

Scalar solitons and the microscopic entropy of hairy black holes in three dimensions

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ABSTRACT: General Relativity coupled to a self-interacting scalar field in three dimensions is shown to admit exact analytic soliton solutions, such that the metric and the scalar field are regular everywhere. Since the scalar field acquires slow fall-off at infinity, the soliton describes an asymptotically AdS spacetime in a relaxed sense as compared with the one of Brown and Henneaux. Nevertheless, the asymptotic symmetry group remains to be the conformal group, and the algebra of the canonical generators possesses the standard central extension. For this class of asymptotic behavior, the theory also admits hairy black holes which raises some puzzles concerning a holographic derivation of their entropy *à la* Strominger. Since the soliton is devoid of integration constants, it has a fixed (negative) mass, and it can be naturally regarded as the ground state of the “hairy sector”, for which the scalar field is switched on. This assumption allows to exactly reproduce the semiclassical hairy black hole entropy from the asymptotic growth of the number of states by means of Cardy formula. Particularly useful is expressing the asymptotic growth of the number of states only in terms of the spectrum of the Virasoro operators without making any explicit reference to the central charges.

KEYWORDS: Black Holes, Classical Theories of Gravity, AdS-CFT Correspondence

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1 Introduction

Scalar fields propagating on a fixed AdS background reveal several interesting features. As shown during the 80's, their energy turns out to be positive provided the squared mass is bounded from below by a negative quantity, according to

$$m^2 \geq m_*^2 := -\frac{(d-1)^2}{4l^2} .$$

This is known as the Breitenlohner-Freedman bound [1–3], which guarantees the stability of an AdS spacetime of radius l against scalar field perturbations. Particularly interesting is the case of scalar fields whose mass is within the range

$$m_*^2 \leq m^2 < m_*^2 + \frac{1}{l^2} , \tag{1.1}$$

since they are able to acquire slow fall-off at infinity. In turn, this generates a strong back reaction of the metric in the asymptotic region, such that in certain cases they cannot be treated as a probe [4–6] (see also [7, 8]). As a consequence, the asymptotic behavior of the metric has to be relaxed as compared with the standard one [9–11] for a localized distribution of matter. This has the effect of enlarging the space of physically admissible configurations, bringing in new classes of solutions including hairy black holes and solitons [4, 12–18]. Finding exact analytic solutions that circumvent no hair theorems is not

an easy task.¹ One of the simplest and instructive examples where this can be achieved corresponds to General Relativity with a minimally coupled self-interacting scalar field in three dimensions [4]. The action is given by

$$I[g_{\mu\nu}, \phi] = \frac{1}{\pi G} \int d^3x \sqrt{-g} \left[\frac{R}{16} - \frac{1}{2}(\nabla\phi)^2 - V(\phi) \right], \quad (1.2)$$

with the following self-interaction potential

$$V(\phi) = -\frac{1}{8l^2} (\cosh^6 \phi + \nu \sinh^6 \phi), \quad (1.3)$$

having a global maximum at $\phi = 0$, such that $V(0) = -\frac{1}{8l^2}$, and a mass term given by $m^2 = V''(0) = -\frac{3}{4l^2}$, which lies within the range (1.1). This potential has a simple interpretation in the conformal (Jordan) frame (see appendix A).

2 Hairy black holes and relaxed AdS asymptotics

When the self-interaction parameter fulfills $\nu \geq -1$, apart from the BTZ black hole [33, 34], which is a solution of this action in vacuum, i.e., with a vanishing scalar field, the field equations admit an exact, static and circularly symmetric hairy black hole [4]. The solution possesses a non trivial scalar field given by

$$\phi(r) = \operatorname{arctanh} \sqrt{\frac{B}{H(r) + B}}, \quad (2.1)$$

where $H(r) = \frac{1}{2} (r + \sqrt{r^2 + 4Br})$, and the metric reads

$$ds^2 = - \left(\frac{H}{H+B} \right)^2 F(r) dt^2 + \left(\frac{H+B}{H+2B} \right)^2 \frac{dr^2}{F(r)} + r^2 d\varphi^2, \quad (2.2)$$

