

Gauge fields in $(A)dS_d$ within the unfolded approach: algebraic aspects

E.D. Skvortsov

*I.E. Tamm Department of Theoretical Physics, P.N. Lebedev Physical Institute,
Leninsky prospect 53, 119991, Moscow, Russia*

E-mail: skvortsov@lpi.ru

ABSTRACT: It has recently been shown that generalized connections of the $(A)dS_d$ symmetry algebra provide an effective geometric and algebraic framework for all types of gauge fields in $(A)dS_d$, both for massless and partially-massless. The equations of motion are equipped with a nilpotent operator called σ_- whose cohomology groups correspond to the dynamically relevant quantities like differential gauge parameters, dynamical fields, gauge invariant field equations, Bianchi identities etc. In the paper the σ_- -cohomology is computed for all gauge theories of this type and the field-theoretical interpretation is discussed. In the simplest cases the σ_- -cohomology is equivalent to the ordinary Lie algebra cohomology.

KEYWORDS: Gauge Symmetry, Field Theories in Higher Dimensions

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Introduction

This paper is devoted to studying gauge fields in the (anti)-de Sitter background within the framework of the unfolded approach. The background (anti)-de Sitter space $(A)dS_d$ can have any dimension $d \geq 4$. The gauge fields under consideration are the fields of the most general spin type, so-called mixed-symmetry fields [1–21], whose spin degrees of freedom are described by tensors having the symmetry of arbitrary Young diagrams.

The gauge fields in Minkowski space are presented only by massless fields with arbitrary spin. By contrast, for given mixed-symmetry field in $(A)dS_d$ there are different types of gauge modes, each type appearing at certain critical value of the mass parameter [22–24], so one may talk about different types of massless fields with the same spin. Only one member of the family of massless fields with the same spin is unitary in AdS_d [22–24]. In addition there are partially-massless fields [16, 18, 25–34], which, due to higher derivative gauge transformation law, have more degrees of freedom than massless fields but less than massive ones and have no counterparts in Minkowski case.

In this paper we study all types of gauge fields in $(A)dS_d$, both unitary and nonunitary, in order to better understand the peculiarity of the former within the framework of the unfolded approach. The goal will be to construct the nonlinear theory with fields of mixed-symmetry type in the spectrum, which is still lacking.

A new object, which can be referred to as generalized Yang-Mills field, shows up naturally in the unfolded approach. Generalized Yang-Mills field (or generalized connection) of the (anti)-de Sitter algebra is a degree- q differential form over (anti)-de Sitter space with values in arbitrary representation of its symmetry algebra, which is $\mathfrak{so}(d, 1)$ for de Sitter and $\mathfrak{so}(d - 1, 2)$ for anti-de Sitter. Since the space-time symmetry algebra is just an orthogonal algebra, which can be realized as a Lie algebra of antisymmetric matrices, the ordinary Yang-Mills field emerges when $q = 1$ and the representation is an irreducible one on rank-two antisymmetric tensors. The ordinary Yang-Mills field of the (anti)-de Sitter algebra is known [35, 36] to describe the (anti)-de Sitter gravity, i.e. the theory of massless spin-two field.

The main statement of [19], extending the results of [37–39], is that each gauge field in $(A)dS_d$ can be described by certain generalized Yang-Mills field (connection) of the (anti)-de Sitter algebra.

The unfolded approach provides an effective framework for field theories. The unfolded approach [40, 41] is a reformulation of differential equations in first order form by making use of the de Rham differential and exterior product of differential forms. The underlying algebraic structure is the Free Differential Algebra [42–45] whereto Lie algebras, their modules and Chevalley-Eilenberg cohomology belong. The main achievements of the unfolded approach are the full classical nonlinear theory of totally-symmetric massless fields of spins $s = 0, 1, \dots, \infty$ [46–48] and the coordinate-free description of black-holes [49, 50].

Every linearized unfolded system comes equipped with a nilpotent operator called σ_- , representing the algebraic part of the generalized covariant derivative acting on the fields. The σ_- -cohomology group $H(\sigma_-)$ contains all information about the dynamically relevant independent quantities of a given unfolded system [14, 51–53]. Differential gauge parameters, dynamical fields, gauge invariant equations of motion and Bianchi identities are the representatives of $H(\sigma_-)$. The σ_- -cohomology is a nice tool which allows to avoid solving differential equations. This paper is written to present the results on the σ_- -cohomology for the unfolded equations describing arbitrary-spin massless and partially-massless fields in (anti)-de Sitter space.

A typical linearized unfolded system consists of two parts coupled together via an appropriate Chevalley-Eilenberg cocycle. The one containing the forms of degree higher than zero is referred to as the gauge module and describes the gauge sector, another one containing zero-degree forms is referred to as the Weyl module and describes the physical degrees of freedom.

The generalized Yang-Mills fields provide the explicit construction for the gauge module of every gauge field in $(A)dS_d$. We compute $H(\sigma_-)$ for the gauge module, the result is presented in section 3.7. The same technique, developed in appendix B, can be applied to the Weyl module, showing that the cohomology matching condition between the gauge module and the Weyl module is fulfilled. Certain special cases were considered in [34, 38, 39, 52].

That massless fields in Minkowski space are the systems with the first class constraints and the massive fields both in Minkowski and $(A)dS_d$ are the systems with the second class constraints is mirrored in certain dualities on $H(\sigma_-)$. The gauge fields in $(A)dS_d$ are the systems with both the first and the second class constraints present. Therefore, the duality on $H(\sigma_-)$ is found to be more complicated.

A generic element of the σ_- -complex $\mathcal{C}(\sigma_-)$ is a differential form with values in some finite-dimensional tensor representation of the Lorentz algebra, the latter results from the restriction of the irreducible module of the (anti)-de Sitter algebra, in which a generalized Yang-Mills field takes values, to its Lorentz subalgebra. The Lorentz algebra commutes with the action of σ_- . Therefore, the representatives of $H(\sigma_-)$ are labelled by Young diagrams of the Lorentz algebra. They correspond to the fields, gauge parameters, etc. of the minimal formulation in terms of metric-like fields, which turns out to be very complicated in contrast to the formulation in terms of generalized Yang-Mills fields.

Quite surprisingly, the symmetry types of the representatives of $H(\sigma_-)$ turn out to be given by what may be called 'the maximally symmetric part' of the tensor product, i.e. the corresponding Young diagrams tend to be as symmetric as it is possible, the rest of diagrams that are less symmetric label acyclic subcomplexes of $\mathcal{C}(\sigma_-)$. Making essential use of Young diagrams allows us to present the results in a simple form.

The paper is organized as follows. We begin in section 1 by presenting the classification of the gauge fields in $(A)dS_d$, including all types of massless and partially-massless fields. Essential facts of the unfolded approach are quickly summarized in section 2, where a linear gauge theory with generalized Yang-Mills fields of the (anti)-de Sitter algebra is defined. The σ_- operator is discussed in section 3, where the results on the σ_- -cohomology are stated, the proof is in appendix B. The field-theoretical interpretation of the σ_- -cohomology is in section 4. In section 5 the example of a two-row massless field is given to illustrate the general formalism. The conclusions are in section 6.

Preliminaries

Whereas only tensor representations are considered in the paper, we do not make any distinction between irreducible finite-dimensional highest weight modules of $sl(d)$ and $so(d)$ with highest weight $(s_1, \dots, s_n, 0_{n+1}, \dots, 0_\nu)$, $\nu = d - 1$ for $sl(d)$, $\nu = [d/2]$ for $so(d)$, and Young diagrams of shape $\mathbb{Y}(s_1, \dots, s_n)$. For simplicity, we ignore the Young diagrams of height close to the maximal admissible height ν , so that it is assumed $n < \nu - 1$, thus for $so(d)$ we will not consider (anti)-self dual representations.

A Young diagram \mathbf{X} can be defined in several ways: (1) by specifying the lengths of its rows $\mathbf{X} = \mathbb{Y}(s_1, \dots, s_n)$ (row notation), n being the number of rows and s_i being the length of the i -th row, $s_i \geq s_{i+1}$; (2) by specifying the widths and heights of its subblocks

$$\mathbb{Y}\{(s_1, p_1), \dots, (s_N, p_N)\} \equiv \mathbb{Y}(\overbrace{s_1, \dots, s_1}^{p_1}, \dots, \overbrace{s_N, \dots, s_N}^{p_N}),$$

where s_i and p_i are the width and the height of the i -th subblock, N is the number of subblocks; (3) by specifying the heights of columns $\mathbb{Y}[h_1, \dots, h_n]$, h_i being the height of the i -th column, $h_i \geq h_{i+1}$.

Let \mathfrak{f} be some orthogonal algebra $\mathfrak{so}(p, q)$, the orthogonal algebras of interest being $\mathfrak{so}(d-1)$, $\mathfrak{so}(d-1, 1)$, $\mathfrak{so}(d, 1)$ and $\mathfrak{so}(d-1, 2)$. A tensor $C^{\mathbf{X}}$ of \mathfrak{f} or $sl(d)$ is said to have the symmetry of some Young diagram \mathbf{X} if its indices realize the irreducible representation of the permutation group labelled by \mathbf{X} , which for $sl(d)$ guarantees the irreducibility of the tensor.

If a tensor $C^{\mathbf{X}}$ having the symmetry of $\mathbf{X} = \mathbb{Y}(s_1, \dots, s_n)$ needs to be written explicitly, it is always taken in the symmetric basis, meaning that ① it has n groups of indices, the k -th group containing s_k indices; ② it is manifestly symmetric with respect to permutations of indices within any group; ③ it satisfies the Young condition

$$C^{a(s_1), \dots, b(s_k), \dots, bc(s_i-1), \dots, u(s_n)} \equiv 0, \quad 1 \leq k < i \leq n,$$

where a group of symmetric indices is denoted by one letter with the number of indices indicated in brackets, e.g. $a(s_1) \equiv a_1 a_2 \dots a_{s_1}$, and the (normalized) sum over all permutations of two or more (groups of) indices denoted by the same letter is implied, e.g. $b(s_k), \dots, bc(s_i-1) \equiv \frac{1}{(s_k+1)!} \sum_{\sigma} b_{\sigma(1)} \dots b_{\sigma(s_k)}, \dots, b_{\sigma(s_k+1)} c_1 \dots c_{s_i-1}$.

A tensor of the orthogonal algebra is said to be an irreducible tensor having the symmetry of \mathbf{X} iff in addition to having the symmetry of \mathbf{X} it is completely traceless, i.e. contraction of the invariant metric tensor with any two indices vanishes identically.

1 Gauge fields in (anti)-de Sitter space

According to [19], which is a generalization of numerous results [22–25, 27] concerned with gauge fields in $(A)dS_d$, any gauge field in $(A)dS_d$ is completely determined by a triple (\mathbf{S}, q, t) , where \mathbf{S} is a finite-dimensional irreducible representation of the (anti)-de Sitter ‘Wigner little algebra’ $\mathfrak{so}(d-1)$, specified by some Young diagram $\mathbf{S} = \mathbb{Y}(s_1, \dots, s_n)$; q is an integer in the range $1, \dots, n$ such that $(s_q - s_{q+1}) > 0$; t is an integer in the range $1, \dots, (s_q - s_{q+1})$, which is equal to the order of derivative in the gauge transformation law.

For a given triple (\mathbf{S}, q, t) the irreducible module $\mathcal{H}(E_0; \mathbf{S}_0 \equiv \mathbf{S})$ of the (anti)-de Sitter algebra that is referred to as a massless ($t = 1$) or partially-massless ($t > 1$) field¹ is defined by the following exact sequence [19, 54]

$$0 \rightarrow \mathcal{D}(E_q; \mathbf{S}_q) \longrightarrow \dots \longrightarrow \mathcal{D}(E_1; \mathbf{S}_1) \longrightarrow \mathcal{D}(E_0; \mathbf{S}_0) \longrightarrow \mathcal{H}(E_0; \mathbf{S}_0) \rightarrow 0, \quad (1.1)$$

where for the anti-de Sitter case $\mathcal{D}(E'; \mathbf{S}')$ is a Verma module freely generated by the positive grade operators of $\mathfrak{so}(d-1, 2)$ from the vacuum $|E', \mathbf{S}'\rangle$ annihilated by the negative grade generators of $\mathfrak{so}(d-1, 2)$, which is an irreducible representation of the maximal compact subgroup $\mathfrak{so}(2) \oplus \mathfrak{so}(d-1)$ of $\mathfrak{so}(d-1, 2)$ defined by the lowest weights E' and \mathbf{S}' of $\mathfrak{so}(2)$ and $\mathfrak{so}(d-1)$, respectively.

¹In the de Sitter case the construction of $\mathfrak{so}(d, 1)$ modules is different because the corresponding representations of $\mathfrak{so}(d, 1)$ are not of the lowest weight type. Nevertheless, the notion of the lowest energy E_0 can be introduced [27, 31]. As for field equations, the situation is more simple inasmuch as the change $\lambda^2 \rightarrow -\lambda^2$ makes the transition from AdS_d to dS_d .

The lowest weights E_i and \mathbf{S}_i of $\mathfrak{so}(2) \oplus \mathfrak{so}(d-1)$ are defined as

$$E_i = \begin{cases} d + s_q - t - q - 1, & i = 0, \\ d + s_{q-i+1} - (q - i + 1) - 1, & i = 1, \dots, q, \end{cases} \quad (1.2)$$

$$\mathbf{S}_i = \begin{cases} \mathbb{Y}(s_1, \dots, s_n) \equiv \mathbf{S}, & i = 0, \\ \mathbb{Y}(s_1, s_2, \dots, s_{q-1}, s_q - t, s_{q+1}, \dots, s_n), & i = 1, \\ \mathbb{Y}(s_1, \dots, s_{q-i}, s_{q-i+2} - 1, \dots, s_q - 1, s_q - t, s_{q+1}, \dots, s_n), & i = 2, \dots, q-1, \\ \mathbb{Y}(s_2 - 1, s_3 - 1, \dots, s_q - 1, s_q - t, s_{q+1}, \dots, s_n), & i = q. \end{cases} \quad (1.3)$$

Given E' and $\mathbf{S}' = \mathbb{Y}(s'_1, \dots, s'_n)$, $\mathcal{D}(E'; \mathbf{S}')$ can be realized on the solutions of

$$(\square + m'^2)C^{a(s'_1), \dots, u(s'_n)} = 0, \quad (1.4)$$

$$D_m C^{a(s'_1), \dots, mc(s'_i-1), \dots, u(s'_n)} = 0, \quad i = 1, \dots, n, \quad (1.5)$$

where $C^{a(s'_1), \dots, u(s'_n)}(x)$ is an irreducible Lorentz tensor field having the symmetry of \mathbf{S}' , $\square \equiv D^m D_m$ and the mass-like parameter m'^2 is related to E' and \mathbf{S}' as

$$m'^2 = \lambda^2 (E'(E' - d + 1) - s'_1 - \dots - s'_n). \quad (1.6)$$

Therefore, a field-theoretical model for $\mathcal{H}(E_0; \mathbf{S}_0)$ is given by the irreducible Lorentz field $\phi^{\mathbf{S}} \equiv \phi^{a(s_1), \dots, u(s_n)}(x)$ having the symmetry of \mathbf{S} and satisfying equations (1.4)–(1.5) with the mass-like parameter determined by E_0 and $\mathbf{S}_0 \equiv \mathbf{S}$. The exactness of $\mathcal{D}(E_1; \mathbf{S}_1) \longrightarrow \mathcal{D}(E_0; \mathbf{S}_0) \longrightarrow \mathcal{H}(E_0; \mathbf{S}_0)$ implies that at E_0 and \mathbf{S}_0 equations (1.4)–(1.5) become invariant under gauge transformations of the form

$$\delta \phi^{a(s_1), \dots, u(s_n)} = \overbrace{D^c \dots D^c}^t \xi^{a(s_1), \dots, b(s_{q-1}), c(s_q-t), d(s_{q+1}), \dots, u(s_n)} + \dots, \quad (1.7)$$

where ‘ \dots ’ stands for certain lower derivative terms and for the terms that project onto the Young symmetry \mathbf{S} . The gauge parameter $\xi^{\mathbf{S}_1}$ is an irreducible Lorentz tensor field having the symmetry of \mathbf{S}_1 and satisfying equations analogous to (1.4)–(1.5). The rest of $\mathcal{D}(E_i; \mathbf{S}_i)$ with $i = 2, \dots, q$ corresponds to reducible gauge symmetries.

The exact sequence (1.1) can be extended to the right with $\mathcal{D}(E_{-1}; \mathbf{S}_{-1}) \longrightarrow \mathcal{D}(E_{-2}; \mathbf{S}_{-2}) \longrightarrow \dots$, implying that for a field $\phi^{\mathbf{S}_0}$ with gauge transformations (1.7) one can construct the generalized Weyl tensor $C^{\mathbf{S}_{-1}}$ having the symmetry of \mathbf{S}_{-1} , [33]. $C^{\mathbf{S}_{-1}}$ is obtained by applying $(s_q - s_{q+1} - t + 1)$ derivatives to $\phi^{\mathbf{S}_0}$. By definition, the generalized Weyl tensor $C^{\mathbf{S}_{-1}}$ is the lowest order nontrivial on-mass-shell gauge invariant under $\xi^{\mathbf{S}_1}$ -transformations (1.7) tensor built from $\phi^{\mathbf{S}_0}$. That the Weyl tensor does exist, its symmetry type and order of derivative follows from the analysis of $\mathbf{H}(\sigma_-)$ and, of course, from the structure of singular vectors in $\mathcal{D}(E'; \mathbf{S}')$.

Consequently, the space of gauge invariant differential expressions constructed from $\phi^{\mathbf{S}_0}$ is generated by (1.4)–(1.5) and by the generalized Weyl tensor.

To illustrate, the Young diagrams \mathbf{S}_1 of the gauge parameter, \mathbf{S}_0 of the spin and \mathbf{S}_{-1} of the generalized Weyl tensor have the form

$$\begin{array}{c}
 \mathbf{S}_1 = \begin{array}{|c|} \hline s_1 \\ \hline \dots \\ \hline s_{q-1} \\ \hline s_{q-t} \\ \hline s_{q+1} \\ \hline s_{q+2} \\ \hline \dots \\ \hline s_n \\ \hline \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \mathbf{S}_0 = \begin{array}{|c|} \hline s_1 \\ \hline \dots \\ \hline s_{q-1} \\ \hline s_q \\ \hline s_{q+1} \\ \hline s_{q+2} \\ \hline \dots \\ \hline s_n \\ \hline \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \mathbf{S}_{-1} = \begin{array}{|c|} \hline s_1 \\ \hline \dots \\ \hline s_{q-1} \\ \hline s_q \\ \hline s_{q-t+1} \\ \hline s_{q+2} \\ \hline \dots \\ \hline s_n \\ \hline \end{array}
 \end{array}$$

2 Unfolded approach

General definition. A set of differential equations is said to have the unfolded form [40, 41, 55] if it can be written as the zero curvature condition

$$R^{\mathcal{A}} \equiv dW^{\mathcal{A}} + F^{\mathcal{A}}(W) = 0, \quad (2.1)$$

where $W^{\mathcal{A}}$ is a set of differential forms on some manifold \mathcal{M}_d with values in vector spaces labelled by \mathcal{A} , so that indices $\mathcal{A}, \mathcal{B}, \dots$ indicate the vector space rather than components in a particular basis. $|\mathcal{A}|$ is the form-degree of $W^{\mathcal{A}}$, d is the exterior differential on \mathcal{M}_d and $F^{\mathcal{A}}(W)$ is a degree- $(|\mathcal{A}| + 1)$ function of W assumed to be expandable in terms of exterior (wedge) products only²

$$F^{\mathcal{A}}(W) = \sum_{n=1}^{\infty} \sum_{|\mathcal{B}_1| + \dots + |\mathcal{B}_n| = |\mathcal{A}| + 1} f^{\mathcal{A}}_{\mathcal{B}_1 \dots \mathcal{B}_n} W^{\mathcal{B}_1} \wedge \dots \wedge W^{\mathcal{B}_n}, \quad (2.2)$$

where $f^{\mathcal{A}}_{\mathcal{B}_1 \dots \mathcal{B}_n}$ are some x -independent elements of $\text{Hom}(\mathcal{B}_1 \otimes \dots \otimes \mathcal{B}_n, \mathcal{A})$. In order to guarantee the formal consistency of (2.1) $F^{\mathcal{A}}(W)$ must satisfy the integrability condition (referred to as either the generalized Jacobi identity or the Bianchi identity) obtained by applying d to (2.1)

$$F^{\mathcal{B}} \frac{\delta F^{\mathcal{A}}}{\delta W^{\mathcal{B}}} \equiv 0. \quad (2.3)$$

Any solution of (2.3) defines a free differential algebra (FDA) [42–45]. If the Jacobi identity (2.3) is satisfied regardless of \mathcal{M}_d dimension, the free differential algebra is referred to as universal [53, 56]. It is the universal algebras only that will be considered henceforth. Equations (2.1) are invariant under the gauge transformations

$$\delta W^{\mathcal{A}} = d\epsilon^{\mathcal{A}} - \epsilon^{\mathcal{B}} \frac{\delta F^{\mathcal{A}}}{\delta W^{\mathcal{B}}}, \quad \text{if } |\mathcal{A}| > 0, \quad (2.4)$$

$$\delta W^{\mathcal{A}} = -\epsilon^{\mathcal{B}'} \frac{\delta F^{\mathcal{A}}}{\delta W^{\mathcal{B}'}} , \quad \mathcal{B}' : |\mathcal{B}'| = 1, \quad \text{if } |\mathcal{A}| = 0, \quad (2.5)$$

where $\epsilon^{\mathcal{A}}$ is a degree- $(|\mathcal{A}| - 1)$ form taking values in the same space as $W^{\mathcal{A}}$.

²The wedge symbol \wedge will be systematically omitted henceforth.

Linearization. In what follows we consider the linearized unfolded systems. Let the base manifold \mathcal{M}_d be a homogeneous space³ $\mathcal{M}_d = \mathfrak{G}/\mathfrak{H}$, with \mathfrak{g} and \mathfrak{h} being the Lie algebras of \mathfrak{G} and \mathfrak{H} . Typically, \mathfrak{h} is the Lorentz algebra, $\mathfrak{so}(d-1, 1)$. The most general unfolded equations linearized over \mathcal{M}_d have the form [53, 59, 60]

$$R^I = d\Omega^I + f_{JK}^I \Omega^J \Omega^K = 0, \quad (2.6)$$

$$R^A = dW^A + f_B^A W^B + \Omega^I f_I^A W^B + \dots + \Omega^{I_1} \dots \Omega^{I_k} f_{I_1 \dots I_k}^A W^B = 0, \quad (2.7)$$

where Ω is a one-form connection of \mathfrak{g} with Ω^I being the components of Ω in some base, f_{JK}^I are the structure constants of \mathfrak{g} ; $f_{I_1 \dots I_k}^A \equiv 0$ unless $|\mathcal{A}|+1 = |\mathcal{B}|+k$. The set of fields consists of two subsets: degree-one forms Ω^I that are assumed to have the zeroth order and the matter fields W^A , which may have various degrees, are assumed to have the first order so that the unfolded equations are linear in W^A . Ω^I can be recognized as the Cartan connection on \mathcal{M}_d , it describes the background geometry by virtue of the flatness condition (2.6).

The Jacobi identity (2.3) implies that $f_{I_1 \dots I_k}^A$ are closely related to the Lie algebra \mathfrak{g} of the space-time symmetry group \mathfrak{G} . Namely, $f_{I_1 \dots I_k}^A$ is a Chevalley-Eilenberg k -cocycle of \mathfrak{g} with coefficients in $\text{Hom}(\mathcal{B}, \mathcal{A})$ [53]. If $k=1$ and $\mathcal{A} = \mathcal{B}$, f_I^A is just a representation of \mathfrak{g} in the vector space \mathcal{A} . Coboundary cocycles can be removed from the equations (2.7) by a nonsingular field redefinition, thus, $f_{I_1 \dots I_k}^A$ can be assumed to be nontrivial representatives of Chevalley-Eilenberg cohomology groups.

Having fixed the background connection Ω^I , it is useful to introduce [14, 59] the generalized⁴ covariant derivative \mathcal{D}

$$\mathcal{D} = f_B^A + (\delta_B^A d + \Omega^I f_I^A) + \dots + \Omega^{I_1} \dots \Omega^{I_k} f_{I_1 \dots I_k}^A, \quad (2.8)$$

which acts on the whole space $\mathcal{W}_{\mathfrak{q}}$ of matter fields

$$\mathcal{W}_{\mathfrak{q}} = \{W^{\mathcal{B}}, W^{\mathcal{C}}, \dots, W^{\mathcal{D}}\}, \quad q = \max_{\mathcal{A}} |\mathcal{A}| \quad (2.9)$$

We define $\mathcal{W}_{\mathfrak{q}\pm i}$ to be the spaces of differential forms with values in the same vector spaces as $\mathcal{W}_{\mathfrak{q}}$ but with the form degrees shifted by $\pm i$. If for some \mathcal{B} and i we have $|\mathcal{B}| - i < 0$ the corresponding element of $\mathcal{W}_{\mathfrak{q}-i}$ becomes trivial.

In this special case, the nilpotency of \mathcal{D} , $\mathcal{D}^2 = 0$, is equivalent to the generalized Jacobi identity (2.3). Then, the gauge transformations for a matter field $W_{\mathfrak{q}}$, $W_{\mathfrak{q}} \in \mathcal{W}_{\mathfrak{q}}$ read $\delta W_{\mathfrak{q}} = \mathcal{D}\xi_{\mathfrak{q}-1}$ with $\xi_{\mathfrak{q}-1} \in \mathcal{W}_{\mathfrak{q}-1}$. The gauge invariant field curvature $R_{\mathfrak{q}+1} = \mathcal{D}W_{\mathfrak{q}}$ belongs to $\mathcal{W}_{\mathfrak{q}+1}$, the rest of $\mathcal{W}_{\mathfrak{q}\pm i}$ corresponds either to reducible gauge symmetries or to reducible Bianchi identities. Therefore, we have the unfolded complex $\mathcal{C}(\mathcal{W}, \mathcal{D})$,

$$\mathcal{W}_0 \xrightarrow{\mathcal{D}} \mathcal{W}_1 \xrightarrow{\mathcal{D}} \dots \xrightarrow{\mathcal{D}} \mathcal{W}_{\mathfrak{q}-1} \xrightarrow{\mathcal{D}} \mathcal{W}_{\mathfrak{q}} \xrightarrow{\mathcal{D}} \dots \quad (2.10)$$

³The homogeneous spaces of interest are given by Minkowski space $\mathfrak{G} = \text{ISO}(d-1, 1)$, $\mathfrak{H} = \text{SO}(d-1, 1)$, anti-de Sitter space $\mathfrak{G} = \text{SO}(d-1, 2)$, $\mathfrak{H} = \text{SO}(d-1, 1)$, de Sitter space $\mathfrak{G} = \text{SO}(d, 1)$, $\mathfrak{H} = \text{SO}(d-1, 1)$, and by the space with $\mathfrak{G} = \text{SP}(8)$ and \mathfrak{H} being the maximal parabolic subgroup of \mathfrak{G} , in which the symmetries of 4d higher-spin fields gets realized geometrically [57, 58].

