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Anomalies and the chiral magnetic effect in the Sakai-Sugimoto model

Anton Rebhan, Andreas Schmitt and Stefan A. Stricker

Institut für Theoretische Physik, Technische Universität Wien, 1040 Vienna, Austria

E-mail: rebhana@hep.itp.tuwien.ac.at, aschmitt@hep.itp.tuwien.ac.at, stricker@hep.itp.tuwien.ac.at

Abstract: In the chiral magnetic effect an imbalance in the number of left- and right-handed quarks gives rise to an electromagnetic current parallel to the magnetic field produced in noncentral heavy-ion collisions. The chiral imbalance may be induced by topologically nontrivial gluon configurations via the QCD axial anomaly, while the resulting electromagnetic current itself is a consequence of the QED anomaly. In the Sakai-Sugimoto model, which in a certain limit is dual to large-N_c QCD, we discuss the proper implementation of the QED axial anomaly, the (ambiguous) definition of chiral currents, and the calculation of the chiral magnetic effect. We show that this model correctly contains the so-called consistent anomaly, but requires the introduction of a (holographic) finite counterterm to yield the correct covariant anomaly. Introducing net chirality through an axial chemical potential, we find a nonvanishing vector current only before including this counterterm. This seems to imply the absence of the chiral magnetic effect in this model. On the other hand, for a conventional quark chemical potential and large magnetic field, which is of interest in the physics of compact stars, we obtain a nontrivial result for the axial current that is in agreement with previous calculations and known exact results for QCD.

Keywords: Gauge-gravity correspondence, Chiral Lagrangians, QCD

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Contents

1 Introduction

Topologically charged gauge field configurations in QCD generate chirality due to the non-abelian axial anomaly. In the presence of a magnetic field, this chirality, i.e., an imbalance in the number of right- and left-handed quarks, has been predicted to generate an electromagnetic current parallel to the applied magnetic field. This is a consequence of the QED axial anomaly and has been termed chiral magnetic effect [1–3]. As a result, electric charge separation may occur in noncentral heavy-ion collisions, where magnetic fields up to 10^{17} G can be generated temporarily, and corresponding experimental evidence has in fact been reported in refs. [4, 5] (see however ref. [6]).

In a simplified picture, one may study the induced current for a static magnetic field. The generalization to time-dependent magnetic fields, as produced in heavy-ion collisions, in principle amounts to computing a frequency-dependent conductivity [3, 7]. However, the observed charge separation is proportional to the zero-frequency limit [3]. In this paper, the currents we compute always correspond to the zero-frequency limits of the conductivities.

Another simplification of the highly nontrivial scenario of a heavy-ion collision is to mimic the (event-by-event) topologically induced chirality by a nonzero axial chemical potential μ_5, the difference of right- and left-handed chemical potentials. The resulting current is a vector current proportional to μ_5. In a more general setup, although negligible in the heavy-ion context, one may also include a quark chemical potential μ, which is the same for right- and left-handed fermions. Again via a nonzero magnetic field, an axial current is generated in this case [8, 9]. This effect may be of relevance for the physics of compact stars [10], where strongly interacting matter can reach densities of several times nuclear ground state density, and (surface) magnetic fields up to 10^{15} G have been measured, indicating the possibility of even higher magnetic fields in the interior. Also the direct high-density analogue of the chiral magnetic effect has been studied in the context of neutron star physics [11].

In the present paper, we apply a strong-coupling approach, based on the AdS/CFT correspondence [12–14], to compute both kinds of currents. We use a general setup to account for nonzero temperatures, relevant in the context of heavy-ion collisions, as well as for nonzero quark chemical potentials, relevant in the astrophysical context. Besides the chirally symmetric phase we also consider the chirally broken phase which is important in both contexts: heavy-ion collisions are expected to probe the region of the QCD chiral phase transition; in quark matter at densities present in compact stars, chiral symmetry may also be spontaneously broken, for example in the color-flavor locked phase [15].

We use the Sakai-Sugimoto model [16, 17], which is sometimes called "holographic QCD" since in the limit of small 't Hooft coupling it provides a string theory dual to large-N_c QCD. However, we work in the opposite, strongly coupled, limit where the simple gravity approximation can be employed but where the model is no longer dual to QCD. The model still yields interesting qualitative predictions especially in view of the strong-coupling nature of both contexts mentioned above, i.e., in QCD at large (but not asymptotically large) temperature and small quark chemical potential and QCD at large (but not asymptotically large) quark chemical potential and small temperature.

The Sakai-Sugimoto model is particularly suited for our purpose since it has a well-defined concept for chirality and the chiral phase transition. It is straightforward to introduce right- and left-handed chemical potentials independently. Several previous works have considered currents in a magnetic field at nonzero chemical potentials in this model [18–21]. The purpose of the present paper is two-fold. The physical motivation is to extend these calculations to the currents relevant for the chiral magnetic effect, and to compare our strong-coupling results to the weak-coupling results [2] as well as the lattice results [22] in the existing literature. There is also a more theoretical purpose of our work, addressing certain fundamental properties of the Sakai-Sugimoto model. We discuss in detail how to implement the covariant QED anomaly into the model in order to obtain physically acceptable predictions. Moreover, we elaborate on an ambiguity in the definition of the chiral currents in the presence of a Chern-Simons term that has been observed previously [18, 20, 21] (see also [23]).

Our paper is organized as follows. We start with a brief introduction into the model and a general discussion of the currents, in particular the appearance of consistent and

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covariant anomalies, in section 2. In section 3 we discuss the solution of the equations of motion in the presence of background magnetic and electric fields. We present analytical solutions for the chirally broken phase, section 3.1, and the chirally symmetric phase, section 3.2. We then discuss the ambiguity of the currents, defined on the one hand via the general definition from section 2, and on the other hand from the thermodynamic potentials obtained in section 3. In section 4 we present our results for the axial and vector currents and give our conclusions in section 5.

2 Anomalies in the Sakai-Sugimoto model

2.1 Brief introduction into the model

The Sakai-Sugimoto model is based on ten-dimensional type-IIA string theory, with a background geometry given by N_c D4-branes. They span four-dimensional space-time $(\tau, \mathbf{x})$ and a fifth extra dimension x_4 compactified on a circle whose circumference is parametrized by the Kaluza-Klein mass $M_{\rm KK}$, $x_4 \equiv x_4 + 2\pi/M_{\rm KK}$. Through this compactified dimension and antisymmetric boundary conditions for fermions supersymmetry is completely broken. Left- and right-handed chiral fermions are introduced by adding N_f D8- and N_f $\overline{\rm D8}$-branes which extend in all dimensions except x_4. In this compact direction, they are separated by a distance $L \in [0, \pi/M_{\rm KK}]$. For more details about the setup of the model and the explicit form of the background metric, including the holographic direction z and a four-sphere S^4, see the original papers by Sakai and Sugimoto, refs. [16, 17]. We employ the probe brane approximation, i.e., the background geometry is assumed to be unaltered by the flavor branes. This is a good approximation for $N_f \ll N_c$.

There are two possible background geometries. One, interpreted as the confined phase, has a cigar-shaped (x_4, z) subspace, ending at the tip $z = 0$, and a cylinder-shaped (τ, z) subspace, where τ is Euclidean time on a circle with circumference given by the inverse temperature, $\tau \equiv \tau + 1/T$. In the other geometry, interpreted as the deconfined phase, x_4 and τ exchange their roles, such that the (x_4, z) subspace is cylinder-shaped while the (τ, z) subspace is cigar-shaped, corresponding to a geometry with black-hole horizon after analytical continuation $i\tau \to t$.

Chiral symmetry breaking is realized in the model as follows. A U(N_f) gauge symmetry on the flavor branes corresponds to a global U(N_f) at the boundary. Therefore, the bulk gauge symmetries on the D8- and $\overline{\rm D8}$-branes can be interpreted as left- and right-handed flavor symmetry groups in the dual field theory. The Chern-Simons term accounts for the axial anomaly of QCD, such that one is left with the chiral group SU$(N_f)_L \times$ SU$(N_f)_R$ and the vector part U$(1)_V$. There is no explicit breaking of this group since the model only contains massless quarks. Spontaneous chiral symmetry breaking is realized when the D8- and $\overline{\rm D8}$-branes connect in the bulk. They always connect in the confined phase, where the (x_4, z) subspace is singly connected. Whether they connect in the deconfined phase depends on the separation L of the D8- and $\overline{\rm D8}$-branes in the extra dimension x_4. Here we shall always consider maximally separated branes, $L = \pi/M_{\rm KK}$. With this choice the flavor branes necessarily extend to the black-hole horizon and thus never connect in the deconfined phase. Consequently, the deconfinement and chiral phase transitions are identical and happen at a critical temperature $T_c = M_{\rm KK}/(2\pi)$, i.e., when the radii of the τ and x_4 circles are equal. We shall always use the terminology of chirally symmetric and chirally broken phases. This is equivalent to speaking about deconfined and confined phases, but more appropriate in our context because we are interested in the interplay of a magnetic field with chirality. In the probe brane approximation, the chiral/deconfinement phase transition is given solely by the background geometry and is not affected by the gauge fields on the flavor branes. In particular, it does not depend on the chemical potential, which, at least for the deconfinement phase transition, is in accordance with expectations for large-N_c QCD [24].

2.2 Action, equations of motion, and currents

In this section we discuss the general equations of the model in the broken phase where the D8- and $\overline{\rm D8}$-branes are connected. The equations for the symmetric phase are very similar and shall be given later where necessary. The D-brane action consists of a Dirac-Born-Infeld (DBI) and a Chern-Simons (CS) part. We approximate the DBI action by the Yang-Mills (YM) action which is a good approximation for small magnetic fields. The use of the YM action greatly simplifies the treatment since the equations of motion then have solutions which can be given in an almost entirely analytical way. Throughout the paper we shall work with one quark flavor, $N_f = 1$. The currents we compute are expected to be simple sums over quark flavors, each flavor contributing in the same way, distinguished only by its electric charge. This is rather obvious in the chirally symmetric phase. In the chirally broken phase, the flavor contributions may be more complicated in the case of charged pion condensation. However, since we work at vanishing isospin chemical potential, there is only neutral pion condensation and the different flavor contributions decouple.

For one quark flavor and the gauge $A_z = 0$ the (Euclidean) action

$$S = S_{\rm YM} + S_{\rm CS} \tag{2.1}$$

is given by [17]

$$S_{\rm YM} = \kappa M_{\rm KK}^2 \int d^4x \int_{-\infty}^{\infty} dz \left[k(z) F_{z\mu}F^{z\mu} + \frac{h(z)}{2M_{\rm KK}^2} F_{\mu\nu}F^{\mu\nu}\right] , \tag{2.2a}$$

$$S_{\rm CS} = \frac{N_c}{24\pi^2} \int d^4x \int_{-\infty}^{\infty} dz \, A_\mu F_{z\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} , \tag{2.2b}$$

with Greek indices running over $\mu, \nu, \ldots = 0, 1, 2, 3$. Our convention for the epsilon tensor is $\epsilon_{0123} = +1$. In eq. (2.2) we have defined the metric functions

$$k(z) \equiv 1 + z^2 , \qquad h(z) \equiv (1 + z^2)^{-1/3} , \tag{2.3}$$

and the dimensionless constant

$$\kappa \equiv \frac{\lambda N_c}{216\pi^3} , \tag{2.4}$$

where λ is the 't Hooft coupling. The integration over the four-sphere has already been done, and we are left with the integral over space-time $(\tau, \mathbf{x})$ and the holographic coordinate

z which extends from the left-handed boundary ($z = +\infty$) over the tip of the cigar-shaped (x^4, z) subspace ($z = 0$) to the right-handed boundary ($z = -\infty$). The coordinate z is dimensionless and is obtained from the dimensionful coordinate z of ref. [20] upon defining $z' = z/u_{\rm KK}$ ($T < T_c$) and then dropping the prime. Here, $u_{\rm KK} = 4R^3M_{\rm KK}^2/9$, with R being the curvature radius of the background metric. Since we work at finite temperature, we need to work in Euclidean space. However, we use Minkowski notation which is more convenient for the following discussion of the anomaly. More precisely, we start from the Euclidean action with imaginary time τ and replace $A_0 \to iA_0$, after which we may write the result using a Minkowski metric with signature $(-,+,+,+)$. The space-time integral is denoted by d^4x for simplicity but actually is an integral $d\tau\, d^3x$ over imaginary time τ and three-dimensional space. For the general form of the action, without any gauge choice and for more flavors, see for instance refs. [16, 20, 25]. The equations of motion for $N_f = 1$ are

$$
\begin{aligned}
\kappa M_{\rm KK}^2\, \partial_z[k(z)F^{z\mu}] + \kappa h(z)\partial_\nu F^{\nu\mu} &= \frac{N_c}{16\pi^2} F_{z\nu}F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} \,, && \text{(2.5a)}\\
\kappa M_{\rm KK}^2\, \partial_\mu[k(z)F^{z\mu}] &= \frac{N_c}{64\pi^2} F_{\mu\nu}F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} \,, && \text{(2.5b)}
\end{aligned}
$$

where the second equation is obtained from varying A_z in the action prior to setting $A_z = 0$.