with $F(r) = \frac{H^2}{l^2} - (1+\nu) \left(\frac{3B^2}{l^2} + \frac{2B^3}{l^2 H} \right)$, and the coordinates range as $-\infty < t < \infty$, $r > 0$, $0 \leq \varphi < 2\pi$. Note that the hairy black hole is well-defined for $\nu \geq -1$, and it depends on a single non-negative integration constant B , such that the scalar field is regular everywhere. The curvature singularity at the origin is enclosed by an event horizon located at

$$r_+ = B\Theta_\nu, \quad (2.3)$$

where

$$\Theta_\nu = 2(z\bar{z})^{\frac{2}{3}} \frac{z^{\frac{2}{3}} - \bar{z}^{\frac{2}{3}}}{z - \bar{z}}, \quad \text{with } z = 1 + i\sqrt{\nu}, \quad (2.4)$$

is a function of the parameter ν appearing in the potential. The Hawking temperature and the mass, are given by

$$T = \frac{3B(1+\nu)}{2\pi l^2 \Theta_\nu}, \quad \text{and} \quad M = \frac{3B^2(1+\nu)}{8Gl^2}, \quad (2.5)$$

¹Numerical solutions have also been found in [19–25]. The case of minimally coupled scalar fields with electric charge has attracted much recent attention concerning holographic superconductivity [26, 27] (for good reviews see, e.g. [28–30]). In this context, hairy black hole solutions have also been found numerically [31, 32].

respectively,² and the entropy reads

$$S = \frac{A}{4G} = \frac{\pi r_+}{2G}. \tag{2.6}$$

As also shown in [4], the hairy black hole describes an asymptotically AdS spacetime in a relaxed sense as compared with the standard one of Brown and Henneaux [9]. Indeed, the asymptotic behavior of the hairy black hole belongs to the following class:

$$\phi = \frac{\chi}{r^{1/2}} + \alpha \frac{\chi^3}{r^{3/2}} + O(r^{-5/2}) \tag{2.7}$$

$$\begin{aligned} g_{rr} &= \frac{l^2}{r^2} - \frac{4l^2\chi^2}{r^3} + O(r^{-4}) & g_{tt} &= -\frac{r^2}{l^2} + O(1) \\ g_{tr} &= O(r^{-2}) & g_{\varphi\varphi} &= r^2 + O(1) \\ g_{\varphi r} &= O(r^{-2}) & g_{t\varphi} &= O(1) \end{aligned} \tag{2.8}$$

where $\chi = \chi(t, \varphi)$, and α is an arbitrary constant. Remarkably, this set of asymptotic conditions is also left invariant under the conformal group in two dimensions, spanned by two copies of the Virasoro algebra. It was found that the effect of relaxing the asymptotic conditions is such that the generators of the asymptotic symmetries acquire a nontrivial contribution from the scalar field. Following the Regge-Teitelboim approach [40], the canonical generators are given by

$$Q(\xi) = \frac{1}{16\pi G} \int d\varphi \left\{ \frac{\xi^\perp}{lr} \left((g_{\varphi\varphi} - r^2) - 2r^2(lg^{-1/2} - 1) \right) + 2\xi^\varphi \pi^r_\varphi + \xi^\perp \frac{2r}{l} \left[\phi^2 - 2l \frac{\phi \partial_r \phi}{\sqrt{g_{rr}}} \right] \right\}, \tag{2.9}$$

where the massless BTZ black hole has been chosen as reference background. The Poisson brackets of these generators were shown to span two copies of the Virasoro algebra with the standard central charges:

$$c^+ = c^- = c = \frac{3l}{2G}. \tag{2.10}$$

3 Microscopic entropy of the hairy black hole: holographic puzzle and its resolution

The existence of hairy black holes, for $\nu \geq -1$, that fit within a set of asymptotic conditions being invariant under the conformal group at the boundary, whose corresponding charge algebra acquires a nontrivial central extension, makes compulsory wondering whether their entropy could be obtained from the asymptotic growth of the number of states by means of Cardy formula [41], as it is the case for the BTZ black hole [42]. Once the hairy black hole entropy (2.6) is expressed in terms of its mass,

$$S = \pi l \Theta_\nu \sqrt{\frac{2M}{3G(1+\nu)}}, \tag{3.1}$$

²The mass of the hairy black hole has also been obtained following different approaches in refs. [35–39].