⁴If \mathcal{D} consists only of the expression in brackets, it reduces to the ordinary covariant derivative.

We can always split $\mathcal{C}(\mathcal{W}, \mathcal{D})$ as

$$\mathcal{C}(\mathcal{W}, \mathcal{D}) = \mathcal{C}(\mathcal{W}^{\text{contr}}, \mathcal{D}) \oplus \mathcal{C}(\mathcal{W}^{\text{gauge}}, \mathcal{D}) \ni \mathcal{C}(\mathcal{W}^{\text{Weyl}}, \mathcal{D}), \quad (2.11)$$

where $\mathcal{W}^{\text{contr}}$ is a contractible part [33, 42], which can be consistently set to zero. The equations $\mathcal{D}\mathcal{W}^{\text{contr}}$ are of the form $dW^1 + W^2 + \dots = 0$ so that W^1 can be gauged away. In what follows contractible parts will never appear. $\mathcal{W}^{\text{gauge}}$ is referred to as the gauge module, it contains the forms of degree greater than zero, which are necessarily gauge fields by virtue of (2.4). The zero degree forms constitute the Weyl module $\mathcal{W}^{\text{Weyl}}$, which carries physical degrees of freedom since the field equations (2.1) can be treated as a cocycle condition, having pure gauge solutions in the sector of k -forms with $k > 0$ by virtue of the Poincare lemma, hence, only zero degree forms parameterize a solution to (2.1). The semidirect sum sign \ni is due to the Chevalley-Eilenberg cocycle that glues the Weyl module to the gauge module.

Generalized Yang-Mills connections of the (anti)-de Sitter algebra. Below we define a family of gauge modules which is natural to consider in (anti)-de Sitter space.

Let Ω be a flat connection of the (anti)-de Sitter algebra \mathfrak{g} , $\mathfrak{g} = \mathfrak{so}(d, 1)$ (de Sitter) and $\mathfrak{g} = \mathfrak{so}(d - 1, 2)$ (anti-de Sitter). Given an arbitrary irreducible representation of the (anti)-de Sitter algebra \mathbf{A} we define the complex $\mathcal{C}(\mathbf{A}, D_\Omega)$

$$\mathcal{W}_0^{\mathbf{A}} \xrightarrow{D_\Omega} \dots \xrightarrow{D_\Omega} \mathcal{W}_{q-1}^{\mathbf{A}} \xrightarrow{D_\Omega} \mathcal{W}_q^{\mathbf{A}} \xrightarrow{D_\Omega} \mathcal{W}_{q+1}^{\mathbf{A}} \xrightarrow{D_\Omega} \mathcal{W}_{q+2}^{\mathbf{A}} \xrightarrow{D_\Omega} \dots,$$

where $\mathcal{W}_i^{\mathbf{A}}$ is a \mathfrak{g} -module of degree- i differential forms with values in \mathbf{A} . The flatness condition implies the nilpotency of D_Ω , $D_\Omega^2 = 0$. When the dimension of $(A)dS_d$ is reached, $\mathcal{W}_i^{\mathbf{A}}$ becomes trivial, i.e. $\mathcal{W}_i^{\mathbf{A}} = \emptyset$ if $i > d$.

Given a distinguished degree $q > 0$, for the gauge field $W_q^{\mathbf{A}} \in \mathcal{W}_q^{\mathbf{A}}$ one can easily define [37–39] the field curvature $R_{q+1}^{\mathbf{A}} = D_\Omega W_q^{\mathbf{A}}$ that is invariant under the gauge transformations $\delta W_q^{\mathbf{A}} = D_\Omega \xi_{q-1}^{\mathbf{A}}$ and satisfies the Bianchi identity $D_\Omega R_{q+1}^{\mathbf{A}} \equiv 0$. The lower degree elements $\xi_{q-i}^{\mathbf{A}}$ of $\mathcal{W}_{q-i}^{\mathbf{A}}$, $i = 2, \dots, q$ of the complex $\mathcal{C}(\mathbf{A}, D_\Omega)$ correspond to the reducible gauge transformations $\delta \xi_{q-i+1}^{\mathbf{A}} = D_\Omega \xi_{q-i}^{\mathbf{A}}$. The higher degree elements $\mathcal{W}_{q+i}^{\mathbf{A}}$, $i = 2, \dots$ correspond to the reducibility of Bianchi identities. Thus, $\mathcal{C}(\mathbf{A}, D_\Omega)$ is a particular realization of $\mathcal{C}(\mathcal{W}^{\text{gauge}}, \mathcal{D})$ with $\mathcal{D} = D_\Omega$.

(A)dS_d-background geometry. The connection Ω can be presented in components by a one-form $\Omega^{A,B} \equiv \Omega_\mu^{A,B} dx^\mu$ antisymmetric in its fiber indices of \mathfrak{g} , $\Omega^{A,B} = -\Omega^{B,A}$, $A, B, \dots = 0, \dots, d$, with the flatness condition having the form

$$d\Omega^{A,B} + \Omega^A{}_C \wedge \Omega^{C,B} = 0. \quad (2.12)$$

To interpret the fields in terms of the Lorentz algebra the manifest local (anti)-de Sitter symmetry must be lost. The local Lorentz algebra $\mathfrak{so}(d - 1, 1)$ is identified with the subalgebra of the local (anti)-de Sitter algebra that annihilates some vector field $V^A(x)$, called compensator [36, 37], which is convenient to normalize as⁵ $V^B V_B = \mp 1$. The

⁵Upper/lower sign corresponds to the de Sitter/anti-de Sitter case hereinafter.

generalized vielbein field $E_\mu^A dx^\mu$

$$\lambda E^A = D_\Omega V^A = dV^A + \Omega^A_{\ B} V^B \quad (2.13)$$

is required to have the maximal rank, thus giving rise to a nonsingular vielbein field h_μ^a . $E^B V_B = 0$ by virtue of (2.13). The connection of the Lorentz algebra

$$\Omega_L^{A,B} = \Omega^{A,B} \mp \lambda(V^A E^B - E^A V^B) \quad (2.14)$$

allows to define the Lorentz covariant derivative $D = d + \Omega_L$. Both the compensator V^A and the generalized vielbein E^A are Lorentz-covariantly constant

$$DV^A = 0, \quad DE^A = 0. \quad (2.15)$$

For the further convenience we introduce

$$\Omega^{A,B} = \Omega_L^{A,B} + \nabla_-^{A,B} - \nabla_+^{A,B}, \quad \nabla_-^{A,B} = \pm \lambda V^A E^B, \quad \nabla_+^{A,B} = \pm \lambda E^A V^B. \quad (2.16)$$

One can always choose the 'standard gauge' for the compensator field $V_A = \delta_{\bullet}^A$, then $\lambda E^A = \Omega^{A,\bullet}$, $E_\mu^\bullet = 0$ and $\Omega_L^{a,b} = \Omega^{a,b}$, so that the vielbein field h_μ^a and the Lorentz spin-connection $\varpi^{a,b}$ are defined as

$$\lambda h^a = \Omega^{a,\bullet}, \quad \varpi^{a,b} = \Omega^{a,b}. \quad (2.17)$$

In terms of h^a and $\varpi^{a,b}$ the flatness condition (2.12) reads

$$dh^a + \varpi^{a,b} \wedge h^b = 0, \quad (2.18)$$

$$d\varpi^{a,b} + \varpi^{a,c} \wedge \varpi^{c,b} \pm \lambda^2 h^a \wedge h^b = 0. \quad (2.19)$$

Having identified the Lorentz algebra together with the Lorentz covariant derivative D , the complex $\mathcal{C}(\mathbf{A}, D_\Omega)$ can be interpreted in terms of (generalized) connections of the Lorentz algebra. Passing to connections of the Lorentz algebra, the manifest (anti)-de Sitter symmetry gets lost. After converting the connections of the Lorentz algebra into fully world or fully fiber (metric-like) tensors with the help of the background vielbein h_μ^a , in terms of metric-like fields the gauge theory acquires a very complicated form due to Young symmetrizers and because of a large number of component metric-like fields, most of which are auxiliary or pure gauge. The σ_- -technique allows us to find out which fields are dynamical and hence to give an interpretation of $\mathcal{C}(\mathbf{A}, D_\Omega)$ in terms of the metric-like fields of section 1.

Unfolding gauge fields in (anti)-de Sitter. We assume \mathfrak{g} is $\mathfrak{so}(d, 1)$ or $\mathfrak{so}(d - 1, 2)$, the Cartan connection Ω^I is presented by a one-form $\Omega^{A,B} \equiv \Omega_\mu^{A,B} dx^\mu$ with the flatness condition (2.6) having the form (2.12). The unfolded equations for a gauge field (\mathbf{S}, q, t) are expected to have the following form, which is a special case of the unfolded complex (2.10),

$$D_\Omega W_{\mathbf{q}}^{\mathbf{A}} = f(E, \dots, E)(C_0), \quad (2.20)$$

$$\widetilde{D}_\Omega C_0 = 0, \quad (2.21)$$

where $W_{\mathfrak{q}}^{\mathbf{A}}$ is a \mathfrak{g} -connection (4.1) associated with (\mathbf{S}, q, t) -field, C_0 is a certain infinite-dimensional \mathfrak{g} -module, the Weyl module, whose restriction to the Lorentz algebra yields a direct sum of an infinite number of irreducible Lorentz tensors. $f(E, \dots, E)$ is a Chevalley-Eilenberg cocycle gluing the gauge module to the Weyl module. \widetilde{D}_Ω is a \mathfrak{g} -covariant derivative in the Weyl module. Note that only finitely many Lorentz modules constituting the Weyl module are glued with the gauge module. These are given by the Weyl tensor together with certain of its descendants.

The explicit constructions known up-to-date are given by a massless spin- s field [37, 40, 47, 48, 55]; partially-massless spin- s fields can be rewritten in the same way since a simple change of variables gives all the coefficients of \widetilde{D}_Ω from those of D_Ω , [39]; for the series $(\mathbf{S}, q_{\min}, 1)$, where q_{\min} is the height of the shortest column of \mathbf{S} , the free unfolded equations were obtained in [33, 34]. However, it is still lacking for arbitrary-spin massless and partially-massless gauge fields in $(A)dS_d$. We expect it may be extracted analogously to the series $(\mathbf{S}, q_{\min}, 1)$ from the unfolded equations for massive arbitrary-spin field in $(A)dS_d$ of [33, 34], which are obtained as the constrained radial reduction of unfolded equations for massless fields in Minkowski space [14]. However, it is not obvious at the moment how the equations can be cast into the form (2.20). The equations for the Weyl module (2.21) were given in [33, 34] for arbitrary case together with the constraints that single out the different cases, i.e. massive, massless or partially-massless.

3 The Sigma-minus operator

The unfolded form of any field-theoretical system has many advantages in that it is formulated in terms of connections of the space-time symmetry algebra. As compared to the minimal formulation in terms of metric-like fields the unfolded form requires more component fields with many of them playing auxiliary role. Therefore, given some unfolded equations whose field-theoretical interpretation is not clear or while unfolding some known field system there comes the question of what fields are the true dynamical ones and what gauge parameters are the true differential ones, etc.

A natural gauge sector of an unfolded complex in (anti)-de Sitter space is presented by the complex $\mathcal{C}(\mathbf{A}, D_\Omega)$ of gauge connections of the (anti)-de Sitter algebra. Actually, every finite-dimensional irreducible gauge module is given by some $W_{\mathfrak{q}}^{\mathbf{A}}$. With the help of the σ_- -cohomology technique [14, 51–53] we classify all dynamically relevant independent quantities in $\mathcal{C}(\mathbf{A}, D_\Omega)$, which gives the full list of dynamical fields contained in $W_{\mathfrak{q}}^{\mathbf{A}}$, differential gauge parameters in $\xi_{\mathfrak{q}-1}^{\mathbf{A}}$ and the gauge invariant equations that can be imposed on $W_{\mathfrak{q}}^{\mathbf{A}}$ in terms of $R_{\mathfrak{q}+1}^{\mathbf{A}}$.

The initial data for σ_- are a flat connection Ω of the (anti)-de Sitter algebra \mathfrak{g} together with an irreducible \mathfrak{g} -module \mathbf{A} . The starting point is that, according to (2.16), the (anti)-de Sitter covariant derivative D_Ω splits as

$$D_\Omega = D + \nabla_- - \nabla_+, \tag{3.1}$$

where (∇_\pm) are nilpotent algebraic operators, $(\nabla_\pm)^2 = 0$. The operators ∇_\pm preserve Young symmetry properties.

First, in 3.1 we introduce the operator σ_- in abstract terms and give an overview of the well-known facts on the interpretation of the σ_- -cohomology [14, 51–53]. Then, to strictly define σ_- for the complex $\mathcal{C}(\mathbf{A}, D_\Omega)$ of gauge connections we need some details about the restriction of irreducible modules, which are in section 3.2. Before giving a formal definition for σ_- in 3.4, two simple examples are considered in 3.3. To present the result on σ_- -cohomology in a simple form, which is done in 3.7 with the proof being in appendix B, we define the highest weight part and the maximally symmetric part of tensor products in 3.5 and introduce in 3.6 a special structure joining restriction of modules with maximally symmetric part of tensor products. A number of useful examples on σ_- -cohomology is given in 3.8.

3.1 Interpretation of cohomology

Suppose that the field content of some unfolded system of equation is given by a graded collection of degree- q differential forms⁶ $\omega_{\mathbf{q}}^g$, $g = 0, 1, \dots$. The corresponding gauge parameters are the forms of degree- $(q - 1)$ with the values in the same spaces as $\omega_{\mathbf{q}}^g$. If $q > 1$ there are reducible gauge symmetries with parameters $\xi_{\mathbf{q}-\mathbf{i}}^g$, $i = 2, \dots, q$. Suppose also that the gauge transformations, the field curvatures and the Bianchi identities have the form, which is a special case of the unfolded complex (2.10),

$$\delta\xi_{\mathbf{1}}^g = D\xi_{\mathbf{0}}^k + \sigma_-(\xi_{\mathbf{0}}^{g+1}), \tag{3.2}$$

$$\dots = \dots, \tag{3.3}$$

$$\delta\xi_{\mathbf{q}-\mathbf{1}}^g = D\xi_{\mathbf{q}-\mathbf{2}}^g + \sigma_-(\xi_{\mathbf{q}-\mathbf{2}}^{g+1}), \tag{3.4}$$

$$\delta\omega_{\mathbf{q}}^g = D\xi_{\mathbf{q}-\mathbf{1}}^g + \sigma_-(\xi_{\mathbf{q}-\mathbf{1}}^{g+1}), \tag{3.5}$$

$$R_{\mathbf{q}+\mathbf{1}}^g = D\omega_{\mathbf{q}}^g + \sigma_-(\omega_{\mathbf{q}}^{g+1}), \tag{3.6}$$

$$0 = DR_{\mathbf{q}+\mathbf{1}}^g + \sigma_-(R_{\mathbf{q}+\mathbf{1}}^{g+1}), \tag{3.7}$$

where σ_- is an algebraic operator that decreases the grade by one and increases the form degree by one. The only differential part is in the Lorentz covariant derivative D . The formal consistency of the system requires (1) $\{D, \sigma_-\} = 0$, which trivially holds in the systems of interest since σ_- is built of the background vielbein and $Dh^a = 0$ is equivalent to (2.18); (2) σ_- is a nilpotent operator, $\sigma_-^2 = 0$.

Having a nilpotent operator suggests the cohomology problem. The σ_- -cohomology turns out to have a very clear field-theoretical meaning, classifying all dynamically relevant independent quantities. Indeed, the degree- k , $k = 1, \dots, q - 1$, gauge parameters $\xi_{\mathbf{k}}^g$ that are σ_- -exact can be gauged away with the help of the reducible algebraic gauge symmetry with $\xi_{\mathbf{k}-\mathbf{1}}^{g+1}$. The leftover gauge symmetry $0 = \delta\xi_{\mathbf{k}}^g = D\xi_{\mathbf{k}-\mathbf{1}}^g + \sigma_-(\xi_{\mathbf{k}-\mathbf{1}}^{g+1})$ just expresses $\xi_{\mathbf{k}-\mathbf{1}}^{g+1}$ modulo σ_- -closed part in terms of $\xi_{\mathbf{k}-\mathbf{1}}^g$. Therefore, the true differential gauge parameters are σ_- -closed and are not σ_- -exact thus being representatives of the σ_- -cohomology groups $H^k(\sigma_-)$. Quite analogously, the dynamical fields, i.e. the fields

⁶The analysis is not affected if the form degree varies with the grade, $\omega_{\mathbf{q}_{\mathbf{g}}}^g$, as it occurs for massless fields in Minkowski space [14].

cohomology group	interpretation
$H^{q-i}, i = 1, \dots, q$	differential gauge parameters at the i -th level of reducibility
H^q	dynamical fields
H^{q+1}	independent gauge invariant equations on dynamical fields
$H^{q+i+1}, i = 1, \dots$	Bianchi identities at the i -th level

Table 1. Interpretation of the σ_- -cohomology.

that cannot be gauged away by some algebraic gauge symmetry and are not expressed in terms of derivatives of some other fields, are the representatives of $H^q(\sigma_-)$. The independent gauge invariant differential expressions are the representatives of $H^{q+1}(\sigma_-)$. The nontrivial (reducible) Bianchi identities are the representatives of $H^{q+j}(\sigma_-)$, $j = 2, \dots$. We sum up the interpretation of $H(\sigma_-)$ in table 1. The maximal number of derivatives connecting two elements of $H_{g_1}^{q'}$ and $H_{g_2}^{q'+1}$ is equal to⁷ $(g_2 - g_1 + 1)$, meaning that if some representatives of $H_{g_1}^{q'}$ and $H_{g_2}^{q'+1}$ correspond to a parameter and a dynamical field then the gauge transformation law contains $(g_2 - g_1 + 1)$ derivatives; if $H_{g_1}^{q'}$ and $H_{g_2}^{q'+1}$ correspond to a dynamical field and a gauge invariant equation then the latter is up to $(g_2 - g_1 + 1)$ -th order in derivative, etc.

3.2 Restriction of irreducible modules

When formulated in terms of Young diagrams the restriction rules are the same⁸ both for $sl(d+1)$ and $so(d+1)$. Let \mathfrak{g} and \mathfrak{h} denote either $sl(d+1)$ and $sl(d)$ or $so(d+1)$ and $so(d)$.

When restricted to the subalgebra $\mathfrak{h} \subset \mathfrak{g}$, irreducible finite-dimensional representations of \mathfrak{g} decompose into a direct sum of irreducible representations of \mathfrak{h} . We denote this functor as $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} \mathbf{X}$, where \mathbf{X} is a Young diagram that determines an irreducible representation of \mathfrak{g} . The result of applying $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}}$ to $\mathbf{X} = \mathbb{Y}(s_1, \dots, s_n)$ reads [61]

$$\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} \mathbf{X} = \bigoplus_{k_1, \dots, k_N} \mathbf{X}_{\{k_1, \dots, k_N\}}, \quad (3.8)$$

where the multiplicity of each irreducible module $\mathbf{X}_{\{k_1, \dots, k_N\}}$ is one and

$$\mathbf{X}_{\{k_1, \dots, k_N\}} = \begin{cases} \mathbb{Y}(k_1, \dots, k_n), & k_1 \in [s_2, s_1], \dots, k_{n-1} \in [s_n, s_{n-1}], k_n \in [0, s_n], \\ \emptyset, & \text{otherwise.} \end{cases} \quad (3.9)$$

So, the result of the restriction is given by various Young diagrams obtained by removing an arbitrary number of cells from the right of rows of \mathbf{X} provided that the length of each shortened row is not less than the length the next row of \mathbf{X} .

⁷In the (anti)-de Sitter case two covariant derivatives might appear in the form of a commutator, which is an algebraic expression and not a second order differential operator. Fortunately, we are able to trace the appearance of such commutator terms.

⁸It is worth stressing that at least one of the weights (s_1, \dots, s_ν) of $\mathfrak{so}(2\nu + 1)$ or $\mathfrak{so}(2\nu)$ must be zero in order to get rid of (anti)-selfdual representations both for the (anti)-de Sitter algebra and its Lorentz subalgebra, which is implied.

$\text{Res}_{\mathfrak{h}}^g \mathbf{X}$ is endowed with a natural structure of a graded vector space if to each element $\mathbf{X}_{\{k_1, \dots, k_N\}}$ we assign the grade $g = k_1 + k_2 + \dots + k_N - s_2 - \dots - s_n$, so that the grade of the minimal rank element of $\text{Res}_{\mathfrak{h}}^g \mathbf{X}$ is 0 and the grade of the maximal rank element of $\text{Res}_{\mathfrak{h}}^g \mathbf{X}$ is s_1 .

It is convenient to define generally reducible \mathfrak{h} -modules \mathbf{X}_g to be a direct sum of irreducible \mathfrak{h} -modules $\mathbf{X}_{\{k_1, \dots, k_N\}}$ having the same grade g (the same rank)

$$\mathbf{X}_g = \bigoplus_{k_1 + \dots + k_n = g} \mathbf{X}_{\{k_1, \dots, k_N\}}, \quad (3.10)$$

so that $\text{Res}_{\mathfrak{h}}^g \mathbf{X} = \bigoplus_g \mathbf{X}_g$.

Among $\mathbf{X}_{\{k_1, \dots, k_N\}}$ there are elements \mathbf{X}^m (those having first $(m-1)$ rows k_1, \dots, k_{m-1} of maximal length with the rest of rows k_m, \dots, k_n having minimal length) that can be referred to as maximally symmetric

$$\mathbf{X}^m = \begin{cases} \mathbb{Y}(s_1, \dots, \widehat{s_m}, \dots, s_n), & m = 1, \dots, n, \\ \mathbb{Y}(s_1, \dots, s_n), & m > n, \end{cases} \quad (3.11)$$

i.e. as if the m -th weight is thrown away. The grade of maximally symmetric elements \mathbf{X}^k is given by

$$g(\mathbf{X}^k) = \begin{cases} s_1 - s_{k+1}, & 1 \leq k \leq n, \\ s_1, & k > n \end{cases} \quad (3.12)$$

3.3 Two simple examples

The simplest σ_- -model for $so(d+1)$. Let \mathbf{A} be a rank- s totally symmetric irreducible module of the (anti)-de Sitter algebra. The action of D_Ω on an element $W_{\mathbf{q}}^{\mathbf{A}}$ reads

$$D_\Omega W_{\mathbf{q}}^{A(s)} = DW_{\mathbf{q}}^{A(s)} + s\lambda V^A E_M W_{\mathbf{q}}^{A(s-1)M} - s\lambda E^A V_M W_{\mathbf{q}}^{A(s-1)M}, \quad (3.13)$$

where we adopt the signs as for the anti-de Sitter case. The result of the restriction of $W_{\mathbf{q}}^{A(s)}$ to the Lorentz subalgebra is given by

$$W_{\mathbf{q}}^{A(s)} \iff \omega_{\mathbf{q}}^{a(k)}, \quad k = 0, \dots, s, \quad (3.14)$$

If we choose the standard gauge (2.17) for the compensator V^A , the field $\omega_{\mathbf{q}}^{a(k)}$ can be identified with the traceless part of $W_{\mathbf{q}}^{a(k)\bullet(s-k)}$. Note that the contraction of two Lorentz indices of $W_{\mathbf{q}}^{a(k)\bullet(s-k)}$ with the metric η_{ab} does not vanish

$$\eta_{bb} W_{\mathbf{q}}^{a(k-2)bb\bullet(s-k)} = W_{\mathbf{q}}^{a(k-2)\bullet(s-k+2)}. \quad (3.15)$$

We refer to such expressions with a part of indices of the (anti)-de Sitter algebra restricted to the Lorentz algebra and with the other indices pointing along V^A as to 'raw'. In terms of the raw fields $W_{\mathbf{q}}^{a(k)\bullet(s-k)}$ (3.13) is rewritten as

$$D_\Omega W_{\mathbf{q}}^{a(k)\bullet(s-k)} = DW_{\mathbf{q}}^{a(k)\bullet(s-k)} + (s-k)\lambda h_m W_{\mathbf{q}}^{a(k)m\bullet(s-k-1)} - k\lambda h^a W_{\mathbf{q}}^{a(k-1)\bullet(s-k+1)},$$

where use is made of $E^a = h^a$, $E^\bullet = 0$, $V^a = 0$, $V^\bullet = 1$. Next, we rewrite the raw expressions in terms of the irreducible connections $\omega_{\mathbf{q}}^{a(k)}$

$$D_{\Omega}\omega_{\mathbf{q}}^{a(k)} = D\omega_{\mathbf{q}}^{a(k)} + \lambda h_m \omega_{\mathbf{q}}^{a(k)m} + \lambda f_k \left(h^a \omega_{\mathbf{q}}^{a(k-1)} - \frac{(k-1)}{d+2k-4} \eta^{aa} h_m \omega_{\mathbf{q}}^{a(k-2)m} \right)$$

where the fields have been rescaled to get rid of some factors and

$$f_k = \frac{k(s-k-1)(d+s+k-2)}{d+2k-2}.$$

In terms of Lorentz connections, ∇_- and ∇_+ of (3.1) or the operators $V^A E_M$ and $E^A V_M$ of (3.13) give rise to the two algebraic operators

$$\sigma_- \left(\omega_{\mathbf{q}}^{a(k+1)} \right) = h_m \omega_{\mathbf{q}}^{a(k)m}, \quad (3.16)$$

$$\sigma_+ \left(\omega_{\mathbf{q}}^{a(k-1)} \right) = f_k \left(h^a \omega_{\mathbf{q}}^{a(k-1)} - \frac{(k-1)}{d+2k-4} \eta^{aa} h_m \omega_{\mathbf{q}}^{a(k-2)m} \right). \quad (3.17)$$

It is straightforward to verify that $\sigma_-^2 = 0$ due to $h^a h^b + h^b h^a \equiv 0$. For the same reason $\sigma_+^2 = 0$.

The simplest σ_- -model for $sl(d+1)$. It is useful to consider fields with relaxed trace constraints, for instance, the unfolded off-shell constraints for symmetric fields of all spins in Minkowski space were found in [62] to have the form of zero curvature and covariant constancy conditions with the fields not subjected to any trace constraints. For this purpose, we take \mathbf{A} to be an irreducible module of $sl(d+1)$. The analogue of the Lorentz algebra is then $sl(d)$.

Again, take \mathbf{A} be a rank- s totally symmetric tensor representation. The result of the restriction of $W_{\mathbf{q}}^{A(s)}$ to the $sl(d)$ -subalgebra is given by the same number of component fields

$$W_{\mathbf{q}}^{A(s)} \iff \omega_{\mathbf{q}}^{a(k)}, \quad k = 0, \dots, s. \quad (3.18)$$

Since no trace constraints are imposed, the 'raw' fields $W_{\mathbf{q}}^{a(k)\bullet(s-k)}$ are directly identified with the irreducible $sl(d)$ -fields $\omega_{\mathbf{q}}^{a(k)}$

$$\omega_{\mathbf{q}}^{a(k)} = W_{\mathbf{q}}^{a(k)\bullet(s-k)}, \quad k = 0, \dots, s. \quad (3.19)$$

∇_- and ∇_+ give rise to

$$\sigma_- \left(\omega_{\mathbf{q}}^{a(k+1)} \right) = h_m \omega_{\mathbf{q}}^{a(k)m}, \quad (3.20)$$

$$\sigma_+ \left(\omega_{\mathbf{q}}^{a(k+1)} \right) = \tilde{f}_k h^a \omega_{\mathbf{q}}^{a(k-1)}, \quad (3.21)$$

where $\tilde{f}_k = (s-k-1)k$. The nilpotency of σ_{\pm} is obvious.

