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BCFW recursion relation with nonzero boundary contribution

Bo Feng,[a] Junqi Wang,[b] Yihong Wang[b] and Zhibai Zhang[c]

[a] *Center of Mathematical Science, Zhejiang University, Hangzhou, China*

[b] *Physics Department, Zhejiang University, Hangzhou, China*

[c] *Mathematics Department, Zhejiang University, Hangzhou, China*

E-mail: b.feng@cms.zju.edu.cn, junqi.wang.87@gmail.com, yihong.w2@gmail.com, zhibai.zhang1987@gmail.com

Abstract: The appearance of BCFW on-shell recursion relation has deepen our understanding of quantum field theory, especially the one with gauge boson and graviton. To be able to write the BCFW recursion relation, the knowledge of boundary contributions is needed. So far, most applications have been constrained to the cases where the boundary contribution is zero. In this paper, we show that for some theories, although there is no proper deformation to annihilate the boundary contribution, its effects can be analyzed in simple way, thus we do able to write down the BCFW recursion relation with boundary contributions. The examples we will present in this paper include the $\lambda\phi^4$ theory and Yukawa coupling between fermions and scalars.

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Contents

1 Motivations

For theory with Lagrangian description, we can calculate amplitudes using Feynman diagrams. Any Feynman diagram is constructed by putting some elements, i.e., the vertex, together through propagators. Thus any higher point amplitude can be constructed recursively from lower point amplitudes with one very important feature: *these lower point amplitudes must be well-defined off-shell.* Comparing to the on-shell amplitudes, which

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have physical meaning, off-shell amplitudes are usually longer and more complicated, especially for gauge theory where gauge freedom renders the expression including many redundant information. Thus it is natural to ask if we can construct any higher point on-shell amplitude recursively from lower point on-shell amplitudes only. If we could, we can call this theory "on-shell constructible" to be distinguish with off-shell constructions by Feynman diagrams.

Initially by Witten's twistor program [1], BCFW recursion relation [2, 3] provides the first concrete example for on-shell constructibility. Let us first review how the goal is achieved. First we pick up two special momenta p_1, p_2 and do the following deformation (BCFW deformation) using an auxiliary momentum q:

$$p_1(z) = p_1 + zq, \qquad p_2(z) = p_2 - zq \; . \tag{1.1}$$

The opposite sign makes the momentum conservation satisfied. Furthermore, if we impose the conditions $q^2 = 0$, $p_1 \cdot q = p_2 \cdot q = 0$, the on-shell conditions of $p_1(z)$ and $p_2(z)$ are also satisfied. In another word, we have a deformed on-shell amplitude $A(z)$ over single complex variable z. Having the deformed $A(z)$ we consider following contour integration

$$B = \oint_C \frac{A(z)}{z} dz \tag{1.2}$$

where contour C is big enough circle around $z = 0$. We can evaluate the integration by two different ways: either by contour around $z = \infty$ or the big contour around the origin. Thus we have

$$A(z=0) = -\sum_{z_\alpha} \text{Res}\left(\frac{A(z)}{z}\right) + B\,, \tag{1.3}$$

where $A(z = 0)$ is the amplitude we want to find and B is the boundary contribution. The residue part can always be calculated using the factorization properties from lower-point on-shell amplitudes. In another word, the expression (1.3) tells us that for any theory, some parts of tree amplitudes are "on-shell constructible". The trouble part comes from the boundary contribution B. It is easy to see that $B \neq 0$ when and only when $A(z)$ is not zero under the limit $z \to \infty$. Thus if there is some deformation such that $A(z) \to 0$ when $z \to \infty$, we will get the wanted on-shell constructibility. Assuming this strong condition, i.e., $B = 0$, some beautiful results can be derived in [4].

From above discussions, we see that for the application of on-shell constructibility, the knowledge of boundary contribution becomes very important. The analysis of the boundary behavior is, in general, an extremely nontrivial task for many theories, especially the one with gauge symmetry, as demonstrated in the beautiful paper [5] as well as others, for examples [6]–[10]. In these papers, it is shown that for gauge theory or gravity theory, with proper choice of BCFW deformation we can make the boundary contribution zero and derive the on-shell recursion relation.

However, there are other theories where we can not find any deformation to set boundary contribution zero. One typical example is the $\lambda\phi^4$ theory. For these theories, the on-shell constructibility is not so easy to answer. In fact, more accurate statement from expression (1.3) is following: *If $B = 0$, the theory is on-shell constructible, but if $B \neq 0$, it can be on-shell constructible or not on-shell constructible.*

In this paper, we will address the problem carefully with $B \neq 0$. We will show that for some theories, although there is no any choice to set boundary contribution zero, the boundary contribution can be analyzed and obtained in fairly simple way through lower point on-shell amplitudes. Thus for these theories, we can still write down the on-shell recursion relations.

The structure of our paper is following. In section 2, we use the $\lambda\phi^4$ theory as our first example to demonstrate the on-shell constructibility with nonzero boundary contributions. In section 2.1, we have identified boundary contributions from Feynman diagram analysis and written down the BCFW recursion relation. Then in section 2.2, we calculated several amplitudes using our BCFW formula and compared with results from Feynman diagrams in appendix A.

There is another way to deal with $\lambda\phi^4$ theory by introducing a massive field as given in [4]. For the new Lagrangian we have a triple deformation with vanishing boundary contributions and similar BCFW recursion relation. In section 3, we use same triple deformation for $\lambda\phi^4$ theory. We showed that how the boundary contributions for $\lambda\phi^4$ theory are mapped to the pole contributions in the new Lagrangian, thus established the equivalent relation between these two methods.

In section 4, we discuss the scalar QCD, i.e., fermion interacts with scalar through the Yukawa coupling. This example is more interesting because this kind of interactions is a major part of standard model. We analyzed the boundary behavior for various helicity configurations in section 4.1 and wrote down the corresponding BCFW recursion relations. In section 4.2, we present explicit calculations to demonstrate our results.

There are two appendixes. In appendix A we have present amplitudes calculated directly by Feynman diagrams. Its role is to check calculations did by BCFW recursion relation with boundary contributions. In appendix B, we have discussed the boundary contributions for general $2l$ fermions in scalar QCD.

2 The $\lambda\phi^4$ theory

In this section, we discuss our first example with nonzero boundary contribution: the $\lambda\phi^4$ theory. We will analyze the boundary behavior first, and then write down the BCFW recursion relation with boundary contribution. As a comparison we have done same calculation using the standard Feynman diagram method in appendix A.