at a first glance, for a reader that is slightly familiarized with the standard form of the Cardy formula,

$$S = 2\pi\sqrt{\frac{c^+}{6}\tilde{\Delta}^+} + 2\pi\sqrt{\frac{c^-}{6}\tilde{\Delta}^-}, \quad (3.2)$$

where $\tilde{\Delta}^\pm = \frac{1}{2}(Ml \pm J)$ are the eigenvalues of the left and right Virasoro operators, it doesn't seem so obvious how this task could be successfully performed. In order to clarify the picture, some remarks are worth to be pointed out:

- For a fixed value of the mass M , there are (at least) two different static and circularly symmetric black holes; namely, the BTZ black hole (in vacuum), and the hairy black hole (when the nontrivial scalar field is switched on).
- Note that, since the hairy black hole depends on a single integration constant, the scalar field cannot be switched off keeping the mass fixed. This means that the hairy and BTZ black holes cannot be smoothly deformed into each other, which suggests that they belong to different disconnected sectors.

Thus, for a fixed value of the mass, despite the fact that there are (at least) two different black hole configurations, eq. (3.2) only reproduces the entropy of the BTZ black hole, which corresponds to the vacuum sector. This raises a puzzle: How can the entropy of the hairy black hole be obtained by means of Cardy formula?

As it is explained in section 5, formula (3.2) implicitly assumes that the ground state is AdS spacetime. Thus, since the entropy obtained from (3.2) does not reproduce (at least) the sum of the entropy of the hairy and vacuum black holes, AdS spacetime should be regarded as a suitable ground state *only* for the vacuum sector. Hence, if holography works, one should expect that the hairy sector possesses a different ground state, disconnected from the vacuum one, so that Cardy formula successfully reproduces the entropy of the hairy black hole when this is taken into account.

The resolution of the puzzle comes from the fact that the suitable ground state for the hairy sector indeed exists and it is described by a soliton. The full picture can then be summarized as follows:

- The BTZ and the hairy black hole belong to disconnected sectors, each with its own ground state. For the vacuum sector, the ground state corresponds to AdS spacetime, while for the hairy sector, the ground state corresponds to the scalar soliton.
- The soliton fulfills what is expected for a ground state: It is smooth and regular everywhere, as well as devoid of integration constants.
- In analogy with what occurs for the vacuum, the energy spectrum of the hairy sector consists of a continuous part bounded from below by zero (hairy black holes), a gap (naked singularities), and a ground state with negative mass fixed by the fundamental constants of the theory (soliton).

It is then reassuring to verify that Cardy formula reproduces the entropy of the hairy black hole in exact agreement with the semiclassical result. This is performed in section 5. In what follows, the precise analytic form of the scalar soliton is discussed.

4 Scalar soliton

The field equations that correspond to the action (1.2) with the self-interaction potential (1.3), in the case of $\nu = 0$ admit the following exact solution:

$$ds^2 = l^2 \left(-\frac{4(1+\rho^2)^4}{(3+2\rho^2)^2} d\tau^2 + \frac{64(1+\rho^2)^3}{(3+2\rho^2)^4} d\rho^2 + \frac{64}{81} \rho^2 (1+\rho^2) d\varphi^2 \right), \quad (4.1)$$

with

$$\phi(\rho) = \operatorname{arctanh} \sqrt{\frac{1}{3+2\rho^2}}, \quad (4.2)$$

where the coordinates range according to $-\infty < \tau < \infty$, $0 \leq \rho < \infty$, and $0 \leq \varphi < 2\pi$. Note that the solution is devoid of integration constants. It is also simple to verify that the soliton is smooth everywhere and it fits within the relaxed set of asymptotically AdS conditions given by eq. (2.8). This latter property can be explicitly checked by performing a coordinate transformation defined through

$$\tau = \frac{8t}{9l}, \quad \rho = \sqrt{\frac{9r}{8l} - \frac{1}{2}}, \quad (4.3)$$

with $r \geq \frac{4l}{9}$, so that the metric and the scalar field read

$$ds^2 = -\frac{(4l+9r)^4}{81l^2(8l+9r)^2} dt^2 + \frac{81l^2(4l+9r)^3}{(9r-4l)(8l+9r)^4} dr^2 + \left(r^2 - \frac{16}{81}l^2 \right) d\varphi^2, \quad (4.4)$$

$$\phi(r) = \operatorname{arctanh} \sqrt{\frac{4l}{8l+9r}}, \quad (4.5)$$

respectively.