3.4 Formal definition

Given some Lie algebra \mathfrak{g} , its representation \mathbf{A} and a commutative subalgebra $\mathfrak{f} \subset \mathfrak{g}$, there is a well-known definition⁹ of (co)homology of the Lie algebra \mathfrak{f} with coefficients in a \mathfrak{g} -module \mathbf{A} taken as an \mathfrak{f} -module.

$$\begin{aligned} \partial : \mathbf{A} \otimes \Lambda^q(\mathfrak{f}) &\longrightarrow \mathbf{A} \otimes \Lambda^{q-1}(\mathfrak{f}), \\ \partial(a \otimes u_1 \wedge \dots \wedge u_q) &= \sum_{i=1}^{i=q} (-)^{i+1} u_i(a) \otimes u_1 \wedge \dots \wedge \hat{u}_i \wedge \dots \wedge u_q, \quad a \in \mathbf{A}, u_i \in \mathfrak{f}, \end{aligned} \quad (3.22)$$

where $u_i(a)$ is the action of $u_i \in \mathfrak{f} \subset \mathfrak{g}$ on a vector $a \in \mathbf{A}$.

The above definition leads to the σ_- -cohomology for the case of $\mathfrak{g} = sl(d+1)$, in which \mathfrak{f} in some base is identified with the subalgebra of matrices having nonvanishing entries in the first row except for the leftmost entry

$$\left(\begin{array}{c|c} 0 & u \\ \hline * & \mathfrak{h} \end{array} \right), \quad u \in \mathfrak{f}. \quad (3.23)$$

Now we turn to orthogonal algebra, $\mathfrak{g} = so(d+1)$, the Lorentz subalgebra is $\mathfrak{h} = so(d)$, then \mathfrak{f} as a vector subspace is given by antisymmetric matrices that have zeros everywhere except for the first row and the first column

$$\left(\begin{array}{c|c} 0 & u \\ \hline -u & \mathfrak{h} \end{array} \right), \quad u \in \mathfrak{f}, \quad (3.24)$$

i.e., \mathfrak{f} is nothing but the translation generators. However, \mathfrak{f} is a subspace and not a subalgebra, so that we cannot use the classical definition. Nevertheless, we will show that in some cases one still can associate with \mathfrak{f} certain nilpotent operator ∂ , $\partial^2 = 0$, acting on \mathbf{A} and hence build a complex.

Suppose we are given a Lie algebra \mathfrak{g} , its module \mathbf{A} and a subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$, which is to be identified with the Lorentz subalgebra. There is a canonical decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus_{\alpha} \mathfrak{g}_{\alpha}$ of \mathfrak{g} as a vector space into irreducible representations \mathfrak{g}_{α} of the subalgebra \mathfrak{g}_0 . Restricting to the subalgebra \mathfrak{g}_0 , the \mathfrak{g} -module \mathbf{A} decomposes into \mathfrak{g}_0 -modules \mathbf{A}_k

$$\text{Res}_{\mathfrak{g}_0}^{\mathfrak{g}} \mathbf{A} = \bigoplus_k \mathbf{A}_k. \quad (3.25)$$

The subalgebra \mathfrak{g}_0 acts diagonally, i.e. $\mathfrak{g}_0(\mathbf{A}_k) \subset \mathbf{A}_k$. In contrast, the action of \mathfrak{g}_{α} takes \mathbf{A}_k to some other \mathbf{A}_i . The morphism $\rho_{\alpha} : \mathfrak{g}_{\alpha} \otimes \mathbf{A} \longrightarrow \mathbf{A}$ defined as $u \otimes a \longrightarrow u(a)$, $u \in \mathfrak{g}_{\alpha}$, $a \in \mathbf{A}$, is a homomorphism of two \mathfrak{g}_0 -modules.

The definition (3.22) rests on \mathfrak{f} being a subalgebra. This may not be the case now for \mathfrak{g}_{α} . Nevertheless we can construct certain commuting operators, which are beyond the representation of \mathfrak{g} on \mathbf{A} . To succeed we need the action of \mathfrak{g}_{α} on \mathbf{A} to be \mathbb{Z} -graded.

⁹We are grateful to E. Feigin for many valuable discussions on Lie algebra cohomology and σ_- and for reference [63].

For classical Lie algebras the notion of rank is well-defined, so let $|\mathbf{A}_k|$ denote the rank of \mathbf{A}_k . Let us define another decomposition $\mathbf{A} = \bigoplus_g \mathbf{A}_g$ of the \mathfrak{g} -module \mathbf{A} into generally reducible \mathfrak{g}_0 -modules such that

$$\mathbf{A}_g = \bigoplus_{k:|\mathbf{A}_k|=g} \mathbf{A}_k \tag{3.26}$$

is a direct sum over \mathbf{A}_k having rank g .

The tensor product $\mathfrak{g}_\alpha \otimes \mathbf{A}$ of the two \mathfrak{g}_0 -modules can be explicitly computed. In view of general properties of tensor products $\mathfrak{g}_\alpha \otimes \mathbf{A}_g$ decomposes into representations with ranks confined in the range $|g - |\mathfrak{g}_\alpha||, g + |\mathfrak{g}_\alpha|$. Therefore, the rank provides us with natural \mathbb{Z} -grading on \mathbf{A} . Let the operators realizing the action of \mathfrak{g} on \mathbf{A} be denoted as $\vartheta_{\mathfrak{g}}$. Then, there is a decomposition of $\vartheta_{\mathfrak{g}_\alpha}$ into the parts with definite grade

$$\vartheta_{\mathfrak{g}_\alpha} = \bigoplus_{i \in \mathbb{Z}} \vartheta_{\mathfrak{g}_\alpha}^i, \quad \vartheta_{\mathfrak{g}_\alpha}^i : \mathbf{A}_g \longrightarrow \mathbf{A}_{g+i}, \tag{3.27}$$

$\vartheta_{\mathfrak{g}_0}$ has by definition zero grade part only, $\vartheta_{\mathfrak{g}_0} = \vartheta_{\mathfrak{g}_0}^0$. Obviously, there is a certain n such that $\vartheta_{\mathfrak{g}_\alpha}^j \equiv 0$ if $j < -n$ for any α and we assume that there is a certain \mathfrak{g}_{\min} among \mathfrak{g}_α such that $\vartheta_{\mathfrak{g}_{\min}}^{-n} \neq 0$. If there are several \mathfrak{g}_α such that $\vartheta_{\mathfrak{g}_\alpha}^{-n} \neq 0$ then \mathfrak{g}_{\min} is a direct sum over such \mathfrak{g}_α . By definition of representation we have

$$[\vartheta_x, \vartheta_y] = \vartheta_{[x,y]}, \quad x, y \in \mathfrak{g} \quad \implies \quad [\vartheta_{x'}^{-n}, \vartheta_{y'}^{-n}] = 0, \quad x', y' \in \mathfrak{g}_{\min}, \tag{3.28}$$

i.e. $\vartheta_{\mathfrak{g}_{\min}}^{-n}$ are commuting operators.

Consequently, given a \mathbb{Z} -graded decomposition of action of some algebra on its representation it is possible to single out commuting operators that belong to the lowest or highest grade. Despite the fact that \mathfrak{g}_{\min} does not form a subalgebra, the operators $\vartheta_{\mathfrak{g}_{\min}}^{-n}$ do form a commutative subalgebra.

The complex $\mathcal{C}(\mathbf{A}, \partial)$ is defined in a standard way:

$$\partial : \mathbf{A} \otimes \Lambda(\mathfrak{g}_{\min}), \quad \partial : \mathbf{A}_g \otimes \Lambda^q(\mathfrak{g}_{\min}) \longrightarrow \mathbf{A}_{g-n} \otimes \Lambda^{q-1}(\mathfrak{g}_{\min}), \tag{3.29}$$

$$\partial(a \otimes u_1 \wedge \dots \wedge u_q) = \sum_{i=1}^{i=q} (-)^{i+1} \vartheta_{u_i}^{-n}(a) \otimes u_1 \wedge \dots \wedge \hat{u}_i \wedge \dots \wedge u_q, \quad a \in \mathbf{A}, u_i \in \mathfrak{g}_{\min}. \tag{3.30}$$

We collect in table 2 some cases that are or may be of interest. Note that the signature of \mathfrak{so} -algebras does not matter, in what follows we assume that appropriate real forms are chosen. Item 1 is the case we investigate in the present paper, it concerns the gauge fields in (anti)-de Sitter space. Item 2: the trace constraints on fields are fully relaxed so that we have the correct pattern of gauge symmetries but there can be imposed no field equations. The decomposition \mathfrak{g}_α consists of a vector, covector and a scalar. In this case, vector (or covector) representation itself forms a commutative subalgebra, so that $\mathbf{H}(\mathbf{A}, \sigma_-)$ coincides with the ordinary Lie algebra cohomology, the answer can be found in [63]. Item 3: \mathfrak{g} can be taken as the conformal algebra, this provides a natural framework for conformal fields, which are studied in [64]. Note that in this case we again meet the ordinary Lie algebra

	\mathfrak{g}	\mathfrak{g}_0	\mathfrak{g}_α	description
1	$so(d+1)$	$so(d)$	\square	$(A)dS_d$ fields on-shell
2	$sl(d+1)$	$sl(d)$	$\square \oplus \square^* \oplus \bullet$	$(A)dS_d$ fields off-shell
3	$so(d+2)$	$so(d)$	$2\square \oplus \bullet$	conformal fields on-shell
4	$sl(d+2)$	$sl(d)$	$2\square \oplus 2\square^* \oplus 4\bullet$	conformal fields off-shell
5	$sl(d)$	$so(d)$	$\square\square$	trace decomposition
6	$sl(d+1)$	$so(d)$	$\square\square \oplus 2\square \oplus \bullet$	unconstrained $(A)dS_d$ fields

Table 2. Decomposition of certain algebras with respect to their subalgebras.

cohomology. Item 4: the same case of conformal fields but the description is off-shell. Item 5 corresponds to the trace decomposition of a tensor with fully relaxed trace constraints in terms of traceless tensors. Item 6 may be related to the unconstrained approach of [4, 65–68] for the case of gauge fields in $(A)dS_d$.

We will study the complex $\mathcal{C}(\mathbf{A}, \sigma_-)$ dual to $\mathcal{C}(\mathbf{A}, \partial)$ for \mathfrak{g} being the (anti)-de Sitter algebra and \mathfrak{g}_0 being its Lorentz subalgebra. The most significant fact for computing σ_- -cohomology is that on account of

$$[\vartheta_w, \vartheta_u^{-n}] = \vartheta_{[w,u]}^{-n}, \quad [w, u] \in \mathfrak{g}_{\min}, \quad w \in \mathfrak{g}_0, \quad u \in \mathfrak{g}_{\min} \quad (3.31)$$

the differential ∂ (or σ_-) commutes with the action of \mathfrak{g}_0 and hence \mathfrak{g}_0 acts on the (co)homology, so that it is convenient to label the representatives of (co)homology by the weights of irreducible \mathfrak{g}_0 -modules or by Young diagrams in the case of interest.

σ_- , specialization to $(A)dS_d$. The σ_- -complex $\mathcal{C}(\mathbf{A}, \sigma_-)$ is associated with the complex $\mathcal{C}(\mathbf{A}, D_\Omega)$. To build $\mathcal{C}(\mathbf{A}, \sigma_-)$ we need an irreducible \mathfrak{g} -module \mathbf{A} and nondegenerate D_Ω , meaning that it yields E^A with the maximal rank (or, in the standard gauge, h^a), so that very little is needed from $\mathcal{C}(\mathbf{A}, D_\Omega)$. The representatives of σ_- -cohomology are irreducible Lorentz modules whose weights are to be found.

The (anti)-de Sitter algebra as a vector space \mathfrak{g} splits as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where \mathfrak{h} is the Lorentz algebra $\mathfrak{so}(d-1, 1)$ and \mathfrak{p} is a vector representation of \mathfrak{h} , namely, the translation generators P_a constitute \mathfrak{p} . The splitting (2.16) $\Omega = \Omega_L + \nabla_- - \nabla_+$ of a \mathfrak{g} -connection Ω implies that $\nabla_- - \nabla_+$ corresponds to the operators $\vartheta_{\mathfrak{p}}$ from the previous subsection that represent the action of the translation generators. So ∇_- and ∇_+ are the operators $\vartheta_{\mathfrak{p}}^{-1}$ and $\vartheta_{\mathfrak{p}}^{+1}$ such that $\vartheta_{\mathfrak{p}} = \vartheta_{\mathfrak{p}}^{-1} + \vartheta_{\mathfrak{p}}^{+1}$. Both ∇_- and ∇_+ are algebraic, and hence algebraic are the induced operators σ_- and σ_+ , we ignore the dependence on the space-time coordinates x^μ .

We define the q -cochain $C^q(\mathbf{A}) = \bigoplus_{\mathfrak{g}} C_{\mathfrak{g}}^q(\mathbf{A})$ with

$$C_{\mathfrak{g}}^q(\mathbf{A}) = \mathbf{A}_{\mathfrak{g}} \otimes \Lambda^q(\mathfrak{p}). \quad (3.32)$$

Then, ∇_{\pm} induce two nilpotent operators σ_{\pm}

$$\sigma_{\pm} : C_{\mathfrak{g}}^q(\mathbf{A}) \longrightarrow C_{\mathfrak{g}}^{q\pm 1}(\mathbf{A}). \quad (3.33)$$

In tensorial terms, we decompose $\mathcal{W}_{\mathbf{q}}^{\mathbf{A}}$ into connections of the Lorentz subalgebra and convert all form indices to fiber ones with the help of the inverse background vielbein $h^{a\mu}$, $h^{a\mu}h_{\mu}^b = \eta^{ab}$, so that the Lorentz algebra \mathfrak{h} starts acting on the former form indices too. $\mathbf{A} \otimes \Lambda^q(\mathfrak{p})$ is an \mathfrak{h} -module with the action of \mathfrak{h} on $\Lambda^q(\mathfrak{p})$ induced from that on \mathfrak{p} . Moreover, \mathfrak{h} commutes both with σ_- and σ_+ , which allows us to decompose $\mathbf{A} \otimes \Lambda^q(\mathfrak{p})$ into irreducible \mathfrak{h} -modules and parameterize the representative of the σ_- -cohomology groups by Young diagram of \mathfrak{h} .

An element of $C_g^q(\mathbf{A})$ is a degree- q exterior form $\omega_{\mathbf{q}}^{\mathbf{X}}$ with values in a generally reducible \mathfrak{h} -module \mathbf{X} belonging to \mathbf{A}_g . As an \mathfrak{h} -module $\omega_{\mathbf{q}}^{\mathbf{X}}$ can be decomposed into irreducible \mathfrak{h} -modules, the decomposition is equivalent to taking \mathfrak{h} -tensor product

$$C_g^q(\mathbf{A}) \sim \mathbf{A}_g \otimes_{\mathfrak{h}} \mathbb{Y}[q] = \bigoplus_{r=0}^{r=q} \bigoplus_{i_r} M_{r,i_r}^{g,q} \mathbf{X}_{r,i_r}^{g,q}, \quad (3.34)$$

where the sum is over irreducible \mathfrak{h} -modules $\mathbf{X}_{r,i_r}^{g,q}$, with $M_{r,i_r}^{g,q}$ being the multiplicity of $\mathbf{X}_{r,i_r}^{g,q}$. By definition, all irreducible modules in \mathbf{A}_g have the same rank, denote it $|\mathbf{A}_g|$. The additional summation index r distinguishes between traces of different orders, so that the rank $|\mathbf{X}_{r,i_r}^{g,q}|$ of $\mathbf{X}_{r,i_r}^{g,q}$ is equal to $|\mathbf{A}_g| + q - 2r$. In tensorial terms, the trace order is half the number of indices that must be contracted to get a tensor with the symmetry of $\mathbf{X}_{r,i_r}^{g,q}$.

The operator σ_- preserves two natural gradings - the total rank of the tensor it acts on (the rank of the fiber tensor g plus the form degree q) and the trace order. In addition, that σ_- preserves the \mathfrak{h} -module structure means that it does not mix different \mathfrak{h} tensors up. Granting this, $\mathcal{C}(\mathbf{A}, \sigma_-)$ decomposes into a direct sum

$$\mathcal{C}(\mathbf{A}, \sigma_-) = \bigoplus_{q+g} \bigoplus_r \bigoplus_{\mathbf{X}} \mathcal{C}(\mathbf{A}, \sigma_-; \mathbf{X}, q+g, r) \quad (3.35)$$

of complexes parameterized by an arbitrary Young diagram \mathbf{X} from $\mathbf{A}_g \otimes \mathbb{Y}[q]$, the total rank $q+g$ and the trace order r . Restricted on $\mathcal{C}(\mathbf{A}, \sigma_-; \mathbf{X}, q+g, r)$, σ_- is given by a set of linear maps $\mathbb{R}^{M_{r,i_r}^{g,q}} \otimes \mathbf{X} \longrightarrow \mathbb{R}^{M_{r,i_r}^{g-1, q+1}} \otimes \mathbf{X}$ acting on the first factor.

Consequently, $\mathcal{C}(\mathbf{A}, \sigma_-)$ is well-defined for any \mathfrak{g} -module \mathbf{A} . The background vielbein E^A and the compensator field V^A provide a field-theoretical realization of $\mathcal{C}(\mathbf{A}, \sigma_-)$, at the condition that the vielbein field E^A has the maximal rank. We will see that $\mathcal{C}(\mathbf{A}, \sigma_-)$ has rich cohomology as distinct from, for example, the de Rham complex.

It is worth stressing that the equation of motion, gauge transformations, Bianchi identities, etc. contain both σ_- and σ_+ , e.g. $D\omega^g + \sigma_-(\omega^{g+1}) + \sigma_+(\omega^{g-1}) = 0$. Hence, we can chose either σ_- or σ_+ to be the operator σ_- of section 3.1 used to interpret the unfolded equations.

We might study the cohomology problem both for σ_- and σ_+ , however, the choice of σ_- , acting from higher rank tensors of \mathbf{A}_{g+1} to lower rank tensors of \mathbf{A}_g , is more natural since equation $D\omega^g + \sigma_-(\omega^{g+1}) + \sigma_+(\omega^{g-1}) = 0$ expresses higher rank auxiliary fields ω^{g+1} in terms of derivatives of lower rank fields ω^g if σ_- has vanishing cohomology at grade g , plus possibly lower derivative terms coming from $\sigma_+(\omega^{g-1})$. Therefore, the chain of auxiliary fields starts from dynamical fields having the lowest possible rank, these fields will be recognized as the field potentials $\phi^{\mathbf{S}}$.

The interpretation of unfolded equations in terms of the σ_+ -cohomology leads to dual formulations, in which the dynamical fields are tensors of a rank higher than that of ϕ^S . An example of such a dual formulation was studied in [69].

3.5 Distinguished parts of tensor products

The highest weight part. Let $\mathbf{X} = \mathbb{Y}(s_1^x, \dots, s_n^x)$ and $\mathbf{Y} = \mathbb{Y}(s_1^y, \dots, s_n^y)$ be two Young diagrams either of $sl(d)$ or $so(d)$. Despite the fact that to decompose the tensor product $\mathbf{X} \otimes_{sl(d)} \mathbf{Y}$ into a direct sum of irreducible modules is a complicated problem, we can be sure that at least one irreducible module is present in $\mathbf{X} \otimes \mathbf{Y}$ whose Young diagram $\mathbf{Z} = \mathbb{Y}(s_1^x + s_1^y, \dots, s_n^x + s_n^y)$ is obtained by row-by-row concatenation. This one is called the highest-weight part, $\text{hwp}(\mathbf{X}, \mathbf{Y})$. The highest weight part of $\mathbf{X} \otimes \mathbf{Y}$ is given by a single Young diagram. In the case of interest the second multiplier is a one-column Young diagram. Let us denote the highest weight part of $\mathbf{X} \otimes \mathbb{Y}[q]$ as

$$\text{hwp}(\mathbf{X}, q) = \begin{cases} \mathbb{Y}(s_1^x + 1, \dots, s_q^x + 1, s_{q+1}^x, \dots, s_n^x), & q < n, \\ \mathbb{Y}(s_1^x + 1, \dots, s_n^x + 1, \underbrace{1, \dots, 1}_{q-n}), & q \geq n. \end{cases} \quad (3.36)$$

Remark on $so(d)$ -tensor products. Roughly speaking, the difference between the tensor products of $sl(d)$ and $so(d)$ is that in the latter case one is able to contract indices with the help of the invariant tensor η_{ab} of $so(d)$, i.e. to take traces. Thus, the $so(d)$ -tensor product rule for $\mathbf{X} \otimes_{so(d)} \mathbf{Y}$ consists of taking traces, which removes pairs of cells (one from \mathbf{X} and another from \mathbf{Y}), and, then, adding the rest of the cells of \mathbf{Y} to a set of diagrams obtained from \mathbf{X} at the first stage. The precise rules can be found in [70], which for the cases of interest are given in appendix A.

In general the tensor product of two irreducible $so(d)$ -modules decomposes into a direct sum of irreducible modules whose multiplicities can be greater than one, because the same diagram can be obtained generally by removing and, then, adding cells from/to different places.

Given some element \mathbf{Z} of $\mathbf{X} \otimes_{so(d)} \mathbf{Y}$, the number of cells that were removed from \mathbf{X} (or \mathbf{Y}) is called the trace order of \mathbf{Z} .

The maximally symmetric part. For the case of $sl(d)$ the highest weight part will also be called the maximally symmetric part. However, for the case of $so(d)$ the two definitions are different.

For the case of $so(d)$, given two irreducible $so(d)$ -modules \mathbf{X} and \mathbf{Y} , the maximally symmetric part of $\mathbf{X} \otimes \mathbf{Y}$ with the trace order r , $\text{msp}(\mathbf{X}, \mathbf{Y}, r)$, is a sum of the form $\bigoplus_{\alpha} \text{hwp}(\mathbf{X}_{\alpha}^r, \mathbf{Y}_{\alpha}^r)$, where \mathbf{X}_{α}^r and \mathbf{Y}_{α}^r are the diagrams obtained from \mathbf{X} and \mathbf{Y} by taking all possible traces of order r , so that \mathbf{X}_{α}^r and \mathbf{Y}_{α}^r each has r cells less than \mathbf{X} and \mathbf{Y} , respectively. The index α runs over all inequivalent traces of order r .

As distinct from the $sl(d)$ -case, the maximally symmetric part of the $so(d)$ -tensor product may contain many irreducible modules, however, each comes with multiplicity one.

In this paper the second multiplier is always a one column diagram, i.e. $\mathbf{Y} = \mathbb{Y}[q]$. Given an $so(d)$ -Young diagram $\mathbf{X} = \mathbb{Y}\{(s_1, p_1), \dots, (s_N, p_N)\}$ (the block notation for Young

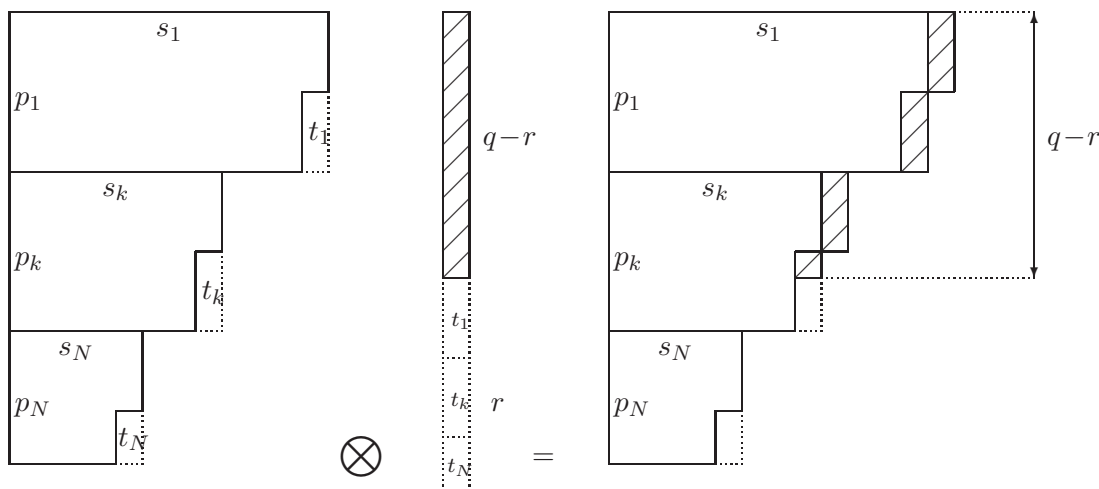


Figure 1. Taking the maximally symmetric part of $so(d)$ -tensor product $\mathbf{X} \otimes \mathbf{Y}$, $\mathbf{X} = \mathbb{Y}\{(s_1, p_1), \dots, (s_N, p_N)\}$, $\mathbf{Y} = \mathbb{Y}[q]$. Since taking the trace must result in a Young diagram, different traces correspond to different partitions of $r = t_1 + \dots + t_N$, such that $t_i \leq p_i$. Then, t_i cells of \mathbf{Y} are removed from the bottom-right of the i -th block of \mathbf{X} . Finally, the rest of cells from $\mathbb{Y}[q]$, i.e. $q - r$ cells, is added to the first rows, which gives the highest weight part. If $r = 0$ then we get the $sl(d)$ -case, for which $\text{hwp}(\mathbf{X}, q) \equiv \text{msp}(\mathbf{X}, q)$.

diagrams is more convenient) and two nonnegative integers q, r such that $q \geq r$ the maximally symmetric part of $\mathbf{X} \otimes_{so(d)} \mathbb{Y}[q]$ with the trace order r is denoted $\text{msp}^{so(d)}(\mathbf{X}, q, r)$. The process of taking msp is illustrated on figure 1. It is evident that different partitions of r give rise to distinct elements of the msp. The sum over the traces of all orders is denoted $\text{msp}(\mathbf{X}, q)$, $\text{msp}(\mathbf{X}, q) = \sum_r \text{msp}(\mathbf{X}, q, r)$.

Thus, the maximally symmetric part for $\mathbf{X} \otimes_{so(d)} \mathbf{Y}$ is obtained by taking all possible traces and, then, adding the rest of the cells according to the hwp-rule. In terms of Young diagrams, we see that $\text{hwp}^{sl(d)} \equiv \text{msp}^{sl(d)} \equiv \text{hwp}^{so(d)}$ and $\text{hwp}^{so(d)} \subset \text{msp}^{so(d)}$.