2.1 The boundary behavior and BCFW relation

Let us consider following BCFW deformation for massless $\lambda\phi^4$ theory

$$\lambda^{(i)} = \lambda^{(i)} + z\lambda^{(j)} \quad \tilde{\lambda}^{(j)} = \tilde{\lambda}^{(j)} - z\tilde{\lambda}^{(i)} \tag{2.1}$$

Using the only nontrivial vertex $\lambda\phi^4$ to construct the tree-level Feynman diagram, we found that all diagrams can be divided into two categories: (A) particles i, j are attached to same

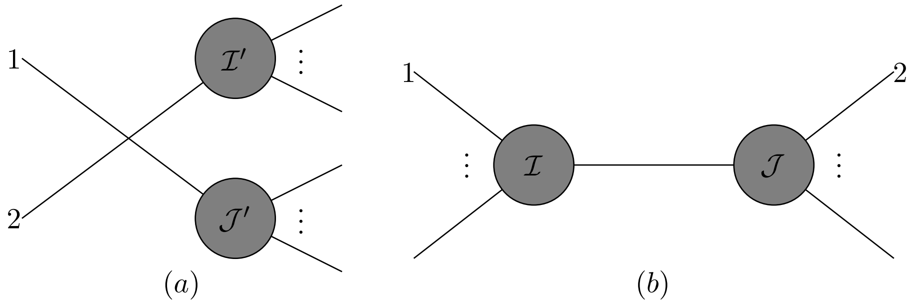

Figure 1. (a) The contribution from boundary. (b) The contribution from pole part.

vertex; (B) particles i, j are attached to different vertexes. For diagrams in category (B), there is at least one propagator on the line connecting i, j depending on z linearly, i.e., we will have factor $\frac{1}{P^2 - z\langle j|P|i]}$ in the expression. Thus under the limit $z \to \infty$, expressions in category (B) will go to zero, so they do not give boundary contributions.

Opposite to the category (B), since i, j are attached to same vertex, the whole expressions in category (A) do not depend on z at all. In another word, there are nonzero boundary contributions from category (A). By this simple analysis, we know that the boundary contribution can be calculated by attaching the lower-point tree level amplitudes to this vertex.

Having above analysis, we can immediately write down the BCFW on-shell recursion relation for this simple theory as

$$A = A_b + A_{\text{pole}} \tag{2.2}$$

where A_b as boundary contribution given by

$$A_b = (-i\lambda) \sum_{\mathcal{I}' \bigcup \mathcal{J}' = \{n\} \backslash \{i,j\}} A_{\mathcal{I}'} (\{K_{\mathcal{I}'})) \frac{1}{P^2_{\mathcal{I}'}} \frac{1}{P^2_{\mathcal{J}'}} A_{\mathcal{J}'} (\{K_{\mathcal{J}'}\}) \tag{2.3}$$

and A_{pole} as contributions from poles given by standard BCFW-form

$$A_{\text{pole}} = \sum_{i \in \mathcal{I}, j \notin \mathcal{I}} A_{\mathcal{I}} (\{K_{\mathcal{I}'}\}, p_i(z_{\mathcal{I}}), -P_{\mathcal{I}}(z_{\mathcal{I}})) \frac{1}{P^2_{\mathcal{I}}} A_{\mathcal{J}} (\{K_{\mathcal{J}'}\}, p_j(z_{\mathcal{I}}), P_{\mathcal{I}}(z_{\mathcal{I}})) \tag{2.4}$$

The expression (2.3) just states the fact that set $\mathcal{I}'$, set $\mathcal{J}'$ and particles i, j are attached to same vertex with coupling constant $-i\lambda$. There two contributions can be represented by figure 1 (a) and (b), where we have set $i, j = 1, 2$:

In following subsection, we will check the formula above using explicit calculations. For simplicity, we will focus on the color ordered case. For ordered case, something new is happening: when particles i, j have distance more than two, they will never be attached to same vertex and the boundary contribution will be zero. Thus we can check our result using $\langle 1|2]$-shifting with boundary contribution against the $\langle 1|4]$-shifting without boundary contribution. We want to emphasize that although for ordered case we can have deformation without boundary contribution, in real calculation, we need to sum up all orderings, so formula with boundary contribution will be unavoidable.

Having above explanation, in following calculations, we will write down results from shifting $\langle 1|2]$ and compare them with the one from shifting $\langle 1|4]$ as well as the one with direct Feynman diagrams in appendix A.

2.2 The $\langle 1|2]$ shifting

Since for $\lambda\phi^4$ theory we have only quadruple vertex, tree-level amplitudes with odd number of scalars are automatically zero, thus we will consider six, eight and ten point amplitudes only.

In this part we will use the $\langle 1|2]$ shifting given by

$$\lambda_1(z) = \lambda_1 + z\lambda_2, \qquad \widetilde{\lambda}_2(z) = \widetilde{\lambda}_2 - z\widetilde{\lambda}_1 \tag{2.5}$$

thus we have following results.

Six-point amplitudes: first we consider the $\langle 1|2]$ shifting. It is easy to see that there is only one figure contributing to pole part

$$A_{6,pole}^{\langle 1|2]}(1,\ldots,6) = A_4(5,6,\widehat{1},-\widehat{P})\frac{1}{P_{561}^2}A_4(\widehat{P},\widehat{2},3,4) = (-i\lambda)^2\left(\frac{1}{P_{156}^2}\right) \tag{2.6}$$

For the boundary part, there are two contributions

$$\begin{aligned} A_{6,b}^{\langle 1|2]}(1,\ldots,6) &= A_4(1,2,-P_1,-P_2)\left(\frac{1}{p_3^2}A_2(P_1,3)\right)\left(\frac{1}{P_{123}^2}A_4(P_2,4,5,6)\right) \\ &\quad + A_4(1,2,-P_1,-P_2)\left(\frac{1}{P_{3,4,5}^2}A_4(P_1,3,4,5)\right)\left(\frac{1}{p_6^2}A_2(P_2,6)\right) \\ &= (-i\lambda)^2\left(\frac{1}{P_{123}^2}+\frac{1}{P_{126}^2}\right) \end{aligned} \tag{2.7}$$

where for simplicity we have defined the notation $A_2(a,b) = \delta^4(p_a - p_b)p_a^2$, $P_{ijk} = p_i + p_j + p_k$. Putting together we have

$$A_6^{\mathrm{FD}}(1,\ldots,6) = (-i\lambda)^2\left(\frac{1}{P_{123}^2}+\frac{1}{P_{126}^2}+\frac{1}{P_{156}^2}\right) \tag{2.8}$$

which agrees with the one from three Feynman diagrams.

