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Sequences of dipole black rings and Kaluza-Klein bubbles

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Abstract: We construct new exact solutions to 5D Einstein-Maxwell equations describing sequences of Kaluza-Klein bubbles and dipole black rings. The solutions are generated by 2-soliton transformations from vacuum black ring - bubble sequences. The properties of the solutions are investigated. We also derive the Smarr-like relations and the mass and tension first laws in the general case for such configurations of Kaluza-Klein bubbles and dipole black rings. The novel moment is the appearance of the magnetic flux in the Smarr-like relations and the first laws.

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1 Introduction

Higher dimensional gravity and especially black holes in higher dimensional spacetimes became established and very active area of research. Higher dimensional black holes exhibit very interesting properties and features some of which are completely absent in four dimensions. Of particular interest are the black holes in spacetimes with compact extra dimensions (with circle topology in most cases) known as Kaluza-Klein black holes. More precisely, a five dimensional spacetime $\mathcal{M}$ (the case we consider here) is called Kaluza-Klein spacetime if it is asymptotically $\mathcal{M}^4 \times S^1$ where $\mathcal{M}^4$ is the 4-dimensional Minkowski spacetime. Very recently the uniqueness theorem for vacuum Kaluza-Klein black holes was established in [1]. This theorem gives complete classification of the possible horizon topologies and classification of the black solutions on the basis of the so-called interval (rod)[1]

[1]With regard to the general concept of rod structure we refer the reader to [2]. More precise mathematical definition of the rod structure (the so called interval structure) can be found in [1].

structure. Some exact Kaluza-Klein black hole solutions[2] have also been constructed [2]–[7]. Among them are the solutions describing sequences of static vacuum black holes and bubbles [3]–[4] which are of particular interest in the context of the present paper. We also refer the reader to the review article [17] where the Kaluza-Klein black holes are considered from different perspectives. Nevertheless, the known exact Kaluza-Klein black hole solutions are far from being exhaustive. The thermodynamics of Kaluza-Klein black holes in the presence of Maxwell field is also interesting to be studied. Especially in the case of black rings the thermodynamics exhibits a novel feature - the (local) dipole charge appears together with its corresponding potential in the first law [19]. Moreover, new terms related to the magnetic flux appear in the Smarr-like relations and the first laws for Kaluza-Klein black holes with Maxwell field along the compact dimension [20].

In the present paper we are dealing with sequences of 5D Kaluza-Klein bubbles and black rings in the presence of self-gravitating Maxwell field. While in the static vacuum case the construction of such configurations can be done in relatively simple way by solving linear equations, in the presence of self-gravitating Maxwell field, the construction of such configurations is much more difficult since we are forced to solve nonlinear equations. In this paper we construct new exact solutions to the 5D Einstein-Maxwell gravity describing sequences of dipole black rings and Kaluza-Klein bubbles. The solutions are generated by 2-soliton transformations from vacuum black ring - bubble configurations. The basic physical quantities characterizing the new solutions are computed. We also derive the Smarr-like relations and the mass and tension first laws for such configurations of dipole black rings and Kaluza-Klein bubbles in the general case.

2 Solution generating method and the exact solutions

2.1 Solution generating method

In five dimensions the Einstein-Maxwell equations read

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}\left(F_{\mu\lambda}F_{\nu}^{\ \lambda} - \frac{1}{6}F_{\sigma\lambda}F^{\sigma\lambda}g_{\mu\nu}\right), \\ \nabla_{\mu}F^{\mu\nu} &= \nabla_{[\mu}F_{\nu\lambda]} = 0. \end{aligned} \tag{2.1}$$

In this paper we consider 5D EM gravity in spacetimes with the symmetry group $R \times \mathrm{U}(1)^2$ generated by the commuting Killing fields ξ, ζ and η. Here ξ is the asymptotically timelike Killing field and ζ and η are the axial Killing fields, respectively. The Killing field η will be associated with the compact dimension. We also assume that all the Killing fields are hypersurface orthogonal. In this case, using adapted coordinates in which $\xi = \partial/\partial t$, $\zeta = \partial/\partial\psi$ and $\eta = \partial/\partial\phi$, the 5D spacetime metric can be written in the form

$$ds^2 = -e^{2\chi-u}dt^2 + e^{-2\chi-u}\rho^2 d\psi^2 + e^{-2\chi-u}e^{2\Gamma}\left(d\rho^2 + dz^2\right) + e^{2u}d\phi^2 \tag{2.2}$$

where all the metric functions depend on the canonical coordinates ρ and z only.

[2]It is worth mentioning that there are interesting Kaluza-Klein black hole solutions which have an asymptotic different from the standard one $\mathcal{M}^4 \times S^1$. For explicit examples we refer the reader to [8]–[16].

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For the electromagnetic field we impose the following conditions

$$\mathcal{L}_\xi F = \mathcal{L}_\zeta F = \mathcal{L}_\eta F = 0, \qquad (2.3)$$
$$i_\xi F = i_\zeta F = i_\eta \star F = 0,$$

where $\star$ is the Hodge dual, $\mathcal{L}_X$ denotes the Lie derivative along the vector field X and i_X is the interior product of the vector field X with an arbitrary form. From a local point of view these conditions mean that the gauge potential[3] A has the local form $A = A_\phi d\phi$.

The 1-form $i_\eta F$ is invariant under the spacetime symmetries and therefore can be considered as 1-form on the factor space $\hat{\mathcal{M}} = \mathcal{M}/R \times \mathrm{U}(1)^2$. Since the factor space is simply connected [1] and $i_\eta F$ is closed ($d i_\eta F = 0$) there exists a globally well-defined potential λ such that

$$i_\eta F = -d\lambda. \qquad (2.4)$$

It is worth mentioning that locally we have $\lambda = A_\phi$ up to a constant.

Further we introduce the complex Ernst potential $\mathcal{E}$ defined by

$$\mathcal{E} = e^u + \frac{i}{\sqrt{3}}\lambda. \qquad (2.5)$$

With the help of the Ernst potentials the dimensionally reduced 5D Einstein-Maxwell equations can be written in the following form

$$\begin{aligned}
(\mathcal{E}+\mathcal{E}^*)\left(\partial_\rho^2\mathcal{E} + \rho^{-1}\partial_\rho\mathcal{E} + \partial_z^2\mathcal{E}\right) &= 2\left(\partial_\rho\mathcal{E}\partial_\rho\mathcal{E} + \partial_z\mathcal{E}\partial_z\mathcal{E}\right), \\
\partial_\rho^2\chi + \rho^{-1}\partial_\rho\chi + \partial_z^2\chi &= 0, \\
\rho^{-1}\partial_\rho\Gamma &= (\partial_\rho\chi)^2 - (\partial_z\chi)^2 + \frac{3}{(\mathcal{E}+\mathcal{E}^*)^2}\left(\partial_\rho\mathcal{E}\partial_\rho\mathcal{E}^* - \partial_z\mathcal{E}\partial_z\mathcal{E}^*\right), \\
\rho^{-1}\partial_z\Gamma &= 2\partial_\rho\chi\partial_z\chi + \frac{6}{(\mathcal{E}+\mathcal{E}^*)^2}\partial_\rho\mathcal{E}\partial_z\mathcal{E}^*.
\end{aligned} \qquad (2.6)$$

The consistency conditions for the last two equations in (2.6), i.e. the equations for the metric function Γ, are guaranteed by the first two equations in (2.6).

In this way we reduced the problem of solving the 5D EM equations to two effective 4D problems, i.e. two Ernst equations. The central and most difficult task is to solve the nonlinear Ernst equation. Here we will not discuss in detail the methods for solving the Ernst equation. Instead we shall present the working formulas we need. Details can be found in [18].

Let us consider a solution to the vacuum 5D Einstein equations

$$ds_E^2 = g_{00}^E dt^2 + g_{\psi\psi}^E d\psi^2 + g_{\rho\rho}^E(d\rho^2 + dz^2) + g_{\phi\phi}^E d\phi^2 \qquad (2.7)$$

with metric function $g_{\phi\phi}^E$ given by

$$g_{\phi\phi}^E = e^{2u_0} = \prod_{i=1}^{N}\left(e^{2\tilde{U}_{\nu_i}}\right)^{\epsilon_i}, \qquad (2.8)$$

[3]In the presence of dipole (magnetic) charges the gauge potential is not globally well-defined [19].

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where

$$e^{2\tilde{U}_{\nu_i}} = R_{\nu_i} + \zeta_{\nu_i} = \sqrt{\rho^2 + \zeta_{\nu_i}^2} + \zeta_{\nu_i} = \sqrt{\rho^2 + (z-\nu_i)^2} + (z-\nu_i) \tag{2.9}$$

and ν_i and ϵ_i are constants.