3.6 Restriction and hwp, msp

Since the Lorentz modules that label the subcomplexes of $\mathcal{C}(\mathbf{A}, \sigma_-)$ come from the tensor products of the restriction of \mathbf{A} by one-column diagrams and the diagrams labelling the σ_- -cohomology tend to be as symmetric as possible, to write out the results we need to combine the maximally symmetric part of a tensor product with the restriction functor.

$sl(d+1)$. Given an $sl(d+1)$ -irreducible module $\mathbf{A} = \mathbb{Y}(s_1, \dots, s_n)$ let $\text{mspr}^{sl(d)}(\mathbf{A}, q)$ be the element from $\text{Res}_{sl(d)}^{sl(d+1)} \mathbf{A} \otimes \mathbb{Y}[q]$ of the form

$$\text{mspr}^{sl(d)}(\mathbf{A}, q) = \mathbb{Y}(s_1 + 1, \dots, s_q + 1, \widehat{s_{q+1}}, s_{q+2}, \dots, s_n), \quad (3.37)$$

i.e., one cell is added to the right of each row in the range $1, \dots, q$ and the $(q+1)$ -th row is thrown away. Therefore, one can rewrite the definition of mspr in terms of hwp or msp

with the argument being the maximally symmetric component \mathbf{A}^{q+1} , defined in (3.11),

$$\text{mspr}^{sl(d)}(\mathbf{A}, q) \equiv \text{msp}^{sl(d)}(\mathbf{A}^{q+1}, q) \equiv \text{msp}^{sl(d)}(\mathbf{A}^{q+1}, \mathbb{Y}[q]) \equiv \text{hwp}^{sl(d)}(\mathbf{A}^{q+1}, q). \quad (3.38)$$

so(d + 1). Let $\mathbf{A} = \mathbb{Y}(s_1, \dots, s_n)$ be a Young diagram of $so(d + 1)$ and such that all weights (s_1, \dots, s_n) are different. Then, $\text{mspr}^{so(d)}(\mathbf{A}, q, r)$ is defined as

$$\text{mspr}^{so(d)}(\mathbf{A}, q, r) = \text{msp}^{so(d)}(\mathbf{A}^{q-r+1}, q, r) \equiv \text{msp}^{so(d)}(\mathbf{A}^{q-r+1}, \mathbb{Y}[q], r), \quad (3.39)$$

i.e. $\text{mspr}^{so(d)}(\mathbf{A}, q, r)$ is a sum over all diagrams that are obtained from the maximally symmetric component \mathbf{A}^{q-r+1} (see (3.11) for the definition) by taking all possible traces of order r and, then, adding one cell to each of the first $(q-r)$ rows. The weight s_{q-r+1} to be thrown away is determined by the number of cells that remain to be added after taking the trace.

If among the weights (s_1, \dots, s_n) some are equal then the block notation is more convenient, so we take $\mathbf{A} = \mathbb{Y}\{(s_1, p_1), \dots, (s_N, p_N)\}$. The process of taking $\text{mspr}(\mathbf{A}, q, r)$ is illustrated on figure 2a. Similar to the case where all weights in \mathbf{A} are different, one first takes all possible traces and then adds the rest of the cells, i.e. $(q-r)$, according to the hwp rules. The difference is that certain diagrams must be deleted while taking the highest weight part. Let some cell be called a filled vacancy if it has been removed (while taking traces) and then restored (while adding the rest of the cells according to the hwp-rules). Let k be the number of the block of \mathbf{A} to which the $(q-r+1)$ -th weight belongs. Then, the diagrams to be deleted are those diagrams for which there is at least one filled vacancy at the k -th block.

The sum over the traces of all orders $r = 0, 1, \dots, q$ such that $s_{q-r+1} = s_{q+1}$ is denoted $\text{mspr}(\mathbf{A}, q)$. The condition $s_{q-r+1} = s_{q+1}$ implies that all elements of $\text{mspr}(\mathbf{A}, q)$ are from $\mathbf{A}_g \otimes \mathbb{Y}[q]$ with g being equal to the grade of \mathbf{A}^{q+1} . If all weights (s_1, \dots, s_n) are different then $\text{mspr}(\mathbf{A}, q) = \text{mspr}(\mathbf{A}, q, r = 0)$ contains a single element.

It is worth stressing that the weight of $so(d + 1)$ is $(s_1, \dots, s_n, 0_{n+1}, \dots, 0_\nu)$, $\nu = [(d + 1)/2]$, and if $(q - r) > n$ these zero rows should be added to \mathbf{A} .

Duality map. Now we define the duality map which takes any element of $\text{mspr}(\mathbf{A}, q, r)$ to some other element of the complex $\mathcal{C}(\mathbf{A}, \sigma_-)$ that is defined by the same Young diagram. Given an $so(d + 1)$ -module \mathbf{A} , let \mathbf{X} be any irreducible $so(d)$ -module that is an element of $\text{mspr}(\mathbf{A}, q, r)$ for some q and r . By definition, \mathbf{X} appears the same time as a trace of order r in the decomposition of $C_{g,r}^q(\mathbf{A})$ (3.32) into irreducible $so(d)$ -modules, where $g = g(\mathbf{A}^{q-r+1})$. Let ϵ' be defined for \mathbf{X} according to figure 2a. Provided that g is not the maximal possible value of the grade, the dual to \mathbf{X} is an $so(d)$ -module $\tilde{\mathbf{X}}$ that is defined by the same Young diagram \mathbf{X} and is the element of $C_{g+1,r'}^{q'}(\mathbf{A})$, where $q' = q + 2\epsilon' + 1$, $r' = r + \epsilon' + 1$. See figure 2b for the illustration.

$$\begin{array}{ccccc} & & C_{g,r}^q(\mathbf{A}) & & C_{g+1,r'}^{q'}(\mathbf{A}) \\ & & \cup & & \cup \\ \omega_{\mathbf{q}}^{\mathbf{A}_g} & \xleftarrow{\pi} & \mathbf{X} & \longrightarrow & \tilde{\mathbf{X}} & \xrightarrow{\tilde{\pi}} & \omega_{\mathbf{q}}^{\mathbf{A}_{g+1}} & (3.40) \\ & & \cap & & & & \\ & & \text{mspr}(\mathbf{A}, q, r) & & & & \end{array}$$

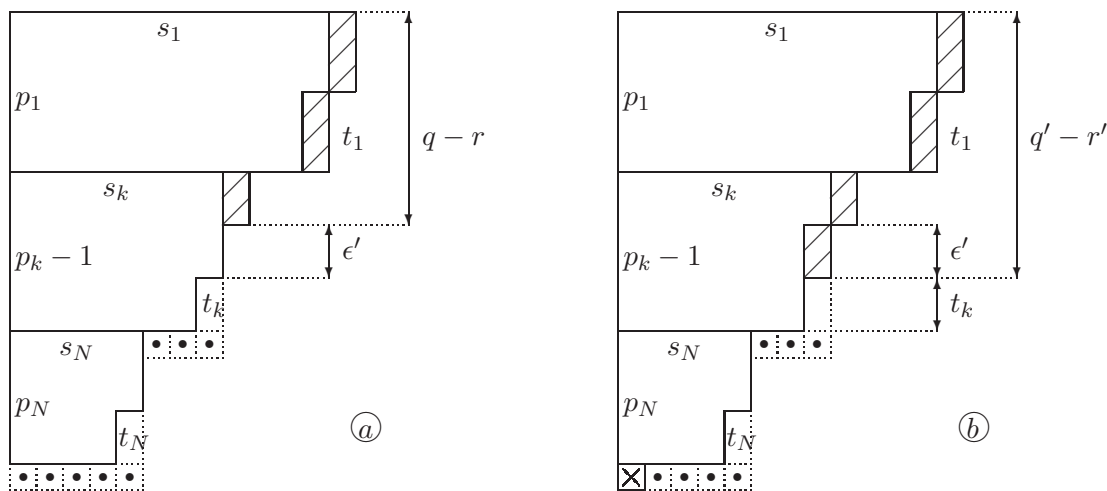


Figure 2. Illustration for $\mathbf{A} = \mathbb{Y}\{(s_1, p_1), \dots, (s_N, p_N)\}$.

(a) Taking $\text{mspr}(\mathbf{A}, q, r)$. That the weight s_{q-r+1} from the k -th block is thrown away implies that the blocks $1, \dots, k-1, k+1, \dots, N$ remain unchanged and the k -th block becomes shorter by one row. In tensor language, in order to project onto \mathbf{A}^{q-r+1} one needs to contract the compensator V^A with the indices corresponding to the cells marked by \bullet and then apply Young symmetry and trace projectors. Any trace is determined by a partition $r = t_1 + \dots + t_N$ such that $t_i \leq p_i$ if $i = 1, \dots, k-1, k+1, \dots, N$ and $t_k \leq p_k - 1$. In taking the trace t_i cells are removed from the bottom-right of the i -th block. The rest of cells, which are drawn hatched, is added to the first rows. The additional condition for diagrams having equal rows implies that the cells being added must not overlap with any of the t_k cells that have been removed from the k -th block while taking traces. Therefore, the gap ϵ' is always nonnegative.

(b) Taking dual. The same diagram is obtained in another way. The diagram $\tilde{\mathbf{A}}^{q-r+1}$ is just \mathbf{A}^{q-r+1} with one extra cell below the last row. Consider one special trace of order $r' = r + \epsilon' + 1$ of a degree- $q' = q + 2\epsilon' + 1$ form $w_{\mathbf{q}'}^{\tilde{\mathbf{A}}^{q-r+1}}$. First, one takes the trace of $\tilde{\mathbf{A}}^{q-r+1}$ of order $r + \epsilon' + 1$: r cells are removed in the same way as in (a), one cell is removed from the last row, ϵ' cells are removed from the k -th block in addition to the t_k cells just removed. Second, the rest of the cells, i.e. $q - r + \epsilon'$ is added to the first rows. Finally, the same diagram as in (a) is obtained.

Indeed, the diagram $\tilde{\mathbf{X}} = \mathbf{X}$ belongs to $\tilde{\mathbf{A}}^{q-r+1} \otimes \mathbb{Y}[q']$, where $\tilde{\mathbf{A}}^{q-r+1} \in \mathbf{A}_{g+1}$, namely

$$\tilde{\mathbf{A}}^{q-r+1} = \mathbb{Y}(s_1, \dots, s_{q-r}, \widehat{s_{q-r+1}}, s_{q-r+2}, \dots, s_n, 1). \quad (3.41)$$

The trace of order $r + \epsilon' + 1$ is to be taken as follows: the trace of order r is taken as for \mathbf{X} , in doing so extra ϵ' cells are removed from the block to which the $(q-r+1)$ -th weight belongs, then the only cell in the last row of $\tilde{\mathbf{A}}^{q-r+1}$ is removed. The rest of $q' - r' = q - r + \epsilon'$ cells is added according to the hwp-rules, which results in the same diagram \mathbf{X} .

It is technically very difficult to define the duality map in terms of irreducible Lorentz tensor because to do so we need to use the explicit form of Young and trace projectors in order to embed an irreducible tensor with the symmetry of \mathbf{X} into the Lorentz connection $\omega_{\mathbf{q}}^{\mathbf{A}^g}$ with the help of certain projector π and into $\omega_{\mathbf{q}}^{\mathbf{A}^{g+1}}$ with the help of projector $\tilde{\pi}$. Fortunately, to make contact between the formulation in terms of generalized Yang-Mills

fields of the (anti)-de Sitter algebra and metric-like fields of section 1 the explicit form of π and $\tilde{\pi}$ is not needed.

3.7 Sigma-minus cohomology, the result

Let us first state the main result on the σ_- -cohomology for the case of $sl(d+1)$, and then for the case of interest $so(d+1)$. The proof is left to the appendix B since it is rather technical.

Theorem 1. *Let $\mathbf{A} = \mathbb{Y}(s_1, \dots, s_n)$ be a Young diagram defining an irreducible $sl(d+1)$ -module and $\mathcal{C}(\mathbf{A}, \sigma_-)$ be the associated σ_- -complex. Then,*

$$H^q(\mathbf{A}, \sigma_-) = \begin{cases} mspr(\mathbf{A}, q), & q = 0, \dots, d-1, \\ \emptyset, & q \geq d, \end{cases} \quad (3.42)$$

where the grade of a single element of $H^q(\mathbf{A}, \sigma_-)$ is $g(\mathbf{A}^{q+1})$.

For $sl(d+1)$ σ_- has a plain algebraic meaning inasmuch as V_0 can be identified with the commutative subalgebra \mathfrak{p} of $sl(d+1)$ that is a covector representation of the $sl(d)$ subalgebra. Then, the definition of σ_- coincides with the Lie cohomology of \mathfrak{p} with values in the $sl(d+1)$ -module \mathbf{A} , the solution can be found, for example, in [63].

Theorem 2. *Let $\mathbf{A} = \mathbb{Y}(s_1, \dots, s_n)$ be a Young diagram defining an irreducible $so(d+1)$ -module (the signature is irrelevant and $so(d+1)$ can be viewed as the (anti)-de Sitter algebra) and $\mathcal{C}(\mathbf{A}, \sigma_-)$ be the associated σ_- -complex, then*

$$H^q(\mathbf{A}, \sigma_-) = H^q(\mathbf{A}, \sigma_-)^{\text{reg}} \oplus H^q(\mathbf{A}, \sigma_-)^{\text{irreg}}, \quad (3.43)$$

where $H^q(\mathbf{A}, \sigma_-)^{\text{reg}}$ is the regular part of the cohomology

$$H^q(\mathbf{A}, \sigma_-)^{\text{reg}} = \sum_{k=0}^{k=q} mspr(\mathbf{A}, q, k), \quad (3.44)$$

and the irregular part $H(\mathbf{A}, \sigma_-)^{\text{irreg}}$ is given by the elementwise dualization of the regular part,

$$H(\mathbf{A}, \sigma_-)^{\text{irreg}} = \{\tilde{\omega} : \omega \in H(\mathbf{A}, \sigma_-)^{\text{reg}}\} = \tilde{H}(\mathbf{A}, \sigma_-)^{\text{reg}}, \quad (3.45)$$

i.e. the representatives of $H(\mathbf{A}, \sigma_-)^{\text{irreg}}$ are obtained by applying the duality map to a representative of each cohomology class of $H^q(\mathbf{A}, \sigma_-)^{\text{reg}}$ - different classes of $H^q(\mathbf{A}, \sigma_-)^{\text{reg}}$ are mapped to different classes in $H(\mathbf{A}, \sigma_-)^{\text{irreg}}$.

Note that the grade of $\tilde{\omega}$ is greater by one than that of ω , with the form degree and trace order depending on the number of equal weights in ω and on its degree and trace order. Therefore, different representatives of $H^q(\mathbf{A}, \sigma_-)^{\text{reg}}$ having the same degree, grade and trace order can give rise to classes of $H(\mathbf{A}, \sigma_-)^{\text{irreg}}$ with different degrees and trace orders but necessarily having the same grade. It is worth noting also that the duality map applied to a representative at the highest grade gives nothing.

$q \backslash g$	0	1	$s - 1$
0	$\boxed{s-1}$	\emptyset	\emptyset
1	$\boxed{s} \oplus \boxed{s-2}$	\emptyset	\emptyset
2	\emptyset	$\boxed{s} \oplus \boxed{s-2}$	\boxed{s}
3	\emptyset	$\boxed{s-1}$	$\begin{array}{c} \boxed{s} \\ \boxed{s} \\ \boxed{s} \end{array} \oplus \begin{array}{c} \boxed{s} \\ \boxed{s-1} \end{array}$

Table 3. Sigm-cohomology for $\mathbf{A} = \mathbb{Y}(s - 1, s - 1)$.

The latter theorem encompasses all the special cases addressed in the literature: in [71] the precise field theoretical meaning was given to $H(\sigma_-)$, and the example of $\mathcal{C}(\mathbb{Y}(k \rightarrow \infty), \sigma_-)$ was investigated in detail; $H^q(\sigma_-)$ at lower degrees $q = 0, 1, 2$ for the complex $\mathcal{C}(\mathbb{Y}(s - 1, s - 1), \sigma_-)$ related to massless spin- $s \geq 2$ field was found in [56], previously the field-theoretical interpretation of this result was known as the Central on-mass-shell theorem [72, 73]; for the purpose of constructing a Lagrangian the cohomology groups corresponding to the dynamical field and to the Weyl tensor for a field $(\mathbf{S}, q, t = 1)$, where q is equal to the length of the shortest column in \mathbf{S} , were found in [38, 74, 75]; $H^q(\mathbb{Y}(s - 1, s - t), \sigma_-)$, $q = 0, 1$ corresponding to a partially-massless spin- s field $(\mathbb{Y}(s), q = 1, t)$ were important for [39]; in [33, 34] $H^q(\sigma_-)$ at lower degrees were found for the following fields $(\mathbb{Y}(2, 1), 1, 1)$, $(\mathbb{Y}(3, 1), 1, 1)$ and $(\mathbf{S}, 1, 1)$ with $\mathbf{S} = \mathbb{Y}(s_1, s_2, \dots, s_n)$ or $\mathbf{S} = \mathbb{Y}(s_1, s_1, s_2, \dots, s_n)$ such that $s_1 - s_2 \geq 4$.

Corollary 1. For \mathbf{A} such that all of the weights s_i are different, the result turns to a very simple form because of

$$mspr(\mathbf{A}, q, r) = msp(\mathbf{A}^{q-r+1}, q, r) \quad (3.46)$$

and hence

$$H^q(\mathbf{A}, \sigma_-)^{\text{reg}} = \sum_{k=0}^{k=q} msp(\mathbf{A}^{q-k+1}, q, k). \quad (3.47)$$

The duality map applied to any representative of a nontrivial cohomology class with some q, g, r except for those at the maximal grade produces a representative of the cohomology class with $q + 1, g + 1, r + 1$ labelled by the same Young diagram.

3.8 Examples

$\mathbf{A} = \mathbb{Y}(s - 1, s - 1)$. The main theorem applied to $\mathbf{A} = \mathbb{Y}(s - 1, s - 1)$ gives the list of σ_- -cohomologies at lower degrees, which is presented in table 3. Note that $\epsilon' = 1$ (see figure 2 for the definition of ϵ') for the only representative of $H_{g=0, r=0}^{q=0}$, hence the duality map takes it to the class at degree $q + 2\epsilon' + 1 = 3$. Analogously, $\epsilon' = 0$ for both in $H_{g=0}^{q=1}$, hence the duality map takes them to the class at degree two. There are no duals for those at the maximal grade $s - 1$.

$q \backslash g$	0	1	$s_1 - s_2$	$s_1 - s_2 + 1$	s_1
0	$\boxed{s_2}$	\emptyset	\emptyset	\emptyset	\emptyset
1	$\boxed{s_2 - 1}$	$\boxed{s_2}$	$\boxed{s_1 + 1}$	\emptyset	\emptyset
2	\emptyset	$\boxed{s_2 - 1}$	$\boxed{s_1}$	$\boxed{s_1 + 1}$	$\begin{array}{c} \boxed{s_1 + 1} \\ \boxed{s_2 + 1} \end{array}$
3	\emptyset	\emptyset	\emptyset	$\boxed{s_1}$	$\begin{array}{c} \boxed{s_1 + 1} \\ \boxed{s_2 + 1} \\ \oplus \end{array} \begin{array}{c} \boxed{s_1 + 1} \\ \boxed{s_2} \end{array}$

Table 4. Sigm-cohomology for $\mathbf{A} = \mathbb{Y}(s_1, s_2)$.

Consider a gauge theory with the gauge field given by a one-form $W_1^{\mathbf{A}}$, which, as is well-known [37], describes a massless spin- s field, i.e. $(\mathbb{Y}(s), 1, 1)$. The interpretation of the σ_- -cohomology is as follows: \mathbf{H}^0 corresponds to a traceless rank- $(s - 1)$ gauge parameter, $\xi^{a(s-1)}$. The dynamical field \mathbf{H}^1 is represented by two traceless tensors of ranks s and $s - 2$, which can be combined into a doubly traceless Fronsdal field $\phi^{a(s)}$. The gauge transformation law is of first order, $\delta\phi^{a(s)} = D^a \xi^{a(s-1)}$. In \mathbf{H}^2 we see the second order equations, which are in one-to-one correspondence with the dynamical fields, suggesting the system admits a Lagrangian [37, 51]. The Weyl tensor $C^{a(s), b(s)}$ is also present in \mathbf{H}^2 , which is the order s derivative of fields. In \mathbf{H}^3 there are Bianchi identities both for Fronsdal equations, corresponding to the gauge symmetry with $\xi^{a(s-1)}$ and for the Weyl tensor, implying that it is constructed out of $\phi^{a(s)}$ rather than being an independent object.

$\mathbf{A} = \mathbb{Y}(s_1, s_2)$. The σ_- -cohomology at lower degrees is given in table 4. Consider a gauge theory with gauge field $W_1^{\mathbf{A}}$, which describes a partially-massless spin- $(s_1 + 1)$ field of depth $t = s_1 - s_2 + 1$ [39]. Indeed, there is a gauge parameter $\xi^{a(s_2)}$ in \mathbf{H}^0 ; the dynamical fields are $\phi^{a(s_2-1)}$, $\phi^{a(s_2)}$ and the primary field $\phi^{a(s_1+1)}$ with the highest rank. The appearance of fields with lower ranks, which cannot be generally associated with the traces of a single field, is because partially-massless fields lie between massless and massive. For a Lagrangian description of a massive spin- $(s_1 + 1)$ field in addition to a traceless field $\phi^{a(s_1+1)}$ one needs supplementary traceless fields of ranks $s_1 - 1, s_1 - 2, \dots, 1, 0$, which vanish on-mass-shell [76, 77]. For partially-massless fields this chain becomes shorter because of disappearance of fields with ranks $s_2 - 2, \dots, 0$. However, not all of the supplementary fields can now be excluded, these are the fields $\phi^{a(s_2-1)}$ and $\phi^{a(s_2)}$. The gauge transformation law has schematically the form $\delta\phi^{a(s_1+1)} = D^a \dots D^a \xi^{a(s_2)} + \dots$. In \mathbf{H}^2 we see the wave equation for $\phi^{a(s_1+1)}$, Weyl tensor and two more constraints on supplementary fields.

That there is no Bianchi identity in \mathbf{H}^3 for the gauge symmetry with $\xi^{a(s_2)}$ is due to the fact that σ_- is an operator that is responsible for expressing fields of higher rank in terms of derivatives of lower rank fields and hence cannot track out the Bianchi identities of the form $D^b \dots D^b G_{b(s_1-s_2+1)a(s_2)} + \dots \equiv 0$, where $G^{a(s_1+1)} = \square\phi^{a(s_1+1)} + \dots$ is the equation on $\phi^{a(s_1+1)}$. In this case Bianchi identities correspond to the reversed situation when lower rank fields are expressed in terms of divergences of higher rank fields.

$q \setminus g$	0	1	$s - 1$
0	$\text{msp}(\mathbf{X}, 0) = \mathbf{X}$	\emptyset	\emptyset
..
$q - 1$	$\text{msp}(\mathbf{X}, q-1)$	\emptyset	\emptyset
q	$\text{msp}(\mathbf{X}, q)$	\emptyset	\emptyset
$q + 1$	\emptyset	$\text{msp}(\mathbf{X}, q)$	$\text{mspr}(\mathbf{A}, q+1) = \mathbf{A}$
$q + 2$	\emptyset	$\text{msp}(\mathbf{X}, q-1)$	$\text{mspr}(\mathbf{A}, q+2)$
..
$2q+1$	\emptyset	$\text{msp}(\mathbf{X}, 0)$	$\text{mspr}(\mathbf{A}, 2q+1)$
$2q+2$	\emptyset	\emptyset	$\text{mspr}(\mathbf{A}, 2q+2)$

Table 5. Sigm-cohomology for $\mathbf{A} = \mathbb{Y}\{(s - 1, q + 1)\}$.

\mathbf{A} is a $(s - 1) \times (q + 1)$ block diagram, $\mathbf{A} = \mathbb{Y}\{(s - 1, q + 1)\}$. Consider now a gauge theory with the field $W_{\mathbf{q}}^{\mathbf{A}}$, which describes a massless field with spin $\mathbf{S} = \mathbb{Y}\{(s, q)\}$. According to [22, 24] it is only for these fields that the number of degrees of freedom in Minkowski space is equal to that in $(A)dS_d$. The fields with $\mathbf{S} = \mathbb{Y}\{(s, q)\}$ are the true massless fields in this sense. With $\mathbf{X} = \mathbb{Y}\{(s - 1, q)\} = \mathbf{A}^1 = \dots = \mathbf{A}^{q+1}$ the σ -cohomology is given in table 5. The gauge parameter at the deepest level of reducibility given by \mathbf{H}^0 is just a traceless tensor with the symmetry of \mathbf{X} . Note that $\text{mspr}(\mathbf{A}, r) = \text{msp}(\mathbf{X}, r)$ if $r \leq q$. The gauge parameter in \mathbf{H}^r , $r = 1, \dots, q - 1$ along with the primary component

$$\mathbb{Y}\{(s, r), (s - 1, q - r)\} = \text{hwp}(\mathbf{X}, r) \subset \text{msp}(\mathbf{X}, r) \quad (3.48)$$

contain certain traces that are needed for the gauge symmetry to be realized off-shell. The explicit expression for $\xi_{\mathbf{r}}^{\mathbf{A}}$ reads

$$\xi^{a(s), \dots, b(s), c(s-1), \dots, u(s-1)} = \xi^{a(s-1), \dots, b(s-1), c(s-1), \dots, u(s-1), \bullet(s-1) | \overbrace{a \dots b}^r} \quad (3.49)$$

Similarly for the dynamical field in \mathbf{H}^q

$$\phi^{a(s), \dots, u(s)} = \xi^{a(s-1), \dots, u(s-1), \bullet(s-1) | \overbrace{a \dots u}^q} \quad (3.50)$$

We see that there is a one-to-one correspondence between the second order equations in \mathbf{H}^{q+1} and the dynamical fields in \mathbf{H}^q . There is also a generalized Weyl tensor in \mathbf{H}^{q+1} , which is an irreducible tensor of the Lorentz algebra with the symmetry \mathbf{A} . By virtue of the definition of mspr for diagrams with equal rows, $\text{mspr}(\mathbf{A}, q+1)$ contains only one irreducible component, which has the symmetry of \mathbf{A} itself. For higher degrees $q + 2, \dots$ $\text{mspr}(\mathbf{A})$ contains also certain traces. It is easy to see the duality of the form $\mathbf{H}_{g=0}^{q-k} \sim \mathbf{H}_{g=1}^{q+k+1}$, $k = 0, \dots, q$, implying that there is a one-to-one correspondence between equations of motion and dynamical fields, gauge symmetries at the level- k and the order- k Bianchi identities.