In the case of $\nu > -1$, the solution generalizes as

$$ds^2 = l^2 \left(1 + \frac{1}{\alpha_\nu(1+\rho^2)} \right)^{-2} \times \left(-(1+\rho^2)^2 d\tau^2 + \frac{4d\rho^2}{2+\rho^2+\frac{c_\nu}{1+\rho^2}} + \left(\frac{2}{2+c_\nu} \right)^2 \rho^2 \left(2+\rho^2 + \frac{c_\nu}{1+\rho^2} \right) d\varphi^2 \right), \quad (4.6)$$

with

$$\phi(\rho) = \operatorname{arctanh} \sqrt{\frac{1}{1+\alpha_\nu(1+\rho^2)}}, \quad (4.7)$$

and where³

$$\alpha_\nu := \frac{1}{2} \left(\Theta_\nu + \sqrt{\Theta_\nu^2 + 4\Theta_\nu} \right) \quad \text{and} \quad c_\nu := \frac{2(1+\nu)}{\alpha_\nu^3}, \quad (4.8)$$

just depend on the self-interaction parameter ν , and Θ_ν is defined in eq. (2.4), so that there are no integration constants. The soliton is regular and its causal structure coincides with the one of AdS spacetime. Applying the following change of coordinates:

$$\tau = \frac{\Theta_\nu \alpha_\nu}{3(1+\nu)l} t, \quad \rho = \sqrt{\frac{3(1+\nu)r}{\alpha_\nu \Theta_\nu l} + \frac{1}{\alpha_\nu} - 1}, \quad (4.9)$$

³The previous case $\nu = 0$ is recovered since $\Theta_0 = \frac{4}{3}$, and thus $\alpha_0 = 2$ and $c_0 = \frac{1}{4}$. For $-1 < \nu < \infty$, these constants range as $0 < \alpha_\nu < \infty$, and $1 > c_\nu > 0$ (see appendix B).

one verifies that the soliton with a nonvanishing value of ν also fulfills the set of relaxed asymptotically AdS conditions defined by (2.8). These coordinates are useful in order to compute the mass of the soliton, which can be readily obtained from the canonical generators defined by the surface integrals in eq. (2.9). The mass is found to be given by

$$M_{\text{sol}} = Q(\partial_t) = -\frac{\Theta_\nu^2}{24G(1+\nu)}, \tag{4.10}$$

and as expected, depends only on the Newton constant and the self-interaction parameter. Note that the soliton mass is manifestly negative, and it turns out to be bounded in between zero and the mass of AdS spacetime, i.e. for the allowed values of the self-interaction parameter, $-1 < \nu < \infty$, the soliton mass ranges according to (see appendix B)

$$-\frac{1}{8G} < M_{\text{sol}} < 0. \tag{4.11}$$

5 Microscopic entropy of the hairy black hole

As mentioned in section 3, the entropy of the hairy black hole can be microscopically computed provided the soliton is regarded as the ground state of the hairy sector. In order to achieve this task, the asymptotic growth of the number of states, given by Cardy formula, has to be recalled from scratch.

If the spectrum of the Virasoro operators L_0^\pm , whose eigenvalues are denoted by Δ^\pm , is such that the lowest eigenvalues Δ_0^\pm are nonvanishing (i.e., for $\Delta_0^\pm \neq 0$), Cardy formula reads [41, 43, 44]

$$S = 2\pi\sqrt{\frac{(c^+ - 24\Delta_0^+)}{6} \left(\Delta^+ - \frac{c^+}{24}\right)} + 2\pi\sqrt{\frac{(c^- - 24\Delta_0^-)}{6} \left(\Delta^- - \frac{c^-}{24}\right)}, \tag{5.1}$$

where it is assumed that the ground state is non degenerate.