The 2-soliton transformation generates the following solution to the 5D Einstein-Maxwell equations from the vacuum solution (2.7)

$$\begin{aligned} ds^2 &= \frac{g_{00}^E}{W}dt^2 + \frac{g_{\psi\psi}^E}{W}d\psi^2 + \frac{\mathcal{Y}^3}{W}g_{\rho\rho}^E(d\rho^2+dz^2) + W^2 g_{\phi\phi}^E d\phi^2, \\ \lambda &= 4\sqrt{3}\Delta k e^{u_0}\frac{[R_{k_1}a(1+b^2)+R_{k_2}b(1+a^2)]}{W_2} + \lambda_0, \end{aligned} \tag{2.10}$$

where k_1, k_2 ($\Delta k = k_1 - k_2$) and λ_0 are constants. Without loss of generality we put $\lambda_0 = 0$. The functions included in (2.10) are presented below. The functions a and b are given by

$$\begin{aligned} a &= \alpha \prod_{i=1}^{N}\left(\frac{e^{2U_{k_1}} + e^{2\tilde{U}_{\nu_i}}}{e^{\tilde{U}_{\nu_i}}}\right)^{\epsilon_i}, \\ b &= \beta \prod_{i=1}^{N}\left(\frac{e^{2U_{k_2}} + e^{2\tilde{U}_{\nu_i}}}{e^{\tilde{U}_{\nu_i}}}\right)^{-\epsilon_i}, \end{aligned}$$

where α and β are constants and

$$e^{2U_{k_i}} = R_{k_i} - \zeta_{k_i} = \sqrt{\rho^2 + (z-k_i)^2} - (z-k_i). \tag{2.11}$$

The function W is presented in the form

$$W = \frac{W_1}{W_2} \tag{2.12}$$

where

$$\begin{aligned} W_1 &= \left[(R_{k_1}+R_{k_2})^2 - (\Delta k)^2\right](1+ab)^2 + \left[(R_{k_1}-R_{k_2})^2 - (\Delta k)^2\right](a-b)^2, \\ W_2 &= \left[(R_{k_1}+R_{k_2}+\Delta k) + (R_{k_1}+R_{k_2}-\Delta k)ab\right]^2 \\ &\quad + \left[(R_{k_1}-R_{k_2}-\Delta k)a - (R_{k_1}-R_{k_2}+\Delta k)b\right]^2. \end{aligned} \tag{2.13}$$

For the function $\mathcal{Y}$ we have

$$\mathcal{Y} = \mathcal{Y}_0 \frac{W_1}{(R_{k_1}+R_{k_2})^2 - (\Delta k)^2}e^{2h} \tag{2.14}$$

where $\mathcal{Y}_0$ is a constant and

$$h = \gamma_{k_1,k_1} - 2\gamma_{k_1,k_2} + \gamma_{k_2,k_2} + \sum_i^N \epsilon_i\left(\gamma_{k_1,\nu_i} - \gamma_{k_2,\nu_i}\right), \tag{2.15}$$

$$\gamma_{k,l} = \frac{1}{2}\tilde{U}_k + \frac{1}{2}\tilde{U}_l - \frac{1}{4}\ln[R_k R_l + (z-k)(z-l) + \rho^2]. \tag{2.16}$$

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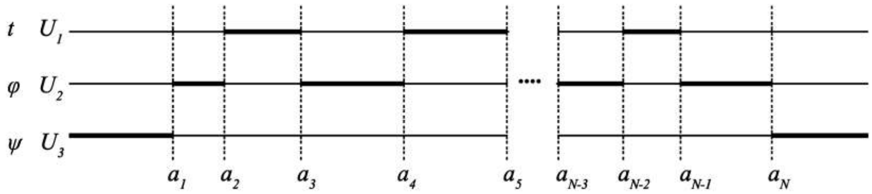

Figure 1. Rod structure of the bubble - black ring sequence

Before closing this subsection we should explain the following. The 2-soliton transformation involves two parameters k_1 and k_2 but one of them can be absorbed by a shift $z \to z + constant$ under which the "electromagnetic part" of the field equations is invariant. We prefer to keep using the two parameters k_1 and k_2 but those readers who feel unconformable with these parameters may think that parameters k_1 and k_2 are appropriate functions only of Δk and examples will be given below (see eq. (3.2)).

2.2 The exact solutions

In order to generate the solutions to the 5D Einstein-Maxwell equations describing sequence of dipole black rings and Kaluza-Klein (KK) bubbles we take as a seed the vacuum solution describing neutral black rings[4] on KK bubbles constructed in [4]. More precisely this solution describes a sequence of q black rings and $p = q + 1$ KK bubbles in the form *bubble – black ring – bubble – black ring – ... – bubble – black ring – bubble* with rod structure shown on figure (1). There are q finite rods $[a_2, a_3], [a_4, a_5], ..., [a_{N-2}, a_{N-1}]$ corresponding to the black ring horizons and p finite rods $[a_1, a_2], [a_3, a_4], ..., [a_{N-1}, a_N]$ corresponding to the KK bubbles, i.e. the axes of the Killing vector $\partial/\partial\phi$. The semi-infinite rods $[-\infty, a_1]$ and $[a_N, +\infty]$ describe the axes of the Killing vector $\partial/\partial\psi$. The number N is even and is given by $N = 2(q + 1)$. For simplicity we will confine here to the particular case of one black ring surrounded by two KK bubbles, i.e. we will consider $q = 1$ and $p = 2$. The generalization to sequences consisting of arbitrary number of bubbles and black rings is straightforward and it is presented in appendix A. The explicit analytical form of the solution in the specified case is the following

$$g^E_{00} = -\frac{(R_{a_2} - \zeta_{a_2})}{(R_{a_3} - \zeta_{a_3})} = -\prod_{i=2}^{3} \left(e^{2U_{a_i}}\right)^{(-1)^i}, \tag{2.17}$$

$$g^E_{\phi\phi} = \frac{(R_{a_1} - \zeta_{a_1})(R_{a_3} - \zeta_{a_3})}{(R_{a_2} - \zeta_{a_2})(R_{a_4} - \zeta_{a_4})} = \prod_{i=1}^{4} \left(e^{2U_{a_i}}\right)^{(-1)^{i+1}}, \tag{2.18}$$

$$g^E_{\psi\psi} = (R_{a_1} + \zeta_{a_1})(R_{a_4} - \zeta_{a_4}), \tag{2.19}$$

$$g^E_{\rho\rho} = \frac{Y_{14}Y_{23}}{4R_{a_1}R_{a_2}R_{a_3}R_{a_4}} \sqrt{\frac{Y_{12}Y_{34}}{Y_{24}Y_{13}}} \frac{R_{a_4} - \zeta_{a_4}}{R_{a_1} - \zeta_{a_1}}, \tag{2.20}$$

[4]It should be noted that the topology of the black rings is $S^2 \times S^1$ where S^1 is not topologically supported, i.e. S^1 is associated with the orbits of ζ and is not the Kaluza-Klein circle. This can be seen from the rod diagram (1).

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where

$$Y_{ij} = R_{a_i} R_{a_j} + \zeta_{a_i} \zeta_{a_j} + \rho^2. \tag{2.21}$$

Taking into account that

$$e^{2U_{a_i}} = \rho^2 e^{-2\tilde{U}_{a_i}} \tag{2.22}$$

and $\sum_{i=1}^{N} (-1)^{i+1} = 0$ for even N, it is not difficult to see that

$$g_{\phi\phi}^{E} = \prod_{i=1}^{4} \left(e^{2\tilde{U}_{a_i}} \right)^{(-1)^i}. \tag{2.23}$$

For the seed solution under consideration we find

$$a = \alpha \prod_{i=1}^{4} \left(\frac{e^{2U_{k_1}} + e^{2\tilde{U}_{a_i}}}{e^{\tilde{U}_{a_i}}} \right)^{(-1)^i}, \tag{2.24}$$

$$b = \beta \prod_{i=1}^{4} \left(\frac{e^{2U_{k_2}} + e^{2\tilde{U}_{a_i}}}{e^{\tilde{U}_{a_i}}} \right)^{(-1)^{i+1}}. \tag{2.25}$$

3 Analysis of the solution

The investigation of the solution shows that the functions W, $\mathcal{Y}$ and λ, under some conditions discussed below, are regular everywhere for the following ordering of the parameters

$$a_{2m-1} < k_2 < k_1 < a_{2m} \tag{3.1}$$

where $m = 1, 2$. In other words the parameters k_1 and k_2 must lie on any of the bubble rods.

A convenient choice for the parameters k_1 and k_2 is the following

$$\begin{aligned} k_1 &= \frac{a_{2m} + a_{2m-1}}{2} + \frac{1}{2}\Delta k, \\ k_2 &= \frac{a_{2m} + a_{2m-1}}{2} - \frac{1}{2}\Delta k. \end{aligned} \tag{3.2}$$

There are potential singularities in the functions W and $\mathcal{Y}$ for $z = k_1$ and $z = k_2$. In order to eliminate these potential singularities we must impose

$$\alpha^2 = \prod_{i=1}^{2m-1} (k_1 - a_i)^{(-1)^{i+1}} \prod_{j=2m}^{4} (a_j - k_1)^{(-1)^{j+1}}, \tag{3.3}$$

$$\beta^2 = \prod_{i=1}^{2m-1} (k_2 - a_i)^{(-1)^{i}} \prod_{j=2m}^{4} (a_j - k_2)^{(-1)^{j}}. \tag{3.4}$$