Next we introduce the chiral currents. The usual way is to define them through the variation of the on-shell action with respect to the boundary values of the gauge fields (see however ref. [23] for a discussion of possible alternatives). We thus replace $A_\mu(x,z) \to A_\mu(x,z) + \delta A_\mu(x,z)$ in the action and keep the terms linear in $\delta A_\mu(x,z)$ to obtain

$$
\begin{aligned}
\delta S_{\rm YM} &= 2\kappa M_{\rm KK}^2 \left\{ \int d^4x\, k(z)F^{z\mu}\delta A_\mu \Big|_{z=-\infty}^{z=\infty} + \int d^3x \int_{-\infty}^{\infty} dz\, \frac{h(z)}{M_{\rm KK}^2} F^{\nu\mu}\delta A_\mu \Big|_{x_\nu} \right.\\
&\qquad \left. - \int d^4x \int_{-\infty}^{\infty} dz \left[\partial_z[k(z)F^{z\mu}] + \frac{h(z)}{M_{\rm KK}^2}\partial_\nu F^{\nu\mu}\right]\delta A_\mu \right\} , \qquad \text{(2.6a)}\\
\delta S_{\rm CS} &= \frac{N_c}{8\pi^2} \left\{ -\frac{1}{3}\int d^4x\, A_\nu F_{\rho\sigma}\delta A_\mu \Big|_{z=-\infty}^{z=\infty} - \frac{2}{3}\int d^3x \int_{-\infty}^{\infty} dz\, A_\sigma F_{z\nu}\delta A_\mu \Big|_{x_\rho} \right.\\
&\qquad \left. + \int d^4x \int_{-\infty}^{\infty} dz\, F_{z\nu}F_{\rho\sigma}\delta A_\mu \right\} \epsilon^{\mu\nu\rho\sigma} \,. \qquad \text{(2.6b)}
\end{aligned}
$$

In the total variation $\delta S = \delta S_{\rm YM} + \delta S_{\rm CS}$ the bulk terms vanish upon using the equation of motion for A_μ (2.5a) and we are left with boundary terms only. According to the holographic correspondence, we keep only the boundary terms at $|z| = \infty$ and drop any terms from space-time infinities. This may seem natural but possibly is problematic in our case as we shall discuss later after we have implemented our specific ansatz. The boundary terms at the holographic boundary $z = \pm\infty$ lead to the left- and right-handed currents

$$
\mathcal{J}^\mu_{L/R} \equiv -\frac{\delta S}{\delta A_\mu(x, z=\pm\infty)} = \mp\left(2\kappa M_{\rm KK}^2 k(z)F^{z\mu} - \frac{N_c}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}\right)_{z=\pm\infty} , \qquad \text{(2.7)}
$$

where the first (second) term is the YM (CS) contribution. This result of the currents is in agreement with refs. [20, 23, 26], see also [27, 28]. The overall minus sign in the definition

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originates from our use of the Euclidean action which is minus the Minkowski action, and the functional derivative is taken with respect to the space-time coordinates x (and not also with respect to the holographic coordinate z plus a subsequent limit $z \to \pm\infty$). The currents (2.7) can also be obtained from

$$\mathcal{J}^{\mu}_{L/R} = \mp \left. \frac{\partial \mathcal{L}}{\partial\, \partial_z A_\mu} \right|_{z=\pm\infty} , \tag{2.8}$$

in accordance with the usual rules of the gauge/gravity correspondence.

As already pointed out in ref. [20], it is only the YM part of the current, i.e., the first term in eq. (2.7), which appears in the asymptotic expansion of the gauge fields. From the definition (2.7) and with $k(z) = 1 + z^2$ we find

$$A_\mu(x,z) \;=\; A_\mu(x, z=\pm\infty) \pm \frac{\mathcal{J}^{L/R}_{\mu,\mathrm{YM}}}{2\kappa M_{\mathrm{KK}}^2} \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) . \tag{2.9}$$

One can also confirm this relation from our explicit results in the subsequent sections.

2.3 Consistent and covariant anomalies

The divergence of the currents (2.7) can be easily computed with the help of the equation of motion for A_z (2.5b). One obtains

$$\partial_\mu \mathcal{J}^{\mu}_{L/R} = \partial_\mu (\mathcal{J}_{\mathrm{YM}} + \mathcal{J}_{\mathrm{CS}})^{\mu}_{L/R} = \mp \frac{N_c}{16\pi^2} \left(1 - \frac{2}{3}\right) F^{L/R}_{\mu\nu} \widetilde{F}^{\mu\nu}_{L/R} , \tag{2.10}$$

with the left- and right-handed field strengths $F^{L/R}_{\mu\nu}(x) \equiv F_{\mu\nu}(x, z=\pm\infty)$, and the left- and right-handed dual field strength tensors $\widetilde{F}^{\mu\nu}_{L/R} = \frac{1}{2} F^{L/R}_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}$. (For notational convenience we use the labels L, R and related labels such as V, A sometimes as superscript, sometimes as subscript.) With the vector and axial currents

$$\mathcal{J}^\mu \equiv \mathcal{J}^\mu_R + \mathcal{J}^\mu_L , \qquad \mathcal{J}^\mu_5 \equiv \mathcal{J}^\mu_R - \mathcal{J}^\mu_L , \tag{2.11}$$

and the vector and axial field strengths introduced as $F^R_{\mu\nu} = F^V_{\mu\nu} + F^A_{\mu\nu}$, $F^L_{\mu\nu} = F^V_{\mu\nu} - F^A_{\mu\nu}$, eq. (2.10) yields the vector and axial anomalies

$$\partial_\mu \mathcal{J}^\mu \;=\; \frac{N_c}{12\pi^2} F^V_{\mu\nu} \widetilde{F}^{\mu\nu}_A , \tag{2.12a}$$

$$\partial_\mu \mathcal{J}^\mu_5 \;=\; \frac{N_c}{24\pi^2} \left(F^V_{\mu\nu} \widetilde{F}^{\mu\nu}_V + F^A_{\mu\nu} \widetilde{F}^{\mu\nu}_A \right) . \tag{2.12b}$$

The coefficients on the right-hand side (which as we saw receive contributions from both the YM and CS parts of the currents) are in accordance with the standard field theoretic results for N_c chiral fermionic degrees of freedom coupled to left and right chiral gauge fields [29]. The above form of the anomaly, which is symmetric in vector and axial-vector gauge fields, is called *consistent* anomaly. If left- and right-handed Weyl spinors are treated separately, this form of the anomaly arises unambiguously. This is explained for instance in ref. [30], where left- and right-handed fields are separated by an extra dimension. This is not unlike

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our present model and it is thus not surprising that the consistent anomaly arises naturally from the above definition of the currents. In QED, however, we must require that the vector current be strictly conserved, even in the presence of axial field strengths. As was first discussed by Bardeen [29], this can be achieved by the introduction of a counterterm that mixes left- and right-handed gauge fields. Having even parity, Bardeen's counterterm is uniquely given by [30]

$$\Delta S = c \int d^4x (A^L_\mu A^R_\nu F^L_{\rho\sigma} + A^L_\mu A^R_\nu F^R_{\rho\sigma})\epsilon^{\mu\nu\rho\sigma} \,, \tag{2.13}$$

where c is a constant determined by requiring a strictly conserved vector current. Because this expression can be naturally written as a (metric-independent) integral over a hypersurface at $|z| = \Lambda \to \infty$ with left- and right-handed fields concentrated at the respective brane locations, ΔS can actually be interpreted as a (finite) counterterm in holographic renormalization. In particular, it does not change the equations of motion.

To obtain the contribution of Bardeen's counterterm to the chiral currents we replace $A^{L/R}_\mu \to A^{L/R}_\mu + \delta A^{L/R}_\mu$ to obtain

$$\begin{aligned} \delta\Delta S = &\pm c \int d^4x \left(A^{R/L}_\nu F^{R/L}_{\rho\sigma} - A^{L/R}_\nu F^{R/L}_{\rho\sigma} + 2A^{R/L}_\nu F^{L/R}_{\rho\sigma} \right) \delta A^{L/R}_\mu \epsilon^{\mu\nu\rho\sigma} \\ &\mp 2c \int d^3x \, A^{R/L}_\nu A^{L/R}_\sigma \delta A^{L/R}_\mu \Big|_{x_\rho} \epsilon^{\mu\nu\rho\sigma} \,. \end{aligned} \tag{2.14}$$

Again dropping the space-time surface terms, the contribution to the currents is therefore

$$\Delta\mathcal{J}^\mu_{L/R} = \mp c \left(A^{R/L}_\nu F^{R/L}_{\rho\sigma} - A^{L/R}_\nu F^{R/L}_{\rho\sigma} + 2A^{R/L}_\nu F^{L/R}_{\rho\sigma} \right) \epsilon^{\mu\nu\rho\sigma} \,, \tag{2.15}$$

and the contribution to the divergence of the currents becomes

$$\partial_\mu \Delta\mathcal{J}^\mu_{L/R} = \mp c \left(F^{R/L}_{\mu\nu} \widetilde{F}^{\mu\nu}_{R/L} + F^{L/R}_{\mu\nu} \widetilde{F}^{\mu\nu}_{R/L} \right) \,. \tag{2.16}$$

Denoting renormalized left- and right-handed currents as

$$\bar{\mathcal{J}}^\mu_{L/R} \equiv \mathcal{J}^\mu_{L/R} + \Delta\mathcal{J}^\mu_{L/R} \,, \tag{2.17}$$

and similarly the renormalized axial and vector currents as $\bar{\mathcal{J}}_\mu$, $\bar{\mathcal{J}}^5_\mu$, we find that the choice

$$c = \frac{N_c}{48\pi^2} \tag{2.18}$$

leads to the *covariant* anomaly

$$\partial_\mu \bar{\mathcal{J}}^\mu = 0 \,, \tag{2.19a}$$

$$\partial_\mu \bar{\mathcal{J}}^\mu_5 = \frac{N_c}{8\pi^2} F^V_{\mu\nu} \widetilde{F}^{\mu\nu}_V + \frac{N_c}{24\pi^2} F^A_{\mu\nu} \widetilde{F}^{\mu\nu}_A \,. \tag{2.19b}$$

Note that the prefactor in front of the first term in the axial anomaly now has changed to $N_c/(8\pi^2)$, from $N_c/(24\pi^2)$ in eq. (2.12b), which is the well-known result for the Adler-Bell-Jackiw anomaly for QED [31, 32] and which is essential for getting the correct pion

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decay rate $\pi^0 \to 2\gamma$. The necessity of adding the counterterm (2.13) to the Sakai-Sugimoto model is in fact completely analogous to the very same and well-known procedure in chiral models where a Wess-Zumino-Witten term accounts for the anomaly [33].

In the literature sometimes the coefficient of the subleading term in the asymptotic behavior of $A_\mu(x, |z| \to \infty)$ and thus the YM part of the current (see eq. (2.9)) is identified with the full current [7], see also [34–37]. Using this identification, it has also been assumed that the equation of motion for A_z (2.5b) represents the anomaly equation [21]. Indeed, from eq. (2.5b) one obtains the apparent anomaly

$$\partial_\mu \mathcal{J}^\mu_{\mathrm{YM},L/R} = \mp \frac{N_c}{16\pi^2} F^{L/R}_{\mu\nu} \widetilde{F}^{\mu\nu}_{L/R} , \tag{2.20}$$

which leads to

$$\partial_\mu \mathcal{J}^\mu_{\mathrm{YM}} = \frac{N_c}{4\pi^2} F^V_{\mu\nu} \widetilde{F}^{\mu\nu}_A , \tag{2.21a}$$

$$\partial_\mu \mathcal{J}^\mu_{\mathrm{YM},5} = \frac{N_c}{8\pi^2} \left(F^V_{\mu\nu} \widetilde{F}^{\mu\nu}_V + F^A_{\mu\nu} \widetilde{F}^{\mu\nu}_A \right) , \tag{2.21b}$$

and this does contain the same coefficient in front of $F^V_{\mu\nu}\widetilde{F}^{\mu\nu}_V$ as the full covariant anomaly (2.19). However, it differs from the latter in the presence of axial gauge fields. In particular, the vector current is then not strictly conserved. The renormalized current $\bar{\mathcal{J}}_{L/R}$ satisfies eq. (2.20) only for $F^L_{\mu\nu} = F^R_{\mu\nu}$.[1] Even when this issue may be ignored, because all axial vector field strengths are set to zero, it appears to be questionable to keep only part of the full current (2.7).

In the remainder of the paper we shall consider the full currents for which Bardeen's counterterm is needed, and study the implications, which indeed differ from keeping only the YM part of the currents. (The effect of truncating to the YM part can be easily read off from the expressions that we shall give.)

3 Background electromagnetic fields and chemical potentials

The discussion in the previous section was general in the sense that we have not specified any gauge fields except for the gauge choice $A_z = 0$. In this section we specify our ansatz according to the physical situation we are interested in. This includes a background magnetic field B as well as separate left- and right-handed chemical potentials $\mu_{L,R} = A_0(z = \pm\infty)$, or, equivalently, the ordinary quark chemical potential $\mu = (\mu_R + \mu_L)/2$ and an axial chemical potential $\mu_5 = (\mu_R - \mu_L)/2$. With these ingredients we can obtain results relevant for the heavy-ion context (nonzero μ_5, negligibly small μ) and for the astrophysical context (vanishing μ_5, large μ). In order to be able to check the axial anomaly explicitly, we also add an electric field E and an "axial electric field" ϵ parallel to the magnetic field. The electric field E is needed because the axial anomaly is proportional to $\mathbf{E} \cdot \mathbf{B}$. The (unphysical) field ϵ shall be used to check the absence of a vector anomaly, i.e., the conservation

[1]The more general validity of eq. (2.20) has been assumed incorrectly in eq. (2.1) of ref. [7] and eq. (36) of ref. [2].

of the vector current, which must be true even in the presence of ϵ, and would be trivial without ϵ. In our final results for the currents, the electric fields are however set to zero.

For previous discussions of background electric and magnetic fields in the Sakai-Sugimoto model see for instance refs. [18–20, 38, 39]. We shall only consider spatially homogeneous systems. This is the simplest case, which might however require generalization when the true ground state is more complicated, for instance when Skyrme crystals are formed [40].

3.1 Chirally broken phase

In our ansatz the nonzero fields are $A_0(t,z)$, $A_1(x_2)$, $A_3(t,z)$, where the dependence on t will only be present for nonvanishing electric fields E and ϵ at the holographic boundary. The temporal component A_0 is needed to account for nonzero (left- and right-handed) chemical potentials which correspond to the values of A_0 at the boundary. The electromagnetic fields are encoded in the boundary values of the spatial components. Since the gauge symmetry in the bulk corresponds to a global symmetry for the dual field theory, the fields at the holographic boundary are not dynamical and merely serve as background fields. This is however sufficient for our purpose. The magnetic field $\mathbf{B}$ is assumed to point into the 3-direction, $\mathbf{B} = (0,0,B)$. Consequently, we can choose

$$A_1(x_2) = -x_2 B \tag{3.1}$$

at the holographic boundary. The equations of motion show that A_1 can be chosen to be constant in z throughout the bulk. (This is different in the presence of an isospin chemical potential [20].) Consequently, $F_{12}(x,z) = B$. For notational convenience we have absorbed the electric quark charge q_f into B, i.e., actually $B \to q_f B$ with $q_f = 2/3\,e$ for $f = u$, and $q_f = -1/3\,e$ for $f = d$. With nonzero A_0 and A_1, accounting for the chemical potential and the magnetic field, a nonzero A_3 is induced, even without electric field. In the broken phase, A_3 develops a nonzero boundary value, corresponding to the gradient of the neutral pion [18–20]. Just as for a usual superfluid, where the gradient of the phase of the order parameter is proportional to the superfluid velocity, this gradient of the pion field can be viewed as an axial supercurrent [20].