For the shifting $\langle 1|4]$, there is no boundary part but there are three terms in pole part: $(123|456)$, $(612|345)$ and $(561|234)$. Adding three terms together we get the same answer.

Eight-point amplitudes: there are two types of trees with three vertexes contributing at this level: (A) type $(123|45|678)$ plus Z_8 cyclic ordering and (B) type $(123|48|567)$ where $4,8$ at at the two sides of propagators, plus Z_4 cyclic ordering. Adding them together we have

$$\begin{aligned} A_8^{\mathrm{FD}}(1,\ldots,8) &= A_8^{FD(a)} + A_8^{FD(b)} \\ &= (-i\lambda)^3\sum_{\sigma\in\mathbb{Z}_8}\left(\frac{1}{P_{\sigma(1)\sigma(2)\sigma(3)}^2P_{\sigma(6)\sigma(7)\sigma(8)}^2}+\frac{1}{2P_{\sigma(1)\sigma(2)\sigma(3)}^2P_{\sigma(5)\sigma(6)\sigma(7)}^2}\right) \end{aligned} \tag{2.9}$$

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where FD means the result from direct Feynman diagrams.

Now we use the $\langle 1|2]$-shifting. The pole contribution is given by sum of following two terms (notice that the tree amplitude is zero with odd number of external lines):

$$\begin{aligned}A^{\langle 1|2]}_{8,pole}(1,2,\ldots,8) &= \widehat{A}_4(7,8,\widehat{1},-\widehat{P})\frac{1}{P^2_{178}}\widehat{A}_6(\widehat{P},\widehat{2},3,4,5,6)\\ &\quad+\widehat{A}_6(5,6,7,8,\widehat{1},-\widehat{P})\frac{1}{P^2_{234}}\widehat{A}_4(\widehat{P},\widehat{2},3,4) \qquad (2.10)\\ &= (-i\lambda)^3\left[\frac{1}{P^2_{178}}\left(\frac{1}{\widehat{P}^2_{\widehat{2}34}}+\frac{1}{P^2_{345}}+\frac{1}{P^2_{456}}\right)+\left(\frac{1}{P^2_{567}}+\frac{1}{P^2_{678}}+\frac{1}{\widehat{P}^2_{\widehat{1}78}}\right)\frac{1}{P^2_{234}}\right]\end{aligned}$$

Using the locations of the poles $z_1=-\frac{P^2_{178}}{\langle 1|P_{178}|2]}$, $z_2=-\frac{P^2_{234}}{\langle 1|P_{234}|2]}$, we can simplify

$$\frac{1}{P^2_{178}\widehat{P}^2_{\widehat{2}34}}+\frac{1}{\widehat{P}^2_{\widehat{1}78}P^2_{234}}=\frac{1}{P^2_{178}P^2_{234}}\left(\frac{1}{1-\frac{z_1}{z_2}}+\frac{1}{1-\frac{z_2}{z_1}}\right)=\frac{1}{P^2_{178}P^2_{234}}, \qquad (2.11)$$

where we have used the identity $\frac{1}{1-\frac{z_1}{z_2}}+\frac{1}{1-\frac{z_2}{z_1}}=1$. In fact, there is a general identity

$$\sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{1-\frac{z_i}{z_j}}=1, \qquad (2.12)$$

which will be useful also in our ten-point calculation.

For boundary part, there are following three splitting (3|45678), (345|678) and (34567|8). Adding them up we have

$$\begin{aligned}A^{\langle 1|2]}_{8,b}(1,2,\ldots,8) = (-i\lambda)^3\Bigg[&\frac{1}{P^2_{123}}\left(\frac{1}{P^2_{456}}+\frac{1}{P^2_{567}}+\frac{1}{P^2_{678}}\right) \qquad (2.13)\\ &+\frac{1}{P^2_{812}}\left(\frac{1}{P^2_{345}}+\frac{1}{P^2_{456}}+\frac{1}{P^2_{567}}\right)+\frac{1}{P^2_{345}P^2_{678}}\Bigg]\end{aligned}$$

It is easy to check that add the pole part and boundary part we indeed reproduce the result from Feynman diagrams.

Now we move to the shifting $\langle 1|4]$. There is no boundary part, but for the pole part, there are following six terms: (123|45678), (812|34567), (781|23456), (78123|456), (67812|345) and (56781|234). Adding them up, we get again the same answer.

It should be interesting to compare terms we have added up in each method. For Feynman diagram method, there are $8+4=12$ terms. For $\langle 1|2]$-shifting there are $2+3=5$ terms while for $\langle 1|4]$-shifting there are 6 terms. Different method has given different combinations of various propagators.

Ten-point amplitudes: for this case, there are several topologies for tree amplitudes as shown in appendix A. The result can be summarized with cyclic ordering as (A.12). There are four kinds of diagrams with Z_{10} cyclic ordering and another three, Z_5 cyclic ordering, so there are total 55 terms.

For the $\langle 1|2]$-shifting, the boundary part has following four terms: $(3|456789(10))$, $(345|6789(10))$, $(34567|89(10))$ and $(3456789|(10))$. The result is given by