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3.1 Asymptotics

In order to study the asymptotic behavior of the solution we introduce the asymptotic coordinates r and θ defined by

$$\rho = r\sin\theta, \;\; z = r\cos\theta.$$

Then in the asymptotic limit we find

$$g^E_{00} \approx -1 + \frac{1}{r}\sum_{s=1}^{q=1}(a_{2s+1} - a_{2s}) \; = -1 + \frac{c^E_t}{r}, \tag{3.5}$$

$$g^E_{\psi\psi} \approx r^2\sin^2\theta, \tag{3.6}$$

$$g^E_{\phi\phi} \approx 1 - \frac{1}{r}\sum_{s=1}^{p=2}(a_{2s} - a_{2s-1}) \; = 1 + \frac{c^E_\phi}{r}, \tag{3.7}$$

$$g^E_{\rho\rho} \approx 1, \tag{3.8}$$

$$W \approx 1 - \frac{1-\alpha\beta}{1+\alpha\beta}\frac{\Delta k}{r}, \tag{3.9}$$

$$\mathcal{Y} \approx \mathcal{Y}_0(1+\alpha\beta)^2, \tag{3.10}$$

$$\lambda \approx \sqrt{3}\frac{\alpha+\beta}{1+\alpha\beta}\frac{\Delta k}{r}. \tag{3.11}$$

In order for our solution to be asymptotically Kaluza — Klein we must impose

$$\mathcal{Y}_0 = \frac{1}{(1+\alpha\beta)^2}. \tag{3.12}$$

3.2 Balance conditions

Let us first consider the semi-infinite rods corresponding to the axes of the Killing vector $\partial/\partial\psi$. The regularity conditions then give the following period of ψ

$$\Delta\psi = 2\pi \lim_{\rho\to 0}\sqrt{\frac{\rho^2 g_{\rho\rho}}{g_{\psi\psi}}} = 2\pi. \tag{3.13}$$

For any bubble rod $[a_{2m-1}, a_{2m}], m = 1, 2$ corresponding to an axis of the Killing vector $\partial/\partial\phi$ the regularity condition gives

$$(\Delta\phi)_{Rod[a_{2m-1},a_{2m}]} = 2\pi \lim_{\rho\to 0}\sqrt{\frac{\rho^2 g_{\rho\rho}}{g_{\phi\phi}}} = \left(\frac{\mathcal{Y}}{W}\right)^{3/2}_{Rod[a_{2m-1},a_{2m}]}(\Delta\phi)^E_{Rod[a_{2m-1},a_{2m}]} \tag{3.14}$$

where

$$\begin{aligned}(\Delta\phi)^E_{Rod[a_1,a_2]} &= 4\pi(a_4 - a_1)\sqrt{\frac{a_2 - a_1}{a_3 - a_1}},\\ (\Delta\phi)^E_{Rod[a_3,a_4]} &= 4\pi(a_4 - a_1)\sqrt{\frac{a_4 - a_3}{a_4 - a_2}},\end{aligned} \tag{3.15}$$

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are the periods for the seed solution corresponding to the two bubble rods, and

$$\left(\frac{\mathcal{Y}}{W}\right)_{Rod[a_1,a_2]} = \left[\frac{1+\alpha\beta\left(\frac{k_2-a_1}{k_1-a_1}\right)}{1+\alpha\beta}\right]^2 \left(\frac{k_1-a_1}{k_2-a_1}\right),$$

$$\left(\frac{\mathcal{Y}}{W}\right)_{Rod[a_3,a_4]} = \left[\frac{1+\alpha\beta\left(\frac{a_4-k_1}{a_4-k_2}\right)}{1+\alpha\beta}\right]^2 \left(\frac{a_4-k_2}{a_4-k_1}\right).$$

Then we obtain two balance conditions

$$(\Delta\phi)_{Rod[a_{2m-1},a_{2m}]} = L, \quad m = 1, 2. \tag{3.16}$$

The parameters can be adjusted in appropriate way so that the balance conditions be satisfied.

3.3 Dipole charge

The dipole charge associated with the black ring is defined as

$$Q = \frac{1}{2\pi}\int_{S^2_{\mathcal{H}}} F \tag{3.17}$$

where $S^2_{\mathcal{H}}$ is the 2-sphere of the black ring horizon.

When we calculate the charge of the black ring horizon for our solution we have to consider two separate cases, i.e. when the parameters of the soliton transformation k_1 and k_2 belong to the first bubble rod $a_1 < k_i < a_2$ or to the second bubble rod $a_3 < k_i < a_4$. We find the following expressions

$$a_1 < k_i < a_2 \tag{3.18}$$

$$Q = \frac{L\sqrt{3}\beta\Delta k(a_3-a_2)(a_2-k_2)^{-1}(a_4-k_2)^{-1}\left[1+\alpha\beta\prod_{i=3}^{4}\left(\frac{a_i-k_1}{a_i-k_2}\right)^{(-1)^i}\right]}{2\pi\left[1+\alpha\beta\left(\frac{a_4-k_1}{a_4-k_2}\right)\right]\left[1+\alpha\beta\prod_{i=2}^{4}\left(\frac{a_i-k_1}{a_i-k_2}\right)^{(-1)^i}\right]},$$

$$a_3 < k_i < a_4 \tag{3.19}$$

$$Q = -\frac{L\sqrt{3}\alpha\Delta k(a_3-a_2)(k_1-a_1)^{-1}(k_1-a_3)^{-1}\left[1+\alpha\beta\prod_{i=1}^{2}\left(\frac{a_i-k_1}{a_i-k_2}\right)^{(-1)^i}\right]}{2\pi\left[1+\alpha\beta\left(\frac{k_2-a_1}{k_1-a_1}\right)\right]\left[1+\alpha\beta\prod_{i=1}^{3}\left(\frac{a_i-k_1}{a_i-k_2}\right)^{(-1)^i}\right]}.$$

3.4 Dipole potential and magnetic fluxes

If we consider the dual field H defined by $H = \star F$ and more precisely

$$i_\zeta i_\xi H = i_\zeta i_\xi \star F, \tag{3.20}$$

we can show (see the next section) that there exists a potential $\mathcal{B}$ such that

$$i_\zeta i_\xi H = d\mathcal{B}. \tag{3.21}$$

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In our case $\mathcal{B}$ can be given explicitly by the expression

$$\mathcal{B} = \sqrt{3}e^{-u_0}\frac{\hat{\omega}_{k_1k_2}}{W_1} + C_B, \tag{3.22}$$

where C_B is a constant. The constant C_B plays no essential role and we set it zero. The function $\omega_{k_1k_2}$ is given by [18]

$$\begin{aligned}\omega_{k_1k_2} =& [(R_{k_1}+R_{k_2})^2-(\Delta k)^2](1+ab)[(R_{k_1}-R_{k_2}+\Delta k)b+(R_{k_1}-R_{k_2}-\Delta k)a] \\ &+[(R_{k_1}-R_{k_2})^2-(\Delta k)^2](b-a)[(R_{k_1}+R_{k_2}+\Delta k)-(R_{k_1}+R_{k_2}-\Delta k)ab]. \end{aligned}\tag{3.23}$$

The asymptotic behaviour of the potential $\mathcal{B}$ is

$$\mathcal{B} \approx \frac{\sqrt{3}\Delta k}{1+\alpha\beta}\left[(1-\cos\theta)\beta - (1+\cos\theta)\alpha\right]. \tag{3.24}$$

Further, we need the value of the potential $\mathcal{B}$ on the black ring horizon and on the axes of the Killing field $\zeta = \partial/\partial\psi$. After some algebra we find

$$\mathcal{B}_{\mathcal{H}} = -\frac{2\sqrt{3}\alpha\Delta k\prod_{i=3}^{4}(a_i-k_1)^{(-1)^i}}{\left[1+\alpha\beta\prod_{i=3}^{4}\left(\frac{a_i-k_1}{a_i-k_2}\right)^{(-1)^i}\right]}, \qquad a_1 < k_i < a_2 \tag{3.25}$$

$$\mathcal{B}_{\mathcal{H}} = \frac{2\sqrt{3}\beta\Delta k\prod_{i=1}^{2}(k_2-a_i)^{(-1)^{i+1}}}{\left[1+\alpha\beta\prod_{i=1}^{2}\left(\frac{a_i-k_1}{a_i-k_2}\right)^{(-1)^i}\right]}, \qquad a_3 < k_i < a_4,$$

$$\mathcal{B}^+ = \mathcal{B}_{Rod[z_4,+\infty]} = -2\sqrt{3}\Delta k\frac{\alpha}{1+\alpha\beta}, \tag{3.26}$$

$$\mathcal{B}^- = \mathcal{B}_{Rod[-\infty,z_1]} = 2\sqrt{3}\Delta k\frac{\beta}{1+\alpha\beta}. \tag{3.27}$$

The magnetic fluxes Ψ^+ and Ψ^- are defined in the next section - see the discussion around eqs. (4.27) and (4.29). For our exact solution we find

$$\Psi^+ = L\frac{\sqrt{3}\Delta k\beta(a_4-k_2)^{-1}}{\left[1+\alpha\beta\frac{a_4-k_1}{a_4-k_2}\right]}, \tag{3.28}$$

$$\Psi^- = -L\frac{\sqrt{3}\Delta k\alpha(k_1-a_1)^{-1}}{\left[1+\alpha\beta\frac{k_2-a_1}{k_1-a_1}\right]}. \tag{3.29}$$

The quantities $\mathcal{B}_{\mathcal{H}}$, $\mathcal{B}^+$, $\mathcal{B}^-$, Ψ^+ and Ψ^- play important role in the Smarr-like relations and the mass and tension first laws as we will see in the next section.