Next we introduce the electric field $\mathbf{E} = (0,0,E)$ parallel to $\mathbf{B}$ and, as explained above, an "axial electric field" $\boldsymbol{\epsilon} = (0,0,\epsilon)$. We thus have to add $-t(E \mp \epsilon)$ to the boundary value of A_3, such that, together with the axial supercurrent $\jmath_t$, we have

$$A_3(t, z = \pm\infty) = -t(E \mp \epsilon) \mp \jmath_t \,. \tag{3.2}$$

Due to the axial electric field we allow the supercurrent to become time-dependent. Strictly speaking the electric fields prevent us from using a thermodynamic description since it introduces a time-dependence and thus non-equilibrium physics. Therefore, our electric field should be considered infinitesimal. This is sufficient for our purpose since we can check the anomaly relations with an arbitrarily small electric field. Moreover, as mentioned above, the physical situations we are interested in do not require finite electric fields anyway.

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With the above ansatz, the YM and CS contributions to the action (2.2) become

$$S_{\rm YM} = \kappa M_{\rm KK}^2 \int d^4x \int_{-\infty}^{\infty} dz\, k(z) \left[-(\partial_z A_0)^2 + (\partial_z A_3)^2\right] , \tag{3.3a}$$

$$\begin{aligned} S_{\rm CS} = \frac{N_c}{12\pi^2} \int d^4x \int_{-\infty}^{\infty} dz \Big\{ &(\partial_2 A_1)\left[A_0(\partial_z A_3) - A_3(\partial_z A_0)\right] \\ &- A_1 \left[(\partial_2 A_0)(\partial_z A_3) - (\partial_2 A_3)(\partial_z A_0)\right] \Big\} . \end{aligned} \tag{3.3b}$$

We have written all terms which are needed to derive the equations of motion, including the ones that vanish on-shell. More specifically, the second line in the CS action (3.3b) vanishes on-shell because neither A_0 nor A_3 depends on x_2, but yields a finite contribution to the equations of motion. The equations of motion are

$$\partial_z(k\partial_z A_0) = 2\beta\partial_z A_3 \,, \tag{3.4a}$$

$$\partial_z(k\partial_z A_3) = 2\beta\partial_z A_0 \,, \tag{3.4b}$$

$$\partial_t(k\partial_z A_0) = 2\beta\partial_t A_3 \,, \tag{3.4c}$$

with the dimensionless magnetic field

$$\beta \equiv \frac{\alpha B}{M_{\rm KK}^2} \,, \tag{3.5}$$

and $\alpha \equiv 27\pi/(2\lambda)$. We defer the details of solving the equations of motion to appendix A.1. The results for the gauge fields and field strengths are

$$\begin{aligned} A_0(t,z) &= \mu_t - \mu_{5,t}\frac{\sinh(2\beta\arctan z)}{\sinh\beta\pi} \\ &\quad -(\jmath_t - \epsilon t)\left[\frac{\cosh(2\beta\arctan z)}{\sinh\beta\pi} - \coth\beta\pi\right] , \end{aligned} \tag{3.6a}$$

$$\begin{aligned} A_3(t,z) &= -tE - \mu_{5,t}\left[\frac{\cosh(2\beta\arctan z)}{\sinh\beta\pi} - \coth\beta\pi\right] \\ &\quad -(\jmath_t - \epsilon t)\frac{\sinh(2\beta\arctan z)}{\sinh\beta\pi} , \end{aligned} \tag{3.6b}$$

and

$$k\partial_z A_0 = -2\beta\left[\mu_{5,t}\frac{\cosh(2\beta\arctan z)}{\sinh\beta\pi} + (\jmath_t - \epsilon t)\frac{\sinh(2\beta\arctan z)}{\sinh\beta\pi}\right] , \tag{3.7a}$$

$$k\partial_z A_3 = -2\beta\left[\mu_{5,t}\frac{\sinh(2\beta\arctan z)}{\sinh\beta\pi} + (\jmath_t - \epsilon t)\frac{\cosh(2\beta\arctan z)}{\sinh\beta\pi}\right] . \tag{3.7b}$$

Here we have denoted

$$\mu_t \equiv \mu + \epsilon t\coth\beta\pi \,, \qquad \mu_{5,t} \equiv \mu_5 + Et\tanh\beta\pi \,, \tag{3.8}$$

i.e., both boundary values of A_0 become time-dependent through the electric fields. As can be seen from the detailed derivation in appendix A.1, this time-dependence is unavoidable.

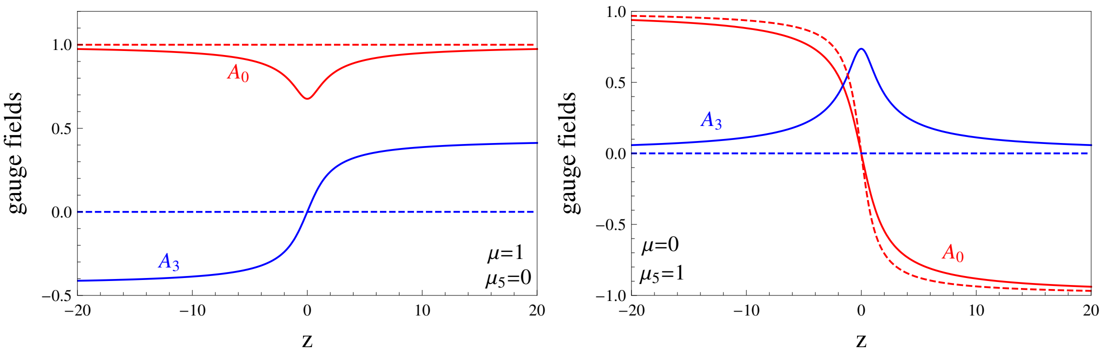

Figure 1. Gauge fields in the chirally broken phase as functions of the holographic coordinate $z \in [-\infty, \infty]$ for a finite quark chemical potential and vanishing axial chemical potential (left) and vice versa (right). Dashed lines: gauge fields with vanishing magnetic field; solid lines: gauge fields with a nonzero magnetic field $\beta = 0.6$. In both plots we have set the electric fields to zero, $E = \epsilon = 0$. The boundary values at $z = \pm\infty$ correspond to left- and right-handed quantities. The magnetic field induces an axial supercurrent (boundary value of A_3) in the case of a nonvanishing quark chemical potential. If both μ and μ_5 are nonvanishing, the gauge fields are neither symmetric nor antisymmetric in z. The analytic expressions for these curves are given in eqs. (3.6).

In figure 3.1 we plot the gauge fields for $E = \epsilon = 0$ at the minimum, i.e., after $\jmath_t$ has been determined to minimize the free energy, see below.

The thermodynamic potential $\Omega = \frac{T}{V} S_{\rm on-shell}$ is obtained from eqs. (A.13), treating t as an external parameter,

$$\Omega = \frac{8\kappa M_{\rm KK}^2}{3} \left\{ \left[(\jmath_t - \epsilon t)^2 - \mu_{5,t}^2\right] \rho(\beta) + \beta \left[\mu_t(\jmath_t - \epsilon t) + tE\mu_{5,t}\right] \right\} , \tag{3.9}$$

where we have abbreviated

$$\rho(\beta) \equiv \beta \coth \beta\pi + \frac{\pi\beta^2}{2\sinh^2 \beta\pi} \simeq \begin{cases} \dfrac{3}{2\pi} + \dfrac{\pi}{6}\beta^2 & \text{for } \beta \to 0 \\ |\beta| & \text{for } |\beta| \to \infty \end{cases} . \tag{3.10}$$

Minimization of Ω with respect to $\jmath_t$ yields the axial supercurrent

$$\jmath_t = -\frac{\beta\mu}{2\rho(\beta)} + \epsilon t \left[1 - \frac{\beta \coth \beta\pi}{2\rho(\beta)}\right] . \tag{3.11}$$

We see that the supercurrent depends neither on μ_5 nor on E. Therefore, at $t = 0$ it is simply the one-flavor limit of the result obtained in ref. [20] (where the D8 and $\overline{\rm D8}$ branes were identified with R and L, not with L and R, respectively, hence the different sign of the supercurrent).

3.2 Chirally symmetric phase

As explained in section 2.1, in the chirally symmetric phase the D8 and $\overline{\rm D8}$-branes are not connected. On both branes the holographic coordinate z now runs from $z = 0$, the

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black hole horizon, to the holographic boundary at $z=\infty$, and both branes yield separate contributions to the action,

$$S = (S^L_{\rm YM}+S^R_{\rm YM}) + (S^L_{\rm CS}-S^R_{\rm CS})\,. \tag{3.12}$$

The CS action assumes different overall signs on the D8- and $\overline{\rm D8}$-branes since its parity is odd. The YM and CS contributions are

$$S^h_{\rm YM} = \kappa M_{\rm KK}^2\theta^3\int d^4x\int_0^\infty dz\,\left[-k_0(z)(\partial_z A_0^h)^2 + k_3(z)(\partial_z A_3^h)^2\right]\,, \tag{3.13a}$$

$$\begin{aligned}S^h_{\rm CS} = \frac{N_c}{12\pi^2}\int d^4x\int_0^\infty dz\,\Big\{&(\partial_2 A_1^h)[A_0^h(\partial_z A_3^h)-A_3^h(\partial_z A_0^h)]\\ &- A_1^h[(\partial_2 A_0^h)(\partial_z A_3^h)-(\partial_2 A_3^h)(\partial_z A_0^h)]\Big\}\,,\end{aligned} \tag{3.13b}$$

with $h=L,R$. Here, we have defined the dimensionless temperature

$$\theta \equiv \frac{2\pi T}{M_{\rm KK}}\,. \tag{3.14}$$

In contrast to the broken phase there are different metric functions for temporal and spatial components of the gauge fields,

$$k_0(z)\equiv\frac{(1+z^2)^{3/2}}{z}\,,\qquad k_3(z)\equiv z(1+z^2)^{1/2}\,. \tag{3.15}$$

Note the slight difference in notation of the gauge fields: while in the broken phase $A_\mu^{L/R}(x)\equiv A_\mu(x,z=\pm\infty)$ always implies evaluation at the holographic boundary, here we label the bulk gauge fields $A_\mu^{L/R}(x,z)$ by L and R to indicate whether they live on the D8- or on the $\overline{\rm D8}$-brane. Since we always discuss broken and symmetric phases separately, this should not cause any confusion.

The equations of motion on the separate branes become

$$\partial_z(k_0\partial_z A_0^{L/R}) = \pm\frac{2\beta}{\theta^3}\partial_z A_3^{L/R}\,, \tag{3.16a}$$

$$\partial_z(k_3\partial_z A_3^{L/R}) = \pm\frac{2\beta}{\theta^3}\partial_z A_0^{L/R}\,, \tag{3.16b}$$

$$\partial_t(k_0\partial_z A_0^{L/R}) = \pm\frac{2\beta}{\theta^3}\partial_t A_3^{L/R}\,. \tag{3.16c}$$

Details of solving the equations of motion are presented in appendix A.2. The final solution for the gauge fields is

$$A_0^{L/R}(t,z) = (\mu_t\mp\mu_{5,t})\left[p(z)-\frac{p_0}{q_0}q(z)\right]\,, \tag{3.17a}$$

$$A_3^{L/R}(t,z) = -t(E\mp\epsilon)\pm\frac{\mu_t\mp\mu_{5,t}}{2\beta/\theta^3}\left[k_0\partial_z p-\frac{p_0}{q_0}(1+k_0\partial_z q)\right]\,, \tag{3.17b}$$

which is plotted in figure 3.2 for $E=\epsilon=0$. Below we shall also need the field strengths on the branes,

$$k_0\partial_z A_0^{L/R} = (\mu_t\mp\mu_{5,t})\left(k_0\partial_z p-\frac{p_0}{q_0}k_0\partial_z q\right)\,, \tag{3.18a}$$

$$k_3\partial_z A_3^{L/R} = \pm\frac{2\beta}{\theta^3}(\mu_t\mp\mu_{5,t})\left[p(z)-\frac{p_0}{q_0}q(z)\right]\,. \tag{3.18b}$$

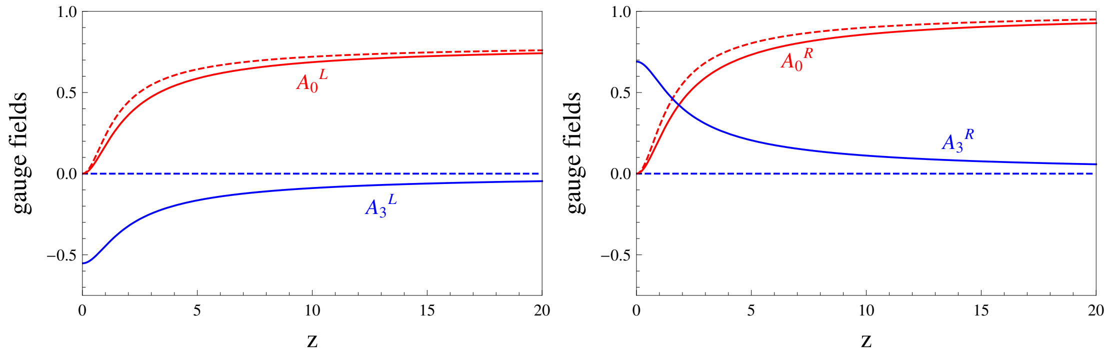

Figure 2. Left- and right-handed gauge fields (left and right panel, respectively) in the chirally symmetric phase as functions of the holographic coordinate $z \in [0, \infty]$ for $\mu = 0.9$, $\mu_5 = 0.1$. We have set the electric fields to zero, $E = \epsilon = 0$. The temporal components $A_0^{L/R}$ approach the chemical potentials $\mu \mp \mu_5$ at the boundary $z = \infty$, while the spatial components $A_3^{L/R}$ vanish at $z = \infty$. A finite magnetic field (solid lines, here $\beta/\theta^3 = 0.6$) distorts the gauge fields compared to the case of vanishing magnetic fields (dashed lines). In particular, the spatial component develops a nonzero value at $z = 0$. The different sign of this value for left- and right-handed fields, i.e., on the D8- and $\overline{\text{D8}}$-branes, ensures the correct parity behavior of the fields. The analytical expressions for these curves are given in eqs. (3.17).