$$
\begin{aligned}
&A^{\langle 1|2]}_{10,b}(1,2,\ldots,10)\\
&\quad= A_4(1,2,-P_1,-P_2)\left(\frac{1}{p_3^2}A_2(P_1,3)\right)\left(\frac{1}{P_{123}^2}A_8(P_2,4,\ldots,10)\right)\\
&\qquad+A_4(1,2,-P_1,-P_2)\left(\frac{1}{P_{345}^2}A_4(P_1,3,4,5)\right)\left(\frac{1}{P_{12345}^2}A_6(P_2,6,\ldots,10)\right)\\
&\qquad+A_4(1,2,-P_1,-P_2)\left(\frac{1}{p_{34567}^2}A_6(P_1,3,\ldots,7)\right)\left(\frac{1}{P_{89(10)}^2}A_4(P_2,8,9,10)\right)\\
&\qquad+A_4(1,2,-P_1,-P_2)\left(\frac{1}{p_{(10)12}^2}A_8(P_1,3,\ldots,9)\right)\left(\frac{1}{p_{10}^2}A_2(P_2,10)\right)\\
&\quad= (-i\lambda)^4\frac{1}{P_{123}^2}\left(\sum_{\sigma\in\mathbb{Z}_8}\left(\frac{1}{P^2_{\sigma(3)\sigma(4)\sigma(5)}P^2_{\sigma(6)\sigma(7)\sigma(8)}}+\frac{1}{2P^2_{\sigma(3)\sigma(4)\sigma(5)}P^2_{\sigma(7)\sigma(8)\sigma(9)}}\right)\right)\\
&\qquad+\frac{1}{P^2_{345}P^2_{12345}}\left(\frac{1}{P^2_{678}}+\frac{1}{P^2_{789}}+\frac{1}{P^2_{89(10)}}\right)+\frac{1}{P^2_{34567}P^2_{89(10)}}\left(\frac{1}{P^2_{345}}+\frac{1}{P^2_{456}}+\frac{1}{P^2_{567}}\right)\\
&\qquad+\frac{1}{P^2_{12(10)}}\sum_{\sigma\in\mathbb{Z}_8}\left(\frac{1}{P^2_{\sigma(3)\sigma(4)\sigma(5)}P^2_{\sigma(6)\sigma(7)\sigma(8)}}+\frac{1}{P^2_{\sigma(3)\sigma(4)\sigma(5)}P^2_{\sigma(7)\sigma(8)\sigma(9)}}\right)
\end{aligned}
\tag{2.14}
$$

The pole part is given by the sum of three terms

$$
\begin{aligned}
&A^{\langle 1|2]}_{10,pole}(1,2,\ldots 10)\\
&\quad= \widehat{A_4}(9,10,\widehat{1},-P)\frac{1}{P^2_{9(10)1}}\widehat{A_8}(P,\widehat{2},3,4,5,6,7,8)\\
&\qquad+\widehat{A_6}(7,8,9,10,\widehat{1},-P)\frac{1}{P^2_{2345}}\widehat{A_6}(P,\widehat{2},3,4,5,6)\\
&\qquad+\widehat{A_8}(5,6,7,8,9,10,\widehat{1},-P)\frac{1}{P^2_{234}}\widehat{A_4}(P,\widehat{2},3,4)\\
&\quad= (-i\lambda)^4\frac{1}{P^2_{9(10)1}}\left(\sum_{\sigma\in\mathbb{Z}_8}\left(\frac{1}{\widehat{P}^2_{\sigma(2)\sigma(3)\sigma(4)}\widehat{P}^2_{\sigma(5)\sigma(6)\sigma(7)}}+\frac{1}{2\widehat{P}^2_{\sigma(2)\sigma(3)\sigma(4)}\widehat{P}^2_{\sigma(6)\sigma(7)\sigma(8)}}\right)\right)\\
&\qquad+\left(\frac{1}{\widehat{P}^2_{789}}+\frac{1}{\widehat{P}^2_{89(10)}}+\frac{1}{\widehat{P}^2_{9(10)1}}\right)\frac{1}{P^2_{23456}}\left(\frac{1}{\widehat{P}^2_{234}}+\frac{1}{\widehat{P}^2_{345}}+\frac{1}{\widehat{P}^2_{456}}\right)\\
&\qquad+\frac{1}{P^2_{234}}\left(\sum_{\sigma\in\mathbb{Z}_8}\left(\frac{1}{\widehat{P}^2_{\sigma(5)\sigma(6)\sigma(7)}\widehat{P}^2_{\sigma(8)\sigma(9)\sigma(10)}}+\frac{1}{2\widehat{P}^2_{\sigma(5)\sigma(6)\sigma(7)}\widehat{P}^2_{\sigma(9)\sigma(10)\sigma(1)}}\right)\right)
\end{aligned}
\tag{2.15}
$$

Using (2.12) we can show following identity:

$$
\begin{aligned}
&(-i\lambda)^4\Bigg[\frac{1}{P^2_{9(10)1}}\left(\frac{1}{\widehat{P}^2_{\widehat{2}34}}\frac{1}{P^2_{567}}+\frac{1}{P^2_{456}}\frac{1}{\widehat{P}^2_{\widehat{2}3456}}+\frac{1}{\widehat{P}^2_{\widehat{2}3456}}\frac{1}{\widehat{P}^2_{\widehat{2}34}}+\frac{1}{\widehat{P}^2_{\widehat{2}34}}\frac{1}{P^2_{567}}+\frac{1}{P^2_{345}}\frac{1}{\widehat{P}^2_{\widehat{2}3456}}\right)\\
&\quad+\frac{1}{P^2_{234}}\left(\frac{1}{P^2_{678}}\frac{1}{\widehat{P}^2_{9(10)\widehat{1}}}+\frac{1}{\widehat{P}^2_{9(10)\widehat{1}}}\frac{1}{\widehat{P}^2_{\widehat{2}3456}}+\frac{1}{\widehat{P}^2_{\widehat{2}3456}}\frac{1}{P^2_{789}}+\frac{1}{P^2_{567}}\frac{1}{\widehat{P}^2_{9(10)\widehat{1}}}+\frac{1}{P^2_{89(10)}}\frac{1}{\widehat{P}^2_{\widehat{2}3456}}\right)\\
&\qquad+\frac{1}{\widehat{P}^2_{9(10)\widehat{1}}}\frac{1}{P^2_{23456}}\left(\frac{1}{\widehat{P}^2_{\widehat{2}34}}\right)\Bigg] \qquad (2.16)\\
&=(-i\lambda)^4\Bigg[\frac{1}{P^2_{9(10)1}}\left(\frac{1}{P^2_{234}}\frac{1}{P^2_{567}}+\frac{1}{P^2_{456}}\frac{1}{P^2_{23456}}+\frac{1}{P^2_{23456}}\frac{1}{P^2_{234}}+\frac{1}{P^2_{234}}\frac{1}{P^2_{567}}+\frac{1}{P^2_{345}}\frac{1}{P^2_{23456}}\right)
\end{aligned}
$$

and then we can show that the result is same as given by $A^{\rm FD}_{10}$.

For shifting $\langle 1|4]$, there are nine terms from the recursion relations. Summing it up with some algebra we see that it reproduce the right answer.

Again, we count terms from different methods. The Feynman diagrams give 55 terms. The $\langle 1|2]$ shifting gives $4+3=7$ terms while the $\langle 1|4]$ shifting gives 9 terms.