4 Interpretation of results: gauge fields vs. gauge connections

According to [19], a gauge field defined by a triple (\mathbf{S}, q, t) can be described by the gauge connection $W_{\mathbf{q}}^{\mathbf{A}}$ of the (anti)-de Sitter algebra \mathfrak{g} , where \mathbf{A}

$$\mathbf{S} = \begin{array}{c} \overline{s_1} \\ \dots \\ \overline{s_{q-1}} \\ \overline{s_q} \\ \overline{s_{q+1}} \\ \dots \\ \overline{s_n} \end{array} \mathfrak{so}(d-1) \quad (\mathbf{S}, q, t) \iff (\mathbf{A}, q), \quad \mathbf{A} = \begin{array}{c} \overline{s_1 - 1} \\ \dots \\ \overline{s_q - 1} \\ \overline{s_q - t} \\ \overline{s_{q+1}} \\ \dots \\ \overline{s_n} \end{array} \mathfrak{g} \quad (4.1)$$

or with the indices written explicitly

$$\phi^{a(s_1), \dots, v(s_n)} \iff W_{\mu_1 \dots \mu_q}^{A(s_1-1), \dots, B(s_q-1), C(s_q-t), D(s_{q+1}), \dots, F(s_p)} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_q} \quad (4.2)$$

A field (\mathbf{S}, q, t) can be described by the connection $W_{\mathbf{q}}^{\mathbf{A}}$ of \mathfrak{g} in the sense that there is the inclusion of exact sequences, discussed in detail below,

$$\begin{array}{ccccccccc}
 \dots & & \dots & & \dots & & \dots & & \dots \\
 \mathfrak{g}\text{-module} & \hookrightarrow & \phi^{\mathbf{S}_0} & \hookrightarrow & \bar{\phi}^{\mathbf{S}_0} & \hookrightarrow & e_{\mathbf{q}}^{\mathbf{L}_0} & \hookrightarrow & W_{\mathbf{q}}^{\mathbf{A}} \\
 \mathcal{D}(E_0; \mathbf{S}_0) & & \text{on-shell} & & \text{off-shell} & & \text{frame-like} & & \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \mathfrak{g}\text{-module} & \hookrightarrow & \xi^{\mathbf{S}_1} & \hookrightarrow & \bar{\xi}^{\mathbf{S}_1} & \hookrightarrow & \xi_{\mathbf{q}-1}^{\mathbf{L}_1} & \hookrightarrow & \xi_{\mathbf{q}-1}^{\mathbf{A}} \\
 \mathcal{D}(E_1; \mathbf{S}_1) & & \text{on-shell} & & \text{off-shell} & & \text{frame-like} & & \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \dots & & \dots & & \dots & & \dots & & \dots
 \end{array}$$

which is exact in vertical arrows, with horizontal arrows denoting the inclusion maps. The leftmost vertical arrows are the arrows from (1.1); the rightmost vertical arrows are realized as the action of D_{Ω} in the complex $\mathcal{C}(\mathbf{A}, D_{\Omega})$; the vertical arrows next to them are realized as the D_{Ω} in the complex $\mathcal{C}(\mathbf{A}, D_{\Omega})$ with \mathbf{A} being treated as an $\mathfrak{so}(d-1, 1)$ -module.

In order to obtain an off-shell formulation for a (\mathbf{S}, q, t) field one has to get rid of differential constraints (1.5) on the dynamical field $\phi^{\mathbf{S}_0}$ and gauge parameters $\xi^{\mathbf{S}_i}$, which implies that the field content needs to be extended. The extended fields $\bar{\phi}^{\mathbf{S}_0}$ and gauge parameters $\bar{\xi}^{\mathbf{S}_i}$ have the same symmetry type but are no longer irreducible Lorentz tensors, satisfying certain trace constraints that are weaker than the full tracelessness. We refer to such fields and gauge parameters with relaxed trace constraints as to *extensions*. The fields $\phi^{\mathbf{S}_0}$, $\xi^{\mathbf{S}_i}$ are the highest weight parts¹⁰ of the decomposition of $\bar{\phi}^{\mathbf{S}_0}$ and $\bar{\xi}^{\mathbf{S}_i}$ into irreducible Lorentz tensors.

The problem with the vertical arrows from the second and third columns, which are realized on irreducible metric-like fields and on their extensions, respectively, is in that the explicit use of Young symmetrizers is needed. Let us now define the horizontal arrows that denote the inclusion maps.

¹⁰see section 3.5 for the definition.

$\mathcal{D}(E_i; \mathbf{S}_i)$ is realized on the solutions of (1.4)–(1.5) imposed on the irreducible tensor field of the Lorentz algebra having the symmetry of \mathbf{S}_i , which for $i = 0$ and for $i = 1, \dots, q$ corresponds to imposing the equations of motion on the field $\phi^{\mathbf{S}_0}$ and to imposing the gauge fixing conditions on the gauge parameters $\xi^{\mathbf{S}_i}$, respectively.

The extended fields are embedded as a maximally symmetric parts¹⁰ into the connections of the Lorentz algebra. The dynamical field $\bar{\phi}^{\mathbf{S}_0}$ and the gauge parameters $\bar{\xi}^{\mathbf{S}_i}$, $i = 1, \dots, q$ are embedded into the generalized frame field $e_{\mathbf{q}}^{\mathbf{L}_0}$ and $\xi_{\mathbf{q}-i}^{\mathbf{L}_i}$, respectively. The modules \mathbf{L}_i , $i = 0, \dots, q$ are certain irreducible Lorentz-modules coming from the restriction of \mathbf{A} , namely, $\mathbf{L}_i = \mathbf{A}^{q-i+1}$.

Written in terms of tensor fields $\bar{\phi}^{\mathbf{S}_0}$, $\bar{\xi}^{\mathbf{S}_i}$ of the Lorentz algebra, all expressions, for example, the gauge transformation law and the equations of motion, are extremely complicated due to the presence of Young symmetrizers and trace projectors. In contrast, when reformulated in terms of the connection $W_{\mathbf{q}}^{\mathbf{A}}$ of \mathfrak{g} , the theory of any gauge field (\mathbf{S}, q, t) has a very simple form.

The main result of [19] is that certain components of $W_{\mathbf{q}}^{\mathbf{A}}$, $\xi_{\mathbf{q}-1}^{\mathbf{A}}$, \dots , $\xi_{\mathbf{0}}^{\mathbf{A}}$ were identified with $\phi^{\mathbf{S}}$, $\xi^{\mathbf{S}_1}$, \dots , $\xi^{\mathbf{S}_q}$ and it was proved that the correct mass-like terms determined by E_i and \mathbf{S}_i arise provided that certain equations in terms of $R_{\mathbf{q}+1}^{\mathbf{A}}$ are imposed on $W_{\mathbf{q}}^{\mathbf{A}}$ and certain gauge fixing conditions in terms of $W_{\mathbf{q}}^{\mathbf{A}}$, $\xi_{\mathbf{q}-1}^{\mathbf{A}}$, \dots , $\xi_{\mathbf{1}}^{\mathbf{A}}$ are imposed on $\xi^{\mathbf{S}_1}$, \dots , $\xi^{\mathbf{S}_q}$.

Joining together the interpretation of $\mathbf{H}(\sigma_-)$ presented in section 3.1 and the theorem on the structure of $\mathbf{H}(\mathbf{A}, \sigma_-)$ yields the following.

The highest grade representatives of \mathbf{H}^q , given by¹¹ $\text{mspr}(\mathbf{A}, q)$ with the grade $g = g(\mathbf{A}^{q+1})$, are to be interpreted as the primary dynamical fields. We see that $\mathbf{S}_0 = \text{hwp}(\mathbf{A}^{q+1}, q)$ belongs to $\text{mspr}(\mathbf{A}, q)$ with the rest of elements of $\text{mspr}(\mathbf{A}, q)$ having smaller rank. Thus, a representative of $\text{mspr}(\mathbf{A}, q)$ is given by a tensor field $\bar{\phi}^{\mathbf{S}_0}$ having the symmetry of \mathbf{S}_0 , but $\bar{\phi}^{\mathbf{S}_0}$ is not generally irreducible, containing certain traces. The on-shell field $\phi^{\mathbf{S}_0}$ is embedded into $\bar{\phi}^{\mathbf{S}_0}$ as the highest weight part, $\phi^{\mathbf{S}_0} = \text{hwp}(\mathbf{A}^{q+1}, q)$. Actually, it is easy to find among the fields $\omega_{\mathbf{q}}^{\mathbf{X}}$ coming from the restriction of $W_{\mathbf{q}}^{\mathbf{A}}$ the one that contains $\bar{\phi}^{\mathbf{S}_0}$. It is the generalized frame field $e_{\mathbf{q}}^{\mathbf{L}_0}$ with $\mathbf{L}_0 = \mathbf{A}^{q+1}$.

Likewise for the level- i gauge parameter $\xi^{\mathbf{S}_i}$. The extension $\bar{\xi}^{\mathbf{S}_i}$ is the representative of \mathbf{H}^{q-i} at the highest grade, $\mathbf{H}^{q-i} = \text{mspr}(\mathbf{A}^{q-i+1}, q-i)$. The on-shell gauge parameter $\xi^{\mathbf{S}_i}$ enters $\bar{\xi}^{\mathbf{S}_i}$ as the highest weight part, $\xi^{\mathbf{S}_i} = \text{hwp}(\mathbf{A}^{q-i+1}, q-i)$.

The number of derivatives connecting the highest grade fields of \mathbf{H}^{q-i} and \mathbf{H}^{q-i-1} is equal to $g(\mathbf{A}^{q-i+1}) - g(\mathbf{A}^{q-i}) + 1$. Substituting the explicit form of \mathbf{A} in terms of (\mathbf{S}, q, t) gives exactly the difference of the lowest energies $E_{i-1} - E_i$, which also counts the number of derivatives for a field-theoretical realization.

Note that for massless unitary fields, i.e. those having $t = 1$ and q equal to the height of the first block of \mathbf{S} , $\mathbf{H}_{g>0}^{q-i} = \emptyset$, $i = 0, \dots, q$ and thus the primary dynamical field $\bar{\phi}^{\mathbf{S}_0}$ appears at the lowest grade, so do its gauge parameters at all levels. For nonunitary massless fields and for partially-massless fields in addition to the primary dynamical field $\bar{\phi}^{\mathbf{S}_0}$ certain other dynamical fields having smaller rank can appear at lower grade.

¹¹See section 3.6 for the definition.

Note also that the highest rank representatives of some $H_g^{q'}$ correspond to the on-shell situation, in which all lower rank representatives of $H_g^{q'}$ are zero by virtue of gauge fixing conditions. So to make contact with the on-shell description in terms of metric-like fields, presented in section 1, it is sufficient to interpret the highest rank representatives only.

As for the field equations the situation is more complicated. Since in a general case of mixed-symmetry field, i.e. the one having $\mathbf{S} = \mathbb{Y}(s_1, s_2, \dots)$ with $s_1 \neq s_2 > 0$, all of the first order constraints (1.5) cannot be achieved via imposing gauge conditions on a single gauge parameter $\bar{\xi}^{\mathbf{S}_1}$, the full system of equations of motion consists both of second and first order equations. Thus we cannot expect the number of equations to be equal to the number of component fields in $\bar{\phi}^{\mathbf{S}_0}$. The representatives of H_g^{q+1} with g equal to the grade of the primary dynamical field, i.e. to $g(\mathbf{A}^{q+1})$ correspond to certain first order gauge invariant equations for $\bar{\phi}^{\mathbf{S}_0}$.

For $t = 1$ we see that the highest rank representatives of H_g^{q+1} have the symmetry of all the gauge parameters¹² for a massless spin- \mathbf{S} field in Minkowski space, except for $\xi^{\mathbf{S}_1}$. The number of representatives of H_g^{q+1} with the highest rank equals the number of first order constraints (1.5) minus one. One constraint of (1.5) can be imposed as a gauge condition for $\xi^{\mathbf{S}_1}$. In the Minkowski case the rest of the constraints (1.5) can be imposed with the help of other gauge parameters, one parameter - one constraint.

For $t > 1$, i.e. for partially-massless fields, the gauge symmetry with $\xi^{\mathbf{S}_1}$ is so weak that none of the constraints (1.5) can be imposed with the help of $\xi^{\mathbf{S}_1}$. Thus, the number of representatives of H_g^{q+1} with the highest rank equals the number of first order constraints in (1.5).

Similar statements can be made about the correspondence of the the highest rank representatives of H_g^{q+i} and level- i gauge parameters of a massless spin- \mathbf{S} field in Minkowski space. This correspondence is not accidental since the first order constraints (1.5) are the same for gauge fields in Minkowski and $(A)dS_d$, with D_m being the covariant derivative in the space of interest. The difference is that for massless fields in Minkowski space these results can be achieved via gauge fixing and for gauge fields in $(A)dS_d$ most of the constraints (all for $t > 1$) are to be imposed as independent equations.

As for second order field equations that are the representatives of H_{g+1}^{q+1} , we see that at least there is a representative in H_{g+1}^{q+1} that has the symmetry of \mathbf{S}_0 . It is for this representative that the mass-like term was calculated in [19] and was shown to coincide with the group-theoretical result (1.6). There is no one-to-one correspondence $H_g^q \leftrightarrow H_{g+1}^{q+1}$ between the second order equations and primary fields $\bar{\phi}^{\mathbf{S}_0}$ since by virtue of Bianchi identities a number of the second order equations corresponding to the traces of $\bar{\phi}^{\mathbf{S}_0}$ can be obtained as the derivative of certain first order constraints from H_g^{q+1} .

The only highest grade representative of H^{q+1} is given by $\text{mspr}(\mathbf{A}, q + 1, 0) = \text{msp}(\mathbf{A}^{q+2}, q + 1) = \text{hwp}(\mathbf{A}^{q+2}, q + 1)$, it has the symmetry \mathbf{S}_{-1} of a Weyl tensor for a field (\mathbf{S}, q, t) .

Let us consider certain higher degree cohomology groups for $t = 1$. The representatives

¹²Recall that a massless spin- \mathbf{S} field in Minkowski space has a number of gauge symmetries with the parameters whose Young diagrams are obtained by removing one cell from \mathbf{S} in all possible ways.

of H_{g+1}^{q+2} correspond to the Bianchi identities. As is expected, there is a representative of H_{g+1}^{q+2} having the symmetry of $\xi^{\mathbf{S}_1}$. Actually, there are also the representatives having the symmetry of all gauge parameters of a massless spin- \mathbf{S} field in Minkowski space. This suggests the enhancement of the gauge symmetry in the flat limit $\lambda^2 \rightarrow 0$ [24].

Physical degrees of freedom. The σ_- -cohomology can be used to directly count the number of physical degrees of freedom, as it was demonstrated in the case of massless fields in Minkowski space in [14]. However instructive it might be, there is no need to count degrees of freedom explicitly. It is sufficient to look at $H^{q+1}(\mathbf{A}, \sigma_-)$.

Firstly, suppose that not all of the equations in $H^{q+1}(\mathbf{A}, \sigma_-)$ are imposed. It implies that, in addition to the Weyl tensor and its descendants coupled to the gauge module, some other components of the field curvature $R_{q+1}^{\mathbf{A}}$ are nonzero on-shell, these can be parameterized by new fields, which are analogous to Weyl tensor. We can analyze the Bianchi identities and solve them with some other fields, and so on. As a result, an infinite-dimensional module grows at each place where some equation was not imposed, which gives rise to new degrees of freedom, thus making the system reducible.

Contrariwise, if the equation $R_{q+1}^{\mathbf{A}} = [\text{Weyl tensor}]$ were describing more physical degrees of freedom than the number of states in the corresponding irreducible representation $\mathcal{H}(\mathbf{S}; E_0)$, there should be a possibility to further impose certain gauge invariant differential equations that would make the system irreducible. This contradicts the statement that all gauge invariant independent equations are classified through σ_- -cohomology.

Consequently, once all components of the field curvature except for the Weyl tensor and its descendants are set to zero, the system automatically describes the correct number of physical degrees of freedom.

Remarks on the Weyl module. Recall that the generalized Weyl tensor is by definition the lowest order gauge-invariant combination of derivatives of the dynamical field $\phi^{\mathbf{S}}$ that is allowed to be nonzero on-mass-shell. The Weyl tensor is a representative of $H^{q+1}(\mathbf{A}, \sigma_-)$ at the highest nontrivial grade.

It is difficult to write down explicit expressions for the Weyl module inasmuch as we are faced with Young symmetrizers since the Weyl tensor and its descendants are tensors of the Lorentz algebra. The problem is to adjust coefficients in front of σ_- and σ_+ acting on the fields from the Weyl module. In somewhat different setup it was done in [33, 34].

The spectrum of fields of the Weyl module is in the results of [19, 33, 34, 38]. To determine this spectrum one can use the following heuristic consideration: given the symmetry types \mathbf{S}_0 and \mathbf{S}_1 of a dynamical field and its gauge parameter, respectively, one marks the extra cells of \mathbf{S}_0 as compared to \mathbf{S}_1 . The marked cells correspond to derivatives in the gauge transformation law. Then one adds cells to \mathbf{S}_1 , emulating various derivatives of the dynamical field, until one of the new cells is found in the same column with a marked cell. The latter situation correspond to implicit antisymmetrization of two derivatives, which is identically zero in Minkowski space or gives a tensor of a lower rank in (anti)-de Sitter space. Therefore, a diagram with a new cell being in the same column with a marked cell corresponds

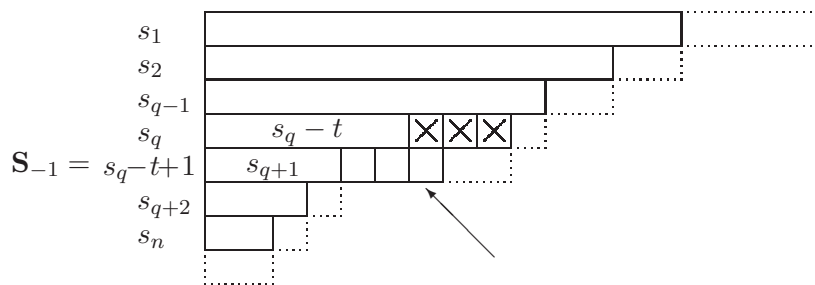


Figure 3. The spectrum of the Weyl module is shown. The cells corresponding to derivatives are marked. The descendants of the Weyl tensor are obtained by adding cells in the places outlined by a dotted line. The arrow shows the place where two covariant derivatives, one from the gauge transformations and another one from the expression of the Weyl tensor in terms of gauge potential $\phi^{\mathbf{S}}$, happens to be in the same column.

to certain gauge invariant expression. The diagram with the smallest number of added cells is the Weyl tensor, all the others are its descendants. See figure 3 for the illustration.

Note that a Weyl tensor alone does not determine (\mathbf{S}, q, t) , as it was the case for massless fields in Minkowski space, but the Weyl module does, of course.

We see that the spectrum of the Weyl module looks almost as the one coming from a restriction of a tensor of the (anti)-de Sitter algebra whose symmetry is given by a Young diagram with the first row tending to infinity. Therefore, the method for calculating the σ_- -cohomology developed in appendix B can be applied to the Weyl module too. Consequently, we can find a rather simple and complete answer for the structure of $H(\sigma_-)$ of the Weyl module in terms of $H(\sigma_-)$ of the gauge module. Namely, there is a one-to-one correspondence $H_i^k(\text{Weyl}, \sigma_-) \leftrightarrow H_{g'+i}^{q+k+1}(\text{gauge}, \sigma_-)$, $i = 0, 1, \dots$ where g' is the grade of the Weyl tensor in $H^{q+1}(\text{gauge}, \sigma_-)$, i.e. $g' = g(\mathbf{A}^{q+2})$, between the (reducible) Bianchi identities for the Weyl tensor in the gauge module and those in the Weyl module. This result confirms that the two modules are glued properly.

5 The simplest mixed-symmetry field

To illustrate we consider a massless unitary field of spin $\mathbb{Y}(s_1, s_2)$, i.e. $\mathbf{S} = \mathbb{Y}(s_1, s_2)$ and $q = t = 1$. The exact sequence (1.1) defining the irreducible representation $\mathcal{H}(E_0; \mathbb{Y}(s_1, s_2))$ with $E_0 = d + s_1 - 3$ given by (1.2) reads

$$0 \longrightarrow \mathcal{D}(E_0 + 1; \mathbb{Y}(s_1 - 1, s_2)) \longrightarrow \mathcal{D}(E_0; \mathbb{Y}(s_1, s_2)) \longrightarrow \mathcal{H}(E_0; \mathbb{Y}(s_1, s_2)) \longrightarrow 0.$$

On-shell metric-like formulation, [22, 23]. The field potential $\phi^{a(s_1), b(s_2)}$ is an irreducible Lorentz tensor field having the symmetry of \mathbf{S} and satisfies (1.4)–(1.5)

$$(\square + m^2)\phi^{a(s_1), b(s_2)} = 0, \tag{5.1}$$

$$D_c \phi^{a(s_1-1)c, b(s_2)} = D_c \phi^{a(s_1), b(s_2-1)c} = 0 \tag{5.2}$$

where the mass-like parameter is determined by $(\mathbf{S}, 1, 1)$ according to (1.6)

$$m^2 = \lambda^2((s_1 - 2)(d + s_1 - 3) - s_1 - s_2) \tag{5.3}$$

The equations are invariant under the gauge transformations $\delta\phi^{a(s_1),b(s_2)} = D^a\xi^{a(s_1-1),b(s_2)}$, where the gauge parameter is an irreducible Lorentz tensor having the symmetry of $\mathbf{S}_1 = \mathbb{Y}(s_1 - 1, s_2)$ and is subjected to (1.4)–(1.5) equations with

$$m_\xi^2 = \lambda^2 ((s_1 - 1)(d + s_1 - 2) - s_1 - s_2 + 1). \quad (5.4)$$

Off-shell metric-like formulation. The extended field content is given by the field potential $\phi^{a(s_1),b(s_2)}$ having the symmetry of \mathbf{S} and satisfying

$$\eta_{cc}\phi^{a(s_1),b(s_2-2)cc} \equiv 0, \quad \eta_{cc}\eta_{dd}\phi^{a(s_1-4)ccdd,b(s_2)} \equiv 0, \quad (5.5)$$

and thus not being irreducible. The gauge parameter $\xi^{a(s_1-1),b(s_2)}$ needs not be extended in this rather simple case, thus being irreducible

$$\eta_{cc}\xi_1^{a(s_1-3)cc,b(s_2)} \equiv \eta_{cc}\xi_1^{a(s_1-2)c,cb(s_2-1)} \equiv \eta_{cc}\xi_1^{a(s_1-1),b(s_2-2)cc} \equiv 0. \quad (5.6)$$

The algebraic constraints (5.5) imposed on the field $\phi^{a(s_1),b(s_2)}$ implies that it consists of three irreducible Lorentz tensors, two of them corresponding to nontrivial traces,

$$\phi^{a(s_1),b(s_2)} \longleftrightarrow \boxed{\begin{array}{c} s_1 \\ s_2 \end{array}} \oplus \boxed{\begin{array}{c} s_1-2 \\ s_2 \end{array}} \oplus \boxed{\begin{array}{c} s_1-1 \\ s_2-1 \end{array}}. \quad (5.7)$$

The gauge parameter is a single irreducible Lorentz tensor,

$$\xi_1^{a(s_1-1),b(s_2)} \longleftrightarrow \boxed{\begin{array}{c} s_1-1 \\ s_2 \end{array}}. \quad (5.8)$$

The gauge transformations have the same form

$$\delta\phi^{a(s_1),b(s_2)} = D^a\xi^{a(s_1-1),b(s_2)}. \quad (5.9)$$

Despite the fact that the gauge parameter $\xi^{a(s_1-1),b(s_2)}$ is no longer required to have vanishing divergences (1.5), one can verify that the algebraic constraints (5.5) and (5.6) are consistent with (5.9).

The field equations consist of the two independent equations

$$D_n D^n \phi^{a(s_1),b(s_2)} - D^a D_n \phi^{a(s_1-1)n,b(s_2)} + \frac{1}{2} D^a D^a \phi^{a(s_1-2)n, b(s_2)} + 2\lambda^2 \eta^{aa} \phi^{a(s_1-2)n, b(s_2)} + 2\lambda^2 \eta^{ab} \phi^{a(s_1-1)n, b(s_2-1)} + m^2 \phi^{a(s_1),b(s_2)} = 0, \quad (5.10)$$

$$D^a \phi^{a(s_1-1)n,b(s_2-1)} - D_n \phi^{a(s_1),b(s_2-1)n} = 0, \quad (5.11)$$

the one that reduces to the wave equation (5.1) after imposing certain gauge, and the other that excludes low spin states with the spins $\mathbb{Y}(s_1, s_2 - i)$, $i = 1, \dots, s_2$ inasmuch as $D_c \phi^{a(s_1),b(s_2-1)c} = 0$ cannot be imposed as a gauge fixing condition, and is, in fact, an independent constraint, whose gauge-invariant implementation is given by (5.11).

To conclude, it can be shown that (5.10), (5.11) together with the gauge transformations (5.9) imply the correct number of physical degrees of freedom, which corresponds to $\mathcal{H}(E_0; \mathbb{Y}(s_1, s_2))$.

Unfolded formulation, (A)dS_d-covariant. According to the statement of [38], the gauge module for a gauge field $(\mathbb{Y}(s_1, s_2), 1, 1)$ is given by a single connection of the (anti)-de Sitter algebra $W_1^{A(s_1-1), B(s_1-1), C(s_2)}$ that takes values in the irreducible representation with the symmetry of $\mathbf{A} = \mathbb{Y}(s_1 - 1, s_1 - 1, s_2)$. The gauge transformations and the field curvature read

$$\delta W_1^{A(s_1-1), B(s_1-1), C(s_2)} = D_\Omega \xi_0^{A(s_1-1), B(s_1-1), C(s_2)}, \quad (5.12)$$

$$R_2^{A(s_1-1), B(s_1-1), C(s_2)} = D_\Omega W_1^{A(s_1-1), B(s_1-1), C(s_2)}. \quad (5.13)$$

The field curvature is manifestly gauge invariant $\delta R_2^{A(s_1-1), B(s_1-1), C(s_2)} = 0$.

The correct field equations are imposed by setting all components of the field curvature to zero except for the Weyl tensor and its descendants. The Weyl tensor is an irreducible Lorentz tensor $C^{a(s_1), b(s_1)}$ having the symmetry of $\mathbb{Y}(s_1, s_1)$. Among its descendants $C^{a(s_1+i), b(s_1), c(j)}$, $i = 0, 1, 2, \dots, j = 0, \dots, s_2$ those with $i = 0$ couple with the gauge module. The set $C^{a(s_1), b(s_1), c(j)}$, $j = 0, \dots, s_2$ can be embedded into the irreducible tensor $C_0^{A(s_1), B(s_1), C(s_2)}$ of the (anti)-de Sitter algebra subjected to

$$V_M \left(C_0^{A(s_1), B(s_1-1)M, C(s_2)} - \frac{s_2}{s_1 - s_2 + 1} C_0^{A(s_1), B(s_1-1)C, C(s_2-1)M} \right) \equiv 0, \quad (5.14)$$

which removes the components of the $\text{Res}(\mathbb{Y}(s_1, s_1, s_2))$ having the symmetry of $\mathbb{Y}(s_1, s_1 - i, j)$ with $i = 1, \dots, s_1 - s_2$. Therefore, the field equations have the form

$$R_2^{A(s_1-1), B(s_1-1), C(s_2)} = E_M E_N C_0^{A(s_1-1)M, B(s_1-1)N, C(s_2)}, \quad (5.15)$$

and we do not consider the constraints on $C_0^{A(s_1), B(s_1), C(s_2)}$ following from the Bianchi identity $D_\Omega R_2^{A(s_1-1), B(s_1-1), C(s_2)} \equiv 0$.