The functions $p(z)$, $q(z)$ are hypergeometric functions which we defined in eqs. (A.19) and which depend on the ratio β/θ^3. Their values at $z = 0$ are denoted by p_0, q_0, see eqs. (A.22), and the ratio p_0/q_0 behaves for small and large magnetic fields as

$$\frac{p_0}{q_0} \simeq \begin{cases} 1 + (2\beta/\theta^3)^2(\ln 4 - 1) & \text{for } \beta/\theta^3 \to 0 \\ 2|\beta|/\theta^3 & \text{for } |\beta|/\theta^3 \to \infty \end{cases} . \tag{3.19}$$

The boundary values of the temporal components are $A_0^{L/R}(t, z = \infty) = \mu_t \mp \mu_{5,t}$ with

$$\mu_t \equiv \mu + 2t\epsilon \frac{\beta}{\theta^3}\frac{q_0}{p_0}, \qquad \mu_{5,t} \equiv \mu_5 + 2tE\frac{\beta}{\theta^3}\frac{q_0}{p_0}. \tag{3.20}$$

It is instructive to compare this behavior of the axial chemical potential with the expected behavior for free fermions in a magnetic field. To this end, consider the lowest Landau level in which the spin of all (say, positively charged) fermions is aligned parallel to the magnetic field. As a consequence, all right- (left-) handed massless fermions move parallel (antiparallel) to the magnetic field. An electric field parallel to the magnetic field now shifts all momenta in the positive 3-direction by an amount Et. Consequently, some of the left- handed fermions are converted into right-handed fermions and a shift Et is induced in the difference of right- and left-handed Fermi momenta, $(p_F^R - p_F^L)/2 = Et$ [2, 41]. Interpreting $\mu_{5,t}$ as $(p_F^R - p_F^L)/2$ (strictly speaking there is no well-defined Fermi momentum in our model), eq. (3.20) reproduces this shift for asymptotically large magnetic fields because in this case $q_0/p_0 \to \theta^3/(2\beta)$. For small magnetic fields $q_0/p_0 \to 1$, and the shift becomes linear in the magnetic field. Since $\beta/\theta^3 \propto B/T^3$, we can in principle also obtain $\mu_{5,t} = \mu_5 + tE$ for sufficiently small temperatures and fixed magnetic field. However, we cannot reduce the temperature arbitrarily in the above expression since below the critical temperature T_c we are in the chirally broken phase. In this case the analogous, temperature-independent relation in eq. (3.8) holds.

The free energy, obtained from the YM and CS contributions (A.33), is

$$\Omega = -\frac{2\kappa M_{\rm KK}^2}{3}\left[\theta^3(\mu_t^2+\mu_{5,t}^2)\eta - 4\beta\, t\,(\mu_t\epsilon+\mu_{5,t}E)\right] , \tag{3.21}$$

where we introduced the function

$$\eta(\beta/\theta^3) \equiv I_0 - (2\beta/\theta^3)^2 I_3 + 2\frac{p_0}{q_0} \simeq \begin{cases} 3+(2\beta/\theta^3)^2(\ln 4-1) & \text{for } \beta/\theta^3\to 0 \\ 4|\beta|/\theta^3 & \text{for } |\beta|/\theta^3\to\infty \end{cases} , \tag{3.22}$$

with integrals I_0 and I_3 defined in eqs. (A.34).

3.3 Ambiguity of currents

In the following discussion we restrict ourselves to the symmetric phase, but one can easily check that all arguments hold for the broken phase as well. Let us first give the analogue of the definition of the currents (2.7) for the symmetric phase,

$$\mathcal{J}^\mu_{L/R} = -\left(2\kappa M_{\rm KK}^2\theta^3 k_{(\mu)} F^{z\mu}_{L/R} \mp \frac{N_c}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}A_\nu^{L/R}F_{\rho\sigma}^{L/R}\right)_{z=\infty} , \tag{3.23}$$

where the notation $k_{(\mu)}$ (no summation over μ) indicates the different metric functions for temporal and spatial components, see eq. (3.15). Equivalently, and in analogy to eq. (2.8), we can write the currents in the symmetric phase as

$$\mathcal{J}^\mu_{L/R} = -\left.\frac{\partial\mathcal{L}}{\partial\,\partial_z A_\mu^{L/R}}\right|_{z=\infty} . \tag{3.24}$$

We shall now show that the currents defined via these equations are different from the ones obtained via taking the derivative of the free energy (3.21) with respect to the corresponding source. We do so for the vector density, i.e., the sum of left- and right-handed 0-components of the currents. One can observe the same ambiguity for the other nonvanishing components. The following arguments do not depend on the electric fields, so we temporarily set $\epsilon = E = 0$ for simplicity (and for a truly equilibrated situation). From the definition (3.23) and the gauge fields (3.17) and field strengths (3.18) we obtain

$$\mathcal{J}^0 = \mathcal{J}^0_R + \mathcal{J}^0_L = 4\kappa M_{\rm KK}^2\theta^3\frac{p_0}{q_0}\mu\,. \tag{3.25}$$

On the other hand, the free energy Ω of the system should yield the number density via the thermodynamic relation

$$n = -\frac{\partial\Omega}{\partial\mu} = \frac{4\kappa M_{\rm KK}^2\theta^3}{3}\mu\,\eta\,. \tag{3.26}$$

This result shows that $n \neq \mathcal{J}^0$ which, given spatial homogeneity, is inconsistent. This inconsistency is absent for vanishing magnetic fields: using the behavior of the functions

p_0/q_0 and η from eqs. (3.19) and (3.22) one sees that for $\beta = 0$ the expressions for $\mathcal{J}^0$ and n are identical. We can formulate this observation in a more general way. To this end we write the left- and right-handed on-shell Lagrangians, i.e., the integrands of the on-shell action (3.12), as $\mathcal{L}_h(A_0^h, \partial_z A_0^h, A_3^h, \partial_z A_3^h)$, where all arguments of $\mathcal{L}_h$ depend on the chemical potentials μ_h with $h = L, R$ and $\mu_{L/R} = \mu \mp \mu_5$. Then, with $\Omega_h = T/V \int d^4x \int_0^\infty dz\, \mathcal{L}_h$ we have

$$
\begin{aligned}
\frac{\partial \Omega_h}{\partial \mu_h} &= \frac{T}{V}\sum_{i=0,3}\int d^4x \int_0^\infty dz \left(\frac{\partial \mathcal{L}_h}{\partial A_i^h}\frac{\partial A_i^h}{\partial \mu_h} + \frac{\partial \mathcal{L}_h}{\partial\, \partial_z A_i^h}\frac{\partial\, \partial_z A_i^h}{\partial \mu_h}\right) \\
&= \frac{T}{V}\sum_{i=0,3}\left[\int d^4x\, \frac{\partial \mathcal{L}_h}{\partial\, \partial_z A_i^h}\frac{\partial A_i^h}{\partial \mu_h}\bigg|_{z=0}^{z=\infty} + \int d^4x \int_0^\infty dz\, \partial_2 \frac{\partial \mathcal{L}_h}{\partial\, \partial_2 A_i^h}\frac{\partial A_i^h}{\partial \mu_h}\right], \qquad (3.27)
\end{aligned}
$$

where we have used partial integration and added and subtracted the derivative term in x_2 in order to make use of the equations of motion. Now we use

$$
\frac{\partial A_0^h}{\partial \mu_h}\bigg|_{z=\infty} = 1\,, \qquad \frac{\partial A_0^h}{\partial \mu_h}\bigg|_{z=0} = \frac{\partial A_3^h}{\partial \mu_h}\bigg|_{z=\infty} = \frac{\partial \mathcal{L}_h}{\partial\, \partial_z A_3^h}\bigg|_{z=0} = 0\,, \qquad (3.28)
$$

which follows from the explicit solutions (3.17), whose behavior at $z = 0$, $z = \infty$ is obtained with the help of eqs. (A.21), (A.22), and (A.23). With these relations and the definition of the current from eq. (3.24) we obtain

$$
-\frac{\partial \Omega}{\partial \mu} = \mathcal{J}^0 - \frac{T}{V}\sum_{h=L,R}\sum_{i=0,3}\int d^4x \int_0^\infty dz\, \partial_2 \frac{\partial \mathcal{L}_h}{\partial\, \partial_2 A_i^h}\frac{\partial A_i^h}{\partial \mu_h}\,. \qquad (3.29)
$$

This is the general form of the difference between the density defined as the 0-component of the current defined via eq. (3.24) and the density defined via the thermodynamic relation (3.26). For an explicit check of this relation one inserts the expressions

$$
\partial_2 \frac{\partial \mathcal{L}_{L/R}}{\partial\, \partial_2 A_0^{L/R}} = \pm \frac{4\kappa M_{\rm KK}^2}{3}\beta \partial_z A_3^{L/R}\,, \qquad (3.30a)
$$

$$
\partial_2 \frac{\partial \mathcal{L}_{L/R}}{\partial\, \partial_2 A_3^{L/R}} = \mp \frac{4\kappa M_{\rm KK}^2}{3}\beta \partial_z A_0^{L/R}\,, \qquad (3.30b)
$$

and

$$
\frac{\partial A_0^{L/R}}{\partial \mu_{L/R}} = p(z) - \frac{p_0}{q_0} q(z)\,, \qquad (3.31a)
$$

$$
\frac{\partial A_3^{L/R}}{\partial \mu_{L/R}} = \pm \frac{\theta^3}{2\beta}\left[k_0 \partial_z p - \frac{p_0}{q_0}(1 + k_0 \partial_z q)\right]\,, \qquad (3.31b)
$$

into eq. (3.29). This yields

$$
-\frac{\partial \Omega}{\partial \mu} = \mathcal{J}^0 + \frac{4\kappa M_{\rm KK}^2 \theta^3}{3}\mu \left[I_0 - \left(\frac{2\beta}{\theta^3}\right)^2 I_3 - \frac{p_0}{q_0}\right]\,. \qquad (3.32)
$$

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With the definition (3.22) this confirms the difference between n and $\mathcal{J}^0$ obtained from eqs. (3.25) and (3.26).

From the general form (3.29) we see that the additional term is a boundary term at the spatial boundary of the system. This suggests that the ambiguity in the currents is related to the terms we have dropped in section 2.2, see eqs. (2.6). These terms correspond to currents at the spatial boundary and disappear in the presence of a homogeneous magnetic field only if the variation $\delta A_\mu(x, z = \pm\infty)$ can be chosen to vanish at this boundary. So this problem might be resolved by considering more complicated, spatially inhomogeneous gauge fields. In our homogeneous ansatz, it is however a priori not clear which definition of the currents corresponds to the correct physics.

A possible solution to this ambiguity was suggested and applied in refs. [18, 21, 42]. In these references, the CS action has been modified according to

$$\begin{aligned} S'^{h}_{\rm CS} = \frac{N_c}{12\pi^2}\int d^4x \int_0^\infty dz\, \bigg\{ &\frac{3}{2}(\partial_2 A_1^h)\left[A_0^h(\partial_z A_3^h) - A_3^h(\partial_z A_0^h)\right] \\ &- \frac{1}{2}(\partial_z A_1^h)\left[A_0^h(\partial_2 A_3^h) - A_3^h(\partial_2 A_0^h)\right]\bigg\}\,. \end{aligned} \tag{3.33}$$

This modified action (marked by a prime) is obtained from the original CS action (3.13b) by adding a boundary term at the holographic and the spatial boundary,

$$S'^{h}_{\rm CS} = S^h_{\rm CS} + S^h_{\rm boundary}\,, \tag{3.34}$$

with

$$\begin{aligned} S^h_{\rm boundary} = \frac{N_c}{24\pi^2}\bigg\{ &\int d^3x \int_0^\infty dz\, A_1^h\left[A_0^h(\partial_z A_3^h) - A_3^h(\partial_z A_0^h)\right]_{x_2} \\ &- \int d^4x\, A_1^h\left[A_0^h(\partial_2 A_3^h) - A_3^h(\partial_2 A_0^h)\right]_{z=0}^{z=\infty}\bigg\}\,. \end{aligned} \tag{3.35}$$

Note that this boundary term cannot be considered as a holographic counterterm since it involves an integration over z. From eq. (3.33) we see that the addition of $S^h_{\rm boundary}$ effectively amounts to a multiplication of the on-shell action by $3/2$ because the second line in eq. (3.33) vanishes on-shell. The benefit of the modified action is that the integrand on the right-hand side of eq. (3.29) vanishes now, i.e., there is no ambiguity in the currents anymore.