3.5 Mass and tension

The ADM mass and the tension can be calculated from the asymptotic expansion of the metric

$$M = \frac{1}{4}L(2c_t^E - c_\phi^E) = \frac{L}{L^E}M^E, \tag{3.30}$$

$$\mathcal{T}L = \frac{1}{4}L(c_t^E - 2c_\phi^E) + \frac{3}{4}L\,\frac{1-\alpha\beta}{1+\alpha\beta}\Delta k = \mathcal{T}^E\frac{L}{L^E} + \frac{3}{4}L\,\frac{1-\alpha\beta}{1+\alpha\beta}\Delta k, \tag{3.31}$$

here $\mathcal{T}^E$ and L^E are the tension and the length of the Kaluza-Klein circle at infinity corresponding to the seed solution.

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3.6 Temperature and entropy

The temperature of the event horizon is given by

$$T = \frac{1}{2\pi}\lim_{\rho\to 0}\sqrt{\frac{-g_{tt}}{\rho^2 g_{\rho\rho}}}\,. \tag{3.32}$$

Applying this formula to our solution we find

$$T = \mathcal{Y}_{\mathcal{H}}^{-3/2}T^E, \tag{3.33}$$

where

$$\mathcal{Y}_{\mathcal{H}} = \left[\frac{1+\alpha\beta\prod_{i=3}^{4}\left(\frac{a_i-k_1}{a_i-k_2}\right)^{(-1)^i}}{1+\alpha\beta}\right]^2 \prod_{i=3}^{4}\left(\frac{a_i-k_2}{a_i-k_1}\right)^{(-1)^i}$$

and

$$T^E = \frac{1}{4\pi}\frac{\sqrt{a_4-a_2}\sqrt{a_3-a_1}}{(a_4-a_1)(a_3-a_2)} \tag{3.34}$$

is the temperature of the event horizon for the seed solution and the metric function $\mathcal{Y}$ is evaluated on the horizon rod $a_2 < z < a_3$.

Further we can find the black ring entropy.

$$S = \mathcal{Y}_{\mathcal{H}}^{3/2}S^E\left(\frac{L}{L^E}\right). \tag{3.35}$$

Again

$$S^E = \frac{L^E}{4T^E}(a_3-a_2) \tag{3.36}$$

is the entropy corresponding to the seed solution.

3.7 Mirror solutions

The solutions with k_i placed on different bubble rods are related by a discrete symmetry described below. For a given solution with parameters $a_1, a_2, a_3, a_4, k_1, k_2$ where $a_1 < k_i < a_2$, we can find a "mirror" solution which has the same mass and opposite charge. The parameters $a'_1, a'_2, a'_3, a'_4, k'_1, k'_2$ of this "mirror" solution are given by the transformations

$$\begin{aligned}
(k_2-a_1) &\to (a'_4-k'_1),\\
(a_2-k_1) &\to (k'_2-a'_3),\\
(a_4-a_3) &\to (a'_2-a'_1),\\
(a_3-a_2) &\to (a'_3-a'_2),\\
(k_1-k_2) &\to (k'_1-k'_2),
\end{aligned} \tag{3.37}$$

where $a'_3 < k'_i < a'_4$. The physical quantities of the mirror solution are given by

$$\begin{aligned}
&M' = M,\ L' = L,\ \mathcal{T}' = \mathcal{T},\ S' = S,\ T' = T,\\
&Q' = -Q,\ \Psi'^{+} = -\Psi^{-},\ \Psi'^{-} = -\Psi^{+}, \mathcal{B}'^{+} = -\mathcal{B}^{-},\ \mathcal{B}'^{-} = -\mathcal{B}^{+}.
\end{aligned} \tag{3.38}$$

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We can also consider a mirror solution of second kind. This solution is obtained from the mirror solution by reversing the sign of the electromagnetic potential, i.e. $\lambda'' = -\lambda' = -\lambda$. For the mirror solution of the second kind we have

$$\begin{aligned} M'' &= M,\ L'' = L,\ \mathcal{T}'' = \mathcal{T},\ S'' = S,\ T'' = T, \\ Q'' &= Q,\ \Psi''^{+} = \Psi^{-},\ \Psi''^{-} = \Psi^{+}, \mathcal{B}''^{+} = \mathcal{B}^{-},\ \mathcal{B}''^{-} = \mathcal{B}^{+}. \end{aligned} \tag{3.39}$$

3.8 Parameter counting

Our solutions are characterized by 4 parameters - the lengths of the three finite rods and Δk. For a given length L of the KK circle at infinity we have 2 constraints coming from the balance conditions. Therefore we are left with 2 parameters for the regular solutions. This means that the regular solutions are characterized by two independent parameters which can be chosen to be the mass M and the dipole charge Q. Let us note, however, that the solutions in the general case are not uniquely specified by the mass and the dipole charge. There are different solutions which can have the same mass and dipole charge — for example, a given solution and its mirror solution of second kind have the same mass and dipole charge.

The full classification of the KK black holes with dipole charges exceeds the scope of this paper. This question was briefly discussed in [20] where it was pointed out that the dipole KK black holes (with electromagnetic field along the compact dimension) can classified by the rod structure, the dipole charges and also by the magnetic flux(es). Detailed consideration of the classification will be presented elsewhere.

4 Smarr-like relations and first laws for the mass and the tension

In this section we derive the Smarr-like relations and first laws for the mass and the tension for the configurations under consideration. Our derivation will be done for the more general case of 5D Einstein-Maxwell-dilaton gravity given by the field equations

$$\begin{aligned} R_{\mu\nu} &= 2\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{2}e^{-2\gamma\varphi}\left(F_{\mu\sigma}F_\nu^{\ \sigma} - \frac{1}{6}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}\right), \\ \nabla_\mu\left(e^{-2\gamma\varphi}F^{\mu\nu}\right) &= 0, \qquad \nabla_{[\sigma}F_{\mu\nu]} = 0, \\ \nabla_\mu\nabla^\mu\varphi &= -\frac{\gamma}{8}e^{-2\gamma\varphi}F_{\sigma\lambda}F^{\sigma\lambda}, \end{aligned} \tag{4.1}$$

where $R_{\mu\nu}$ is the Ricci tensor for the spacetime metric $g_{\mu\nu}$, $F_{\mu\nu}$ is the Maxwell tensor, φ is the dilaton field and γ is the dilaton coupling parameter. For $\gamma = 0$ (and $\varphi = 0$) we obtain the 5D Einstein-Maxwell equations.

4.1 Smarr-like relations

In order to derive the Smarr-like relations we shall use the generalized Komar integrals [21] in the form presented in [20]. The generalized Komar integrals are given by

$$M = -\frac{L}{16\pi}\int_{S^2_\infty}\left[2i_\eta\star d\xi - i_\xi\star d\eta\right], \tag{4.2}$$

$$\mathcal{T} = -\frac{1}{16\pi}\int_{S^2_\infty}\left[i_\eta\star d\xi - 2i_\xi\star d\eta\right], \tag{4.3}$$

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where the integration is performed over the 2-dimensional sphere at the spatial infinity of $\mathcal{M}^4$.

The generalized Komar integrals allow us to define the intrinsic mass of each object in the configuration [7]. The intrinsic mass of each black hole is given by

$$M_i^{\mathcal{H}} = -\frac{L}{16\pi}\int_{\mathcal{H}_i}\left[2 i_\eta \star d\xi - i_\xi \star d\eta\right] \tag{4.4}$$

where $\mathcal{H}_i$ is the 2-dimensional surface which is an intersection of the i-th horizon with a constant t and ϕ hypersurface. Analogously the intrinsic mass of each bubble is

$$M_j^{\mathcal{B}} = -\frac{L}{16\pi}\int_{\mathcal{B}_j}\left[2 i_\eta \star d\xi - i_\xi \star d\eta\right]. \tag{4.5}$$

One can show that the intrinsic masses of the black holes and bubbles are given by

$$M_i^{\mathcal{H}} = \frac{1}{2}L l_i^{\mathcal{H}}, \quad M_j^{\mathcal{B}} = \frac{1}{4}L l_j^{\mathcal{B}}, \tag{4.6}$$

where $l_i^{\mathcal{H}}$ and $l_j^{\mathcal{B}}$ are the lengths of the horizon and bubble rods, respectively. It was also shown in [7] that

$$M_i^{\mathcal{H}} = \frac{1}{4\pi}\kappa_{\mathcal{H}_i}\mathcal{A}_{\mathcal{H}_i}, \quad M_j^{\mathcal{B}} = \frac{L}{8\pi}\kappa_{\mathcal{B}_j}\mathcal{A}_{\mathcal{B}_j}, \tag{4.7}$$

where $\kappa_{\mathcal{H}_i}$ and $\mathcal{A}_{\mathcal{H}_i}$ are the surface gravity and the area of the i-th horizon and the surface gravity and area of j-th bubble. The surface gravity and the area for a bubble were first introduced in [22]. The bubble surface gravity is defined by

$$\kappa_{\mathcal{B}}^2 = \frac{1}{2}\nabla_{[\mu}\eta_{\nu]}\nabla^{[\mu}\eta^{\nu]} \tag{4.8}$$

where the right hand side is evaluated on the bubble. The reader might consult [22] for other equivalent definitions. The bubble area is given by

$$\mathcal{A}_{\mathcal{B}} = \int_{\mathcal{B}}\sqrt{|g_{tt}|g_{\rho\rho}g_{\psi\psi}}\, dz d\psi. \tag{4.9}$$