σ_- -map. In the table 6 we draw the diagrams corresponding to the elements of $H_g^q(\mathbf{A}, \sigma_-)$ for low q , which are relevant for the field $\phi^{a(s_1), b(s_2)}$.

Unfolded formulation, Lorentz-covariant. The Lorentz-covariant frame-like formulation is constructed by reducing the representation of the (anti)-de Sitter algebra \mathfrak{g} down to the representations of the Lorentz algebra

$$\text{Res}_{\mathfrak{so}(d-1,1)}^{\mathfrak{g}} \left(\begin{array}{|c|} \hline s_1 - 1 \\ \hline s_1 - 1 \\ \hline s_2 \\ \hline \end{array} \right) = \bigoplus_{j=0}^{j=s_1-s_2-1} \bigoplus_{i=0}^{i=s_2} \begin{array}{|c|} \hline s_1 - 1 \\ \hline s_2 + j \\ \hline i \\ \hline \end{array}, \quad (5.16)$$

hence, the gauge field $W_1^{\mathbf{A}}$ decomposes into the following set of $\mathfrak{so}(d-1, 1)$ -connections

$$W_1^{A(s_1-1), B(s_1-1), C(s_2)} \longleftrightarrow \omega_1^{a(s_1-1), b(s_1-i), c(j)}, \quad i \in [1, s_1 - s_2], \quad j \in [0, s_2]. \quad (5.17)$$

The gauge parameter and the field curvature decompose in a similar way.

The physical field $\phi^{a(s_1), b(s_2)}$, which is a representative of $H_{g=0}^{q=1}(\sigma_-)$, is embedded into the generalized frame field

$$e_1^{a(s_1-1), b(s_2)} = W_1^{a(s_1-1), b(s_2) \bullet (s_1-s_2-1), \bullet (s_2)} \quad (5.18)$$

$q \setminus g$	0	1	$s_1 - s_2 - 1$	$s_1 - s_2$	$s_1 - 1$
0	$\begin{array}{ c } \hline s_1 - \\ \hline s_2 \\ \hline \end{array}$	\emptyset	\emptyset	\emptyset	
1	$\begin{array}{ c } \hline s_1 \\ \hline s_2 \\ \hline \end{array} \oplus \begin{array}{ c } \hline s_1 - 2 \\ \hline s_2 \\ \hline \end{array} \oplus \begin{array}{ c } \hline s_1 - \\ \hline s_2 - 1 \\ \hline \end{array}$		\emptyset	\emptyset	\emptyset
2	$\begin{array}{ c } \hline s_1 \\ \hline s_2 - 1 \\ \hline \end{array} \oplus \begin{array}{ c } \hline s_1 - 2 \\ \hline s_2 - 1 \\ \hline \end{array}$	$\begin{array}{ c } \hline s_1 \\ \hline s_2 \\ \hline \end{array} \oplus \begin{array}{ c } \hline s_1 - 2 \\ \hline s_2 \\ \hline \end{array}$	$\begin{array}{ c } \hline s_1 \\ \hline \\ \hline \end{array}$	\emptyset	\emptyset
3	\emptyset	$\begin{array}{ c } \hline s_1 \\ \hline s_2 - 1 \\ \hline \end{array} \oplus \begin{array}{ c } \hline s_1 - 2 \\ \hline s_2 - 1 \\ \hline \end{array} \oplus \begin{array}{ c } \hline s_1 - 1 \\ \hline s_2 \\ \hline \end{array}$	$\begin{array}{ c } \hline s_1 \\ \hline s_1 - 1 \\ \hline \end{array}$	$\begin{array}{ c } \hline s_1 \\ \hline \\ \hline \end{array}$	$\begin{array}{ c } \hline s_1 \\ \hline s_1 \\ \hline s_1 + 1 \\ \hline \end{array}$
4	\emptyset	$\begin{array}{ c } \hline s_1 - \\ \hline s_2 - 1 \\ \hline \end{array}$	$\begin{array}{ c } \hline s_1 - \\ \hline s_1 - \\ \hline \end{array}$	$\begin{array}{ c } \hline s_1 \\ \hline s_1 - \\ \hline \end{array}$	$\begin{array}{ c } \hline s_1 \\ \hline s_1 \\ \hline s_2 + 1 \\ \hline \end{array} \oplus \begin{array}{ c } \hline s_1 \\ \hline s_1 \\ \hline s_2 \\ \hline \end{array} \oplus \begin{array}{ c } \hline s_1 \\ \hline s_1 - \\ \hline s_2 + 1 \\ \hline \end{array}$

Table 6. Sigm-cohomology for $\mathbf{A} = \mathbb{Y}(s_1 - 1, s_1 - 1, s_2)$.

as the msp, which is given by

$$\phi^{a(s_1), b(s_2)} = e^{a(s_1-1), b(s_2)}|a, \quad e^{a(s_1-1), b(s_2)}|c = e_\mu^{a(s_1-1), b(s_2)} h^{\mu c}. \quad (5.19)$$

The decomposition of $e^{a(s_1-1), b(s_2)}|c$ into irreducible $\mathfrak{so}(d-1, 1)$ -tensors reads

$$\begin{array}{|c|} \hline s_1 - 1 \\ \hline s_2 \\ \hline \end{array} \otimes \square = \left[\begin{array}{|c|} \hline s_1 \\ \hline s_2 \\ \hline \end{array} \oplus \begin{array}{|c|} \hline s_1 - 2 \\ \hline s_2 \\ \hline \end{array} \oplus \begin{array}{|c|} \hline s_1 - 1 \\ \hline s_2 - 1 \\ \hline \end{array} \right] \oplus \left[\begin{array}{|c|} \hline s_1 - 1 \\ \hline s_2 \\ \hline \end{array} \oplus \begin{array}{|c|} \hline s_1 \\ \hline s_2 + 1 \\ \hline \end{array} \right]. \quad (5.20)$$

The terms in the first brackets correspond to $H_{g=0}^{q=1}(\sigma_-)$, the field components in the second brackets are σ_- -exact inasmuch as they can be gauged away by the gauge parameters that have the same symmetry type.

The gauge parameter $\xi^{a(s_1-1), b(s_2)}$, which is a representative of $H_{g=0}^{q=0}(\sigma_-)$, is defined as

$$\xi^{a(s_1-1), b(s_2)} = \xi_0^{a(s_1-1), b(s_2)} = \xi_0^{a(s_1-1), b(s_2) \bullet (s_1 - s_2 - 1), \bullet (s_2)}. \quad (5.21)$$

From frame-like to metric-like, field equations. Having identified the representatives of $H_g^{q=1}(\sigma_-)$ and $H_g^{q=0}(\sigma_-)$ corresponding to the dynamical field $\phi^{a(s_1), b(s_2)}$ and to the gauge parameter $\xi^{a(s_1-1), b(s_2)}$, let us now turn to $H_g^{q=2}(\sigma_-)$ whose representatives give the field equations.

The 'raw' field curvature at the lowest grade, i.e. the field curvature for the generalized frame field $e_1^{a(s_1-1), b(s_2)}$, and the two 'raw' field curvatures at grade-one for the auxiliary fields $\omega_1^{a(s_1-1), b(s_2+1)}$ and $\omega_1^{a(s_1-1), b(s_2), c}$ read

$$R^{a(s_1-1), b(s_2)}|aa = D^a e^{a(s_1-1), b(s_2)}|a + (s_1 - s_2 - 1)\omega^{a(s_1-1), b(s_2)}|a + s_2 \omega^{a(s_1-1), b(s_2), a}|a \quad (5.22)$$

$$\begin{aligned} R^{a(s_1-1), b(s_2+1)}|aa &= D^a \omega^{a(s_1-1), b(s_2+1)}|a + (s_1 - s_2 - 2)W^{a(s_1-1), b(s_2+1) \bullet (s_1 - s_2 - 3) a, \bullet (s_2)}|a + \\ &+ s_2 W^{a(s_1-1), b(s_2+1) \bullet (s_1 - s_2 - 2), \bullet (s_2 - 1) a}|a - \eta^{aa} W^{a(s_1-2) \bullet, b(s_2+1) \bullet (s_1 - s_2 - 2), \bullet (s_2)}|a + \\ &- \eta^{ba} e^{a(s_1-1), b(s_2)}|a \end{aligned} \quad (5.23)$$

$$\begin{aligned}
 R^{a(s_1-1),b(s_2);c|aa} &= D^a \omega^{a(s_1-1),b(s_2),c|a} + (s_1 - s_2 - 1) W^{a(s_1-1),b(s_2)\bullet(s_1-s_2-2)a,c\bullet(s_2-1)|a} + \\
 &+ (s_2 - 1) W^{a(s_1-1),b(s_2)\bullet(s_1-s_2-1),c\bullet(s_2-2)a|a} - \eta^{aa} W^{a(s_1-2)\bullet,b(s_2)\bullet(s_1-s_2-1),c\bullet(s_2-1)|a} + \\
 &- \eta^{ba} W^{a(s_1-1),b(s_2-1)\bullet(s_1-s_2),c\bullet(s_2-1)|a} - \eta^{ca} e^{a(s_1-1),b(s_2)|a}, \tag{5.24}
 \end{aligned}$$

where we have substituted (5.18) and

$$\omega^{a(s_1-1),b(s_2+1)|m} = W^{a(s_1-1),b(s_2+1)\bullet(s_1-s_2-2),\bullet(s_2)|m}, \tag{5.25}$$

$$\omega^{a(s_1-1),b(s_2);c|m} = W^{a(s_1-1),b(s_2)\bullet(s_1-s_2-1),c\bullet(s_2-1)|m}. \tag{5.26}$$

The 'raw' field $\omega^{a(s_1-1),b(s_2);c|m}$ have partial Young symmetry properties,

$$(s_2 + 1) \omega^{a(s_1-1),b(s_2);b|m} = -(s_1 - s_2 - 1) \omega^{a(s_1-1),b(s_2+1)|m}. \tag{5.27}$$

When expressed in terms of $\phi^{a(s_1),b(s_2)}$, the representative of $\mathbf{H}_{g=0}^{q=2}$

$$R^{a(s_1-1),b(s_2-2)m|a} = D^a e^{a(s_1-1),b(s_2-1)n|} - D_n e^{a(s_1-1),b(s_2-1)n|a} \tag{5.28}$$

give the same expression as (5.11). The simplest way to take the representative of $\mathbf{H}_{g=1}^{q=2}$ and derive (5.10) is to compute

$$(s_1 - s_2 - t) R^{a(s_1-1),b(s_2)\bullet(s_1-s_2-t-1)n,\bullet(s_2)|a} + s_2 R^{a(s_1-1),b(s_2)\bullet(s_1-s_2-t),\bullet(s_2-1)n|a} = 0,$$

where the terms with the derivative of $\omega^{a(s_1-1),b(s_2+1)|m}$ and $\omega^{a(s_1-1),b(s_2),c|m}$ can be expressed from

$$\begin{aligned}
 R^{a(s_1-1),b(s_2)\bullet(s_1-s_2-1),\bullet(s_2)|an} &= D^a e^{a(s_1-1),b(s_2)|n} - D_n e^{a(s_1-1),b(s_2)|a} + \\
 &- (s_1 - s_2 - 1) \omega^{a(s_1-1),b(s_2)n|a} - s_2 \omega^{a(s_1-1),b(s_2);n|a} = 0, \tag{5.29}
 \end{aligned}$$

that is obtained from (5.22) by symmetrizing one index \mathbf{a} with $a(s_1 - 1)$.

Consequently, formulated in terms of a single connection $W_{\mathbf{q}}^{\mathbf{A}}$ (5.12), (5.13), (5.15) the theory has a very simple form and the representatives of the σ_- -cohomology give all relevant quantities. Technical complications arise when passing to a metric-like formulation, in which the origin of the trace constraints (5.5)–(5.6) is not self-evident and the field equations are more involved.

6 Conclusions

In this paper we have presented the results on the σ_- -cohomology for the algebraic complex $\mathcal{C}(\mathbf{A}, \sigma_-)$ associated with the differential complex $\mathcal{C}(\mathbf{A}, D_{\Omega})$ of gauge connections generated by an arbitrary irreducible finite-dimensional representation \mathbf{A} of the (anti)-de Sitter algebra. The complex $\mathcal{C}(\mathbf{A}, \sigma_-)$ arises if we would like to reinterpret the fields of the (anti)-de Sitter algebra in terms of the Lorentz one.

As distinct from the rich complex $\mathcal{C}(\mathbf{A}, \sigma_-)$, by virtue of the Poincare lemma $\mathcal{C}(\mathbf{A}, D_{\Omega})$ is locally exact in degrees greater than zero and provides no interesting information.

The results on the σ_- -cohomology are important for constructing Lagrangians inasmuch as the Lagrangian equations must set to zero the representatives of $H(\sigma_-)$ that are associated with the equations of motion, as it is established by direct method, for example, for massless symmetric spin- s fields in (anti)-de Sitter space in [51, 78], for massless arbitrary-spin fields in Minkowski space in [12] and for massless two-column fields in (anti)-de Sitter space in [7].

Restriction to the σ_- -cohomology in the sector of fields and gauge parameters yields the minimal formulation for a given gauge field, so that all algebraic gauges are imposed and all auxiliary fields are expressed in terms of dynamical ones. The differential form structure is lost in $H(\sigma_-)$ in that the representatives of $H(\sigma_-)$ are certain Lorentz tensors, which may be embedded into the forms of the degree dictated by $H(\sigma_-)$, however, the constraints involving both the form and the fiber indices arise (these are formulated in terms of the background frame and its inverse). Therefore, the minimal formulation operates with a collection of metric-like fields, which may look strange, e.g. having complicated trace constrains.

That the σ_- -cohomology of the gauge module in the sector of the Weyl tensor together with its Bianchi identities is perfectly glued to the σ_- -cohomology of the Weyl module suggests that the spectrum of fields of the Weyl module is correct. Thus, the next problem is to find the higher-spin algebras [79, 80] for mixed-symmetry fields and construct the corresponding nonlinear equations, which are believed to exist [81–84] and are likely to be constructed within the unfolded approach.

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A Tensor products

Let \mathbf{X} and \mathbf{Y} be two irreducible representations of some Lie algebra \mathfrak{g} , then we can take the tensor product of $\mathbf{X} \otimes_{\mathfrak{g}} \mathbf{Y}$ and decompose it into the direct sum

$$\mathbf{X} \otimes_{\mathfrak{g}} \mathbf{Y} = \bigoplus_{\mathbf{Z}} C_{\mathbf{X}, \mathbf{Y}}^{\mathbf{Z}} \mathbf{Z} \tag{A.1}$$

over irreducible modules \mathbf{Z} with multiplicities given by the Littlewood-Richardson coefficients $C_{\mathbf{X}, \mathbf{Y}}^{\mathbf{Z}}$. For the reader's convenience we present below the tensor product rules for \mathfrak{g} being $sl(d)$ or $so(d)$ and for \mathbf{Y} being a one column Young diagram.

$sl(d)$ -tensor product. Let $\mathbf{X} = \mathbb{Y}\{(s_i, p_i)\}$ and $\mathbf{Y} = \mathbb{Y}[q]$ be irreducible representations of $sl(d)$, then the decomposition of the tensor product $\mathbf{X} \otimes \mathbb{Y}[q]$ is of the form

$$\mathbf{X} \otimes_{sl(d)} \mathbb{Y}[q] = \bigoplus_{\alpha_1 + \dots + \alpha_{N+1} = q} \mathbf{X}^{\{\alpha_j\}}, \tag{A.2}$$

where the multiplicity of each irreducible representation $\mathbf{X}_{\{\alpha_j\}}$ is 1 and the sum is over all Young diagrams $\mathbf{X}^{\{\alpha_j\}}$

$$\mathbf{X}^{\{\alpha_1, \dots, \alpha_{N+1}\}} = \begin{array}{|c|} \hline \begin{array}{c} s_1 \\ p_1 \end{array} \\ \hline \begin{array}{c} s_2 \\ p_2 \end{array} \\ \hline \begin{array}{c} s_N \\ p_N \end{array} \\ \hline \alpha_{N+1} \\ \hline \end{array} : \alpha_i \leq p_i, \quad \text{for } i=1, \dots, N, \quad (\text{A.3})$$

with $\alpha_1 + \dots + \alpha_{N+1} = q$. The diagrams of total height greater than d correspond to identically zero tensors and must be discarded. The first column must be removed from the diagrams of height d , so that the resulted diagram has height $(d - 1)$ at most.

Since $\mathbb{Y}[q]$ corresponds to a representation on antisymmetric tensors, two cells of $\mathbb{Y}[q]$ must not appear in the same row in \mathbf{X} , which determines the shape of $\mathbf{X}^{\{\alpha_j\}}$.

so(d)-tensor product. To decompose the tensor product $\mathbf{X} \otimes_{so(d)} \mathbb{Y}[q]$ of two $so(d)$ representations $\mathbf{X} = \mathbb{Y}\{(s_i, p_i)\}$ and $\mathbf{Y} = \mathbb{Y}[q]$ is a more complicated problem because of ability to take traces with the help of the invariant tensor η_{ab} . The decomposition of $\mathbf{X} \otimes_{so(d)} \mathbb{Y}[q]$ has the form

$$\mathbf{X} \otimes_{so(d)} \mathbb{Y}[q] = \bigoplus_{\{\alpha_j, \beta_i\}} N_{\{\alpha_j, \beta_i\}} \mathbf{Y}^{\{\alpha_j, \beta_i\}}, \quad (\text{A.4})$$

where the sum is over all Young diagrams $\mathbf{Y}^{\{\alpha_j, \beta_i\}}$

$$\mathbf{Y}^{\{\alpha_j, \beta_i\}} = \begin{array}{|c|} \hline \begin{array}{c} s_1 \\ p_1 \end{array} \\ \hline \begin{array}{c} s_2 \\ p_2 \end{array} \\ \hline \begin{array}{c} s_N \\ p_N \end{array} \\ \hline \alpha_{N+1} \\ \hline \end{array} : \alpha_i + \beta_i \leq p_i, \quad \text{for } i=1, \dots, N, \quad (\text{A.5})$$

provided there is a nonnegative integer ρ such that

$$q = \sum_{i=1}^{i=N} (\alpha_i + \beta_i) + \alpha_{N+1} + 2\rho, \tag{A.6}$$

The multiplicity $N_{\{\alpha_j, \beta_i\}}$ of $\mathbf{Y}^{\{\alpha_j, \beta_i\}}$ is given by the number of integer partitions

$$N_{\{\alpha_j, \beta_i\}} = P(\epsilon_1, \dots, \epsilon_N | \rho), \quad \epsilon_i = p_i - \alpha_i - \beta_i, \tag{A.7}$$

of ρ into the sum of N integers $k_1 + \dots + k_N = \rho$ such that $0 \leq k_i \leq \epsilon_i$. The trace order r for $\mathbf{Y}_{\{\alpha_j, \beta_i\}}$ is

$$r = \sum_{i=1}^{i=N} \beta_i + \rho. \tag{A.8}$$

The meaning of the above is as follows. Before adding the cells of $\mathbb{Y}[q]$ to \mathbf{X} , one can take traces, i.e. to remove pairs of cells, one from \mathbf{Y} and another one from \mathbf{X} . Since \mathbf{Y} corresponds to antisymmetric tensor representations, two cells cannot be removed from the same row of \mathbf{X} . Therefore, each trace of order r corresponds to some integer partition of r into the sum $t_1 + \dots + t_N$ provided that $t_i \leq p_i$. A subcolumn of height t_i is removed from the bottom-right of the i -th block. Then, the rest of the cells from \mathbf{Y} , i.e. $(q - r)$, can be added to what \mathbf{X} has turned into after taking traces. Recall that the i -th block consists now of two subblocks $\mathbb{Y}\{s_i, p_i - t_i\}$ and $\mathbb{Y}\{s_i - 1, t_i\}$. There are two types of places to which cells can now be added: α_i cells are added to the top-right of the subblock $\mathbb{Y}\{s_i, p_i - t_i\}$; $(t_i - \beta_i)$ cells are added to the top-right of the subblock $\mathbb{Y}\{s_i - 1, t_i\}$. The latter leads to the possibility to get the same diagrams in many different ways, i.e. results in the multiplicity greater than one. Different partitions of ρ into $\sum_i (t_i - \beta_i)$ provided the trace order $r = \sum_i t_i$ and all β_i are fixed results in identical Young diagrams. So ρ is the number of cells that were first removed and then restored.

When the height of some diagram $\mathbf{Y}^{\{\alpha_j, \beta_i\}}$ is greater than $[d/2]$, the antisymmetric invariant tensor $\epsilon_{a_1 \dots a_d}$ has to be used to transform it to a diagram with height less than $[d/2]$ or to impose (anti)selfduality conditions when the height is $[d/2]$ for d even. We implicitly assume that the rules described above are applicable to all tensors products considered in the paper.

B σ_- -cohomology

The case of $\mathfrak{sl}(d+1)$

Below we compute the σ_- -cohomology for $sl(d+1)$ and $so(d+1)$, starting with the case of $sl(d+1)$. As it has been already mentioned, the σ_- -cohomology in the case of $sl(d+1)$ is closely related to the ordinary Lie algebra cohomology. This is not so in the case of $so(d+1)$, for which a different method should be developed. We first apply the new method to the case of $sl(d+1)$, so that one can check the results. The method is to embed $\mathcal{C}(\mathbf{A}, \sigma_-)$ into the tensor product of much more simple complexes associated to one-row

Young diagrams, $\mathbf{A} = \mathbb{Y}(s)$, then the cohomology of $\mathcal{C}(\mathbf{A}, \sigma_-)$ with \mathbf{A} of general shape can be obtained with the help of certain projectors, whose kernels we are able to find.

Set $\mathfrak{g} = sl(d+1)$ and $\mathfrak{h} = sl(d)$. Let V be a fundamental (vector) representation of \mathfrak{g} . For any nonzero (compensator) vector¹³ $v \in V$, we have the decomposition $V = V_0 \oplus V_1$, where V_1 is a one-dimensional subspace spanned by the vector v and V_0 is a vector representation of \mathfrak{h} . We define a complex

$$\sigma_- : T^m(V) \otimes \Lambda^q(V_0) \longrightarrow T^m(V) \otimes \Lambda^{q+1}(V_0), \quad (\text{B.1})$$

where $T^m(V)$ is the m -th tensor power of V , $\Lambda^q(V_0)$ is the q -th exterior power of V_0 and σ_- is a nilpotent operator

$$\begin{aligned} \sigma_- : (X_1 \otimes \dots \otimes X_m) \otimes z_1 \wedge \dots \wedge z_q &\longrightarrow \\ &\longrightarrow \sum_{i=1}^{i=m} (X_1 \otimes \dots \otimes \widehat{X_{i-1}} \otimes v \otimes X_{i+1} \otimes \dots \otimes X_m) \otimes \rho_v(X_i) \wedge z_1 \wedge \dots \wedge z_q, \end{aligned} \quad (\text{B.2})$$

where $X_1, \dots, X_m \in V$, $z_1, \dots, z_q \in V_0$ and $\rho_v(X)$ is a projector onto V_0 , i.e. $\rho_v(v) = 0$, $\rho_v(x) = x$ for $x \in V_0$.

In order to single out from $T^m(V)$ an irreducible \mathfrak{g} -module \mathbf{A} with rank $m = \text{rank}(\mathbf{A})$, the Young symmetrizer $\pi_{\mathbf{A}}$ is needed. A Young symmetrizer is a weighted sum over all permutations of $m = \text{rank}(\mathbf{A})$ factors

$$\pi_{\mathbf{A}}[X_1 \otimes \dots \otimes X_m] = \sum_{\{\sigma\}} f(\sigma) X_{\sigma_1} \otimes \dots \otimes X_{\sigma_m}, \quad (\text{B.3})$$

where the weight function $f(\sigma)$ is determined by \mathbf{A} , e.g. for $\mathbf{A} = \mathbb{Y}(m)$ $f(\sigma) = (m!)^{-1}$. It is not hard to see that the Young symmetrizer $\pi_{\mathbf{A}}$ commutes with the action of σ_- . Therefore, given any irreducible \mathfrak{g} -module \mathbf{A} the nilpotent operator

$$\sigma_- : \mathbf{A} \otimes \Lambda^q(V_0) \longrightarrow \mathbf{A} \otimes \Lambda^{q+1}(V_0), \quad (\text{B.4})$$

is well-defined, so is the corresponding complex on $\mathbf{A} \otimes \Lambda(V_0)$, which we denote $\mathcal{C}(\mathbf{A}, \sigma_-)$. The definition just given is intermediate in a sense that it does not deal with explicit indices as in section 3.3, but seems to depend on the choice of the compensator as compared to the invariant definition of section 3.4.

On account of the embedding $\mathbf{A} \hookrightarrow T^m(V)$ with $m = \text{rank}(\mathbf{A})$, any element of \mathfrak{h} -module \mathbf{X} from $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} \mathbf{A}$ can be written as a sum over elements of the form

$$\pi_{\mathbf{A}}(\overbrace{v \otimes \dots \otimes v}^{m-k} \otimes \pi_{\mathbf{X}}(x_1 \otimes \dots \otimes x_k)), \quad (\text{B.5})$$

where $x_1, \dots, x_k \in V_0$ and $\mathbf{X} \in \text{Res}_{\mathfrak{h}}^{\mathfrak{g}} \mathbf{A}$.

As it has been already mentioned, $\mathbf{A} \otimes \Lambda(V_0)$ can be considered as an \mathfrak{h} -module, with the action of \mathfrak{h} on $\Lambda(V_0)$ induced from that on V_0 . Due to $\mathfrak{h}v = 0$, σ_- commutes with the action of \mathfrak{h} and, hence, both the elements of the complex and the representatives of σ_- -cohomology can be considered as \mathfrak{h} -modules so that one can deal with irreducible \mathfrak{h} -modules only.

¹³To be strict, a vector from the dual space is also needed, we skip obvious details since we can talk about tensors of $so(d+1)$ modulo traces rather than of $sl(d+1)$ -tensors.

De Rham complex. Consider the de Rham complex \mathcal{R} on the polynomials in d commuting variables y^a with the action of the de Rham differential ∂ defined by

$$\partial(\omega(y^a|\theta^b)) = \theta^c \frac{\partial}{\partial y^c} \omega(y^a|\theta^b), \tag{B.6}$$

where Grassmann variables θ^a are the analogs of dx^a . Rewritten in components, the action of ∂ on the component of degree k and q in y^a and θ^b , respectively, reads

$$\partial(\omega^{a^{(k)}|\mu[q]}) = k\omega^{a^{(k-1)}\mu|\mu[q]}, \tag{B.7}$$

where the antisymmetrization over the form indices μ is implied. As is well known, the cohomology of the de Rham complex is concentrated in the constant polynomials in y^a and θ^b , i.e.