Modifications of a CS action by boundary terms are in fact sometimes necessary in order to ensure validity of the variational principle in the presence of nontrivial boundary values [43]. However, this is not what the above modification is achieving. Instead, it leads to gauge invariance under the residual gauge transformations $A_1 \to A_1 + \partial_1\Lambda(x_1)$ which are compatible with the boundary conditions of our ansatz and which do not vanish at spatial infinity $x_2 = \pm\infty$ [18]. In fact, by this modification one loses all anomalies for the (now uniquely defined) currents, as we show now. To this end, we switch on the electric fields again. Then, the currents of the original action in the symmetric phase are

$$T > T_c: \quad \mathcal{J}^0_{L/R} = 2\kappa M_{\rm KK}^2 \frac{p_0}{q_0}\left[\theta^3(\mu_t \mp \mu_{5,t}) \pm \frac{2}{3}\frac{q_0}{p_0}\beta(E\mp\epsilon)t\right], \tag{3.36a}$$

$$\mathcal{J}^1_{L/R} = 0\,, \tag{3.36b}$$

$$\mathcal{J}^2_{L/R} = \pm\frac{4\kappa M_{\rm KK}^2}{3}\beta\, x_2(E\mp\epsilon)\,, \tag{3.36c}$$

$$\mathcal{J}^3_{L/R} = \mp 4\kappa M_{\rm KK}^2\beta\,(\mu_t\mp\mu_{5,t})\left(1-\frac{1}{3}\right), \tag{3.36d}$$

where we have used the definition (3.23) and the gauge fields (3.17) and field strengths (3.18). All terms containing a 1/3 originate from the CS contribution of the current, i.e., from the second term in eq. (3.23). All other terms are YM contributions. In particular, the 2-component of the current is a pure CS term. This component is unphysical because it depends on our choice to introduce the magnetic field via the gauge field A_1. We could have introduced the same magnetic field via A_2 or a combination of A_1 and A_2, in which case the 1- and 2-components of the currents would have been different. We shall see below that Bardeen's counterterm solves this problem by canceling the 2-component. Here, however, it gives a nonzero contribution to the anomaly. Namely, the divergence of the (unmodified) currents becomes

$$\partial_\mu \mathcal{J}^\mu_{L/R} = \partial_t \mathcal{J}^0_{L/R} + \partial_2\mathcal{J}^2_{L/R} = \mp\frac{N_c}{12\pi^2}B(E\mp\epsilon)\,, \tag{3.37}$$

where we have used the definition of $\mu_{5,t}$ (3.20) and $\kappa M_{\rm KK}^2\beta \equiv N_c B/16\pi^2$. This is exactly the consistent anomaly (2.10), because

$$\mp\frac{N_c}{48\pi^2}F_{\mu\nu}^{L/R}\widetilde{F}^{\mu\nu}_{L/R}\bigg|_{z=\infty} = \mp\frac{N_c}{12\pi^2}B(E\mp\epsilon)\,. \tag{3.38}$$

The new currents $\mathcal{J}'^\mu_{L/R}$ from the modified action are simply obtained by multiplying the CS contribution of the currents (3.36) by 3/2. Doing so in the explicit results (3.36), this yields

$$\partial_\mu \mathcal{J}'^\mu_{L/R} = \partial_t\mathcal{J}'^0_{L/R} + \partial_2\mathcal{J}'^2_{L/R} = 0\,, \tag{3.39}$$

which can also be inferred in generality from (2.10). Consequently, the anomaly has disappeared. In other words, the new vector and the axial currents are both conserved. Nevertheless, one finds nonzero currents in the direction of the magnetic field. Multiplying the CS contribution in eq. (3.36d) by 3/2 one obtains $\mathcal{J}'^3_R + \mathcal{J}'^3_L = N_c/(4\pi^2)\,B\mu_5$ and $\mathcal{J}'^3_R - \mathcal{J}'^3_L = N_c/(4\pi^2)\,B\mu$ [18], both of which are 1/2 times the results of refs. [2] and [9, 44], respectively (cf. section 4.1 below).

In the remainder of the paper we shall again consider the full, unmodified chiral currents (2.7) which contain the complete covariant anomaly upon inclusion of the counterterm (2.13).

4 Axial and vector currents

In this section we shall use the results from the previous sections to compute the vector and axial currents in the presence of a magnetic field and a quark chemical potential μ as well as an axial chemical potential μ_5. We have seen that a consistent definition of the currents is not obvious in the given setup. We shall focus on the definition of the currents presented in section 2.2 since they reproduce, together with Bardeen's counterterm, the correct anomaly. Before going into the details, let us explain the expected physics behind the vector current, in other words the chiral magnetic effect.

4.1 The chiral magnetic effect

A (noncentral) heavy-ion collision, where the chiral magnetic effect is expected to occur, is more complicated than we can capture with our thermodynamic description. The physical situation and its simplified description within a thermodynamic approach is as follows [2]. In the high-temperature phase gluonic sphaleron configurations with nonzero winding number should be produced with relatively high probability, inducing an imbalance in left- and right-handed quarks due to the QCD anomaly and thus a nonzero axial number density n_5. In the simple picture applied in ref. [2], such chirality changing transitions are assumed to have taken place in a nonequilibrium situation, after which in equilibrium a finite n_5 is no longer changed by the QCD anomaly. The QED anomaly on the other hand does not change n_5 as long as only a magnetic field is present, so n_5 can be considered a conserved quantity for which we may introduce μ_5 as the corresponding chemical potential. (We have introduced also electric fields above for the sake of checking the axial anomaly, but shall set them to zero in the final results.) Nonzero quark masses and/or nonzero chiral condensates can be expected to lead to a decay of n_5. In the given context, it is thus questionable to apply the equilibrium description also to the chirally broken phase, and strictly speaking our approach should be extended to a nonequilibrium calculation.

Let us now briefly recapitulate the physics behind the occurrence of the vector current which constitutes the chiral magnetic effect in terms of a (quasi)particle picture [1, 2]. Suppose the magnetic field leads to a spin polarization of all fermions, i.e., the spins of all quarks are aligned parallel or antiparallel to the magnetic field depending on their charge being positive or negative. Massless right-handed fermions, which have positive helicity, have momenta parallel to their spin, so they move parallel to the magnetic field if they have positive charge, and antiparallel otherwise. For left-handed fermions with negative helicity, the situation is exactly reversed. If there are more right-handed than left-handed fermions, $n_5 > 0$, there is a resulting net electromagnetic current parallel to the magnetic field. (Antifermions have helicity opposite to chirality but also opposite charge, so they give a current in the same direction.) For weakly-coupled fermions this picture applies since in the lowest Landau level indeed all fermions have their spins aligned in the direction of the magnetic field according to their charge. The chiral magnetic effect then results solely from the lowest Landau level. The contribution of fermions in higher Landau levels, where both parallel and antiparallel spin projections are populated, cancels out. This can be seen explicitly upon using the thermodynamic potential of free fermions in a magnetic field, and

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the resulting current is [2]

$$J=\frac{N_c}{2\pi^2}\mu_5 B\,. \tag{4.1}$$

In our model we cannot see any Landau levels directly. Therefore, let us also repeat another, apparently more general, derivation of the chiral magnetic effect. It is based on an energy conservation argument originally pointed out by Nielsen and Ninomiya [45] and applied in ref. [2]. It states that an energy $2\mu_5$ is needed to replace a fermion at the left-handed Fermi surface μ_L with a fermion at the right-handed Fermi surface μ_R. This conversion changes the axial number density by $dN_5 = 2$, i.e., the energy actually is $\mu_5 dN_5$. Such a change in N_5 is possible through the QED anomaly in the presence of an electric and a (non-orthogonal) magnetic field. The energy can thus be provided by an electric current. Hence, the change in N_5 per unit volume and time is given by the electric power per unit volume $\mathbf{J}\cdot\mathbf{E}$,

$$\mathbf{J}\cdot\mathbf{E}=\mu_5\frac{dn_5}{dt}\,. \tag{4.2}$$

Now we know from the (covariant) axial anomaly (2.19b) that, with $\nabla\cdot\mathbf{J}_5=0$ and $n_5=\mathcal{J}_5^0$,

$$\frac{dn_5}{dt}=\frac{N_c}{2\pi^2}\mathbf{B}\cdot\mathbf{E}\,. \tag{4.3}$$

Inserting this into eq. (4.2) and taking $\mathbf{B}$ and $\mathbf{E}$ parallel yields a current $\mathbf{J}$ in the direction of $\mathbf{B}$, given by eq. (4.1). Note that this argument only works for a nonzero, although arbitrarily small, electric field.

Besides the vector current we shall also compute the axial current for which the analogous topological result is [9, 44]

$$J_5=\frac{N_c}{2\pi^2}\mu B\,, \tag{4.4}$$

which is proportional to the ordinary quark chemical potential and thus of potential interest in neutron and quark star physics.

4.2 Currents with consistent anomaly

We have already computed the currents in the symmetric phase, see eqs. (3.36). The analogue for the broken phase, obtained from the definition (2.7) and the gauge fields and field strengths (3.6) and (3.7) is

$$T<T_c:\quad \mathcal{J}^0_{L/R}=\pm 4\kappa M_{\rm KK}^2\beta\left[-\mu_{5,t}\coth\beta\pi\mp(\jmath_t-\epsilon t)+\frac{Et\pm(\jmath_t-\epsilon t)}{3}\right]\,, \tag{4.5a}$$

$$\mathcal{J}^1_{L/R}=0\,, \tag{4.5b}$$

$$\mathcal{J}^2_{L/R}=\pm\frac{4\kappa M_{\rm KK}^2}{3}\beta\, x_2\left[E\mp\epsilon\frac{\beta\coth\beta\pi}{2\rho(\beta)}\right]\,, \tag{4.5c}$$

$$\mathcal{J}^3_{L/R}=\mp 4\kappa M_{\rm KK}^2\beta\left[\mp\mu_{5,t}-(\jmath_t-\epsilon t)\coth\beta\pi-\frac{\mu_t\mp\mu_{5,t}}{3}\right]\,. \tag{4.5d}$$

Again, to make the origin of the various terms transparent we have written the CS contributions separately. All terms containing a $1/3$ come from the CS action. As for the

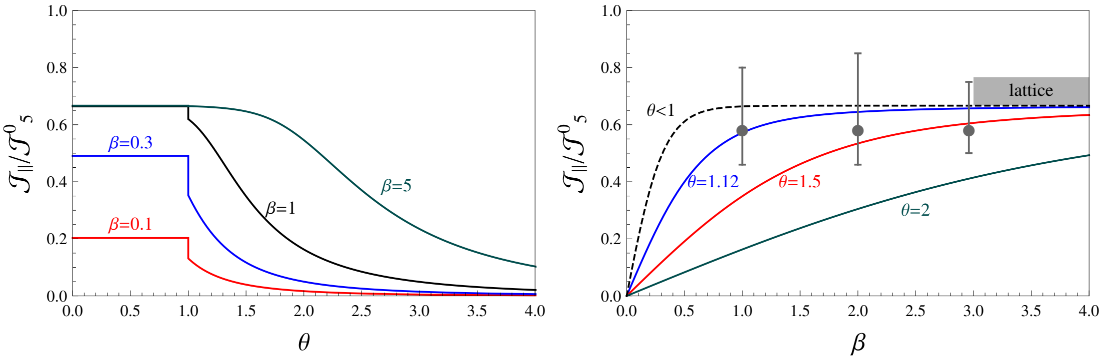

Figure 3. Vector current $\mathcal{J}_\parallel = \mathcal{J}_3$ per imbalance of right- and left-handed fermions $n_5 = \mathcal{J}_5^0$ as a function of the dimensionless temperature $\theta = 2\pi T/M_{\rm KK}$ for different values of the dimensionless magnetic field $\beta = \alpha B/M_{\rm KK}^2 \simeq B/(0.35\,{\rm GeV}^2) \simeq B/(2\cdot 10^{19}\,{\rm G})$ (left panel) and as a function of β for different values of θ (right panel). The critical temperature for the chiral phase transition is $T_c = M_{\rm KK}/(2\pi)$, i.e., $\theta_c = 1$. The currents in this plot are obtained using the consistent anomaly, i.e., *before* adding Bardeen's counterterm to fulfill the covariant anomaly. After this term is added, the vector current vanishes exactly. The left plot shows the discontinuity at the first order chiral phase transition. This discontinuity vanishes for asymptotically large magnetic fields. The right panel shows that the current saturates at a value of $\mathcal{J}_\parallel = \frac{2}{3}\mathcal{J}_5^0$, in very good agreement with the lattice data for the root mean square value of fluctuations of vector currents and axial densities [22]. The three lattice data points are taken from from figures 4 and 8 of ref. [22] and correspond to a temperature $T = 1.12\,T_c$. The shaded area indicates the results read off from figure 11 of ref. [22] for the cleaner case of a ($T = 0$) instanton-like configuration, where the corresponding points lie between $\mathcal{J}_\parallel/\mathcal{J}_5^0 \simeq 0.66 - 0.77$ for magnetic fields of $\beta \simeq 3.0$ and higher.

symmetric phase, we can easily check the consistent anomaly (2.10). Using the expression for the supercurrent (3.11) and $\kappa M_{\rm KK}^2\beta \equiv N_c B/16\pi^2$ we find

$$\partial_t \mathcal{J}^0_{L/R} + \partial_2 \mathcal{J}^2_{L/R} = \mp \frac{N_c}{12\pi^2} B\left(E \mp \epsilon \frac{\beta\coth\beta\pi}{2\rho(\beta)}\right) \tag{4.6}$$

and

$$\mp \frac{N_c}{48\pi^2} F^{L/R}_{\mu\nu}\widetilde{F}^{\mu\nu}_{L/R} = \mp \frac{N_c}{12\pi^2} B\left(E \mp \epsilon \frac{\beta\coth\beta\pi}{2\rho(\beta)}\right), \tag{4.7}$$

which confirms eq. (2.10). The axial electric field seems to be modified by a complicated function of the dimensionless magnetic field. This originates from the mixing of the electric field with the supercurrent, which both enter the boundary value of A_3. We shall see that this somewhat strange structure disappears after adding Bardeen's counterterm.