For regular (smooth) bubbles (i.e. bubbles without conical singularities), the case we consider here, one can show that

$$\kappa_{\mathcal{B}} = \frac{2\pi}{L}. \tag{4.10}$$

Using Stokes theorem the tension can be represented as a bulk integral over a constant t and ϕ hypersurface Σ and surface integrals over the black hole horizons and bubbles

$$\begin{aligned}\mathcal{T}L = &-\frac{L}{16\pi}\sum_i\int_{\mathcal{H}_i}\left(i_\eta \star d\xi - 2 i_\xi \star d\eta\right) - \frac{L}{16\pi}\sum_j\int_{\mathcal{B}_j}\left(i_\eta \star d\xi - 2 i_\xi \star d\eta\right) \\ &-\frac{L}{16\pi}\int_\Sigma d\left(i_\eta \star d\xi - 2 i_\xi \star d\eta\right)\end{aligned} \tag{4.11}$$

where we have taken into account that $\partial\Sigma = S^2_\infty - \sum_i \mathcal{H}_i - \sum_j \mathcal{B}_j$. Using the definitions (4.4) and (4.5), the Killing symmetries and the identity $d\star d\xi = 2\star R[\xi]$ for an arbitrary Killing field, we have

$$\mathcal{T}L = \frac{1}{2}\sum_i M_i^{\mathcal{H}} + 2\sum_j M_j^{\mathcal{B}} + \frac{L}{8\pi}\int_\Sigma (i_\eta \star R[\xi] - 2i_\xi \star R[\eta]) \tag{4.12}$$

where $R[X]$ is the Ricci 1-form[5] with respect to the vector field X. Making advantage of the field equations (4.1) we obtain

$$\star R[\xi] = -\frac{1}{2}e^{-2\gamma\varphi}\left(-\frac{2}{3}i_\xi F \wedge \star F + \frac{1}{3}F \wedge i_\xi \star F\right) \tag{4.13}$$

and the same expression for $\star R[\eta]$, however with ξ replaced by η. Hence we find

$$i_\eta \star R[\xi] - 2i_\xi \star R[\eta] = \frac{1}{2}e^{-2\gamma\varphi} i_\eta F \wedge i_\xi \star F \tag{4.14}$$

and therefore

$$\mathcal{T}L = \frac{1}{2}\sum_i M_i^{\mathcal{H}} + 2\sum_j M_j^{\mathcal{B}} + \frac{L}{16\pi}\int_\Sigma e^{-2\gamma\varphi} i_\eta F \wedge i_\xi \star F. \tag{4.15}$$

Using now the invariance under the Killing field ζ we can write

$$\int_\Sigma e^{-2\gamma\varphi} i_\eta F \wedge i_\xi \star F = 2\pi \int_{\hat{\mathcal{M}}} i_\zeta \left[e^{-2\gamma\varphi} i_\eta F \wedge i_\xi \star F\right] \tag{4.16}$$

where $\hat{\mathcal{M}} = \mathcal{M}/\mathrm{U}(1)^2 \times R = \Sigma/\mathrm{U}(1)$ is the factor space. For $i_\zeta i_\eta F$ we have

$$d i_\zeta i_\eta F = i_\zeta i_\eta dF + i_\zeta \mathcal{L}_\eta F - i_\eta \mathcal{L}_\zeta F = 0 \tag{4.17}$$

and taking into account that $i_\zeta i_\eta F$ vanishes on the axes of η and ζ we conclude that $i_\zeta i_\eta F = 0$ everywhere. Hence we find

$$\begin{aligned}\int_\Sigma e^{-2\gamma\varphi} i_\eta F \wedge i_\xi \star F &= 2\pi \int_{\hat{\mathcal{M}}} i_\zeta \left[e^{-2\gamma\varphi} i_\eta F \wedge i_\xi \star F\right] \\ &= -2\pi \int_{\hat{\mathcal{M}}} e^{-2\gamma\varphi} i_\eta F \wedge i_\zeta i_\xi \star F.\end{aligned} \tag{4.18}$$

As a consequence of the field equations and the spacetime symmetries we have $d[e^{-2\alpha\varphi} i_\zeta i_\xi \star F] = 0$. Since the factor space is simply connected [1] there exists a globally well-defined potential $\mathcal{B}$ on $\hat{\mathcal{M}}$ such that $e^{-2\alpha\varphi} i_\zeta i_\xi \star F = d\mathcal{B}$. With this in mind we obtain

$$\begin{aligned}\int_\Sigma e^{-2\gamma\varphi} i_\eta F \wedge i_\xi \star F &= -2\pi \int_{\hat{\mathcal{M}}} e^{-2\gamma\varphi} i_\eta F \wedge i_\zeta i_\xi \star F \\ &= 2\pi \int_{\hat{\mathcal{M}}} d[\mathcal{B} i_\eta F] = 2\pi \int_{\partial\hat{\mathcal{M}}} \mathcal{B} i_\eta F.\end{aligned} \tag{4.19}$$

[5] We recall that the Ricci 1-form $R[X]$ is defined by $R[X] = R_{\mu\nu}X^\mu dx^\nu$.

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The next step is to calculate the integral on the boundary of the factor space which formally can be presented in the following way $\partial\hat{\mathcal{M}} = Arc(\infty) + Rod_\psi(-\infty, z_1] + \sum_i Rod-Horizon_i + \sum_j Rod-Bubble_j + Rod_\psi[z_{2N}, +\infty)$, where $Arc(\infty)$ is the upper infinite semi-circle. It can be shown that for an upper semi-circle with radius R we have

$$\int_{Arc(R)} \mathcal{B} i_\eta F \sim \frac{1}{R} \tag{4.20}$$

which means that this integral does not give any contribution in the limit $R \to \infty$. The same is true for the integral on the bubble rods since the Killing field η vanishes on them. In this way we find

$$\int_{\partial\hat{\mathcal{M}}} \mathcal{B} i_\eta F = \int_{Rod_\psi(-\infty,z_1]} \mathcal{B} i_\eta F + \int_{Rod_\psi[z_{2N},\infty]} \mathcal{B} i_\eta F + \sum_i \int_{Rod-Horizon_i} \mathcal{B} i_\eta F. \tag{4.21}$$

One can prove that the potential $\mathcal{B}$ is constant on the horizon rods, therefore

$$\sum_i \int_{Rod-Horizon_i} \mathcal{B} i_\eta F = \sum_i \mathcal{B}_i \int_{Rod-Horizon_i} i_\eta F = \sum_i \frac{\mathcal{B}_i}{L} \int_{S^2_{H_i}} F = \sum_i \frac{2\pi\mathcal{B}_i}{L} Q_i \tag{4.22}$$

where

$$Q_i = \frac{1}{2\pi} \int_{S^2_{H_i}} F \tag{4.23}$$

is the magnetic (dipole) charge associated with the i-th horizon. In order to prove that $\mathcal{B}$ is constant on the horizons we consider the following chain of equalities

$$\begin{aligned} &< e^{-2\gamma\varphi} i_\zeta i_\xi \star F, e^{-2\gamma\varphi} i_\zeta i_\xi \star F >= -e^{-4\gamma\varphi} < \xi \wedge (F \wedge \zeta), \xi \wedge (F \wedge \zeta) >= \\ &\quad -e^{-4\gamma\varphi} < \xi, \xi >< F \wedge \zeta, F \wedge \zeta > + e^{-4\gamma\varphi} < i_\xi(F \wedge \zeta), i_\xi(F \wedge \zeta) > \end{aligned} \tag{4.24}$$

where $<,>$ denotes the inner product of two forms of the same degree. Further we should take into account that $i_\xi(F \wedge \zeta) = 0$ since Killing fields ξ and ζ are orthogonal and in our case $i_\xi F = 0$. In this way we find

$$\begin{aligned} < e^{-2\gamma\varphi} i_\zeta i_\xi \star F, e^{-2\gamma\varphi} i_\zeta i_\xi \star F > &= -e^{-4\gamma\varphi} < \xi, \xi >< F \wedge \zeta, F \wedge \zeta > \\ &= -e^{-4\gamma\varphi} < \xi, \xi >< \zeta, \zeta >< F, F > \end{aligned} \tag{4.25}$$

which shows that $e^{-2\gamma\varphi} i_\zeta i_\xi \star F$ is null on the horizons where $< \xi, \xi >= g(\xi, \xi) = 0$. Taking into account also that $e^{-2\gamma\varphi} i_\zeta i_\xi \star F$ is orthogonal to ξ (by definition) we conclude that $e^{-2\gamma\varphi} i_\zeta i_\xi \star F$ is proportional to ξ on the horizons, i.e. $e^{-2\gamma\varphi} i_\zeta i_\xi \star F = \Lambda\xi$ (on the horizons). For arbitrary vector field u tangent to a given horizon we have $i_u d\mathcal{B} = \mathcal{L}_u \mathcal{B} = \Lambda i_u \xi = 0$ which shows that $\mathcal{B}$ is indeed constant on the horizons.