3 Boundary BCFW and auxiliary field

As we have discussed, for $\lambda\phi^4$ theory, the boundary contribution is not zero for BCFW deformation (here we means the general unordered case). However, as presented in [4], by introducing a massive auxiliary field χ, it is possible to rewrite the $\lambda\phi^4$ theory into another form where three-particle BCFW deformation without boundary contribution does exists. In this section, we will explore the relation between auxiliary field method and the boundary BCFW method for $\lambda\phi^4$ theory.

The new Lagrangian with auxiliary field is given by

$$\mathcal{L}(\phi,\chi) = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)+\frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi)-\frac{1}{2}m^2_\chi\chi^2-g\chi\phi^2. \qquad (3.1)$$

The theory is not the $\lambda\phi^4$ theory, but under some limit, we can recover late. The limit is the large mass limit, where χ is not excited, so we can use the equation of motion (where the kinematic part has been set to zero) $m^2_\chi\chi+g\phi^2=0$ to solve χ and then put it back to the Lagrangian to get

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\frac{\lambda}{4!}\phi^4. \qquad (3.2)$$

where to match up the coupling constant, we need to set $\frac{g^2}{2m^2_\chi}=\frac{\lambda}{4!}$.

The Lagrangian $\mathcal{L}(\phi,\chi)$ is on-shell constructible without boundary contribution under following three-particle BCFW deformation

$$
\begin{aligned}
\tilde\lambda^{(1)}(z) &= \tilde\lambda^{(1)}-z\left(\frac{[1,3]}{[2,3]}\tilde\lambda^{(2)}+\frac{[1,3]}{[3,4]}\tilde\lambda^{(4)}\right)\\
\lambda^{(2)}(z) &= \lambda^{(2)}+z\frac{[1,3]}{[2,3]}\lambda^{(1)},\qquad \lambda^{(4)}(z)=\lambda^{(4)}+z\frac{[1,3]}{[3,4]}\lambda^{(1)}. \qquad (3.3)
\end{aligned}
$$

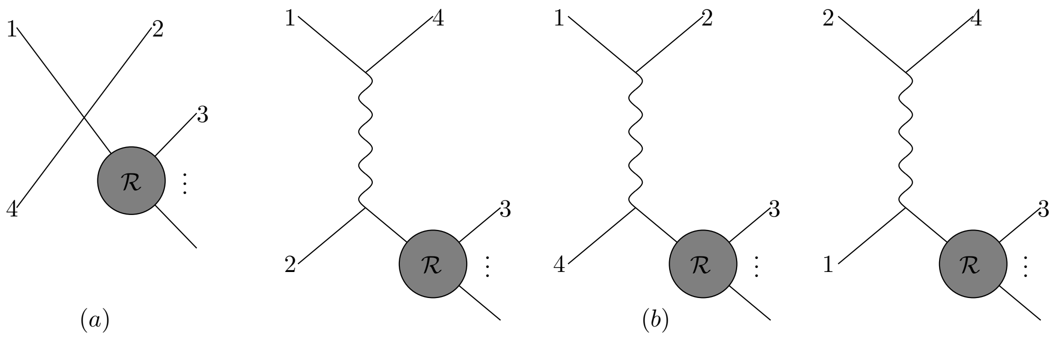

Figure 2. Pure scalar case: (a) boundary term figure (b) boundary term figure in view of auxiliary field.

but $\mathcal{L}(\phi)$ has boundary contribution under same deformation. In another word, for $\mathcal{L}(\phi,\chi)$ theory the BCFW recursion relation for n ϕ scalars is given by

$$\begin{aligned}\widetilde{A}_n = & \sum_{i\in I,\ 2\ or\ 4\in J} \widetilde{A}_I(I(z),P_\phi)\frac{1}{P^2}\widetilde{A}_J(J(z),-P_\phi) \\ & + \sum_{i\in I,\ 2\ or\ 4\in J} \widetilde{A}_I(I(z),P_\chi)\frac{1}{P^2-m_\chi^2}\widetilde{A}_J(J(z),-P_\chi)\,, \end{aligned} \tag{3.4}$$

where the first term has $\langle\phi\phi\rangle$ propagator in middle and the second term, has $\langle\chi\chi\rangle$ propagator. For $\mathcal{L}(\phi)$ theory the corresponding recursion relation is modified to

$$A_n = \sum_{i\in I,\ 2\ or\ 4\in J} A_I(I(z),P_\phi)\frac{1}{P^2}A_J(J(z),-P_\phi) + A_{n,b} \tag{3.5}$$

where A_b is the contribution from boundary. Comparing these two formula (3.4) and (3.5), we find that first term of both formula is, in fact, identical. Thus $\widetilde{A}_n = A_n$ is equivalent to the condition that the second term of $\widetilde{A}_n$, which is provided by the auxiliary propagator, is equal to the boundary part of A_n. Now we show this is true in remaining part of this section.

The boundary part of (3.5): just by checking the Feynman diagrams, it is easy to identify which kind of Feynman diagrams contributes the boundary term. It is nothing, but the one where particles $1,2,4$ are attached to same vertex, as shown in figure 2 (a).

The second term of (3.4): for this part, we need to use the amplitudes with one χ field, so it is important to know the g, m_χ power dependence of these amplitudes. First it is easy to know that the amplitude of m ϕ scalars and one χ scalar is zero when $m = odd$ while when $m = even$, it is not zero. Assuming there are V triple-vertex, I_1 $\langle\phi\phi\rangle$ propagators and I_2 $\langle\chi\chi\rangle$ propagators, by some simple arguments we have

$$I_1 = I_2 = \frac{m}{2} - 1, \qquad V = m - 1\,, \tag{3.6}$$

thus the large mass limit is given by

$$A(m\phi,\chi)\sim\frac{g^V}{(m_\chi^2)^{m/2-1}}\sim\lambda^{\frac{m-1}{2}}m_\chi\,. \tag{3.7}$$

Now we consider the amplitude under z-deformation where z is solved by $P^2(z)-m_\chi^2=0$, i.e., $z_\alpha\sim m_\chi^2$ for large mass limit. With the appearance of z, propagators will have different large mass behavior than the one without z-deformation. For $\langle\chi\chi\rangle$ propagator since $P^2(z_\alpha)-m_\chi^2\sim m_\chi^2$, the large mass behavior is same before and after z-deformation. Opposite to that, the $\langle\phi\phi\rangle$ propagator will be $P^2(z)\sim m_\chi^2$ after z-deformation and $P^2\to(m_\chi^2)^0$ before the z-deformation. Putting this back, we have

$$A(m\phi,\chi,z_\alpha)\sim\lambda^{\frac{m-1}{2}}m_\chi^{1-2t}\,, \tag{3.8}$$

where t is the number of $\langle\phi\phi\rangle$ propagators affected by z-deformation. Applying (3.8) to the second term of (3.4) we have

$$\lambda^{\frac{m_L-1}{2}}m_\chi^{1-2t_L}\frac{1}{m_\chi^2}\lambda^{\frac{m_R-1}{2}}m_\chi^{1-2t_R}\sim\lambda^{\frac{n-2}{2}}(m_\chi^2)^{-2(t_L+t_R)}\,, \tag{3.9}$$

which is not zero under large mass limit when and only when $t_L=t_R=0$.