From the results (3.36) and (4.5) we may compute the vector currents in the chirally symmetric and broken phases. For the following results we set $E = \epsilon = 0$. We find the same result for both phases which is

$$\mathcal{J}_3 = (\mathcal{J}_{\rm YM} + \mathcal{J}_{\rm CS})_3 = \left(1 - \frac{1}{3}\right)\frac{N_c}{2\pi^2} B\mu_5\,. \tag{4.8}$$

This differs by a factor 2/3 from the topological result (4.1). This difference is not surprising since we have not implemented the covariant anomaly yet. To this end we must add

Bardeen's counterterm. Before doing so we point out an interesting result which we obtain by considering the ratio of the vector current over the axial density. From eqs. (3.36) and (4.5) we obtain

$$\frac{\mathcal{J}_3}{\mathcal{J}_5^0} = \frac{2}{3}\begin{cases} \dfrac{2\beta}{\theta^3}\dfrac{q_0}{p_0} & \text{for } T > T_c \\ \tanh\beta\pi & \text{for } T < T_c \end{cases}, \tag{4.9}$$

which is displayed in figure 4.2. In the left panel we see that the first order chiral phase transition manifests itself in the discontinuity of the ratio $\mathcal{J}_3/\mathcal{J}_5^0$. Interestingly, the jump vanishes for asymptotically large magnetic fields. The curves for the symmetric phase are in qualitative agreement with the weak-coupling results in figure 2 of ref. [2]. The right panel shows an intriguing agreement of our result for the ratio $\mathcal{J}_3/\mathcal{J}_5^0$ with recent lattice results [22] for the root mean square values of electric currents and chiral densities at large magnetic fields. While the very good numerical agreement might be a coincidence, the lattice results as well as our result clearly show an asymptotic value significantly smaller than 1. If it were 1, the entire imbalance $\mathcal{J}_5^0$ in right-and left-handed fermions, i.e., all excess right-handed fermions, would contribute to the current for asymptotically large magnetic fields. This is expected at least at weak coupling. In this case, for sufficiently large magnetic fields, all fermions populate the lowest Landau level. Consequently, since the current originates solely from the lowest Landau level, as explained above, one expects $\mathcal{J}_3/\mathcal{J}_5^0 \to 1$. This is confirmed in the weak-coupling calculation of ref. [2], see figure 6 in this reference. The lattice result suggests that at strong coupling there may be important modifications to the Landau level picture. We emphasize, however, that figure 4.2 is not yet our final physical prediction. The model has not yet been appropriately renormalized in order to exhibit the covariant anomaly.

We also remark that the scale of our magnetic fields is very large such that for all physical applications, be it in heavy-ion collisions or in magnetars, the limit of weak magnetic fields is sufficient. In fact, a dimensionless magnetic field $\beta = 1$ corresponds roughly to a magnetic field $B \simeq 2 \cdot 10^{19}$ G if one follows refs. [16, 17] and sets $N_c = 3$, $M_{\rm KK} \simeq 949$ MeV, $\kappa \simeq 0.00745$, which fits the experimental values for the pion decay constant and the rho meson mass. Therefore, for all applications we have in mind, $\beta \ll 1$. Moreover, one should recall that we have used the YM approximation for the DBI action. This is of course a good approximation for small magnetic fields, but our extrapolation to larger magnetic fields may be subject to modification when the full DBI action is employed. On the other hand, in the limit $\beta \gg 1$, the results for the on-shell action, eqs. (A.13) and (A.33) exhibit a strong suppression of the YM action compared to the CS action. This suggests that our approximation is reliable also for asymptotically large magnetic fields.

4.3 Currents with covariant anomaly and absence of the chiral magnetic effect

The next step is to include Bardeen's counterterm (2.13) in order to implement the covariant anomaly. In the broken phase there is a slight complication because the counterterm should only involve genuine background gauge fields, and not those boundary values of the bulk gauge fields that due to the gauge choice $A_z = 0$ represent gradients of the pion

	$\bar{\mathcal{J}}^0$	$\bar{\mathcal{J}}^0_5$	$\bar{\mathcal{J}}_\parallel$	$\bar{\mathcal{J}}^5_\parallel$
$T > T_c$	$\frac{N_c}{4\pi^2}\mu B\,\frac{\theta^3}{\beta}\frac{p_0}{q_0}$	$\frac{N_c}{4\pi^2}\mu_5 B\,\frac{\theta^3}{\beta}\frac{p_0}{q_0}$	0	$\frac{N_c}{2\pi^2}\mu B$
$T < T_c$	$\frac{N_c}{6\pi^2}\mu B\,\frac{\beta}{\rho}$	$\frac{N_c}{2\pi^2}\mu_5 B\,\coth\beta\pi$	0	$\frac{N_c}{4\pi^2}\mu B\,\frac{\beta\coth\beta\pi}{\rho}$

Table 1. Vector and axial densities $\bar{\mathcal{J}}^0$, $\bar{\mathcal{J}}^0_5$, and vector and axial currents $\bar{\mathcal{J}}_\parallel$, $\bar{\mathcal{J}}^5_\parallel$ in the direction of the magnetic field B after adding Bardeen's counterterm. All results are given as functions of the dimensionless temperature $\theta = 2\pi T/M_{\rm KK}$ and the dimensionless magnetic field $\beta = \alpha B/M_{\rm KK}^2$. The densities in the chirally symmetric phase ($T > T_c$) depend on temperature; the ratio p_0/q_0 behaves as $p_0/q_0 \to 1$ for $\beta/\theta^3 \to 0$ and $p_0/q_0 \to 2\beta/\theta^3$ for $\beta/\theta^3 \to \infty$. In the chirally broken phase ($T < T_c$), all quantities are independent of temperature; the function ρ behaves as $\rho \to 3/(2\pi)$ for $\beta \to 0$ and $\rho \to \beta$ for $\beta \to \infty$. The vector current vanishes exactly in both symmetric and broken phases; this indicates the absence of a chiral magnetic effect in the Sakai-Sugimoto model, see discussion in the text. For the axial current, the temperature-independent topological result is reproduced in the symmetric phase. See figure 4.3 for the comparison of the axial currents in the symmetric and broken phases.

field. This means that we have to subtract the time-independent part of the supercurrent $\jmath = -\beta\mu/2\rho$ from the boundary values of the $A_3^{L/R}$. Then, with eq. (2.15) and the value of c from eq. (2.18), the contributions of the counterterm to the currents are

$$\begin{aligned}
T < T_c: \quad \Delta\mathcal{J}^0_{L/R} &= \pm\frac{2\kappa M_{\rm KK}^2}{3}\beta[3(A_3^{R/L} \mp \jmath) - (A_3^{L/R} \pm \jmath)] \\
&= \mp\frac{4\kappa M_{\rm KK}^2}{3}\beta t\left[E \pm 2\epsilon\frac{\beta\coth\beta\pi}{2\rho(\beta)}\right], && \text{(4.10a)} \\
\Delta\mathcal{J}^1_{L/R} &= 0\,, && \text{(4.10b)} \\
\Delta\mathcal{J}^2_{L/R} &= \mp\frac{4\kappa M_{\rm KK}^2}{3}\beta\, x_2\left[E \mp \frac{\beta\coth\beta\pi}{2\rho(\beta)}\right], && \text{(4.10c)} \\
\Delta\mathcal{J}^3_{L/R} &= \mp\frac{2\kappa M_{\rm KK}^2}{3}\beta(3A_0^{R/L} - A_0^{L/R}) = \mp\frac{4\kappa M_{\rm KK}^2}{3}\beta(\mu_t \pm 2\mu_{5,t})\,. && \text{(4.10d)}
\end{aligned}$$

The first observation is that the 2-component of the current vanishes after adding the counterterm. As mentioned above, this 2-component was unphysical anyway. The cancellation of this component is therefore, besides the covariant anomaly, another sign for the necessity of the counterterm. The covariant anomaly is now correctly contained in the renormalized currents $\bar{\mathcal{J}}^\mu_{L/R} = \mathcal{J}^\mu_{L/R} + \Delta\mathcal{J}^\mu_{L/R}$. This is clear by construction, and can also be verified explicitly: adding eqs. (4.10) to eqs. (4.5), yields

$$\partial_\mu\bar{\mathcal{J}}^\mu = 0\,, \qquad \partial_\mu\bar{\mathcal{J}}^\mu_5 = \partial_t\bar{\mathcal{J}}^0_5 = \frac{N_c}{2\pi^2}BE\,, \tag{4.11}$$

with the vector and axial currents $\bar{\mathcal{J}}^\mu$, $\bar{\mathcal{J}}^\mu_5$.

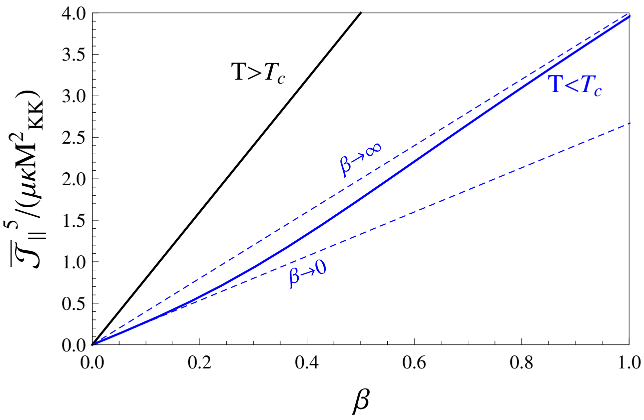

Figure 4. Axial current $\bar{\mathcal{J}}_\parallel^5$ in chirally symmetric ($T > T_c$) and chirally broken ($T < T_c$) phases in the presence of chemical potential μ as functions of the dimensionless magnetic field $\beta = \alpha B/M_{\rm KK}^2$, i.e., the magnetic field in units of $M_{\rm KK}^2/\alpha \simeq 2 \cdot 10^{19}\,$G. In the symmetric phase the current is linear in B, while the current in the broken phase is linear only for small B and asymptotically large B as indicated by the dashed lines. Due to the huge scale for the magnetic field, the small-B approximation for the axial current is sufficient for astrophysical applications. In this case the current in the broken phase is simply 1/3 of the current in the symmetric phase. The analytical results are given in table 1 where it is also shown that the vector current vanishes.

The contributions of the counterterm to the currents in the symmetric phase are

$$T > T_c: \quad \Delta\mathcal{J}^0_{L/R} = \pm\frac{2\kappa M_{\rm KK}^2}{3}\beta(3A_3^{R/L} - A_3^{L/R}) = \mp\frac{4\kappa M_{\rm KK}^2}{3}\beta t(E \pm 2\epsilon)\,, \tag{4.12a}$$

$$\Delta\mathcal{J}^1_{L/R} = 0\,, \tag{4.12b}$$

$$\Delta\mathcal{J}^2_{L/R} = \mp\frac{4\kappa M_{\rm KK}^2}{3}\beta\, x_2(E \mp \epsilon)\,, \tag{4.12c}$$

$$\Delta\mathcal{J}^3_{L/R} = \mp\frac{2\kappa M_{\rm KK}^2}{3}\beta(3A_0^{R/L} - A_0^{L/R}) = \mp\frac{4\kappa M_{\rm KK}^2}{3}\beta(\mu_t \pm 2\mu_{5,t})\,. \tag{4.12d}$$

These counterterms have to be added to the currents (3.36). Again, the 2-component of the currents is canceled, and the covariant anomaly can again be verified explicitly.

The results for the currents after setting $E = \epsilon = 0$ are given in table 1 for both the symmetric and the chirally broken phase. For the axial current we find that the counterterm exactly cancels the CS part of the current,

$$\bar{\mathcal{J}}^3_5 = \mathcal{J}^3_5 + \Delta\mathcal{J}^3_5 = (\mathcal{J}_{\rm YM})^3_5\,. \tag{4.13}$$

In the chirally symmetric phase, this yields exactly the expected topological result (4.4). In the broken phase, the current is suppressed (but nonvanishing, in contrast to the results obtained with a modified CS action [18]). To lowest order in the magnetic fields as well as for asymptotically large magnetic fields this suppression is simply given by a numerical factor. For intermediate magnetic fields the difference to the symmetric phase is given by a complicated function of B. We plot this result in figure 4.3.

	$\mathcal{J}_{\rm YM}$	$\mathcal{J}_{\rm YM+CS}$	$\mathcal{J}_{\rm YM+CS}+\Delta\mathcal{J}$	$\mathcal{J}'_{\rm YM+CS}$
anomaly	"semi-covariant":	consistent:	<u>covariant:</u>	absent:
$\partial_\mu\mathcal{J}_5^\mu/\frac{N_c}{24\pi^2}$	$\underline{3F_V\widetilde{F}_V+3F_A\widetilde{F}_A}$	$F_V\widetilde{F}_V+F_A\widetilde{F}_A$	$\underline{3F_V\widetilde{F}_V+F_A\widetilde{F}_A}$	0
$\partial_\mu\mathcal{J}^\mu/\frac{N_c}{24\pi^2}$	$6F_V\widetilde{F}_A$	$2F_V\widetilde{F}_A$	$\underline{0}$	0
$(\mathcal{J}_\parallel^5/\frac{\mu BN_c}{2\pi^2})\Big\vert_{T>T_c}$	1	$\frac{2}{3}$	1	$\frac{1}{2}$
$\mathcal{J}_\parallel/\frac{\mu_5 BN_c}{2\pi^2}$	1	$\frac{2}{3}$	0	$\frac{1}{2}$
$(\mathcal{J}_\parallel/\mathcal{J}_5^0)\big\vert_{B\to\infty}$	1	$\frac{2}{3}$	0	$\frac{1}{2}$

Table 2. Summary of results for the different (parts of the) chiral currents: the Yang-Mills part $\mathcal{J}_{\rm YM}$ (exclusively considered in ref. [7]), the complete current prior to renormalization $\mathcal{J}_{\rm YM+CS}$ ($\equiv\mathcal{J}$ in the text), the complete current plus Bardeen's counterterm $\mathcal{J}_{\rm YM+CS}+\Delta\mathcal{J}\equiv\bar{\mathcal{J}}$, and the chiral current obtained by modifying the Chern-Simons action according to ref. [18, 21], $\mathcal{J}'_{\rm YM+CS}$. The correct result for the covariant anomaly is underlined. A "1" in the results for the axial current $\mathcal{J}_\parallel^5$ means agreement with the exact QCD result of ref. [9, 44]; a "1" in the results for the electromagnetic current $\mathcal{J}_\parallel$ means agreement with the weak-coupling approach of ref. [2].

The most striking of our results shown in table 1 is that for both phases the renormalized vector current is zero for all magnetic fields,

$$\bar{\mathcal{J}}_3=(\mathcal{J}_{\rm YM}+\mathcal{J}_{\rm CS}+\Delta\mathcal{J})_3=\left(1-\frac{1}{3}-\frac{2}{3}\right)\frac{N_c}{2\pi^2}B\mu_5=0, \tag{4.14}$$

i.e., the chiral magnetic effect has completely disappeared after adding Bardeen's counterterm. The vector current has been computed in the Sakai-Sugimoto model before, and both existing (but differing) results are nonvanishing. One of the results [18] is 1/2 of the topological result (4.1). This result, however, has been obtained with the modified action discussed in section 3.3 which amounts to multiplying the CS contribution by 3/2 (and leaving out the counterterm). As we have seen, this modified action leads to a vanishing anomaly. Another result has recently been presented in ref. [7] as a limit case of a more general frequency-dependent calculation, but using only the YM part of the current. This gives the topological result (4.1), as can also be seen from eq. (4.14). However, as we have shown, this does not produce the complete covariant anomaly, see eqs. (2.21).