$q \backslash g$	0	> 0	(B.8)
0	•	\emptyset	
> 0	\emptyset	\emptyset	

Decomposing $\omega^{a^{(k)}|\mu[q]}$ into irreducible \mathfrak{h} -modules, we have

$$\omega^{a^{(k)}|\mu[q]} \sim \begin{array}{|c|} \hline \text{k} \\ \hline \text{q} \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \text{k} \\ \hline \text{q} \\ \hline \end{array}. \tag{B.9}$$

Evidently, the second, the less antisymmetric component of (B.9) is exact inasmuch as the component with the same symmetry type is in $\omega^{a^{(k+1)}|\mu[q-1]}$, the two components forming a so-called contractible pair. Therefore, the total space of \mathcal{R} is decomposed into a direct sum of contractible pairs plus constants ω^1 that represent the only nontrivial cohomology class.

De Rham complex \mathcal{R}^s with constraints or $\mathcal{C}(\mathbb{Y}(s), \sigma_-)$. Consider now the de Rham complex \mathcal{R}^s on polynomials in y^a having degree not greater than s . Obviously, it can be realized as the space of degree s polynomials in $d + 1$ variables y^a and y^\bullet , ∂ is defined by the same formula. Therefore, \mathcal{R}^s is the simplest example of the σ_- -complex $\mathcal{C}(\mathbf{A}, \sigma_-)$ with $\mathbf{A} = \mathbb{Y}(s)$, and $\omega^{a^{(k)}|\mu[q]}$ is identified with the projection $W^{a^{(k)}\bullet^{(s-k)}|\mu[q]}$ of a single form $W^{A^{(s)}|\mu[q]}$ valued in $sl(d + 1)$ -module $\mathbb{Y}(s)$, c.f. (3.20).

It is easy to find the cohomology of \mathcal{R}^s : since it is the restriction of the de Rham complex, in addition to the de Rham cohomology we will have new cohomology classes with representatives coming from those contractible pairs at grade- s that get broken over restriction - these former exact forms represent now nontrivial cohomology classes since the $s + 1$ grade becomes trivial. Thus,

$q \backslash g$	0	s	(B.10)
0	$\mathcal{O}^0 = \bullet$	\emptyset	
1	\emptyset	$\mathcal{B}^1 = \begin{array}{ c } \hline s+1 \\ \hline \end{array}$	
2	\emptyset	$\mathcal{B}^2 = \begin{array}{ c } \hline s+1 \\ \hline \end{array}$	

where the notation $\mathcal{O}^0, \mathcal{B}^q$ was introduced to label the cohomology classes.

\mathcal{R}^{s_1, s_2} complex. Of use for us will be also the complex \mathcal{R}^{s_1, s_2} , $s_2 > 0$, obtained from \mathcal{R}^{s_1} by restricting further polynomials in y^a to have degree not less than s_2 , or more formally

$$0 \longrightarrow \mathcal{R}^{s_2} \longrightarrow \mathcal{R}^{s_1, s_2} \longrightarrow \mathcal{R}^{s_1} \longrightarrow 0. \quad (\text{B.11})$$

The need for \mathcal{R}^{s_1, s_2} is due to Young symmetrizers, which confine the rows of Young diagrams coming from $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} \mathbf{A}$ to be between two integers, the smallest of which being generally greater than zero.

Again, the cohomology of \mathcal{R}^{s_1, s_2} is easy to find - it is sufficient to find contractible pairs in \mathcal{R}^{s_1} at grade s_2 that get broken, yielding new cohomology

$q \backslash g$	s_2	s_1
0	$\mathcal{A}^0 = \begin{array}{ c } \hline \boxed{s_2} \\ \hline \end{array}$	\emptyset
1	$\mathcal{A}^1 = \begin{array}{ c } \hline \boxed{s_2} \\ \hline \square \\ \hline \end{array}$	$\mathcal{B}^1 = \begin{array}{ c } \hline \boxed{s_1+1} \\ \hline \end{array}$
2	$\mathcal{A}^2 = \begin{array}{ c } \hline \boxed{s_2} \\ \hline \square \\ \hline \square \\ \hline \end{array}$	$\mathcal{B}^2 = \begin{array}{ c } \hline \boxed{s_1+1} \\ \hline \square \\ \hline \end{array}$

(B.12)

It is worth mentioning that the representatives at higher degrees are obtained by adding cells to the bottom-left of \mathcal{A}^0 and \mathcal{B}^1 .

$\mathcal{R}(\mathbf{A}, \partial)$ complex. Given any diagram $\mathbf{A} = \mathbb{Y}(s_1, \dots, s_n)$ we define the complex $\mathcal{R}(\mathbf{A}, \partial)$ as a tensor product of \mathcal{R}^{s_i} ,

$$\mathcal{R}(\mathbf{A}, \partial) = \mathcal{R}^{s_1} \otimes \mathcal{R}^{s_2} \otimes \dots \otimes \mathcal{R}^{s_{n-1}} \otimes \mathcal{R}^{s_n}, \quad (\text{B.13})$$

with the action of the total differential ∂ defined through the action of ∂ on each multiplier $\omega_{\mathbf{q}_i}^i \in \mathcal{R}^{s_i}$ as

$$\begin{aligned} \partial (\omega_{\mathbf{q}_1}^1 \otimes \omega_{\mathbf{q}_2}^2 \otimes \dots \otimes \omega_{\mathbf{q}_n}^n) = & \partial(\omega_{\mathbf{q}_1}^1) \otimes \omega_{\mathbf{q}_2}^2 \otimes \dots \otimes \omega_{\mathbf{q}_n}^n \\ & + (-)^{q_1} \omega_{\mathbf{q}_1}^1 \otimes \partial(\omega_{\mathbf{q}_2}^2) \otimes \dots \otimes \omega_{\mathbf{q}_n}^n + \dots \end{aligned} \quad (\text{B.14})$$

A simple fact from the spectral sequences theory tells us that the cohomology $H^q(\mathbf{A}, \partial)$ of $\mathcal{R}(\mathbf{A}, \partial)$ is just the tensor product of cohomology groups at each factor. However, the complex $\mathcal{R}(\mathbf{A}, \partial)$ is still far from $\mathcal{C}(\mathbf{A}, \sigma_-)$ inasmuch as (1) no Young conditions are imposed; (2) each of the factors possesses its own copy $\Lambda(V_0)$, i.e. the elements of the complex are differential multi-forms¹⁴ rather than just forms. The complex in question $\mathcal{C}(\mathbf{A}, \sigma_-)$ can be extracted from $\mathcal{R}(\mathbf{A}, \partial)$ by applying two projectors $\pi_{\mathbf{A}}$ and π_{Λ} , where π_{Λ} singles out the most antisymmetric part of a multiform with no effect on coefficients, i.e.

$$\pi_{\Lambda} : \Lambda^{q_1} \otimes \dots \otimes \Lambda^{q_n} \longrightarrow \Lambda^{q_1+q_2+\dots+q_n}. \quad (\text{B.15})$$

It is not hard to see that the projectors $\pi_{\mathbf{A}}$ and π_{Λ} commute both with ∂ and with each other, the latter is evident since π_{Λ} affects only the coefficients of multi-forms while $\pi_{\mathbf{A}}$

¹⁴Multi-form is an element of the direct product of several copies of the exterior algebra. With application to higher-spin theories multi-forms were studied in [4, 9, 85, 86].

affects only multi-forms. However, in getting the cohomology $H(\mathbf{A}, \sigma_-)$ of $\mathcal{C}(\mathbf{A}, \sigma_-)$ by applying $\pi_{\mathbf{A}}$ and π_{Λ} to the cohomology $H(\mathbf{A}, \partial)$ of $\mathcal{R}(\mathbf{A}, \partial)$ we may meet two obstructions: (1) certain cohomology classes can fall into the kernel of π_{α} , $\alpha = \mathbf{A}, \Lambda$; (2) there can be contractible pairs $E = \partial(F)$ such that F does not belong to $\ker(\pi_{\alpha})$ but $E \in \ker(\pi_{\alpha})$ and thus F becomes a representative of a nontrivial cohomology class for $\mathcal{C}(\mathbf{A}, \sigma_-)$. It turns out that it is rather simple to find the kernel of π_{α} and we can also track the appearance of new cohomology through (2).

The properties of $\pi_{\mathbf{A}}$. To begin with, let us note that given a Young diagram, say \mathbf{X} , and an irreducible tensor with the symmetry of \mathbf{X} , say $C^{\mathbf{X}}$, written in symmetric basis, it is not necessary for the number of indices of some sort over which the total symmetrization in $C^{\mathbf{X}}$ is performed to equal the length of the corresponding row in \mathbf{X} . We refer to the irreducible \mathfrak{h} -tensors with the number of indices of each sort being equal to the length of the corresponding row as to the tensors with canonical arrangement of indices and noncanonical otherwise. For instance, given $\mathbf{X} = \mathbb{Y}(k, m)$ then $C^{a(k), b(m)}$ is a canonical arrangement and $C^{a(k-i)b(i), b(m)}$ with $i > 0$ is not. Also note that an \mathfrak{h} -tensor obtained by contracting a number of compensators with an irreducible \mathfrak{g} -tensor is not generally irreducible. For the example of a \mathfrak{g} -tensor with the symmetry of $\mathbf{A} = \mathbb{Y}(s_1, s_2)$ we have

$$W^{a(k)\bullet(s_1-k), b(m)\bullet(s_2-k)} = \sum_{j=0}^{k+j \leq s_1} \alpha_j^{k,m} \theta(s_1 - k - j) C^{a(k)b(j), b(m-j)}, \quad (\text{B.16})$$

where $\theta(k \geq 0) = 1$, $\theta(k < 0) = 0$ and it is natural to set $\alpha_{j=0}^{k,m} = 1$, so that

$$\pi_{\mathbf{X}} \left(W^{a(k)\bullet(s_1-k), b(m)\bullet(s_2-m)} \right) = C^{a(k), b(m)} \quad (\text{B.17})$$

for some $\mathbf{X} = \mathbb{Y}(k, m)$ provided that $\mathbf{X} \in \text{Res}_{\mathfrak{h}}^{\mathfrak{g}} \mathbf{A}$. Extraction from (B.16) of irreducible components with the symmetry different from \mathbf{X} we may call noncanonical projection. Symmetrization of all 'a'-indices with one index 'b' in (B.17) does not yield zero except for the term with $\alpha_{j=0}^{k,m}$, rather it gives a recurrent equation for $\alpha_j^{k,m}$. The solution is $\alpha_j^{k,m} = \frac{(-)^j (s_1-k)! (k-m+j)!}{(s_1-k-j)! (k-m+2j)!}$, which does not degenerate in the range of definition. Since each irreducible module in $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} \mathbf{A}$ appears once, there is no confusion with noncanonical projections.

Let $C^{a(k), b(m)}$ be an irreducible tensor with the symmetry of $\mathbf{X} = \mathbb{Y}(k, m)$. It can be embedded into the elements $\omega^{a(k+i)|b(m-i)}$ of \mathcal{R}^{s_1, s_2} in a canonical way if $i = 0$ and noncanonically if $i > 0$. Since each element of $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} \mathbf{A}$ comes with multiplicity one,¹⁵ the projector $\pi_{\mathbf{A}}$ maps all components having the symmetry of \mathbf{X} to the same element of $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} \mathbf{A}$, possibly modulo an overall factor, if $\mathbf{X} \in \text{Res}_{\mathfrak{h}}^{\mathfrak{g}} \mathbf{A}$ and to zero otherwise. Therefore, it is sufficient to deal with irreducible tensors with canonical arrangements of indices.

The kernel of $\pi_{\mathbf{A}}$ is easy to find: if for some canonical F we have $\partial(F) \neq 0$ and $\partial(F) \in \ker(\pi_{\mathbf{A}})$, then it implies that in $\partial(F)$ the number of indices of some sort is less than the length of the corresponding row in \mathbf{A} , i.e. $\partial(F)$ does not belong to $\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} \mathbf{A}$. This being said, new cohomology appears when passing from $\mathcal{R}^{s_1, \dots, s_n}$ to $\mathcal{C}(\mathbf{A}, \sigma_-) = \pi_{\mathbf{A}} \pi_{\Lambda} [\mathcal{R}^{s_1, \dots, s_n}]$, the new cohomologies are given by those new in $\mathcal{R}^{s_i, s_{i+1}}$ as compared to \mathcal{R}^{s_i} .

¹⁵This is true for the cases considered in the paper with $(\mathfrak{g}, \mathfrak{h})$ being $(sl(d+1), sl(d))$ or $(so(d+1), so(d))$.

Tensor product cohomology. Consider the projection $\pi_{\mathbf{A}}\pi_{\Lambda}[\mathbf{H}(\mathbf{A}, \partial)]$ of the cohomology $\mathbf{H}(\mathbf{A}, \partial)$, which is given by taking the tensor product $\mathbf{H}(\mathcal{R}^{s_1, s_2}) \otimes \mathbf{H}(\mathcal{R}^{s_2, s_3}) \otimes \dots$ and applying then $\pi_{\mathbf{A}}\pi_{\Lambda}$. Consider the tensor product $\omega = \omega^1 \otimes \dots \otimes \omega^{n-1} \otimes \omega^n$ of the representatives ω^i of nontrivial cohomology classes of $\mathcal{R}^{s_i, s_{i+1}}$. It turns out that if at least one of the representatives $\omega^1, \dots, \omega^{n-1}$ corresponds to a cohomology class that is characterized by a Young diagram with more than one row then $\pi_{\mathbf{A}}\pi_{\Lambda}[\omega]$ is a representative of the trivial cohomology class in $\mathcal{C}(\mathbf{A}, \sigma_-)$. Recalling the notation of (B.10) and (B.12), we can put it differently by saying that left multipliers of the form $\mathcal{A}^{q>0} \otimes$ and $\mathcal{B}^{q>1} \otimes$ yield trivial cohomology in $\mathcal{C}(\mathbf{A}, \sigma_-)$. Indeed, let us write down all four options for the tensor product of cohomology where the first multiplier is represented by a Young diagram with more than one row (two for simplicity)

$$\begin{array}{ll}
 \mathcal{A}^{q>0} \otimes \mathcal{A} & \sim \begin{array}{|c|} \hline s_2 \\ \hline \times \\ \hline \end{array} \otimes \begin{array}{|c|} \hline s_3 \\ \hline \end{array} & \xrightarrow{\pi_{\mathbf{A}}\pi_{\Lambda}} \begin{array}{|c|} \hline s_2 \\ \hline \times \\ \hline s_3 \\ \hline \end{array}, \\
 \mathcal{A}^{q\geq 0} \otimes \mathcal{B} & \sim \begin{array}{|c|} \hline s_2 \\ \hline \times \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \times \\ \hline s_2 \\ \hline \end{array} & \xrightarrow{\pi_{\mathbf{A}}\pi_{\Lambda}} \emptyset, \\
 \mathcal{B}^{q>1} \otimes \mathcal{A} & \sim \begin{array}{|c|} \hline \times \\ \hline \times \\ \hline s_1 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline s_3 \\ \hline \end{array} & \xrightarrow{\pi_{\mathbf{A}}\pi_{\Lambda}} \begin{array}{|c|} \hline \times \\ \hline \times \\ \hline s_1 \\ \hline s_3 \\ \hline \end{array}, \\
 \mathcal{B}^{q>1} \otimes \mathcal{B} & \sim \begin{array}{|c|} \hline \times \\ \hline \times \\ \hline s_1 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \times \\ \hline s_2 \\ \hline \end{array} & \xrightarrow{\pi_{\mathbf{A}}\pi_{\Lambda}} \begin{array}{|c|} \hline \times \\ \hline \times \\ \hline \times \\ \hline s_1 \\ \hline s_2 \\ \hline \end{array} \sim 0,
 \end{array}$$

where checked cells correspond to the form indices in tensor language, e.g. $\begin{array}{|c|} \hline s_2 \\ \hline \times \\ \hline \end{array}$ correspond to a representative of the form $C^{a(s_2), \mu}$, which is closed inasmuch as tensors with less than s_2 indices 'a' belong to the kernel of π_{Λ} . So, $\mathcal{A}^{q\geq 0} \otimes \mathcal{B}$ is mapped to zero since the first row in a Young diagram cannot be shorter than the second one; $\mathcal{B}^{q>1} \otimes \mathcal{B}$ is also mapped to zero because two form indices of different sorts appear in the same group of symmetric indices, which gives zero after applying π_{Λ} ; both $\mathcal{A}^{q>0} \otimes \mathcal{A}$ and $\mathcal{B}^{q>1} \otimes \mathcal{B}$ are mapped to exact forms in $\mathcal{C}(\mathbf{A}, \sigma_-)$ inasmuch as one form index in the second group of indices can now result from applying $\pi_{\mathbf{A}}\pi_{\Lambda}\partial$ to $C^{a(s_2), b(s_3+1)}$ and $C^{a(s_1)\mu, b(s_2+1)}$, respectively - roughly speaking on account of definite Young symmetry and the fact that forms of different sorts become identical via π_{Λ} , certain arrangements of indices in a tensor can now be obtained through $\pi_{\mathbf{A}}\pi_{\Lambda}\partial$, which is impossible in $\mathcal{R}(\mathbf{A}, \partial)$. Note that the disappearance of classes represented by a tensor product of Young diagrams with more than one row concerns left multipliers in the tensor product and has no effect on the last multiplier \mathcal{R}^{s_n} in $\mathcal{R}^{s_1, \dots, s_n}$, which is always the rightmost one.

The properties of π_{Λ} . Since a component with some definite Young symmetry can enter more than one element of \mathcal{R}^{s_1, s_2} even for $q = 0$, one can adjust coefficients in front of them to get a closed form after applying π_{Λ} , e.g.

$$\omega = \sum_{i=0}^{i=n} \frac{(-)^i}{(n-i)!i!} C^{a(n-i)b(i)}, \quad \pi_{\Lambda}[\partial(\omega)] = 0, \quad (\text{B.18})$$

where it has been taken into account that the action ∂ on some multi-form $B_{\mathbf{q}_1, \mathbf{q}_2}^{a(k)|b(m)}$ reads as

$$\partial \left(B_{\mu[q_1]|\nu[q_2]}^{a(k)|b(m)} \right) = k B_{\mu[q_1]|\nu[q_2]}^{a(k-1)\mu|b(m)} + m B_{\mu[q_1]|\nu[q_2]}^{a(k)|b(m-1)\nu}, \quad (\text{B.19})$$

where indices μ and ν correspond to two different sorts of forms. The projector $\pi_{\mathbf{A}}$ roughly speaking replaces all form indices of different sorts with indices of just one sort, say μ , with further antisymmetrization. It is important that ∂ consists of the two parts in this example, with one decreasing the number of a 's and another one decreasing the number of b 's. Therefore, having some $B_{\mathbf{q}_1, \mathbf{q}_2}^{a(k)|b(m)}$ possessing only one \mathfrak{h} -irreducible component and trying to solve the closeness condition we can meet two situations: (1) $\pi_{\mathbf{A}} \left[\partial \left(B_{\mathbf{q}_1, \mathbf{q}_2}^{a(k)|b(m)} \right) \right] \neq 0$ and to compensate we can introduce $B_{\mathbf{q}_1, \mathbf{q}_2}^{a(k-1)|b(m+1)}$ and $B_{\mathbf{q}_1, \mathbf{q}_2}^{a(k+1)|b(m-1)}$ — (B.18) is an example of this type. However, the process runs out with $B_{\mathbf{q}_1, \mathbf{q}_2}^{a(0)|b(m+k)}$ and $B_{\mathbf{q}_1, \mathbf{q}_2}^{a(k+m)|b(0)}$. Taking into account the properties of $\pi_{\mathbf{A}}$ and the range of tensor ranks of \mathcal{R}^{s_1, s_2} we see that $k+m = s_2$, i.e. the tensor has the lowest possible grade and hence must be totally symmetric, $B_{\mathbf{q}_1, \mathbf{q}_2}^{a(k)|b(m)} \equiv B_{\mathbf{q}_1, \mathbf{q}_2}^{a(k)b(m)}$ thus representing no new cohomology class; (2) $\pi_{\mathbf{A}} \left[\partial \left(B_{\mathbf{q}_1, \mathbf{q}_2}^{a(k)|b(m)} \right) \right] = 0$ implies that $B_{\mathbf{q}_1, \mathbf{q}_2}^{a(k)|b(m)}$ is of the form (with canonical arrangement of indices)

$$B_{\mathbf{q}_1, \mathbf{q}_2}^{a(k)|b(m)} \sim \begin{array}{|c|c|} \hline \times & \overbrace{\hspace{1cm}}^k \\ \hline \times & \overbrace{\hspace{1cm}}^m \\ \hline \times & \\ \hline \end{array} \xrightarrow{\partial} \begin{array}{|c|c|} \hline \times \times & \overbrace{\hspace{1cm}}^{k-1} \\ \hline \times & \overbrace{\hspace{1cm}}^m \\ \hline \times & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \times & \overbrace{\hspace{1cm}}^k \\ \hline \times \times & \overbrace{\hspace{1cm}}^{m-1} \\ \hline \times & \\ \hline \end{array},$$

where checked cells correspond to form indices of different sorts such that ∂ gives zero only after applying $\pi_{\mathbf{A}}$, e.g. $\partial \left(C^{a(k)\nu, b(m)\mu} \right) \neq 0$ and $\pi_{\mathbf{A}} \left[\partial \left(C^{a(k)\nu, b(m)\mu} \right) \right] = 0$. However, all these candidates for new cohomology either are exact analogously to the case of $\mathcal{A}^{q>0} \otimes \mathcal{A}$ and $\mathcal{B}^{q>1} \otimes \mathcal{B}$, or are equivalent to the old classes of the form $\mathcal{A} \otimes \mathcal{A}$, $\mathcal{B} \otimes \mathcal{A}$ or $\mathcal{B} \otimes \mathcal{B}$.

To sum up, nontrivial cohomology classes are generated by various tensor products of the most symmetric representatives of $\mathbf{H}(\mathcal{R}^{s_i, s_{i+1}})$, $i < n$, and $\mathbf{H}(\mathcal{R}^{s_n})$. Since any row in a Young diagram cannot be longer than the previous one, products of the form $\mathcal{A} \otimes \mathcal{B}$ are forbidden. Consequently,

$$\mathbf{H}^q(\mathbf{A}, \boldsymbol{\sigma}_-) = \begin{cases} \pi_{\mathbf{A}} \pi_{\mathbf{A}} \left[\overbrace{\mathcal{B}^1 \otimes \dots \otimes \mathcal{B}^1}^q \otimes \overbrace{\mathcal{A}^0 \otimes \dots \otimes \mathcal{A}^0}^{n-1-q} \otimes \mathcal{O}^0 \right], & q < n \\ \pi_{\mathbf{A}} \pi_{\mathbf{A}} \left[\overbrace{\mathcal{B}^1 \otimes \dots \otimes \mathcal{B}^1}^{n-1} \otimes \mathcal{B}^{q-n+1} \right], & q \geq n, \end{cases} \quad (\text{B.20})$$

which coincides both with the well-known result of Lie algebra cohomology theory, see e.g. [63], and with the first theorem in section 3.7. It is easy to remove the brackets - the diagram of \mathfrak{h} is obtained by concatenation of diagrams of each multiplier.

The case of $\mathfrak{so}(d+1)$

In the case of $\mathfrak{so}(d+1)$, $\mathfrak{g} = \mathfrak{so}(d+1)$ and $\mathfrak{h} = \mathfrak{so}(d)$, we follow along the same path, representing $\mathcal{C}(\mathbf{A}, \boldsymbol{\sigma}_-)$ as a projection of a tensor product of complexes associated with

one-row $so(d+1)$ -diagrams. It allows us to easily find candidates for cohomology since the projectors $\pi_{\mathbf{A}}$, $\pi_{\mathbf{\Lambda}}$ and one new projector $\pi_{\mathbf{cr}}$ either do not lead to new cohomology ($\pi_{\mathbf{A}}$, $\pi_{\mathbf{cr}}$) or it is simple to track the appearance of new cohomology ($\pi_{\mathbf{\Lambda}}$), neither is it difficult to find their kernels. At the final step we compute the Euler characteristic to find the dimension of $\ker(\pi_{\mathbf{A}}\pi_{\mathbf{\Lambda}}\pi_{\mathbf{cr}})$.

Harmonic de Rham complex. Consider the de Rham complex $\mathfrak{h}\mathcal{R}$ on harmonic polynomial in d variables y^a , i.e. $\frac{\partial^2}{\partial y^c \partial y_c} \omega(y^a | \theta^b) \equiv 0$. The action of the differential ∂ is given by the same formula (B.6), or in components by (B.7). The harmonicity condition in terms of components $\omega^{a(k)|\mu[q]}$ is equivalent to the vanishing trace condition for indices a . The cohomology of the harmonic de Rham complex is well-known, for example in the framework of the unfolded approach it was found in [52],

$q \backslash g$	0	1	> 1	(B.21)
0	•	\emptyset	\emptyset	
1	\emptyset	•	\emptyset	
> 1	\emptyset	\emptyset	\emptyset	

There is one new cohomology class as compared to the de Rham complex \mathcal{R} , which is due to the tracelessness condition.

Harmonic de Rham complex with constraints. Analogously, we define $\mathfrak{h}\mathcal{R}^s$ to be the harmonic de Rham complex on polynomials with degree not greater than s . As it was the case for $sl(d+1)$, the complex $\mathfrak{h}\mathcal{R}^s$ turns out to be the simplest example of the σ_- -complex $\mathcal{C}(\mathbb{Y}(s), \sigma_-)$. The component $\omega^{a(k)|\mu[q]}$ is identified with the traceless part of the projection $W^{a(k)\bullet(s-k)|\mu[q]}$ of a single form $W^{A(s)|\mu[q]}$ valued in $so(d+1)$ -module $\mathbb{Y}(s)$, c.f. (3.16). The cohomology of $\mathfrak{h}\mathcal{R}^s$ is given by that of $\mathfrak{h}\mathcal{R}$ plus some new cohomology classes because of breaking certain contractible pairs at grade s due to the degree constraint

$q \backslash g$	0	1	s	(B.22)
0	$\mathcal{O}^0 = \bullet$	\emptyset	\emptyset	
1	\emptyset	$\mathcal{O}^1 = \bullet$	$\mathcal{B}^1 = \boxed{s+1}$	
2	\emptyset	\emptyset	$\mathcal{B}^2 = \begin{array}{ c } \hline \boxed{s+1} \\ \hline \end{array}, \mathcal{C}^2 = \boxed{s}$	
3	\emptyset	\emptyset	$\mathcal{B}^3 = \begin{array}{ c } \hline \boxed{s+1} \\ \hline \end{array}, \mathcal{C}^3 = \begin{array}{ c } \hline \boxed{s} \\ \hline \end{array}$	

Note that the representatives having the symmetry of one-row Young diagrams occur at degree up to two.