Let us now consider the integrals on the semi-infinite rods of the axis of ζ. The potential $\mathcal{B}$ is constant on the axes of ζ and this follows directly from the definition of the potential $\mathcal{B}$. To be specific we will consider $Rod_\psi[a_{2N}, +\infty)$. In the next step we follow [20] and define C^+ to be the 2-dimensional surface generated from the path $[a_{2N}, \infty)$ by acting with

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the isometry generated by η. Since $\eta|_{a_{2N}} = 0$ the 2-surface C^+ has disk topology. Then we have

$$\int_{Rod_\psi[z_{2N},\infty]} \mathcal{B} i_\eta F = \mathcal{B}^+ \int_{Rod_\psi[a_{2N},\infty)} i_\eta F = \frac{\mathcal{B}^+}{L} \int_{C^+} F = \frac{\mathcal{B}^+}{L}\Psi^+ \tag{4.26}$$

where $\mathcal{B}^+ = \mathcal{B}_{|_{Rod_\psi[a_{2N},+\infty)}}$ and we have introduced the magnetic flux through the 2-surface C^+ defined by

$$\Psi^+ = \int_{C^+} F. \tag{4.27}$$

Analogously we obtain

$$\int_{Rod_\psi(-\infty,a_1]} \mathcal{B} i_\eta F = \frac{\mathcal{B}^-}{L}\Psi^- \tag{4.28}$$

where

$$\Psi^- = \int_{C^-} F \tag{4.29}$$

is the magnetic flux through the 2-dimensional surface C^- generated from $(-\infty, a_1]$ by acting with the isometry generated by η and $\mathcal{B}^- = \mathcal{B}_{|_{Rod_\psi[-\infty,a_1)}}$.

Summarizing the results so far we obtain

$$\int_\Sigma e^{-2\alpha\varphi} i_\eta F \wedge i_\xi \star F = 2\pi \int_{\partial\hat{\mathcal{M}}} \mathcal{B} i_\eta F = \sum_i 4\pi^2 \mathcal{B}_i \frac{Q_i}{L} + 2\pi\mathcal{B}^+\frac{\Psi^+}{L} + 2\pi\mathcal{B}^-\frac{\Psi^-}{L} \tag{4.30}$$

or equivalently

$$\mathcal{T}L = \frac{1}{2}\sum_i M_i^{\mathcal{H}} + 2\sum_j M_j^{\mathcal{B}} + \sum_i \frac{\pi}{4}\mathcal{B}_i Q_i + \frac{1}{8}\mathcal{B}^+\Psi^+ + \frac{1}{8}\mathcal{B}^-\Psi^-. \tag{4.31}$$

The magnetic fluxes Ψ^+ and Ψ^- are related via the equation

$$\Psi^+ + \Psi^- = -2\pi\sum_i Q_i. \tag{4.32}$$

This follows from the following chain of equalities

$$\begin{aligned} 0 = L\int_{\hat{\mathcal{M}}} d i_\eta F = L\int_{\partial\hat{\mathcal{M}}} i_\eta F = L\int_{Rod(-\infty,a_1]} i_\eta F + L\sum_i \int_{Rod_{Horizon_i}} i_\eta F \\ + L\sum_j \int_{Rod_{Bubble_j}} i_\eta F + L\int_{Rod[a_{2N},+\infty)} i_\eta F = \Psi^+ + 2\pi\sum_i Q_i + \Psi^-. \end{aligned} \tag{4.33}$$

Using this constraint and defining

$$\Psi = \frac{1}{2}(\Psi^+ - \Psi^-) \tag{4.34}$$

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we find

$$\frac{L}{16\pi}\int_{\Sigma} e^{-2\gamma\varphi} i_\eta F \wedge i_\xi \star F = 2\pi \int_{\partial\hat{\mathcal{M}}} \mathcal{B} i_\eta F \qquad (4.35)$$
$$= \sum_i \frac{\pi}{4}\left(\mathcal{B}_i - \frac{1}{2}\mathcal{B}^+ - \frac{1}{2}\mathcal{B}^-\right) Q_i + \frac{1}{8}(\mathcal{B}^+ - \mathcal{B}^-)\Psi$$

which substituted in (4.15) gives the desired Smarr-Like relation for the tension, namely

$$\mathcal{T}L = \frac{1}{2}\sum_i M_i^H + 2\sum_j M_j^B + \sum_i \frac{\pi}{4}\left(\mathcal{B}_i - \frac{1}{2}\mathcal{B}^+ - \frac{1}{2}\mathcal{B}^-\right) Q_i + \frac{1}{8}(\mathcal{B}^+ - \mathcal{B}^-)\Psi. \qquad (4.36)$$

Following the same method as for the tension we find

$$M = \sum_i M^{\mathcal{H}} + \sum_j M_j^{\mathcal{B}} - \frac{L}{16\pi}\int_{\Sigma} e^{-2\alpha\varphi} i_\xi F \wedge i_\eta \star F. \qquad (4.37)$$

which in view of the fact that $i_\xi F = 0$ gives the Smarr-like relation for the mass

$$M = \sum_i M_i^{\mathcal{H}} + \sum_j M_j^{\mathcal{B}}. \qquad (4.38)$$

The Smarr-like relations for the mass and the tension were derived for the rod structure shown on figure 1. It is not difficult one to show that the derived relations also hold for more general rod structures containing black rings and bubbles.

We have checked explicitly that the derived Smarr-like relations are satisfied for the exact solution constructed in the present paper.

4.2 Mass and tension first law

Our next goal is to derive the mass and tension first laws for the black configurations under consideration. In our derivation we shall follow in part our previous work [20] based on Wald's approach [23].

Our diffeomorphism covariant theory is derived from the Lagrangian

$$\mathbf{L} = \star R - 2d\varphi \wedge \star d\phi - \frac{1}{2}e^{-2\gamma\varphi} F \wedge \star F. \qquad (4.39)$$

When the field equations are satisfied, the first order variation of the Lagrangian is given by

$$\delta\mathbf{L} = d\Theta \qquad (4.40)$$

where

$$d\Theta = d\star\, \upsilon - 4(d\star d\varphi)\delta\varphi - \left(e^{-2\gamma\varphi}\star F\right)\wedge\delta F \qquad (4.41)$$

and

$$\upsilon_\mu = \nabla^\nu \delta g_{\mu\nu} - g^{\alpha\beta}\nabla_\mu \delta g_{\alpha\beta}. \qquad (4.42)$$

Here δ denotes the first order variation of the corresponding quantity.

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The Noether current $\mathcal{I}^X$ associated with a diffeomorphism generated by an arbitrary smooth vector field X, as it has been shown in [23], is

$$\mathcal{I}^X = \Theta(\mathcal{P}, \mathcal{L}_X \Gamma) - i_X \mathbf{L}, \tag{4.43}$$

where the fields $g_{\mu\nu}, F, \varphi$ are collectively denoted by $\mathcal{P}$. The current $\mathcal{I}^X$ satisfies $d\mathcal{I}^X = 0$ when the field equations are satisfied. Since $\mathcal{I}^X$ is closed there exists a 3-form $\mathcal{N}^X$ (Noether charge 3-form) such that $\mathcal{I} = d\mathcal{N}^X$.

Now, let $\mathcal{P}$ be a solution to the field equations (4.1) and let $\delta\mathcal{P}$ be a linearized perturbation satisfying the linearized equations of the Einstein-Maxwell-dilaton gravity. For simplicity we will also assume that $\mathcal{L}_\xi \delta\mathcal{P} = \mathcal{L}_\eta \delta\mathcal{P} = \mathcal{L}_\zeta \delta\mathcal{P} = 0$. Then, choosing X to be a Killing field one can show that [23]

$$\delta d\mathcal{N}^X = d i_X \Theta. \tag{4.44}$$

In the case under consideration we need the Noether forms $\mathcal{N}^\xi$ and $\mathcal{N}^\eta$. After some calculations it can be shown that they are given by

$$\mathcal{N}^\xi = - \star d\xi, \tag{4.45}$$

$$\mathcal{N}^\eta = - \star d\eta - \lambda \left(e^{-2\gamma\varphi} \star F \right), \tag{4.46}$$

where the potential λ is defined by $i_\eta F = -d\lambda$ (see eq. (2.4)).

In fact what we need are the 2-forms $i_\eta \mathcal{N}^\xi$ and $i_\xi \mathcal{N}^\eta$ [20]. For them one can show that

$$\delta \left(d i_\eta \mathcal{N}^\xi \right) = d i_\eta i_\xi \Theta, \qquad \delta \left(d i_\xi \mathcal{N}^\eta \right) = -d i_\eta i_\xi \Theta. \tag{4.47}$$

It turns out useful to combine (4.47) to a single equality

$$\delta \left(2 d i_\eta \mathcal{N}^\xi - d i_\xi \mathcal{N}^\eta \right) = 3 d i_\eta i_\xi \Theta. \tag{4.48}$$

Integrating on Σ we have

$$\delta \int_\Sigma \left(2 d i_\eta \mathcal{N}^\xi - d i_\xi \mathcal{N}^\eta \right) = 3 \int_\Sigma d i_\eta i_\xi \Theta. \tag{4.49}$$

Calculations very similar to those in deriving the Smarr-like relations give the following result for the integral on the left hand side of (4.49)

$$\int_\Sigma \left(2 d i_\eta \mathcal{N}^\xi - d i_\xi \mathcal{N}^\eta \right) = -4\pi^2 \sum_i \mathcal{B}_i \frac{Q_i}{L} - 2\pi \mathcal{B}^+ \frac{\Psi^+}{L} - 2\pi \mathcal{B}^- \frac{\Psi^-}{L} \tag{4.50}$$

Respectively, for the integral of the right hand side of (4.49), we obtain (see also [20])

$$\begin{aligned} \int_\Sigma d i_\eta i_\xi \Theta = & -4\pi(\delta c_t - \delta c_\phi) + 16\pi \mathcal{D} \delta\varphi_\infty - 2 \sum_i \frac{\mathcal{A}_{H_i}}{L} \delta\kappa_{H_i} - 2 \sum_j \mathcal{A}_{B_j} \delta\kappa_{B_j} \\ & -4\pi^2 \sum_i \mathcal{B}_i \delta \left(\frac{Q_i}{L} \right) - 2\pi \mathcal{B}^+ \delta \left(\frac{\Psi^+}{L} \right) - 2\pi \mathcal{B}^- \delta \left(\frac{\Psi^-}{L} \right) . \end{aligned} \tag{4.51}$$

