GLSM charge -1. Since the overall GLSM charge of the one-forms is fixed to be Q, this implies $\deg G = -Q-2$, so that for $Q \geq -1$, the cohomology group vanishes, and for $Q < -2$, the dimension of the cohomology group is $-Q-1$.

We now turn to the fuzzy cohomology groups. The quantized analogue of du_i is the fermionic oscillator C_i. To distinguish from the cohomology of the commutative geometry, we refer to our $\mathbb{P}^1$ as X. Fixing a degree Q line bundle $L^{(Q)}$, the space $H^0(X, L^{(Q)})$ is spanned by the operators:

$$H^0(X, L^{(Q)}) = \{G(U_1, U_2) : [D, G] = -QG\} \tag{A.13}$$

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[21]The set $\{u_1 u_2\}$ for $\mathbb{P}^1$ generates the Stanley-Reisner ideal for $\mathbb{P}^1$. More generally, for a toric variety X, we can form a collection of minimal monomials such that for each monomial, not all of the factors can simultaneously vanish. The main utility for our purposes is that such elements provide a convenient basis of monomials to construct differential forms from. Stanley-Reisner ideals figure prominently in the proposed algorithmic approach to line bundle cohomology in [42]. It would be interesting to use this algorithmic approach to line bundle cohomology to find explicit representatives of fuzzy cohomology groups.

Indeed, acting on the bra vacuum state $\langle 0|$ by elements of $H^0(X, L^{(Q)})$ generates a $Q + 1$ dimensional vector space which is isomorphic to the dual vector space of points of a fuzzy $\mathbb{P}^1$ of size Q:

$$\mathcal{F}_X^*(Q) = \{\langle 0|\, G(U_1, U_2) : [D, G] = -QG\}. \tag{A.14}$$

Note that having non-trivial $H^0(X, L^{(Q)})$ requires $Q \geq 0$.

Next consider the elements of the first cohomology group. The fuzzy analogue of Serre duality is provided by acting with elements of $H^0(X, L^{(Q)})$ on the bra vacuum state $\langle 0|$, and elements of $H^1(X, L^{(Q)})$ on the ket vacuum state $|0\rangle$. Indeed, there is a canonical pairing between U and $U^\dagger$. Thus, elements of $H^1(X, L^{(Q)})$ correspond to operators built exclusively from creation operators. We restrict our discussion to cases where the commutative theory cohomology group is non-trivial so that $Q < -1$. Exploiting the presence of this canonical pairing, we can make the identification:

$$H^1(X, L^{(Q)}) = \{G(U_1^\dagger, U_2^\dagger) \times (U_1^\dagger C_2^\dagger + U_2^\dagger C_1^\dagger) : [D, G] = (Q + 2)G\}. \tag{A.15}$$

Indeed, acting on the ket vaccum state $|0\rangle$ by degree $-Q - 2$ polynomials in the $U^\dagger$'s, we see that each of the states of $\mathcal{F}_X(-Q - 2)$ naturally pairs with a state of $\mathcal{F}_X^*(-Q - 2)$ associated with acting on the bra vacuum state $\langle 0|$ with operators of the cohomology group $H^0(X, L^{(-Q-2)})$.

Let us note that in our discussion the $U^\dagger$'s play two roles: first as anti-holomorphic coordinates for functions, and second as Serre dual coordinates for cohomology groups.

B Intersection theory on fuzzy $\mathbb{P}^1 \times \mathbb{P}^1$

In this subsection we develop intersection theory on fuzzy $\mathbb{P}^1 \times \mathbb{P}^1$. To set notation, we introduce oscillators U_i and V_i for the respective factors $\mathbb{P}_I^1$ and $\mathbb{P}_{II}^1$ of $\mathbb{P}_I^1 \times \mathbb{P}_{II}^1$. We denote the vector space of bosonic points for $\mathbb{P}_I^1 \times \mathbb{P}_{II}^1$ at Kähler classes N_I and N_{II} by $\mathcal{F}_X(N_I, N_{II})$.

In the commutative theory, divisors are defined as the vanishing locus of a bihomogeneous polynomial in the u's and v's. In the non-commutative geometry, this locus is associated with the linear subspace of $\mathcal{F}_X(N_I, N_{II})$ annihilated by the corresponding operator in U and V.

In the commutative geometry, divisors will typically intersect at some number of points. We now discuss the analogous intersection number for fuzzy divisors. Given two operators P and Q of respective bidegrees (p_1, p_2) and (q_1, q_2), the number of points in the intersection is:

$$\#(P = 0) \cap (Q = 0) = \dim(\ker P \cap \ker Q). \tag{B.1}$$

Intersection theory in the commutative theory is fully determined by the bidegrees of the bihomogeneous polynomials P and Q. Indeed, letting σ_1 and σ_2 denote the divisors corresponding to the two $\mathbb{P}^1$ factors of $\mathbb{P}^1 \times \mathbb{P}^1$ such that $\sigma_i \cap \sigma_j = 1 - \delta_{ij}$, the divisor classes for P and Q are:

$$\begin{aligned} [P] &= p_1\sigma_2 + p_2\sigma_1 \\ [Q] &= q_1\sigma_2 + q_2\sigma_1. \end{aligned} \tag{B.2}$$

The intersection number of P and Q in the commutative theory is then given by:

$$[P] \cap [Q] = (p_1\sigma_2 + p_2\sigma_1) \cap (q_1\sigma_2 + q_2\sigma_1) = p_1q_2 + p_2q_1. \tag{B.3}$$

We now establish a similar relation in the non-commutative geometry, with the intersection number given by equation (B.1). In order to keep our discussion as "generic" as possible, we assume that each linear map induced by acting by either P or Q has trivial cokernel.[22]

At the level of set theory, the mutual intersection of the two kernels is:

$$\dim(\ker P \cap \ker Q) = \dim\ker P + \dim\ker Q - \dim\ker PQ. \tag{B.4}$$

Since we are assuming trivial cokernel for each operator, the rank nullity theorem implies:

$$\begin{aligned}
\dim\ker P &= N_I N_{II} - (N_I - p_1)(N_{II} - p_2) \\
\dim\ker Q &= N_I N_{II} - (N_I - q_1)(N_{II} - q_2) \\
\dim\ker PQ &= N_I N_{II} - (N_I - p_1 - q_1)(N_{II} - p_2 - q_2).
\end{aligned} \tag{B.5}$$

Returning to equation (B.4), we find:

$$\dim(\ker P \cap \ker Q) = p_1q_2 + p_2q_1, \tag{B.6}$$

which we recognize as the intersection number for divisors in the commutative geometry. Let us note that in less generic situations where some of the maps are not onto, or in cases where the action by an operator annihilates all points, the effects of the non-commutativity will be more prominent. It would be interesting to develop intersection theory for more general fuzzy toric spaces.

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[22]Recall that given a linear map $T : A \to B$ between vector spaces A and B, the cokernel is defined as $\mathrm{coker}(T) \equiv B/\mathrm{im}(T)$. Having trivial cokernel means that the map T is onto, so that for every vector b in B, there exists a vector a in A such that $T(a) = b$.

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