What are these nonzero contributions with $t_L=t_R=0$ under our triple deformation? They are nothing, but the one given by figure 2 (b). It is also easy to see that they correspond exactly to the boundary part of (3.5).

Thus we have shown that how the boundary contribution can be transferred into contribution from auxiliary fields under triple deformation.

4 The scalar QCD theory

In this section, we consider another example where scalar and fermion interacting by Yukawa coupling form $\overline{\psi}\phi\psi$. Similar interaction terms are presented in Standard Model, so our example will have potential applications for practical calculations.

As in previous section, we will discuss the boundary behavior first and then write down the BCFW recursion relation with boundary contributions. After that we do several concrete calculations to demonstrate our method. For simplicity, our attention will focus on color ordered amplitudes with exactly two fermions and n scalars. Other situations can be discussed similarly.

4.1 The analysis of Feynman diagrams

For ordered amplitudes with two fermions and n scalars, we use q_1,q_2 to denote momenta of fermions and $p_1,\ldots,p_n$, the momenta of scalars, thus the ordered amplitude is denoted by $A(q_1,p_1,\ldots,p_n,q_2)$. By inspecting the general Feynman diagram given in figure 3, we see that there is one common feature: *a single fermionic line connecting these two fermions while other scalars are attached through Yukawa coupling at same side.* Using the fermionic propagator $\frac{i\not{p}}{p^2}$, the amplitude can be written as

$$A=\sum_{\text{diagrams}}\mathcal{S}_i\mathcal{Q}_i \tag{4.1}$$

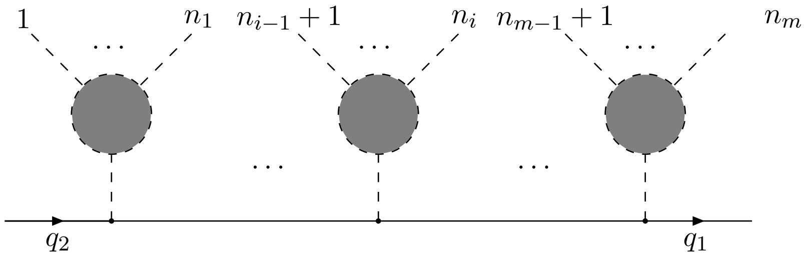

Figure 3. General Feynman diagrams.

where S_i is contribution from scalar part and Q_i is the form

$$\mathcal{Q}(q_1^-, q_2^+; R_1, \ldots, R_m) \sim i^m \frac{\langle 1|R_1|R_2|\ldots|R_m|2]}{R_1^2 R_2^2 \ldots . R_m^2} \tag{4.2}$$

where we have assumed the helicity of q_1, q_2 is $(-,+)$ and there are m fermionic propagators along the line. In fact, it is easy to see that when $h_{q_1} = h_{q_2}$, to get nonzero amplitudes we must have even number of fermionic propagators (i.e., $m = even$) while when $h_{q_1} = -h_{q_2}$, to get nonzero amplitudes we must have odd number of fermionic propagators (i.e., $m = odd$).

Now we introduce following $\langle 1|2]$-deformation between the two fermions

$$\lambda^{(1)} = \lambda^{(1)} + z\lambda^{(2)} \quad \tilde{\lambda}^{(2)} = \tilde{\lambda}^{(2)} - z\tilde{\lambda}^{(1)} \tag{4.3}$$

It is easy to see from Fig 3 that all $\mathcal{S}_i$ factor in (4.1) do not depend on z and the only z-dependence is inside $\mathcal{Q}_i$. With different helicity configurations, the discussion will be a little different, so we consider it case by case.

The helicity $(h_{q_1}, h_{q_2}) = (+,+)$: in this case the number of propagator should be even and we have following two cases: (A) $m = 0$; (B) $m \geq 2$. For case (B), we have

$$\begin{aligned}
&\mathcal{Q}(q_1^+, q_2^+; R_1, \ldots, R_m) \\
&\sim i^m \frac{\left[1|(q_1+R_1+z\lambda_2\widetilde{\lambda}_1)(\prod_{j=1}^{m-2}(q_1+R_j+z\lambda_2\widetilde{\lambda}_1))(q_1+R_m+z\lambda_2\widetilde{\lambda}_1)|\widetilde{\lambda}_2 - z\widetilde{\lambda}_1\right]}{(q_1+R_1+z\lambda_2\widetilde{\lambda}_1))^2(\prod_{j=1}^{m-2}(q_1+R_j+z\lambda_2\widetilde{\lambda}_1)^2)(q_1+R_m+z\lambda_2\widetilde{\lambda}_1)^2} \\
&= i^m \frac{\left[1|(q_1+R_1)(\prod_{j=1}^{m-2}(q_1+R_j+z\lambda_2\widetilde{\lambda}_1))(q_1+R_m+z\lambda_2\widetilde{\lambda}_1)|\widetilde{\lambda}_2 - z\widetilde{\lambda}_1\right]}{(q_1+R_1+z\lambda_2\widetilde{\lambda}_1))^2(\prod_{j=1}^{m-2}(q_1+R_j+z\lambda_2\widetilde{\lambda}_1)^2)(q_1+R_m+z\lambda_2\widetilde{\lambda}_1)^2} \\
&= i^m \frac{\left[1|(q_1+R_1)(\prod_{j=1}^{m-2}(q_1+R_j+z\lambda_2\widetilde{\lambda}_1))(q_1+R_m)|\widetilde{\lambda}_2 - z\widetilde{\lambda}_1\right]}{(q_1+R_1+z\lambda_2\widetilde{\lambda}_1))^2(\prod_{j=1}^{m-2}(q_1+R_j+z\lambda_2\widetilde{\lambda}_1)^2)(q_1+R_m+z\lambda_2\widetilde{\lambda}_1)^2} \\
&\quad + i^m \frac{\left[1|(q_1+R_1)(\prod_{j=1}^{m-2}(q_1+R_j+z\lambda_2\widetilde{\lambda}_1))(z\lambda_2\widetilde{\lambda}_1)|\widetilde{\lambda}_2\right]}{(q_1+R_1+z\lambda_2\widetilde{\lambda}_1))^2(\prod_{j=1}^{m-2}(q_1+R_j+z\lambda_2\widetilde{\lambda}_1)^2)(q_1+R_m+z\lambda_2\widetilde{\lambda}_1)^2}
\end{aligned} \tag{4.4}$$