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where $\mathcal{D}$ is the dilaton charge defined by

$$\mathcal{D} = \frac{1}{4\pi} \int_{S^2_\infty} i_\eta i_\xi \star d\varphi. \tag{4.52}$$

It is worth noting that the dilaton charge is not an independent characteristic. One can show that the dilaton charge can be expressed in the form

$$\begin{aligned} \mathcal{D} &= -\frac{\gamma}{4\pi L}\left[2\pi \sum_i \mathcal{B}_i \mathcal{Q}_i + \mathcal{B}^+\Psi^+ + \mathcal{B}^-\Psi^-\right] \\ &= -\frac{\gamma}{4\pi L}\left[2\pi \sum_i \left(\mathcal{B}_i - \frac{1}{2}\mathcal{B}^+ - \frac{1}{2}\mathcal{B}^-\right)\mathcal{Q}_i + \left(\mathcal{B}^+ - \mathcal{B}^-\right)\Psi\right]. \end{aligned} \tag{4.53}$$

Substituting these results in (4.49) we obtain

$$\begin{aligned} \frac{3}{4}(\delta c_t - \delta c_\phi) =& \ 3\mathcal{D}\delta\varphi_\infty - \frac{3}{8\pi}\sum_i \frac{\mathcal{A}_{\mathcal{H}_i}}{L}\delta\kappa_{\mathcal{H}_i} - \frac{3}{8\pi}\sum_j \frac{\mathcal{A}_{\mathcal{B}_j}}{L}\delta\kappa_{\mathcal{B}_j} \\ & -\frac{\pi}{2}\sum_i \mathcal{B}_i\delta\left(\frac{Q_i}{L}\right) + \frac{\pi}{4}\sum_i \frac{Q_i}{L}\delta\mathcal{B}_i - \frac{1}{4}\mathcal{B}^+\delta\left(\frac{\Psi^+}{L}\right) + \frac{1}{8}\frac{\Psi^+}{L}\delta\mathcal{B}^+ \\ & -\frac{1}{4}\mathcal{B}^-\delta\left(\frac{\Psi^-}{L}\right) + \frac{1}{8}\frac{\Psi^-}{L}\delta\mathcal{B}^-. \end{aligned} \tag{4.54}$$

The next step is to take into account that $3/4(\delta c_t - \delta c_\phi) = \delta(M/L) + \delta\mathcal{T}$ and to express $\delta\mathcal{T}$ from the Smarr-like relation (4.31) which gives

$$\begin{aligned} \delta\left(\frac{M}{L}\right) =& \ 3\mathcal{D}\delta\varphi_\infty - \frac{1}{2\pi}\sum_i \frac{\mathcal{A}_{\mathcal{H}_i}}{L}\delta\kappa_{\mathcal{H}_i} - \frac{1}{8\pi}\sum_i \kappa_{\mathcal{H}_i}\delta\left(\frac{\mathcal{A}_{\mathcal{H}_i}}{L}\right) - \frac{5}{8\pi}\sum_j \mathcal{A}_{\mathcal{B}_j}\delta\kappa_{\mathcal{B}_j} \\ & -\frac{1}{4\pi}\sum_j \kappa_{\mathcal{B}_j}\delta\mathcal{A}_{\mathcal{B}_j} - \frac{3}{4}\pi\sum_i \mathcal{B}_i\delta\left(\frac{Q_i}{L}\right) - \frac{3}{8}\mathcal{B}^+\delta\left(\frac{\Psi^+}{L}\right) - \frac{3}{8}\mathcal{B}^-\delta\left(\frac{\Psi^+}{L}\right). \end{aligned} \tag{4.55}$$

Now using again the Smarr-like relation (4.38) we also find

$$\begin{aligned} \delta\left(\frac{M}{L}\right) =& \ \frac{1}{4\pi}\sum_i \frac{\mathcal{A}_{\mathcal{H}_i}}{L}\delta\kappa_{\mathcal{H}_i} + \frac{1}{4\pi}\sum_i \kappa_{\mathcal{H}_i}\delta\left(\frac{\mathcal{A}_{\mathcal{H}_i}}{L}\right) + \frac{1}{8\pi}\sum_j \mathcal{A}_{\mathcal{B}_j}\delta\kappa_{\mathcal{B}_j} \\ & +\frac{1}{8\pi}\sum_j \kappa_{\mathcal{B}_j}\delta\mathcal{A}_{\mathcal{B}_j}. \end{aligned} \tag{4.56}$$

Combining the above equalities (4.55) and (4.56) we obtain

$$\begin{aligned} \delta\left(\frac{M}{L}\right) =& \ \mathcal{D}\delta\varphi_\infty + \sum_i \frac{\kappa_{\mathcal{H}_i}}{8\pi}\delta\left(\frac{\mathcal{A}_{\mathcal{H}_i}}{L}\right) - \frac{1}{8\pi}\sum_j \mathcal{A}_{\mathcal{B}_j}\delta\kappa_{\mathcal{B}_j} - \frac{\pi}{4}\sum_i \mathcal{B}_i\delta\left(\frac{Q_i}{L}\right) \\ & -\frac{1}{8}\mathcal{B}^+\delta\left(\frac{\Psi^+}{L}\right) - \frac{1}{8}\mathcal{B}^-\delta\left(\frac{\Psi^-}{L}\right) \end{aligned} \tag{4.57}$$

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which in view of the relation (4.31) gives

$$\delta M = LD\delta\varphi_\infty + \sum_i \frac{\kappa_{\mathcal{H}_i}}{8\pi}\delta\mathcal{A}_{\mathcal{H}_i} - \frac{1}{8\pi}\sum_j \mathcal{A}_{\mathcal{B}_j}\delta(\kappa_{\mathcal{B}_j}L) - \sum_i \frac{\pi}{4}\mathcal{B}_i\delta Q_i$$
$$-\frac{1}{8}\mathcal{B}^+\delta\Psi^+ - \frac{1}{8}\mathcal{B}^-\delta\Psi^- + \mathcal{T}\delta L. \tag{4.58}$$

Further taking into account (4.32) and (4.34) we find

$$\delta M = LD\delta\varphi_\infty + \sum_i \frac{\kappa_{\mathcal{H}_i}}{8\pi}\delta\mathcal{A}_{\mathcal{H}_i} - \frac{1}{8\pi}\sum_j \mathcal{A}_{\mathcal{B}_j}\delta(\kappa_{\mathcal{B}_j}L) - \sum_i \frac{\pi}{4}\left(\mathcal{B}_i - \frac{1}{2}\mathcal{B}^+ - \frac{1}{2}\mathcal{B}^-\right)\delta Q_i$$
$$-\frac{1}{8}\left(\mathcal{B}^+ - \mathcal{B}^-\right)\delta\Psi + \mathcal{T}\delta L. \tag{4.59}$$

This is the desired form of the mass first law. For smooth bubbles (the case we consider here) $\delta(\kappa_{\mathcal{B}_j}L) = 0$ and the third term in (4.59) gives no contribution. Once having the mass first law, the tension first law can be easily found and the result is

$$\delta\mathcal{T} = \mathcal{D}\delta\varphi_\infty + \frac{1}{8\pi}\sum_j \kappa_{\mathcal{B}_j}\delta\mathcal{A}_{\mathcal{B}_j} - \frac{1}{8\pi}\sum_i \frac{\mathcal{A}_{\mathcal{H}_i}}{L}\delta\kappa_{\mathcal{H}_i}$$
$$+\sum_i \frac{\pi}{4}\frac{Q_i}{L}\delta\left(\mathcal{B}_i - \frac{1}{2}\mathcal{B}^+ - \frac{1}{2}\mathcal{B}^-\right) + \frac{\Psi}{L}\delta\left[\frac{1}{8}(\mathcal{B}^+ - \mathcal{B}^-)\right]. \tag{4.60}$$

The mass and the tension first laws we derived in this subsection also hold in the general case not only for the rod structure shown in figure 1. We have checked explicitly that the mass and tension first laws are satisfied for our exact solutions.

Summarizing, in this section we have derived the mass and tension Smarr-like relations and the mass and tension first laws. The explicit expressions show that not only the dipole charge appears together with its (effective) potential [19] but also new terms related to the magnetic flux are present.

5 Conclusion

In the present paper we have constructed new exact solutions to 5D Einstein-Maxwell gravity describing sequences of Kaluza-Klein bubbles and dipole black rings. The basic properties and characteristics of the solutions were calculated and discussed. We also derived the Smarr-like relations and the mass and tension first laws. The novel feature is the appearance of the magnetic flux in the Smarr like relations and the first laws. Another interesting feature is the fact that the effective potential associated with the magnetic flux involves the values of $\mathcal{B}$ on the axes of the non-compact direction.

Future work may involve the inclusion of rotation.