We would like pointing out here that, in terms of the shifted Virasoro operators

$$\tilde{L}_0^\pm := L_0^\pm - \frac{c^\pm}{24}, \tag{5.2}$$

this formula can be rewritten as follows:

$$S = 4\pi\sqrt{-\tilde{\Delta}_0^+ \tilde{\Delta}^+} + 4\pi\sqrt{-\tilde{\Delta}_0^- \tilde{\Delta}^-}, \tag{5.3}$$

where $(\tilde{\Delta}_0^\pm) \tilde{\Delta}^\pm$ correspond to the (lowest) eigenvalues of \tilde{L}_0^\pm . Thus, remarkably, the asymptotic growth of the number of states can also be obtained if one only knows the spectrum of \tilde{L}_0^\pm without making any explicit reference to the central charges.

Note that unitarity, and the fact that expression (5.3) makes sense only for negative lowest eigenvalues $\tilde{\Delta}_0^\pm$, impose the following bounds⁴ (see figure 1)

$$-\frac{c^\pm}{24} \leq \tilde{\Delta}_0^\pm < 0. \tag{5.4}$$

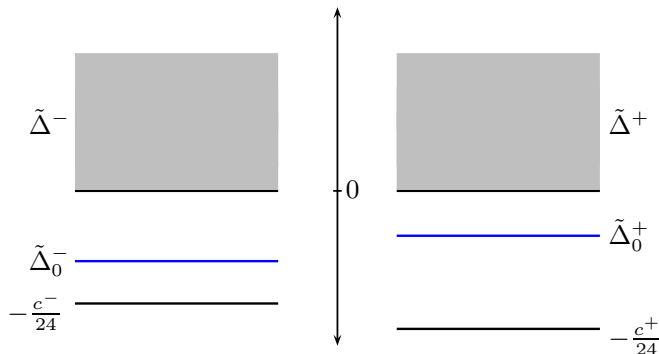


Figure 1. The asymptotic growth of the number of states can be written exclusively in terms of the spectrum of the shifted Virasoro operators \tilde{L}_0^\pm . Then, it is given by $\rho(\tilde{\Delta}^+, \tilde{\Delta}^-) = \rho(\tilde{\Delta}^+) \rho(\tilde{\Delta}^-)$, with $\rho(\tilde{\Delta}^\pm) = \rho(\tilde{\Delta}_0^\pm) \exp\left(4\pi\sqrt{-\tilde{\Delta}_0^\pm \tilde{\Delta}^\pm}\right)$, where $\tilde{\Delta}_0^\pm$ and $\rho(\tilde{\Delta}_0^\pm)$ correspond to the lowest eigenvalues of \tilde{L}_0^\pm and their degeneracies, respectively.

Formula (5.3), or equivalently (5.1), provide the suitable ground in order to obtain the black hole entropy for both sectors, namely vacuum and hairy, from the asymptotic number of states in the microcanonical ensemble. This is performed assuming that the eigenvalues of \tilde{L}_0^\pm are given by their corresponding canonical generators expressed by the surface integrals in eq. (2.9), which are related with the mass and the angular momentum according to

$$\tilde{\Delta}^\pm = \frac{1}{2}(Ml \pm J). \tag{5.5}$$

Explicitly, this can be seen as follows:

Vacuum sector. As it was explained in section 3, for the vacuum sector, the ground state corresponds to AdS spacetime, whose lowest eigenvalue is given by

$$\tilde{\Delta}_0^\pm = \frac{l}{2}M_{\text{AdS}} = -\frac{l}{16G}.$$

Hence, since in the vacuum sector $\tilde{\Delta}_0^\pm$ can be expressed in terms of the central charges, i.e., $\tilde{\Delta}_0^\pm = -\frac{c^\pm}{24}$, one verifies that formula (5.3) reduces to (3.2), which by virtue of (5.5) exactly reproduces the entropy of the BTZ black hole [42].

Hairy sector. In this case, the ground state is described by the scalar soliton, for which

$$\tilde{\Delta}_0^\pm = \frac{l}{2}M_{\text{sol}} = -\frac{l\Theta_\nu^2}{48G(1+\nu)}. \tag{5.6}$$

Note that from eq. (4.11), the lowest eigenvalues of the Virasoro operators in (5.6) fulfill the bounds given by eq. (5.4).⁵ Therefore, since for the hairy black hole $\tilde{\Delta}^\pm = \frac{l}{2}M$, formula (5.3), with (5.6) gives

$$S = 4\pi l \sqrt{-M_{\text{sol}}M} = \pi l \Theta_\nu \sqrt{\frac{2M}{3G(1+\nu)}},$$

⁴The lowest bounds hold provided $c^\pm > 1$ (see e.g. [45]).