which goes zero under the $z \to \infty$ limit since each term has $(m-1)$ z in numerator and m z in denominator. For the case (A) we have

$$\mathcal{Q}(q_1^+, q_2^+) = [1|2 - z1] = [1|2] \tag{4.5}$$

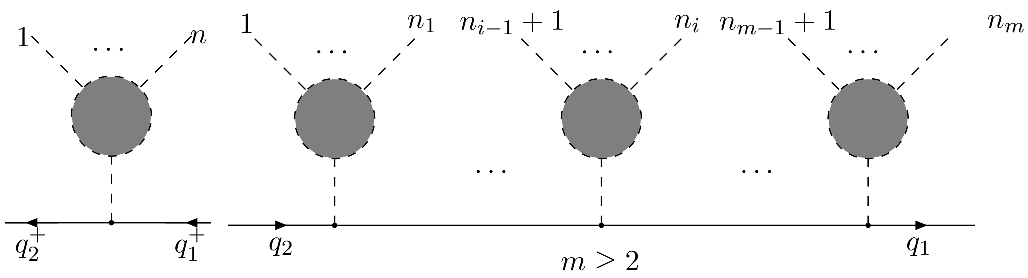

Figure 4. Amplitude in total.

which is independent of z.

From above analysis we see that under our $\langle 1|2]$ shifting, there is nonzero boundary contribution and it is purely given by diagrams of case (A). Thus it is easy to write down the BCFW recursion relation with boundary term as

$$\begin{aligned} &A_{n+2}(q_1^+;p_1,\ldots,p_n;q_2^+) \\ &\quad = \sum_{i=1,h=\pm}^{n-1} A_{i+2}(q_1^+(z_i);p_1,\ldots,p_i;q_i^h(z_i)) \frac{1}{(q_1+\sum_{j=1}^i p_j)^2} \times \\ &\qquad\quad \times A_{n-i+2}(-q_i^{-h}(z_i);p_{i+1},\ldots,p_n;q_2^+(z_i)) + \frac{(-ig)\,[1|2]}{(\sum_{i=1}^n p_i)^2} A_{n+1}(p_1,\ldots,p_n,P_\phi) \end{aligned} \tag{4.6}$$

where A_{n+1} is the amplitude of $(n+1)$ pure scalars. The expression can also be represented by following figure 4

The helicity $(h_{q_1},h_{q_2}) = (-,-)$: in this case the number of propagator should again be even and we have following two cases: (A) $m=0$; (B) $m \geq 2$. For case (B), we have

$$\begin{aligned} &\mathcal{Q}(q_1^-,q_2^-;R_1,\ldots,R_m) \\ &\quad \sim i^m \frac{\left\langle \lambda_1+z\lambda_2|(q_1+R_1+z\lambda_2\widetilde{\lambda}_1)(\prod_{j=1}^{m-2}(q_1+R_j+z\lambda_2\widetilde{\lambda}_1))(q_1+R_m+z\lambda_2\widetilde{\lambda}_1)|\lambda_2\right\rangle}{(q_1+R_1+z\lambda_2\widetilde{\lambda}_1))^2(\prod_{j=1}^{m-2}(q_1+R_j+z\lambda_2\widetilde{\lambda}_1)^2)(q_1+R_m+z\lambda_2\widetilde{\lambda}_1)^2} \\ &\quad = i^m \frac{\left\langle \lambda_1+z\lambda_2|(q_1+R_1+z\lambda_2\widetilde{\lambda}_1)(\prod_{j=1}^{m-2}(q_1+R_j+z\lambda_2\widetilde{\lambda}_1))(q_1+R_m)|\lambda_2\right\rangle}{(q_1+R_1+z\lambda_2\widetilde{\lambda}_1))^2(\prod_{j=1}^{m-2}(q_1+R_j+z\lambda_2\widetilde{\lambda}_1)^2)(q_1+R_m+z\lambda_2\widetilde{\lambda}_1)^2} \\ &\quad = i^m \frac{\left\langle \lambda_1+z\lambda_2|(q_1+R_1)(\prod_{j=1}^{m-2}(q_1+R_j+z\lambda_2\widetilde{\lambda}_1))(q_1+R_m)|\lambda_2\right\rangle}{(q_1+R_1+z\lambda_2\widetilde{\lambda}_1))^2(\prod_{j=1}^{m-2}(q_1+R_j+z\lambda_2\widetilde{\lambda}_1)^2)(q_1+R_m+z\lambda_2\widetilde{\lambda}_1)^2} \\ &\qquad + i^m \frac{\left\langle \lambda_1|(z\lambda_2\widetilde{\lambda}_1)(\prod_{j=1}^{m-2}(q_1+R_j+z\lambda_2\widetilde{\lambda}_1))(q_1+R_m)|\lambda_2\right\rangle}{(q_1+R_1+z\lambda_2\widetilde{\lambda}_1))^2(\prod_{j=1}^{m-2}(q_1+R_j+z\lambda_2\widetilde{\lambda}_1)^2)(q_1+R_m+z\lambda_2\widetilde{\lambda}_1)^2} \end{aligned} \tag{4.7}$$

which goes zero under the $z\to\infty$ limit since each term has $(m-1)$ z in numerator and m z in denominator. For the case (A) we have

$$\mathcal{Q}(q_1^-,q_2^-) = \langle 1+z1|2\rangle = \langle 1|2\rangle \tag{4.8}$$

which is independent of z. Thus under our $\langle 1|2]$ shifting, there is nonzero boundary contribution and it is purely given by diagrams of case (A). Thus it is easy to write down the BCFW recursion relation with boundary term as

$$\begin{aligned} & A_{n+2}(q_1^-; p_1, \ldots, p_n; q_2^-) \\ & = \sum_{i=1,h=\pm}^{n-1} A_{i+2}(q_1^-(z_i); p_1, \ldots, p_i; q_i^h(z_i)) \frac{1}{(q_1 + \sum_{j=1}^i p_j)^2} \times \\ & \qquad \times A_{n-i+2}(-q_i^{-h}(z_i); p_{i+1}, \ldots, p_n; q_2^-(z_i)) + \frac{(-ig)\langle 1|2\rangle}{(\sum_{i=1}^n p_i)^2} A_{n+1}(p_1, \ldots, p_n, P_\phi) \end{aligned} \tag{4.9}$$

where A_{n+1} is the amplitude of $(n+1)$ pure scalars. It is obvious that $(+,+)$ is conjugated to $(-,-)$.