One of the purposes of our paper is to point out the differences of these results and the problems of the various approaches regarding the correct anomaly. A summary of our findings is given in table 2. Although in our approach the correct anomaly is ensured, we do not claim to have the final answer since the problem of the ambiguity of the currents, see section 3.3, remains. Our approach shows that the CS part of the currents is important for two reasons. First, as realized already in earlier works [20, 23, 26], it naturally gives a nonzero contribution when the currents are defined by varying the full action. Second, and maybe more importantly, only by including the CS contribution does one reproduce the standard result for the consistent anomaly. And only then one can completely implement the covariant anomaly (i.e., a conserved vector current even in the presence of axial gauge fields) by adding an appropriate counterterm as a holographic renormalization. We have explained why this counterterm, even in the absence of axial field strengths, but in the presence of an axial chemical potential, changes the result for the vector current. We do not, however, see a general reason why the counterterm must render the vector current zero, i.e., why by requiring the current to be conserved the current itself should disappear as it turned out to be the case in our explicit calculation.

After having understood the difference of our result to previous results in the same model, let us discuss its significance in view of the apparent contradiction to the result (4.1). As explained above, this result can be derived by using the Landau-level structure of fermions in a magnetic field. One might thus view our result as an indication that there are no fermionic quasiparticles filling Landau levels in the Sakai-Sugimoto model. This may be particularly interesting in view of the recent attempts to see Landau-level-like structures in holographic models [46, 47]. However, as we have pointed out, the derivation of the chiral magnetic effect via the energy conservation argument by Nielsen and Ninomiya appears to be more general. Obviously, the energy conservation (4.2) does not hold with our results because the left-hand side is zero while the right-hand side yields the expected nonzero result from the anomaly, see eq. (4.11). More precisely, one can check that eq. (4.2) holds before adding Bardeen's counterterm while the counterterm itself violates eq. (4.2). However, the form of the counterterm seems to be uniquely determined by the requirements of parity and the possibility of accommodating it in holographic renormalization. This raises the question whether the apparently general energy argument actually uses properties of the system which are different in our strong-coupling approach. Clearly, also in our system, chirality is converted by a rate simply given by the anomaly. Possibly the energy needed for this conversion cannot be written as in eq. (4.2). A reason might be that this energy makes use of the existence of Fermi surfaces for the right- and left-handed particles which are absent in our model. It is tempting to speculate that the chiral magnetic effect indeed vanishes in the strong-coupling limit and that the weak-coupling results together with the recent observations of charge separation in heavy-ion collisions suggest that the quark-gluon plasma generated in such a collision is sufficiently weakly coupled to exhibit the chiral magnetic effect. A deeper understanding of our result, however, seems required before drawing this conclusion.

We recall that in the context of heavy-ion collisions the magnetic field clearly is time-dependent, in contrast to our assumption of a constant magnetic field. Therefore, in order to compute the induced current, one has to consider the frequency-dependent chiral conductivity [3, 7], whereas our result only corresponds to the zero-frequency limit.[2] In other words, even if the conductivity at zero frequency vanishes, a nonzero (time-dependent) current can be expected if there is a nonvanishing conductivity at nonzero frequencies. However, this does not imply electric charge separation because the separation of charges is proportional to the zero-frequency limit of the conductivity [3]. This is easy to under-

[2] Judging from the calculation of the chiral magnetic conductivity in ref. [7] (where only the YM part was taken into account), one might expect that the full result, taking into account also the (frequency-independent) CS part and Bardeen's counterterm, leads to a nonzero conductivity for asymptotically large frequency [48]. This seems curious, although we do not see a fundamental reason for this to be unphysical.

stand in analogy to a capacitor which cannot be charged with an alternating current, i.e., integrating the induced current over time will lead to a vanishing charge separation as long as there is no direct current.

5 Summary and conclusions

We have studied the strong-coupling behavior of chiral fermions in the presence of a chemical potential and a background magnetic field in the chirally broken and the chirally symmetric phases. To this end, we have used the Sakai-Sugimoto model which is the model that at present comes closest to providing a gravity dual to (large-N_c) QCD. Our focus was the calculation of (topological) axial and vector currents, which are direct consequences of the anomalies in the model.

In particular, we have investigated the chiral magnetic effect, which has been studied previously in a weak-coupling approach [2] and on the lattice [22]. We have pointed out that a reliable calculation within the Sakai-Sugimoto model requires a careful discussion of the QED anomaly in the model. The standard value of the *consistent* anomaly arises naturally in the model for the most straightforward definition of the current. For this result it is crucial to include the contributions from the CS term which are sometimes ignored in the literature. The *covariant* anomaly can then be implemented by adding Bardeen's counterterm [29], which is also known to be required in chiral models with a Wess-Zumino-Witten term [33], and we have pointed out that this (finite) counterterm has a form that appears consistent with the procedure of holographic renormalization.

After these general discussions we have solved the equations of motion for the chirally broken and the chirally symmetric phases explicitly. In our approximation of the DBI action to lowest order in the gauge fields, the solutions are completely analytical. Electric (vector and axial) fields parallel to the magnetic field have been considered in order to check the anomaly explicitly, but they are not needed and set to zero for our physical (equilibrium) results, which only require magnetic fields in the presence of chemical potentials.

In the presence of a quark chemical potential and a large magnetic field, we have calculated the axial current, which may be of interest for astrophysical phenomena such as pulsar kicks [10]. In the chirally symmetric phase we have reproduced the known topological result [9, 44], while in the chirally broken phase, the current has turned out to be smaller but nonvanishing. These results can also be obtained by using only the YM part of the current, i.e., in the case of the axial current the CS contribution and the contribution of Bardeen's counterterm cancel each other.

This is different for the vector current. In this case, only the YM contribution yields a result in agreement with ref. [2]. With the full current, and after adding Bardeen's counterterm, the vector current becomes zero, indicating the absence of the chiral magnetic effect. This is in no obvious contradiction to the result obtained in the weak-coupling approach [2], since at weak coupling the chiral magnetic effect is a phenomenon that originates solely from fermionic quasiparticles in the lowest Landau level, and it is not clear whether this structure persists at strong coupling.

In comparison to the result from recent lattice calculations [22] we have pointed out an intriguing agreement *before* adding Bardeen's counterterm, i.e., within the consistent anomaly. There the vector current per chirality approaches approximately the value 2/3 for asymptotically large magnetic field. This is clearly different from the weak-coupling approach where this ratio approaches 1. This raises the question whether this asymptotic value can distinguish between strong and weak coupling. It also raises the question whether the lattice result relates to the consistent, as opposed to the covariant, anomaly.

Although we have implemented the correct covariant anomaly, also in our approach a problem remains. Namely, upon computing the free energy explicitly and then taking the derivative with respect to the appropriate source, the currents turn out to be different from the straightforward definition via the gauge/gravity correspondence. This somewhat disturbing discrepancy can be attributed to boundary terms at spatial infinity. We have discussed that a previously suggested fix of this problem by modifying the action [18] seems to be not acceptable because it entirely eliminates the axial anomaly from the correspondingly modified currents. Only the YM part of these currents are anomalous, but those suffer from the same thermodynamic inconsistency that this modification was meant to fix. Because of these ambiguities, further studies are clearly needed (see also ref. [23] for other issues concerning the definition of chiral currents in the Sakai-Sugimoto model).

Quantitative improvements could be achieved by extending our calculation to the full DBI action, though they should be minor for magnetic field strengths of practical interest. More critical, but also considerably more difficult, would be the generalization of our ansatz to allow for inhomogeneous field configurations and/or inhomogeneous solutions. This might be required to resolve the ambiguities in the definition of the chiral currents that we have discussed, since those are related to spatial surface contributions in the CS action. With our present definition of the chiral currents we have been led to question the very existence of the chiral magnetic effect in the strong-coupling regime of the Sakai-Sugimoto model (which is gravity dual to large-N_c QCD only in its inaccessible weak coupling limit). In this context it would be important to understand whether in the strong-coupling regime one has Landau-level-like structures, as conjectured in ref. [19]. This is interesting also in view of recent studies in different gauge/gravity models [46, 47].

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A Solving the equations of motion

A.1 Chirally broken phase

In this appendix we solve the equations of motion in the broken phase, eqs. (3.4). The equation of motion for A_z (3.4c) is trivially integrated with respect to time t to yield

$$k\partial_z A_0 = -2\beta t e(z) + k\partial_z \widetilde{A}_0 \,, \tag{A.1}$$

where we have denoted $e(z) \equiv -\partial_t A_3$ and where we have written the t-independent integration constant as $k\partial_z \widetilde{A}_0$, to be determined below. Inserting this into eqs. (3.4a) and (3.4b) yields

$$\partial_z(k\partial_z \widetilde{A}_0) = 2\beta\partial_z A_3 + 2\beta t\partial_z e\,, \tag{A.2a}$$
$$\partial_z(k\partial_z A_3) = 2\beta\partial_z \widetilde{A}_0 - (2\beta)^2 t\frac{e(z)}{k(z)}\,. \tag{A.2b}$$

Since the left-hand side of eq. (A.2a) does not depend on t, the right-hand side must be independent of t too which implies

$$\partial_z A_3 = -t\partial_z e + \partial_z \widetilde{A}_3\,, \tag{A.3}$$

where we have written the t-independent part as $\partial_z \widetilde{A}_3$. Consequently, eqs. (A.2a) and (A.2b) become

$$\partial_z(k\partial_z \widetilde{A}_0) = 2\beta\partial_z \widetilde{A}_3\,, \tag{A.4a}$$
$$\partial_z(k\partial_z \widetilde{A}_3) = 2\beta\partial_z \widetilde{A}_0 - t\left[(2\beta)^2\frac{e(z)}{k(z)} - \partial_z(k\partial_z e)\right]. \tag{A.4b}$$

Now the square bracket on the right-hand side of eq. (A.4b) must vanish because all other terms in the equation do not depend on t. This yields a differential equation for $e(z)$. Since $e(z) = -\partial_t A_3$, the boundary conditions for A_3 (3.2) imply $e(\pm\infty) = E \mp (\epsilon - \jmath_1)$, where we have decomposed the supercurrent as

$$\jmath_t = \jmath + \jmath_1 t\,, \tag{A.5}$$

with $\jmath$, $\jmath_1$ being t-independent. With these boundary conditions the equation for $e(z)$ is solved by

$$e(z) = E\frac{\cosh(2\beta\arctan z)}{\cosh\beta\pi} - (\epsilon - \jmath_1)\frac{\sinh(2\beta\arctan z)}{\sinh\beta\pi}\,. \tag{A.6}$$

To find the solution for A_0 and A_3 we first conclude from eqs. (A.1) and (A.3),

$$\begin{aligned} A_0(t,z) &= \widetilde{A}_0(z) + g_0(t) \\ &\quad -t\left[E\frac{\sinh(2\beta\arctan z)}{\cosh\beta\pi} - (\epsilon - \jmath_1)\frac{\cosh(2\beta\arctan z)}{\sinh\beta\pi}\right], \end{aligned} \tag{A.7a}$$
$$A_3(t,z) = \widetilde{A}_3(z) - t\left[E\frac{\cosh(2\beta\arctan z)}{\cosh\beta\pi} - (\epsilon - \jmath_1)\frac{\sinh(2\beta\arctan z)}{\sinh\beta\pi}\right]. \tag{A.7b}$$

From the z-integration in eq. (A.1) we have obtained a t-dependent integration constant $g_0(t)$. Such a constant is not permissible in A_3 because of the constraint $e(z) = -\partial_t A_3$. Integration constants independent of z *and* t are included in $\widetilde{A}_0(z)$, $\widetilde{A}_3(z)$. We shall fix $g_0(t) = \jmath_1 t\coth\beta\pi$. This removes the supercurrent from the vector boundary value of $A_0(t,z)$. We cannot at the same time remove the axial field ϵ from this boundary value. This becomes clear in hindsight after determining $\jmath_t$ from minimization of the free energy.

Only with the given choice of $g_0(t)$ this minimization leads to a consistent, i.e., time-independent, result for $\jmath$, $\jmath_1$. It is thus unavoidable for the boundary values of $A_0(t,z)$ to become time-dependent,

$$A_0(t, z=\pm\infty) = \mu_t \mp \mu_{5,t}\,, \tag{A.8}$$

where we defined

$$\mu_t \equiv \mu + t\epsilon \coth\beta\pi\,, \qquad \mu_{5,t} \equiv \mu_5 + tE\tanh\beta\pi\,. \tag{A.9}$$

We have now reduced the equations of motion (A.4) to equations for $\widetilde{A}_0(z)$, $\widetilde{A}_3(z)$ which are simply the gauge fields in the absence of an electric field. These equations can be solved in general,

$$\widetilde{A}_0(z) = a_0 - \frac{c}{2\beta}e^{-2\beta\arctan z} + \frac{d}{2\beta}e^{2\beta\arctan z}\,, \tag{A.10a}$$

$$\widetilde{A}_3(z) = a_3 + \frac{c}{2\beta}e^{-2\beta\arctan z} + \frac{d}{2\beta}e^{2\beta\arctan z}\,, \tag{A.10b}$$

with integration constants a_0, a_3, c, and d which are fixed by the boundary conditions $\widetilde{A}_0(\pm\infty) = \mu \mp \mu_5$, $\widetilde{A}_3(\pm\infty) = \mp\jmath$. The resulting gauge fields $\widetilde{A}_0(z)$, $\widetilde{A}_3(z)$ are then inserted into the gauge fields $A_0(t,z)$ $A_3(t,z)$ from eqs. (A.7) to obtain the final solution

$$\begin{aligned} A_0(t,z) &= \mu_t - \mu_{5,t}\frac{\sinh(2\beta\arctan z)}{\sinh\beta\pi} \\ &\quad -(\jmath_t - \epsilon t)\left[\frac{\cosh(2\beta\arctan z)}{\sinh\beta\pi} - \coth\beta\pi\right]\,, \end{aligned} \tag{A.11a}$$

$$\begin{aligned} A_3(t,z) &= -tE - \mu_{5,t}\left[\frac{\cosh(2\beta\arctan z)}{\sinh\beta\pi} - \coth\beta\pi\right] \\ &\quad -(\jmath_t - \epsilon t)\frac{\sinh(2\beta\arctan z)}{\sinh\beta\pi}\,. \end{aligned} \tag{A.11b}$$

For the free energy we also need the field strengths (times $k(z)$),

$$k\partial_z A_0 = -2\beta\left[\mu_{5,t}\frac{\cosh(2\beta\arctan z)}{\sinh\beta\pi} + (\jmath_t - \epsilon t)\frac{\sinh(2\beta\arctan z)}{\sinh\beta\pi}\right]\,, \tag{A.12a}$$

$$k\partial_z A_3 = -2\beta\left[\mu_{5,t}\frac{\sinh(2\beta\arctan z)}{\sinh\beta\pi} + (\jmath_t - \epsilon t)\frac{\cosh(2\beta\arctan z)}{\sinh\beta\pi}\right]\,. \tag{A.12b}$$

As a check, we can perform a parity transformation on the gauge fields. With $\mu \to +\mu$, $\mu_5 \to -\mu_5$, $\jmath_t \to +\jmath_t$, $B \to +B$, $E \to -E$, $\epsilon \to +\epsilon$ and $z \to -z$ we find $A_0(t,z) \to A_0(t,z)$ and $A_3(t,z) \to -A_3(t,z)$, i.e., the fields have the correct behavior under parity transformations for each t and z.