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A Multi black rings and Kaluza-Klein bubbles sequences

The general form of the vacuum solution describing sequences of q dipole black rings and $p = q + 1$ Kaluza-Klein bubbles is [4]

$$g^E_{00} = -\prod_{i=2}^{N-1} (R_{a_i} - \zeta_{a_i})^{(-1)^i} = -\prod_{i=2}^{N-1} \left(e^{2U_{a_i}}\right)^{(-1)^i}, \tag{A.1}$$

$$g^E_{\phi\phi} = \prod_{i=1}^{N} (R_{a_i} - \zeta_{a_i})^{(-1)^{i+1}} = \prod_{i=1}^{N} \left(e^{2\tilde{U}_{a_i}}\right)^{(-1)^i}, \tag{A.2}$$

$$g^E_{\psi\psi} = (R_{a_1} + \zeta_{a_1})(R_{a_N} - \zeta_{a_N}), \tag{A.3}$$

$$g^E_{\rho\rho} = \frac{Y_{1N}}{2^{N/2}} \left(\prod_{i=1}^{N} \frac{1}{R_i}\right) \left(\prod_{2\leq i<j\leq N-1} Y_{ij}^{(-1)^{i+j+1}}\right) \sqrt{\prod_{i=2}^{N-1} \left(\frac{Y_{1i}}{Y_{iN}}\right)^{(-1)^i}} \frac{R_{a_N} - \zeta_{a_N}}{R_{a_1} - \zeta_{a_1}}, \tag{A.4}$$

where

$$Y_{ij} = R_{a_i} R_{a_j} + \zeta_{a_i}\zeta_{a_j} + \rho^2. \tag{A.5}$$

$$a = \alpha \prod_{i=1}^{N} \left(\frac{e^{2U_{k_1}} + e^{2\tilde{U}_{a_i}}}{e^{\tilde{U}_{a_i}}}\right)^{(-1)^i} \tag{A.6}$$

$$b = \beta \prod_{i=1}^{N} \left(\frac{e^{2U_{k_2}} + e^{2\tilde{U}_{a_i}}}{e^{\tilde{U}_{a_i}}}\right)^{(-1)^{i+1}} \tag{A.7}$$

The 2-soliton transformation applied to this seed solution produces a solution to the 5D Einstein-Maxwell equations describing a sequence of dipole black rings and KK bubbles.

The functions W, $\mathcal{Y}$ and λ are regular everywhere provided parameters k_1 and k_2 lie on any of the bubble rods

$$a_{2m-1} < k_2 < k_1 < a_{2m}, \tag{A.8}$$

where $m = 1, 2, ..., N/2$, and α and β satisfy

$$\alpha^2 = \prod_{i=1}^{2m-1} (k_1 - a_i)^{(-1)^{i+1}} \prod_{j=2m}^{N} (a_j - k_1)^{(-1)^{j+1}}, \tag{A.9}$$

$$\beta^2 = \prod_{i=1}^{2m-1} (k_2 - a_i)^{(-1)^i} \prod_{j=2m}^{N} (a_j - k_2)^{(-1)^j}. \tag{A.10}$$

Conical singularities are avoided satisfying p balance conditions on each of the bubble rods

$$(\Delta\phi)_{Rod[a_{2s-1},a_{2s}]} = L, \quad s = 1, 2, ..., N/2. \tag{A.11}$$

where

$$
\begin{aligned}
(\Delta\phi)_{Rod[a_{2s-1},a_{2s}]} &= 2\pi \lim_{\rho\to 0}\sqrt{\frac{\rho^2 g_{\rho\rho}}{g_{\phi\phi}}} = \left(\frac{\mathcal{Y}}{W}\right)^{3/2}_{Rod[a_{2s-1},a_{2s}]}(\Delta\phi)^E_{Rod[a_{2s-1},a_{2s}]} \\
\left(\frac{\mathcal{Y}}{W}\right)_{Rod[a_{2s-1},a_{2s}]} &= \left[\frac{1+\alpha\beta\prod_{i=1}^{2s-1}\left(\frac{k_1-a_i}{k_2-a_i}\right)^{(-1)^i}}{1+\alpha\beta}\right]^2 \prod_{i=1}^{2s-1}\left(\frac{k_2-a_i}{k_1-a_i}\right)^{(-1)^i}
\end{aligned}
$$

$(\Delta\phi)^E$ is the period for the seed solution given by

$$
\begin{aligned}
(\Delta\phi)^E_{Rod[a_{2s-1},a_{2s}]} &= 4\pi(a_N-a_1)\prod_{i=2}^{2s-1} \\
&\quad \times \prod_{j=2s}^{N-1}(a_j-a_i)^{(-1)^{i+j+1}}\prod_{i=2}^{2s-1}[\sqrt{a_N-a_i}]^{(-1)^{i+1}}\prod_{i=2s}^{N-1}[\sqrt{a_i-a_1}]^{(-1)^i}.
\end{aligned}
$$

The general expressions for the dipole charge and dipole potential characterizing the s-th black ring ($m < s$) are respectively.

$$
\begin{aligned}
\mathcal{Q}_s &= \frac{L\sqrt{3}\beta\Delta k\Delta_s(a_{2s}-k_2)^{-1}\prod_{i=2s+2}^{N}(a_i-k_2)^{(-1)^{i+1}}\left[1+\alpha\beta\prod_{i=2s+1}^{N}\left(\frac{a_i-k_1}{a_i-k_2}\right)^{(-1)^i}\right]}{2\pi\left[1+\alpha\beta\prod_{i=2s}^{N}\left(\frac{a_i-k_1}{a_i-k_2}\right)^{(-1)^i}\right]\left[1+\alpha\beta\prod_{i=2s+2}^{N}\left(\frac{a_i-k_1}{a_i-k_2}\right)^{(-1)^i}\right]}, \\
\mathcal{B}_s &= -\frac{2\sqrt{3}\alpha\Delta k\prod_{i=2s+1}^{N}(a_i-k_1)^{(-1)^i}}{\left[1+\alpha\beta\prod_{i=2s+1}^{N}\left(\frac{a_i-k_1}{a_i-k_2}\right)^{(-1)^i}\right]}
\end{aligned}
\tag{A.12}
$$

where Δ_s denotes the horizon rod length $a_{2s+1}-a_{2s}$.

The general expressions for the dipole charge and dipole potential characterizing the s-th black ring ($m > s$) are respectively.

$$
\begin{aligned}
\mathcal{Q}_s &= -\frac{L\sqrt{3}\alpha\Delta k\Delta_s(k_1-a_{2s+1})^{-1}\prod_{i=1}^{2s-1}(k_1-a_i)^{(-1)^i}\left[1+\alpha\beta\prod_{i=1}^{2s}\left(\frac{a_i-k_1}{a_i-k_2}\right)^{(-1)^i}\right]}{2\pi\left[1+\alpha\beta\prod_{i=1}^{2s-1}\left(\frac{a_i-k_1}{a_i-k_2}\right)^{(-1)^i}\right]\left[1+\alpha\beta\prod_{i=1}^{2s+1}\left(\frac{a_i-k_1}{a_i-k_2}\right)^{(-1)^i}\right]}, \\
\mathcal{B}_s &= \frac{2\sqrt{3}\beta\Delta k\prod_{i=1}^{2s}(k_2-a_i)^{(-1)^{i+1}}}{\left[1+\alpha\beta\prod_{i=1}^{2s}\left(\frac{a_i-k_1}{a_i-k_2}\right)^{(-1)^i}\right]}
\end{aligned}
\tag{A.13}
$$

where Δ_s denotes the horizon rod length $a_{2s+1}-a_{2s}$.

Similarly, we find for the temperature and entropy of the s-th black ring.

$$
\begin{aligned}
T_s &= \mathcal{Y}^{-3/2}_{\mathcal{H}_s}T^E_s \\
S_s &= \mathcal{Y}^{3/2}_{\mathcal{H}_s}S^E_s\left(\frac{L}{L^E}\right),
\end{aligned}
\tag{A.14}
$$

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where

$$\mathcal{Y}_{\mathcal{H}_s} = \left[\frac{1+\alpha\beta\prod_{i=2s+1}^{N}\left(\frac{a_i-k_1}{a_i-k_2}\right)^{(-1)^i}}{1+\alpha\beta}\right]^2 \prod_{i=2s+1}^{N}\left(\frac{a_i-k_2}{a_i-k_1}\right)^{(-1)^i}$$

and

$$T_s^E = \frac{1}{4\pi}(a_N-a_1)^{-1}\prod_{i=2}^{2s}\prod_{j=2s+1}^{N-1}(a_j-a_i)^{(-1)^{i+j}}\prod_{i=2}^{2s}[\sqrt{a_N-a_i}]^{(-1)^i}\prod_{i=2s+1}^{N-1}[\sqrt{a_i-a_1}]^{(-1)^{i+1}}, \tag{A.15}$$

$$S_s^E = \frac{L^E}{4T_s^E}(a_{2s+1}-a_{2s}), \tag{A.16}$$

are the temperature and entropy corresponding to the s-th event horizon of the seed solution and the metric function $\mathcal{Y}$ is evaluated on the horizon rod $a_{2s} < z < a_{2s+1}$.

The magnetic fluxes are given by

$$\Psi^+ = L\frac{\sqrt{3}\Delta k\beta(a_N-k_2)^{-1}}{\left[1+\alpha\beta\frac{a_N-k_1}{a_N-k_2}\right]}, \tag{A.17}$$

$$\Psi^- = -L\frac{\sqrt{3}\Delta k\alpha(k_1-a_1)^{-1}}{\left[1+\alpha\beta\frac{k_2-a_1}{k_1-a_1}\right]}. \tag{A.18}$$

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