⁵Indeed, in the semiclassical regime $c^\pm \gg 1$, since $l \gg G$.

in exact agreement with its semiclassical entropy (3.1).

Thus, for fixed values of the global charges, it is confirmed that the total entropy comes from the contribution of each sector.

6 Discussion and comments

Scalar solitons have been shown to exist in General Relativity with minimally and conformally coupled self-interacting scalar field in three dimensions. The soliton is regular everywhere, has a finite negative mass and fulfills the same set of asymptotically AdS conditions as the hairy black hole which are relaxed as compared with the ones of Brown-Henneaux. It is worth pointing out that the soliton and the hairy black hole also obey the same boundary conditions, since the constant α that appears in eq. (2.7) takes the same value, $\alpha = -\frac{2}{3}$, in both cases.

The very existence of the scalar soliton appears to be essential in order to reproduce the hairy black hole entropy from a microscopic counting. Indeed, the soliton turns out to be the suitable nondegenerate ground state that is required to obtain the entropy from the asymptotic growth of the number states given by Cardy formula.

These results can also be extended for the (grand) canonical ensemble (see appendix C), as well as for the rotating case by applying a boost in the “ $t - \varphi$ ” cylinder.

The lack of integration constants for the soliton can also be understood from the fact that the solution (4.6), (4.7) can be recovered from the hairy black hole (2.1), (2.2) performing a “double Wick rotation” of the form $\varphi \rightarrow i\tau$, $t \rightarrow i\varphi$, with suitable rescalings and redefining the radial coordinate.⁶ In fact, the Euclidean black hole solution is regular provided the Euclidean time period is fixed as the inverse of the Hawking temperature in eq. (2.5). Hence, in order to obtain a different Lorentzian solution without closed timelike curves, the (former) angle has to be unwrapped, and as a consequence the integration constant disappears by a simple rescaling. Analogously, in the vacuum sector, one recovers AdS spacetime from the BTZ black hole.

Note that since this construction is purely geometrical, it should apply quite generically to obtain solitons from black holes in three dimensions.⁷ This could also be interpreted as a manifestation of modular invariance of the dual theory at the boundary.

As a final remark, one can note that, as it occurs for the BTZ black hole, in the hairy sector there is also a gap between the ground state and the hairy black holes, and thus it is natural to wonder whether the hairy black hole could be obtained from some sort of identifications of the soliton. Their possible local equivalence is suggested by the fact their scalar invariants constructed out from the curvature and its derivatives coincide. This can be easily seen as follows: Since the hairy black hole is static and spherically symmetric, the invariants can only depend on l , ν , and the rescaled radial coordinate $\rho = r/B$; therefore, since the soliton is obtained from this geometry by a double Wick rotation which only

⁶The following identity, $\Theta_\nu \alpha_\nu (1 + (1 + \nu)\alpha_\nu^{-3}) = 3(1 + \nu)$ also turns out to be useful.

⁷Indeed, as shown in [46], this mechanism successfully generates a soliton from a black hole with “gravitational hair” in vacuum within the context of the BHT massive gravity theory [47], regardless the value of the cosmological constant.

interchanges the role of time and the angle, their scalar invariants are necessarily the same. Nonetheless, it is simple to verify that the soliton admits no additional Killing vectors apart from the manifest ones, generated by ∂_t and ∂_ϕ , and hence the possible identifications cannot be along an isometry.