$\mathfrak{h}\mathcal{R}^{s_1, s_2}$ complex. We also need the complex $\mathfrak{h}\mathcal{R}^{s_1, s_2}$ that is \mathcal{R}^{s_1} on polynomials whose degree is not less than s_2 , or more formally

$$0 \longrightarrow \mathfrak{h}\mathcal{R}^{s_2} \longrightarrow \mathfrak{h}\mathcal{R}^{s_1, s_2} \longrightarrow \mathfrak{h}\mathcal{R}^{s_1} \longrightarrow 0. \tag{B.23}$$

The cohomology of \mathfrak{hR}^{s_1, s_2} reads as

$q \backslash g$	s_2	s_1
0	$\mathcal{A}^0 = \boxed{s_2}$	\emptyset
1	$\mathcal{A}^1 = \begin{array}{ c } \hline \boxed{s_2} \\ \hline \end{array}$, $\mathcal{D}^1 = \boxed{s_2-1}$	$\mathcal{B}^1 = \boxed{s_1+1}$
2	$\mathcal{A}^2 = \begin{array}{ c } \hline \boxed{s_2} \\ \hline \end{array}$, $\mathcal{D}^2 = \begin{array}{ c } \hline \boxed{s_2-1} \\ \hline \end{array}$	$\mathcal{B}^2 = \begin{array}{ c } \hline \boxed{s_1+1} \\ \hline \end{array}$, $\mathcal{C}^2 = \boxed{s_1}$

(B.24)

$\mathfrak{hR}(\mathbf{A}, \partial)$ complex. Given a Young diagram $\mathbf{A} = \mathbb{Y}(s_1, \dots, s_n)$ the complex $\mathfrak{hR}(\mathbf{A}, \partial)$ is

$$\mathfrak{hR}(\mathbf{A}, \partial) = \mathfrak{hR}^{s_1} \otimes \mathfrak{hR}^{s_2} \otimes \dots \otimes \mathfrak{hR}^{s_{n-1}} \otimes \mathfrak{hR}^{s_n}, \quad (\text{B.25})$$

where the total differential ∂ is defined as in the $sl(d+1)$ -case. With regard to $\mathfrak{hR}(\mathbf{A}, \partial)$, its cohomology is just the tensor product of multipliers' cohomology.

Projectors $\pi_{\mathbf{A}}$ and π_{Λ} . Concerning $\pi_{\mathbf{A}}$ and π_{Λ} , the following statements are still true: (1) it is sufficient to consider only irreducible \mathfrak{h} -tensors with canonical arrangements of indices; (2) the Young symmetry conditions via $\pi_{\mathbf{A}}$ induce the appearance of the new cohomology out of \mathfrak{hR}^{s_i} at grade s_{i+1} such that the whole cohomology is equivalent to that of $\mathfrak{hR}^{s_i, s_{i+1}}$; (3) multiplying from the left by a cohomology represented by a Young diagram with more than one row makes the corresponding tensor product cohomology class trivial, so that $\mathcal{A}^{q>0} \otimes \dots$, $\mathcal{D}^{q>1} \otimes \dots$, $\mathcal{B}^{q>1} \otimes \dots$, $\mathcal{C}^{q>2} \otimes \dots$ result in trivial cohomology after imposing the projectors; (4) the products of the form $\mathcal{A} \otimes \mathcal{B}$, $\mathcal{D} \otimes \mathcal{B}$, $\mathcal{A} \otimes \mathcal{C}$ and $\mathcal{D} \otimes \mathcal{C}$ are mapped to zero by $\pi_{\mathbf{A}}$ if $s_1 > s_2$, if $s_1 = s_2$, then from the above-listed only $\mathcal{A} \otimes \mathcal{C}$ is allowed. In general, the presence of equal rows in \mathbf{A} complicates the answer greatly.

Projector $\pi_{\mathbf{cr}}$. We need one more projector $\pi_{\mathbf{cr}}$ that removes cross-traces since each $\omega(y_i^a | \theta_i^b)$ of \mathfrak{hR}^{s_i} is harmonic, however, $\omega(y_i^a, y_j^a | \theta_i^b, \theta_j^b)$ of $\mathfrak{hR}^{s_i} \otimes \mathfrak{hR}^{s_j}$ is not harmonic in $y_i^a y_j^b$ if $i \neq j$, $\frac{\partial^2}{\partial y_i^c \partial y_j^c} \omega(y_i^a, y_j^a | \theta_i^b, \theta_j^b) \neq 0$. In terms of components it implies that the cross-traces - the traces contracting indices from different groups - do not vanish, e.g.

$$\mathcal{C}^{a(k)|b(m)} = \mathcal{C}_0^{a(k)|b(m)} + \left(\eta^{ab} \mathcal{C}_1^{a(k-1)|b(m-1)} + \dots \right) + \left(\eta^{ab} \eta^{ab} \dots \right) + \dots \quad (\text{B.26})$$

To arrive at $\mathcal{C}(\mathbf{A}, \sigma_-)$, cross-traces must be factored out, this is what $\pi_{\mathbf{cr}}$ does. The action (B.19) of ∂ consists of replacing one index from each group with the form index, hence ∂ cannot increase the cross-trace order. Therefore, if some ω is not a cross-trace itself then $\partial(\omega)$ also is not. Hence it is impossible to have $\partial\omega \neq 0$ and $\pi_{\mathbf{cr}}[\partial\omega] = 0$ if ω does not belong to $\ker(\pi_{\mathbf{cr}})$. Consequently, $\pi_{\mathbf{cr}}$ does not give rise to new cohomology.

Result if $s_i \neq s_{i+1}$. If all weights in \mathbf{A} are different the answer is very simple: the representatives of cohomology classes are of two types. Recalling the notation of (B.22) and (B.24), the ones of the first type have the form

$$\mathbf{H}_{r,g}^q(\mathbf{A}, \sigma_-) = \pi_{\text{tot}} \left[\overbrace{\mathcal{B}^1(\mathcal{C}^2) \otimes \dots \otimes \mathcal{B}^1(\mathcal{C}^2)}^{q-r} \otimes \mathcal{A}^0(\mathcal{D}^1) \dots \otimes \mathcal{A}^0(\mathcal{D}^1) \otimes \mathcal{O}^0 \right], \quad q-r < n \quad (\text{B.27})$$

where $\mathcal{B}^1(\mathcal{C}^2)$ implies that either \mathcal{B}^1 or \mathcal{C}^2 can appear, analogously for $\mathcal{A}^0(\mathcal{D}^1)$ and $\pi_{tot} = \pi_{\mathbf{A}}\pi_{\Lambda}\pi_{cr}$. The degree q , grade g and trace order r are

$$q = \#\mathcal{B}^1 + 2\#\mathcal{C}^2 + \#\mathcal{D}^1, \quad g = s_1 - s_{q-r+1}, \quad r = \#\mathcal{C}^2 + \#\mathcal{D}^1 \quad (\text{B.28})$$

If $q - r \geq n$ then

$$H_{r,g=s_1}^q(\mathbf{A}, \sigma_-) = \pi_{tot} \left[\overbrace{\mathcal{B}^1(\mathcal{C}^2) \otimes \dots \otimes \mathcal{B}^1(\mathcal{C}^2) \otimes \mathcal{B}^{q'}(\mathcal{C}^{q'+1})}^n \right], \quad (\text{B.29})$$

where $q' = q - r - n + 1$. The representatives of the second type have the form

$$H_{r+1,g+1}^{q+1}(\mathbf{A}, \sigma_-) = \pi_{tot} \left[\overbrace{\mathcal{B}^1(\mathcal{C}^2) \otimes \dots \otimes \mathcal{B}^1(\mathcal{C}^2)}^{q-r} \otimes \mathcal{A}^0(\mathcal{D}^1) \dots \otimes \mathcal{A}^0(\mathcal{D}^1) \otimes \mathcal{O}^1 \right], \quad (\text{B.30})$$

where \mathcal{O}^0 is replaced with \mathcal{O}^1 , which shifts by one the degree, grade and trace order. The representative (B.30) is obtained from (B.27) by the duality map.

All above-stated can be reformulated in terms of $\text{mspr}(\mathbf{A}, q, r)$ as in the second theorem of section 3.7.

Some weights in \mathbf{A} are equal. In the case where some rows in \mathbf{A} are equal the block notation is more convenient, so let $\mathbf{A} = \mathbb{Y}\{(s_1, p_1), \dots, (s_N, p_N)\}$. At first sight we have a degeneracy so that the multiplicity of some diagram in $H_{g,r}^q$ can differ from 0 and 1. Indeed, if $s_i = s_{i+1}$ for some i then \mathcal{A}^0 equals \mathcal{C}^2 as diagrams and hence if there is more than one group of equal rows in \mathbf{A} then different partitions $q = q_1 + q_2 + \dots$ corresponding to

$$\begin{aligned} & \dots \otimes \dots \otimes \underbrace{\mathcal{A}^0 \otimes \dots \otimes \mathcal{A}^0 \otimes \overbrace{\mathcal{C}^2 \otimes \dots \otimes \mathcal{C}^2}^{q_1}}}_{p_i} \otimes \dots \\ & \dots \otimes \dots \otimes \underbrace{\mathcal{A}^0 \otimes \dots \otimes \mathcal{A}^0 \otimes \overbrace{\mathcal{C}^2 \otimes \dots \otimes \mathcal{C}^2}^{q_2}}}_{p_j} \dots \dots, \end{aligned} \quad (\text{B.31})$$

result in representatives of different cohomology classes with identical Young diagrams. It seems to be not enough to specify q , g , r and a Young diagram in order to distinguish between different cohomology classes. This will be proved not to be the case.

Because of the decomposition (3.35) for $\mathcal{C}(\mathbf{A}, \sigma_-)$, the cohomology of $\text{h}\mathcal{R}(\mathbf{A}, \partial)$ plus those new induced by Young conditions tells us not only which subcomplexes $\mathcal{C}(\mathbf{A}, \sigma_-; \mathbf{X}, q + g, r)$,

$$0 \longrightarrow V_0 \xrightarrow{\sigma_-} \dots \xrightarrow{\sigma_-} V_{q-1} \xrightarrow{\sigma_-} V_q \xrightarrow{\sigma_-} V_{q+1} \xrightarrow{\sigma_-} \dots \quad (\text{B.32})$$

can have nonvanishing cohomology but also determines the degree q , grade g and by definition the trace order r , where cohomology can be nontrivial. We see that for each $\mathcal{C}(\mathbf{A}, \sigma_-; \mathbf{X}, q + g, r)$ that may have nontrivial cohomology there is a unique place in terms

of q and g where it can happen. Due to the degeneracy, the multiplicity of \mathbf{X} in $H_{g,r}^q$ can be greater than one. Making use of the fact that Euler characteristic of $\mathcal{C}(\mathbf{A}, \boldsymbol{\sigma}_-; \mathbf{X}, q+g, r)$ can be computed either as $\chi = \sum_q (-)^q \dim(H^q)$ or $\chi' = \sum_q (-)^q \dim(V_q)$, we can determine the multiplicity since all but one summands in χ are equal to zero - in this case the Euler characteristic determines the dimension of cohomology of $H(\mathbf{A}, \boldsymbol{\sigma}_-; \mathbf{X}, q+g, r)$ modulo sign factor, of course. It presents no difficulty to compute χ' , we just count the multiplicity of \mathbf{X} appearing as the trace of order r in $\mathbf{A}_g \otimes \mathbb{Y}[q]$ for different q and g while keeping $q+g$ fixed.

so(d)-tensor product of restricted representations. Let us address the question of which Young diagrams can appear in the tensor product $\mathbf{A}_{\{k_1, \dots, k_N\}} \otimes \mathbb{Y}[q]$ with $\mathbf{A}_{\{k_1, \dots, k_N\}} \in \text{Res}_{so(d)}^{so(d+1)} \mathbf{A}$ and $\mathbb{Y}[q]$ being a one-column diagram of height q . It is useful to define $\Delta_i = s_i - s_{i+1}$. Applying tensor product rules, given in appendix A, one gets

$$\mathbf{A}_{\{k_1, \dots, k_N\}} \otimes_{so(n)} \mathbb{Y}[q] = \bigoplus_{\{\alpha_j, \beta_i, \gamma_i\}} N_{\{k_1, \dots, k_N\}}^{\{\alpha_j, \beta_i\}} \mathbf{A}_{\{k_1, \dots, k_N\}}^{\{\alpha_j, \beta_i, \gamma_i\}} \tag{B.33}$$

$$\begin{cases} \alpha_i + \beta_i \leq p_i - 1, \text{ for } i = 1 \dots N, \\ \gamma_i \in [-1, 0, \dots, \Delta_i + 1] \end{cases}$$

(B.34)

The explicit formula for the multiplicity $N_{\{k_1, \dots, k_N\}}^{\{\alpha_j, \beta_i\}}$ can be written in terms of integer partitions. Important is the very 'geometry' of $\mathbf{A}_{\{k_1, \dots, k_N\}}^{\{\alpha_j, \beta_i, \gamma_i\}}$, e.g. $\gamma_i \in [-1, \Delta_i + 1]$ if $i < N$ and $\gamma_i \in [0, \Delta_i + 1]$ if $i = N$, $\gamma_i = \Delta_i + 1$ implies $\alpha_i = p_i - 1$, etc.

We choose some $\mathbf{X} = \mathbf{A}_{\{k_1, \dots, k_N\}}^{\{\alpha_j, \beta_i, \gamma_i\}}$ and do not fix any $\mathbf{A}_{\{k_1, \dots, k_N\}}$, counting contributions of all from \mathbf{A}_g that lead to \mathbf{X} in the tensor product.

The multiplicity $N_{\{k_1, \dots, k_N\}}^{\{\alpha_j, \beta_i\}}$ can be greater than one inasmuch as there are in general many ways to remove cells (take traces) and then add them back in order to get the same diagram \mathbf{X} . The source of cells is $\mathbb{Y}[q]$, of course. Let us refer to a cell in \mathbf{X} , that can be obtained by removing a number of cells, including this one, and then adding the same number of cells back as to a vacancy. More than one cell may be needed because it can be that to remove some cell, according to the tensor product rules, one has first to remove a

number of adjacent cells. At least two cells are required to fill a vacancy, one to take trace and one to restore the original cell. The vacancies correspond to ϵ_i and to the rightmost cell in γ_i provided $\gamma_i \in [0, \Delta_i]$. Note that there is only one way to get the cells corresponding to α_i and $\gamma_k = \Delta_k + 1$ (or β_i and $\gamma_k = -1$), these cells have to be added (or removed), not to mention the cells in the 'interior' of \mathbf{X} that are not affected by the tensor product rules. It is convenient to single the constant parts Q, R and G out of q, r and g that do not vary when passing from one $\mathbf{A}_{\{k_1, \dots, k_N\}}$ to another,

$$\begin{aligned}
 q &= Q + q', & Q &= \sum_{j=1}^{N+1} \alpha_j + \sum_{i=1}^N \beta_i + N_{-1} + N_{\Delta+1}, & q' &= N_{\gamma > k} + N_{\gamma < k} + 2\rho, \\
 g &= G + g', & G &= \sum_{i=1}^{i=N} (\gamma_i + \delta_{\gamma_i, -1} - \delta_{\gamma_i, \Delta_i + 1}), & g' &= N_{\gamma < k} - N_{\gamma > k} \\
 r &= R + r', & R &= \sum_{i=1}^N \beta_i + N_{-1}, & r' &= N_{\gamma > k} + \rho
 \end{aligned}$$

$$\begin{aligned}
 N_{-1} &= \#\{i : \gamma_i = -1\}, & N_{\Delta+1} &= \#\{i : \gamma_i = \Delta_i + 1\}, \\
 N_{\gamma < k} &= \#\{i : \gamma_i < k_i, \gamma_i \neq -1, \Delta_i + 1\}, & N_{\gamma > k} &= \#\{i : \gamma_i > k_i, \gamma_i \neq -1, \Delta_i + 1\},
 \end{aligned}$$

and $\rho, N_{\gamma < k}, N_{\gamma > k}$ are the parts that depend on a particular $\mathbf{A}_{\{k_1, \dots, k_N\}}$. For example, $N_{\gamma < k}$ is equal to the number of those k_i in $\mathbf{A}_{\{k_1, \dots, k_N\}}$ that are greater than γ_i . Note that by virtue of the tensor product rules $\gamma_i = k_i, k_i \pm 1$. It is ρ that governs the multiplicity, ρ is equal to the number of vacancies to be filled. Therefore, the multiplicity of \mathbf{X} in $\mathbf{A}_{\{k_1, \dots, k_N\}} \otimes \mathbb{Y}[q]$ is given by the number of partitions of ρ among the vacancies given by ϵ_i and those γ_i that equal k_i (modulo certain subtleties to be considered below).

Euler characteristic. Let us proceed to the computation of the Euler characteristic. It makes no difference to compute it only for \mathbf{X} of special form dictated by $\mathfrak{h}\mathcal{R}(\mathbf{A}, \partial)$ or in the general case, so we do it for arbitrary \mathbf{X} . Firstly, we construct for \mathbf{X} the generating function $F(z, t)$ such that the coefficient of $z^g t^{q'}$ is equal to the number of ways to get \mathbf{X} in $\mathbf{A}_{\{k_1, \dots, k_N\}} \otimes \mathbb{Y}[Q + q']$ with $\sum_i k_i = G + g'$. Thus, z counts the excess over the base level G , and t counts the number q' of cells needed to get \mathbf{X} .

It is important for the computations to be simple that the whole diagram \mathbf{X} can be cut into pieces such that the generating function can be first constructed for each of the pieces and then the total $F(z, t)$ is just the product of generating functions over the pieces. In the table 1 below we collect all different types of such pieces together with generating functions, where $f_\epsilon(t) = (1 - t^{2\epsilon+2})/(1 - t^2)$. It is easy to see that the generating function for the Euler characteristics of $\mathcal{C}(\mathbf{A}, \boldsymbol{\sigma}_-; \mathbf{X}, q + g, r)$ is just $F(-t, t)$, where the degree of t is equal to $2r'$ and the coefficient of $t^{2r'}$ up to a sign equals the Euler characteristic of $\mathcal{C}(\mathbf{A}, \boldsymbol{\sigma}_-; \mathbf{X}, q + g, R + r')$. Note that $F(-t, t)$ can be a polynomial rather than a monomial because the same diagram \mathbf{X} can appear in the tensor product at different values of r .

At the table 1 below, we collected all possible types of pieces, into which the diagram \mathbf{X} is decomposed. Looking at the table, we see that \mathbf{X} such that at least one of γ_i does not

take an extremal value $\{-1, 0, \Delta_i, \Delta_i + 1\}$ results in vanishing Euler characteristic, which exactly correspond to the fact the cohomology of $\mathfrak{hR}^{s_i, s_i-1}$ is concentrated in the lowest and highest grade. Surprising is that $\chi = 0$ if the subsequence $\gamma_{i-1} = 0, \gamma_i = \Delta_i$ occurs in \mathbf{X} and hence the piece no. 2 can occur only once in \mathbf{X} .

Given \mathbf{X} , depending on whether the piece no. 2 is present or not, the function $F(-t, t)$ can have one of the two forms

$$I : \quad F(-t, t) = \prod_{i:\gamma_i=\Delta_i} t^{2(\epsilon_i+1)}, \quad (\text{B.35})$$

$$II : \quad F(-t, t) = \prod_{i:\gamma_i=\Delta_i} t^{2(\epsilon_i+1)}(1 - t^{2\epsilon'+2}), \quad (\text{B.36})$$

where ϵ' is the ϵ_i that correspond to the piece no. 2 from the table.

I. In the first case we have all γ_i taking one of the maximal values $\{\Delta_i, \Delta_i + 1\}$, i.e. $g = s_1$. The diagram \mathbf{X} consists of blocks of the form

$$\mathcal{BC}_j \sim \pi_{\mathbb{Y}\{s_j, p_j\}} \pi_{\Lambda} \pi_{\mathbf{cr}} \left[\underbrace{\overbrace{\mathcal{B}^1 \otimes \dots \otimes \mathcal{B}^1}^{\alpha_j} \otimes \underbrace{\overbrace{\mathcal{C}^2 \otimes \dots \otimes \mathcal{C}^2}^{\epsilon_j+1}}}_{p_j} \right]. \quad (\text{B.37})$$

That $r' = \sum_i (\epsilon_i + 1)$ takes the maximal value implies that no \mathcal{A}^0 can occur, only \mathcal{C}^2 can. The first case correspond to $q - r > n$ and to the maximal grade so that none of the representatives has a dual pair.

II. In the second case $F(-t, t)$ consists of two monomials, each corresponding to a cohomology class with the same \mathbf{X} but with different r' , the difference is $\epsilon' + 1$. The second class is obtained via the duality map. Here we refer to figure 3 illustrating both classes. The representatives are given by diagrams of the form

$$\pi_{\mathbf{tot}} [\mathcal{BC}_1 \otimes \dots \otimes \mathcal{BC}_{k-1} \otimes \mathcal{BAD}_k \otimes \mathcal{AD}_{k+1} \otimes \dots \otimes \mathcal{AD}_N^0], \quad (\text{B.38})$$

where

$$\begin{aligned} \mathcal{BAD}_k &\sim \pi_{\mathbb{Y}\{s_k, p_k\}} \pi_{\Lambda} \pi_{\mathbf{cr}} \left[\underbrace{\overbrace{\mathcal{B}^1 \otimes \dots \otimes \mathcal{B}^1}^{\alpha_k} \otimes \underbrace{\overbrace{\mathcal{A}^0 \otimes \dots \otimes \mathcal{A}^0}^{\epsilon_k} \otimes \overbrace{\mathcal{D}^1 \otimes \dots \otimes \mathcal{D}^1}^{\beta_k} \otimes \mathcal{A}^0}_{p_k}} \right], \\ \mathcal{AD}_j &\sim \pi_{\mathbb{Y}\{s_j, p_j\}} \pi_{\Lambda} \pi_{\mathbf{cr}} \left[\underbrace{\overbrace{\mathcal{A}^0 \otimes \dots \otimes \mathcal{A}^0}^{\epsilon_j} \otimes \overbrace{\mathcal{D}^1 \otimes \dots \otimes \mathcal{D}^1}^{\beta_j} \otimes \mathcal{A}^0}_{p_j} \right], \end{aligned} \quad (\text{B.39})$$

and the last block \mathcal{AD}_N^0 ends with \mathcal{O}^0 instead of \mathcal{A}^0 . The integer k corresponds to the block that has the form of the piece no. 2, i.e. $\epsilon_k = \epsilon'$. Thus, all γ_j in the range $j = 1, \dots, k - 1$ take one of the maximal values $\{\Delta_j, \Delta_j + 1\}$, the rest of γ_i with $i = k, \dots, N$ take one of the minimal values $\{-1, 0\}$.

The representative of the second cohomology class, which is dual to the first, has the form

$$\pi_{\mathbf{tot}} [\mathcal{BC}_1 \otimes \dots \otimes \mathcal{BC}_{k-1} \otimes \mathcal{BCD}_k \otimes \mathcal{AD}_{k+1} \otimes \dots \otimes \mathcal{AD}_N^1], \quad (\text{B.40})$$

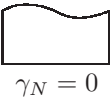
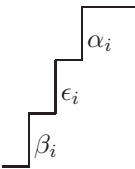
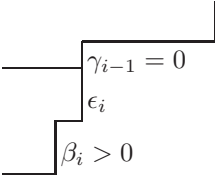
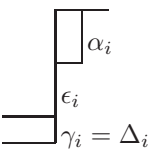
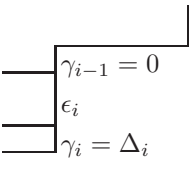
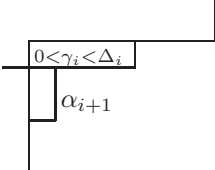
	illustration	$F(z, t)$	$F(-t, t)$	description
1	 $\gamma_N = 0$	$1 + zt$	$1 - t^2$	The last block. There are two ways to get $\gamma_N = 0$: (1) take a diagram with $k_N = 0$ and do nothing; (2) take a diagram with $k_N = 1$ (z), and then take a trace (t)
2		$f_{\epsilon_i}(t)$	$f_{\epsilon_i}(t)$	A group of ϵ_i 'isolated' vacancies. One can remove k cells and then add them back with any $k \in [0, \epsilon_i]$, which yields $1 + t^2 + \dots + t^{2\epsilon_i}$
3		$f_{\epsilon_i+1}(t) + zt f_{\epsilon_i}(t)$	1	A group of ϵ_i vacancies that are not 'isolated', being linked to $\gamma_i = 0$. If $k_i = 0$ then ϵ_i effectively increases to $\epsilon_i + 1$; if $k_i = 1$, i.e. $g' = +1$, then this one extra cell must be removed
4		$f_{\epsilon_i+1}(t) + z^{-1}t f_{\epsilon_i}(t)$	$t^{2(\epsilon_i+1)}$	A group of ϵ_i vacancies is linked to $\gamma_i = \Delta_i$. If $k_i = \Delta_i$ we have f_{ϵ_i+1} ; in the second case of $k_i = \Delta_i - 1$, i.e. $g' = -1$, one extra cell must be added
5		$f_{\epsilon_i+2}(t) + t^2 f_{\epsilon_i}(t) + f_{\epsilon_i+1}(t)(z + z^{-1})t$	0	This case include parts of the previous two cases; if $k_{i-1} = 0$ and $k_i = \Delta_i$ then the 'effective' ϵ_i is equal to $\epsilon_i + 2$
6		$1 + t^2 + t(z + z^{-1})$	0	In this cases γ_i does not take extreme values. If $k_i = \gamma_i$ then we can either do nothing or remove one cell and then add it back; if $k_i = \gamma_i \pm 1$ then one cell must be added (removed)

Table 7. Independent pieces constituting the diagram $\mathbf{A}_{\{k_1, \dots, k_N\}}^{\{\alpha_j, \beta_i, \gamma_i\}}$, together with generating functions and Euler characteristics

$$BCD_k \sim \pi_{\mathbb{Y}\{s_k, p_k\}} \pi_{\Lambda} \pi_{\mathbf{cr}} \left[\underbrace{\overbrace{\mathcal{B}^1 \otimes \dots \otimes \mathcal{B}^1}^{\alpha_k} \otimes \overbrace{\mathcal{C}^2 \otimes \dots \otimes \mathcal{C}^2}^{\epsilon_k} \otimes \overbrace{\mathcal{D}^1 \otimes \dots \otimes \mathcal{D}^1}^{\beta_k}}_{p_k} \right], \quad (\text{B.41})$$

and the last block \mathcal{AD}_N^1 ends with \mathcal{O}^1 instead of \mathcal{A}^0 . The degree q , grade g and trace order r are shifted by $2\epsilon' + 1$, 1 and $\epsilon' + 1$, respectively.

In conclusion, let us note that despite the possibility of great degeneracy mentioned at the beginning, the projectors somehow remove degeneracy so that the multiplicity of any \mathbf{X} in $H_{g,r}^q$ is either zero or one. To be strict, only the shape of (B.38) and (B.40) is relevant, the same diagram can be obtained in many different ways generally (we can replace some \mathcal{C}^2 with \mathcal{A}^0 in blocks $1, \dots, k$, and then make the same number of inverse replacements in blocks $k + 1, \dots, N$). What has been proved is that the multiplicity of \mathbf{X} determined by (B.38) and (B.40) is equal to one. Nevertheless, it can be shown that the suggested representatives (B.38) and (B.40) are indeed not exact. It is not hard to see that the answer just obtained coincides with that in terms of $\text{mspr}(\mathbf{A}, q, r)$ given in the second theorem of section 3.7.

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