We can now insert the gauge fields and field strengths into the action (3.3) to obtain the thermodynamic potential $\Omega = \frac{T}{V}S_{\text{on-shell}}$. The YM and CS contributions are

$$\Omega_{\text{YM}} = \kappa M_{\text{KK}}^2\frac{4\pi\beta^2}{\sinh^2\beta\pi}\left[(\jmath_t - \epsilon t)^2 - \mu_{5,t}^2\right]\,, \tag{A.13a}$$

$$\begin{aligned} \Omega_{\text{CS}} &= \frac{8\kappa M_{\text{KK}}^2}{3}\left(\beta\coth\beta\pi - \frac{\pi\beta^2}{\sinh^2\beta\pi}\right)\left[(\jmath_t - \epsilon t)^2 - \mu_{5,t}^2\right] \\ &\quad + \frac{8\kappa M_{\text{KK}}^2}{3}\beta\left[\mu_t(\jmath_t - \epsilon t) + tE\mu_{5,t}\right]\,, \end{aligned} \tag{A.13b}$$

where the real time parameter t is treated as an external parameter, unrelated to the imaginary time τ, whose integration is assumed to just give a factor $1/T$. In the YM part we have dropped the terms $\propto B^2, E^2$. This vacuum subtraction can be understood in terms of holographic renormalization and follows from the renormalization condition that the thermodynamic potential be zero for vanishing chemical potentials; for the explicit procedure see ref. [20].

A.2 Chirally symmetric phase

Here we solve the equations of motion for the chirally symmetric phase, eqs. (3.16). For notational convenience, let us, in this subsection, denote

$$\beta' \equiv \frac{\beta}{\theta^3} \,. \tag{A.14}$$

The time-dependence of the gauge fields is treated analogously to the broken phase. Thus, eqs. (3.16c) and (3.16b) imply

$$k_0 \partial_z A_0^{L/R} = \mp 2\beta' t e_{L/R}(z) + k_0 \partial_z \widetilde{A}_0^{L/R} \,, \tag{A.15}$$

and

$$\partial_z A_3^{L/R} = -t \partial_z e_{L/R}(z) + \partial_z \widetilde{A}_3^{L/R} \,, \tag{A.16}$$

where $\widetilde{A}_0^{L/R}$, $\widetilde{A}_3^{L/R}$ are constant in t, and where $e_{L/R} \equiv -\partial_t A_3^{L/R}$. Then, eqs. (3.16a) and (3.16b) read

$$\begin{aligned}
\partial_z (k_0 \partial_z \widetilde{A}_0^{L/R}) &= \pm 2\beta' \partial_z \widetilde{A}_3^{L/R} \,, && \text{(A.17a)} \\
\partial_z (k_3 \partial_z \widetilde{A}_3^{L/R}) &= \pm 2\beta' \partial_z \widetilde{A}_0^{L/R} - t \left[(2\beta')^2 \frac{e_{L/R}(z)}{k_0(z)} - \partial_z (k_3 \partial_z e_{L/R}) \right] . && \text{(A.17b)}
\end{aligned}$$

This is analogous to eq. (A.4), the only difference being the two functions $k_0(z)$ and $k_3(z)$ instead of the single function $k(z)$. Again the square bracket in eq. (A.17b) has to vanish. This yields a differential equation for $e_{L/R}(z)$ which is solved as follows. With $\widetilde{e}_{L/R} = k_3 \partial_z e_{L/R}$ one can rewrite this differential equation as

$$\partial_z (k_0 \partial_z \widetilde{e}_{L/R}) = (2\beta')^2 \frac{\widetilde{e}_{L/R}}{k_3} \,. \tag{A.18}$$

This equation has the two independent solutions

$$\begin{aligned}
p(z) &= {}_2F_1 \left[-\frac{\sqrt{1-16\beta'^2}+1}{4}, \frac{\sqrt{1-16\beta'^2}-1}{4}, \frac{1}{2}, \frac{1}{1+z^2} \right] , && \text{(A.19a)} \\
q(z) &= \frac{1}{\sqrt{1+z^2}} \, {}_2F_1 \left[-\frac{\sqrt{1-16\beta'^2}-1}{4}, \frac{\sqrt{1-16\beta'^2}+1}{4}, \frac{3}{2}, \frac{1}{1+z^2} \right] . && \text{(A.19b)}
\end{aligned}$$

Consequently, $\widetilde{e}_{L/R}(z) = P_{L/R}\, p(z) + Q_{L/R}\, q(z)$, with constants $P_{L/R}$, $Q_{L/R}$, and thus

$$e_{L/R}(z) = \frac{1}{(2\beta')^2} \left(P_{L/R}\, k_0 \partial_z p + Q_{L/R}\, k_0 \partial_z q \right) . \tag{A.20}$$

In the following we need the behavior of the functions $p(z)$, $q(z)$, $k_0\partial_z p$, $k_0\partial_z q$ at $z=\infty$ and $z=0$. At $z=\infty$ we have

$$p(\infty) = -k_0\partial_z q(\infty) = 1\,, \qquad q(\infty) = k_0\partial_z p(\infty) = 0\,. \tag{A.21}$$

At $z=0$ one finds

$$p_0 \equiv p(0) = \frac{\sqrt{\pi}}{\Gamma\left[\left(3-\sqrt{1-16\beta'^2}\right)/4\right]\Gamma\left[\left(3+\sqrt{1-16\beta'^2}\right)/4\right]}\,, \tag{A.22a}$$

$$q_0 \equiv q(0) = \frac{\sqrt{\pi}}{2\Gamma\left[\left(5-\sqrt{1-16\beta'^2}\right)/4\right]\Gamma\left[\left(5+\sqrt{1-16\beta'^2}\right)/4\right]}\,, \tag{A.22b}$$

and

$$k_0\partial_z p(z\to 0) = (2\beta')^2 p_0\,\ln z\,, \qquad k_0\partial_z q(z\to 0) = (2\beta')^2 q_0\,\ln z\,. \tag{A.23}$$

The boundary conditions $e_{L/R}(z=\infty) = E\mp\epsilon$ yield $Q_{L/R} = -(2\beta')^2(E\mp\epsilon)$. Inserting this constant into eq. (A.20), the result into eqs. (A.15), (A.16), and integrating the resulting equations over z yields the gauge fields

$$A_0^{L/R}(t,z) = \mp 2\beta' t\left[\frac{P_{L/R}}{(2\beta')^2}\,p(z) - (E\mp\epsilon)\,q(z)\right] + g_0^{L/R}(t) + \tilde{A}_0^{L/R}(z)\,, \tag{A.24a}$$

$$A_3^{L/R}(t,z) = -t\left[\frac{P_{L/R}}{(2\beta')^2}\,k_0\partial_z p(z) - (E\mp\epsilon)\,k_0\partial_z q\right] + \tilde{A}_3^{L/R}(z)\,. \tag{A.24b}$$

Here, $g_0^{L/R}(t)$ are time-dependent integration constants from the z integration. We proceed by solving eqs. (A.17) for $\tilde{A}_0^{L/R}$, $\tilde{A}_3^{L/R}$. Recalling that $p(z)$, $q(z)$ fulfill the differential equation (A.18) one easily checks that the functions

$$\tilde{A}_0^{L/R}(z) = a_0^{L/R} \pm 2\beta'\left[C_{L/R}\,p(z) + D_{L/R}\,q(z)\right]\,, \tag{A.25a}$$

$$\tilde{A}_3^{L/R}(z) = a_3^{L/R} + C_{L/R}\,k_0\partial_z p + D_{L/R}\,k_0\partial_z q\,, \tag{A.25b}$$

with integration constants $a_0^{L/R}$, $a_3^{L/R}$, $C_{L/R}$ and $D_{L/R}$, are solutions of eqs. (A.17). One now inserts these functions into eqs. (A.24) and determines the integration constants as follows. First we recall that all constants except for $g_0^{L/R}(t)$ must not depend on t. This will be used repeatedly in the following. Then we require the boundary condition $A_3^{L/R}(t,z=\infty) = -t(E\mp\epsilon)$ which implies $D_{L/R} = a_3^{L/R}$. Next, we require regularity of $A_3^{L/R}(t,z)$ at $z=0$. With eq. (A.23) we find that $A_3^{L/R}(t,z\to 0)$ diverges logarithmically. Requiring the factor in front of the $\ln z$ term to vanish yields the conditions

$$C_{L/R} = -\frac{q_0}{p_0}D_{L/R}\,, \qquad P_{L/R} = (2\beta')^2\frac{q_0}{p_0}(E\mp\epsilon)\,. \tag{A.26}$$

For the temporal component we need to require $A_0^{L/R}(t,z=0)=0$ [49] which yields $a_0^{L/R} = g_0^{L/R}(t) = 0$. With these results the boundary value of $A_0^{L/R}(t,z)$ becomes

$$A_0^{L/R}(t,z=\infty) = \mp 2\beta'\frac{q_0}{p_0}\left[D_{L/R} + t(E\mp\epsilon)\right]. \tag{A.27}$$

This result shows that, as in the broken phase, the boundary values of axial and vector parts of A_0 necessarily become time-dependent. In other words, in the presence of an electric field one cannot fix these boundary values to be time-independent chemical potentials. At $t=0$ we require $A_0^{L/R}(t=0,z=\infty)=\mu\mp\mu_5$. With these initial values we find

$$D_{L/R} = \mp\frac{p_0}{q_0}\frac{\mu\mp\mu_5}{2\beta'}\,, \tag{A.28}$$

and the time-dependent chemical potentials become

$$A_0^{L/R}(t,z=\infty) = \mu_t \mp \mu_{5,t}\,, \tag{A.29}$$

with

$$\mu_t \equiv \mu + 2\beta' t\epsilon\frac{q_0}{p_0}\,, \qquad \mu_{5,t} \equiv \mu_5 + 2\beta' tE\frac{q_0}{p_0}\,. \tag{A.30}$$

Collecting all the integration constants, we obtain from eqs. (A.24) and (A.25) the final solution for the gauge fields,

$$A_0^{L/R}(t,z) \;=\; (\mu_t\mp\mu_{5,t})\left[p(z)-\frac{p_0}{q_0}\,q(z)\right]\,, \tag{A.31a}$$

$$A_3^{L/R}(t,z) \;=\; -t(E\mp\epsilon)\pm\frac{\mu_t\mp\mu_{5,t}}{2\beta'}\left[k_0\partial_z p-\frac{p_0}{q_0}(1+k_0\partial_z q)\right]\,. \tag{A.31b}$$

Again we can check the behavior of the gauge fields under parity transformations. In contrast to the broken phase, we have separate right- and left-handed fields which transform as $A_0^{L/R}(t,z)\to A_0^{R/L}(t,z)$, and $A_3^{L/R}(t,z)\to -A_3^{R/L}(t,z)$, as it should be. The field strengths become

$$k_0\partial_z A_0^{L/R} \;=\; (\mu_t\mp\mu_{5,t})\left(k_0\partial_z p-\frac{p_0}{q_0}k_0\partial_z q\right)\,, \tag{A.32a}$$

$$k_3\partial_z A_3^{L/R} \;=\; \pm 2\beta'(\mu_t\mp\mu_{5,t})\left[p(z)-\frac{p_0}{q_0}q(z)\right]\,. \tag{A.32b}$$

Inserting these results into the action, given by eqs. (3.12) and (3.13), yields the YM and CS contributions to the free energy,

$$\Omega_{\rm YM} \;=\; -2\kappa\theta^3 M_{\rm KK}^2(\mu_t^2+\mu_{5,t}^2)[I_0-(2\beta')^2 I_3]\,, \tag{A.33a}$$

$$\Omega_{\rm CS} \;=\; \frac{4\kappa M_{\rm KK}^2\theta^3}{3}\left\{(\mu_t^2+\mu_{5,t}^2)\left[I_0-(2\beta')^2 I_3-\frac{p_0}{q_0}\right]+2\beta'\,t\,(\mu_t\epsilon+\mu_{5,t}E)\right\}, \tag{A.33b}$$

where we abbreviated the integrals

$$I_0 \equiv \int_0^\infty \frac{dz}{k_0}\left(k_0\partial_z p-\frac{p_0}{q_0}k_0\partial_z q\right)^2\,, \tag{A.34}$$

$$I_3 \equiv \int_0^\infty \frac{dz}{k_3}\left[p(z)-\frac{p_0}{q_0}q(z)\right]^2\,. \tag{A.35}$$

In the limit $\beta\gg 1$, the combination $I_0-(2\beta')^2 I_3\to 0$, so that for very strong magnetic fields $\Omega_{\rm CS}\gg\Omega_{\rm YM}$, as is also the case in the chirally broken phase, see eqs. (A.13).

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