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A Scalar soliton in the conformal frame

The self-interaction potential in eq. (1.3) has a natural interpretation through the relation between the conformal and Einstein frames. This can be seen performing a precise conformal transformation, followed by a scalar field redefinition of the form

$$\hat{g}_{\mu\nu} = \left(1 - \hat{\phi}^2\right)^{-2} g_{\mu\nu} \quad \text{and} \quad \hat{\phi} = \tanh(\phi), \quad (\text{A.1})$$

so that the action given by (1.2) with (1.3) reduces to the one for General Relativity with cosmological constant and a conformally coupled scalar field, given by

$$I[\hat{g}, \hat{\phi}] = \frac{1}{\pi G} \int d^3x \sqrt{-\hat{g}} \left(\frac{\hat{R} + 2l^{-2}}{16} - \frac{1}{2}(\nabla\hat{\phi})^2 - \frac{1}{16}\hat{R}\hat{\phi}^2 + \frac{\nu}{8l^2}\hat{\phi}^6 \right). \quad (\text{A.2})$$

Note that in this frame, the self-interaction potential turns out to be singled out requiring the matter piece of the action to be conformally invariant; i.e., unchanged under local rescalings of the form $\hat{g}_{\mu\nu} \rightarrow \lambda^2(x)\hat{g}_{\mu\nu}$, and $\hat{\phi} \rightarrow \lambda^{-1/2}(x)\hat{\phi}$.

In the conformal frame the soliton acquires a simple form. In the case of $\nu = 0$ the metric and the scalar field are given by

$$l^{-2}d\hat{s}^2 = -(1 + \rho^2)^2 d\tau^2 + \frac{16(1 + \rho^2)}{(3 + 2\rho^2)^2} d\rho^2 + \left(\frac{4}{9}\right)^2 \rho^2 \frac{(3 + 2\rho^2)^2}{(1 + \rho^2)} d\varphi^2, \quad (\text{A.3})$$

$$\hat{\phi} = \sqrt{\frac{1}{3 + 2\rho^2}}, \quad (\text{A.4})$$

respectively; and when the self-interaction coupling is switched on, for $\nu > -1$, the solution generalizes according to

$$l^{-2}d\hat{s}^2 = -(1 + \rho^2)^2 d\tau^2 + \frac{4d\rho^2}{2 + \rho^2 + \frac{c_\nu}{1 + \rho^2}} + \left(\frac{2}{2 + c_\nu}\right)^2 \rho^2 \left(2 + \rho^2 + \frac{c_\nu}{1 + \rho^2}\right) d\varphi^2, \quad (\text{A.5})$$

$$\hat{\phi} = \sqrt{\frac{1}{1 + \alpha_\nu(1 + \rho^2)}}, \quad (\text{A.6})$$

where the constants α_ν and c_ν are defined in eq. (4.8).

Note that the map between both frames (A.1) is invertible along the whole spacetime, since the conformal factor $(1 - \hat{\phi}^2)^{-2}$ is positive. Thus, in the conformal frame the soliton is also smooth and regular everywhere.

In the case of $\nu = 0$, the corresponding black hole solution was found in [48], which extends for $\nu > -1$ as in ref. [4].

B Ground state mass bounds

Here it is shown that a ground state of mass M_0 is bounded according to

$$-\frac{1}{8G} \leq M_0 < 0, \tag{B.1}$$

which by virtue of (5.6), agrees with the bound in eq. (5.4) for the lowest eigenvalues of the shifted Virasoro operators. The lowest bound is saturated in vacuum, since in this case $M_0 = M_{\text{AdS}} = -\frac{1}{8G}$. Besides, for the range of the self-interaction parameter $-1 < \nu < \infty$, for which the soliton and the hairy black hole exist, the mass of the soliton

$$M_{\text{sol}} = -\frac{\Theta_\nu^2}{24G(1 + \nu)},$$

fulfills

$$-\frac{1}{8G} < M_{\text{sol}} < 0. \tag{B.2}$$

This can be seen as follows. The function Θ_ν defined as

$$\Theta_\nu = 2(z\bar{z})^{\frac{2}{3}} \frac{z^{\frac{2}{3}} - \bar{z}^{\frac{2}{3}}}{z - \bar{z}}, \quad \text{with } z = 1 + i\sqrt{\nu}, \tag{B.3}$$

is monotonically increasing. It vanishes for $\nu = -1$ according to

$$\Theta_\nu \xrightarrow{\nu \rightarrow -1} 2^{2/3}(1 + \nu)^{2/3} - \frac{(1 + \nu)^{4/3}}{2^{2/3}} + \mathcal{O}\left((1 + \nu)^{5/3}\right), \tag{B.4}$$

and then grows so that $\Theta_0 = \frac{4}{3}$, while for large ν , asymptotically behaves as

$$\Theta_\nu \xrightarrow{\nu \rightarrow \infty} \sqrt{3\nu} - \frac{2}{3} + \mathcal{O}\left(\nu^{-1/2}\right). \tag{B.5}$$

Therefore, the behavior of the soliton mass reads

$$GM_{\text{sol}} = \begin{cases} -2^{-2/3} \frac{(1+\nu)^{1/3}}{6} + \frac{1+\nu}{12} + \mathcal{O}\left((1 + \nu)^{4/3}\right) & : \nu \rightarrow -1, \\ -\frac{1}{8} + \frac{\nu^{-1/2}}{6\sqrt{3}} + \mathcal{O}\left(\nu^{-1}\right) & : \nu \rightarrow \infty, \end{cases} \tag{B.6}$$

so that the bound (B.2) is fulfilled.

C Cardy formula and black hole entropy in the (grand) canonical ensemble

Conformal symmetry at the boundary is described by two commuting copies of the Virasoro algebra, so that left and right movers are decoupled, and hence they can be at equilibrium at different temperatures $T_{\pm} = \beta_{\pm}^{-1}$. Therefore, the total free energy is given by the sum of each of their free energies in the canonical ensemble, so that the Euclidean action reads

$$I_E = \left(\beta_+ \tilde{\Delta}^+ + \beta_- \tilde{\Delta}^- \right) l^{-1} - S, \tag{C.1}$$

where S is given by (5.3). Therefore, at the equilibrium the entropy can be written in terms of the lowest eigenvalues of the shifted Virasoro operators and the temperatures of left and right movers according to

$$S = -8\pi^2 l \left(\tilde{\Delta}_0^+ T_+ + \tilde{\Delta}_0^- T_- \right). \tag{C.2}$$

In turn, from the knowledge of the entropy and left and right temperatures, what one extracts from (C.2) are the lowest eigenvalues of the shifted Virasoro operators instead of the central charges. Indeed, by virtue of eq. (5.2), this formula is equivalently expressed in terms of the “effective central charges” $c_{\text{eff}}^{\pm} := c^{\pm} - 24\Delta_0^{\pm}$ as follows⁸

$$S = \frac{\pi^2}{3} \left(c_{\text{eff}}^+ T_+ + c_{\text{eff}}^- T_- \right). \tag{C.3}$$

Formula (C.2), or equivalently (C.3), also allows to obtain the black hole entropy for the vacuum and hairy sectors in the canonical ensemble. Indeed, since for the black holes the free energy is given by

$$I_E = \beta F = \beta M + \beta \Omega_+ J - S,$$

where $\beta = T^{-1}$, and Ω_+ stands for the angular velocity of the horizon, then from eqs. (C.1) and (5.5), the corresponding left and right temperatures are found to be

$$T_{\pm} = \frac{T}{1 \pm l\Omega_+}. \tag{C.4}$$

Hence, the black hole entropy for the vacuum and hairy sectors can be recovered from (C.2) with (C.4) as follows:

Vacuum sector. In this case the ground state corresponds to AdS spacetime, whose lowest eigenvalue is given by

$$\tilde{\Delta}_0^{\pm} = \frac{l}{2} M_{\text{AdS}} = -\frac{l}{16G} = -\frac{c^{\pm}}{24}.$$

Thus, (C.2) reduces to

$$S = \frac{\pi^2}{3} \left(c^+ T_+ + c^- T_- \right),$$

which exactly reproduces the entropy of the BTZ black hole [42].

⁸This might be related to the discrepancy in the central charges found in the context of warped AdS black holes [49, 50].

Hairy sector. The ground state is described by the scalar soliton, for which

$$\tilde{\Delta}_0^\pm = \frac{l}{2} M_{\text{sol}} = -\frac{l\Theta_\nu^2}{48G(1+\nu)}. \quad (\text{C.5})$$

In the absence of angular momentum, by virtue of (2.5), left and right temperatures coincide, i.e., $T_\pm = T$. Therefore, eq. (C.2) reduces to

$$S = -8\pi^2 l^2 M_{\text{sol}} T = \pi l \Theta_\nu \sqrt{\frac{2M}{3G(1+\nu)}}, \quad (\text{C.6})$$

in exact agreement with the semiclassical entropy given by (3.1).

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