The helicity $(h_{q_1}, h_{q_2}) = (+,-)$: in this case the number of propagator should be odd, i.e., we have at least one propagator. This case is, in fact, simpler than previous two cases. The general expression of $\mathcal{Q}$ should be

$$\begin{aligned} & \mathcal{Q}(q_1^+, q_2^-; R_1, \ldots, R_m) \\ & \quad \sim i^m \frac{\left[\widetilde{\lambda}_1|(q_1 + R_1 + z\lambda_2\widetilde{\lambda}_1)(\prod_{j=1}^{m-1}(q_1 + R_j + z\lambda_2\widetilde{\lambda}_1))|\lambda_2\right\rangle}{(q_1 + R_1 + z\lambda_2\widetilde{\lambda}_1))^2(\prod_{j=1}^{m-2}(q_1 + R_j + z\lambda_2\widetilde{\lambda}_1)^2)(q_1 + R_m + z\lambda_2\widetilde{\lambda}_1)^2} \\ & \quad = i^m \frac{\left[\widetilde{\lambda}_1|(q_1 + R_1)(\prod_{j=1}^{m-1}(q_1 + R_j + z\lambda_2\widetilde{\lambda}_1))|\lambda_2\right\rangle}{(q_1 + R_1 + z\lambda_2\widetilde{\lambda}_1))^2(\prod_{j=1}^{m-2}(q_1 + R_j + z\lambda_2\widetilde{\lambda}_1)^2)(q_1 + R_m + z\lambda_2\widetilde{\lambda}_1)^2} \end{aligned} \tag{4.10}$$

which goes zero under the $z \to \infty$ limit since each term has $(m-1)$ z in numerator and m z in denominator. In other word, with this helicity configuration and the choice of BCFW deformation, the boundary contribution is zero and we have familiar BCFW recursion relation which is given by

$$\begin{aligned} & A_{n+2}(q_1^+; p_1, \ldots, p_n; q_2^-) \\ & \quad = \sum_{i=1,h=\pm}^{n-1} A_{i+2}(q_1^+(z_i); p_1, \ldots, p_i; q_i^h(z_i)) \frac{1}{(q_1 + \sum_{j=1}^i p_j)^2} \times \\ & \qquad \qquad \times A_{n-i+2}(-q_i^{-h}(z_i); p_{i+1}, \ldots, p_n; q_2^-(z_i)) \, . \end{aligned} \tag{4.11}$$

It is worth to emphasize that although $(h_1, h_2) = (+,-)$ case does not have boundary contributions, the sub-amplitudes in the recursive calculation will meet the helicities configurations $(+,+)$, $(-,-)$ and $(-,+)$, thus the boundary contributions have been included implicitly through these sub-amplitudes.

The helicity $(h_{q_1}, h_{q_2}) = (-,+)$: in this case the number of propagator should be odd and we should have at least one propagator. However, unlike the previous case where boundary contribution is zero, current one is the most complicated one and we should divide diagrams into three cases: (A) $m=1$; (B) $m=3$; (C) $m \geq 5$. Let us first show that the case (C) does not give boundary contributions. The observation we will use is that when $z\lambda_2\widetilde{\lambda}_1$ are nearby, its contribution is zero, i.e., $\left\langle\alpha|z\lambda_2\widetilde{\lambda}_1|z\lambda_2\widetilde{\lambda}_1|\beta\right\rangle = 0$. Using this, when we have five $(q_1+R_i+z\lambda_2\widetilde{\lambda}_1)$ factors in a row, we can expand it into the power of z as (it is worth to remember that there are five corresponding propagators with $\frac{1}{z}$ dependence)

$$\begin{aligned}&(q_1+R_1+z\lambda_2\widetilde{\lambda}_1)(q_1+R_2)(q_1+R_3+z\lambda_2\widetilde{\lambda}_1)(q_1+R_4)(q_1+R_5+z\lambda_2\widetilde{\lambda}_1)\\&\quad+(q_1+R_1)(q_1+R_2+z\lambda_2\widetilde{\lambda}_1)(q_1+R_3)(q_1+R_4+z\lambda_2\widetilde{\lambda}_1)(q_1+R_5)+\mathcal{O}(z)\end{aligned}$$

For the first term, when we contract with spinor as $\left\langle\lambda_1+z\lambda_2|(q_1+R_1+z\lambda_2\widetilde{\lambda}_1)|\beta\right]$, we have $\langle\lambda_1+z\lambda_2|(q_1+R_1)|\beta]+\left\langle\lambda_1|(z\lambda_2\widetilde{\lambda}_1)|\beta\right]$, thus all terms are at most $\frac{1}{z}$ order.

Having established that the case (C) does not give boundary contribution, we move to case (A) and (B). For case (A) the general result should be

$$I_A = (-ig)^2\sum_{i=1}^{n-1}\frac{\left\langle\lambda_1+z\lambda_2|(q_1+R_i+z\lambda_2\widetilde{\lambda}_1)|\widetilde{\lambda}_2-z\widetilde{\lambda}_1\right]}{(q_1+R_i+z\lambda_2\widetilde{\lambda}_1)^2}\frac{A_{i,\phi}}{R_i^2(q_1+q_2+R_i)^2}\,, \tag{4.12}$$

where $R_i = p_1+\ldots+p_i$ and $A_{i,\phi}$ is the contribution from scalars as

$$A_{i,\phi} = A_{i+1}(p_1,\ldots,p_i,p_{R_i})A_{n-i+1}(p_{i+1},\ldots,p_n,p_{R_2}) \tag{4.13}$$

Taking the residue of I_A around $z=\infty$ we will get

$$B[I_A] = (-ig)^2\sum_{i=1}^{n-1}\frac{(R_i^2+2R_i\cdot q_2)}{\langle 2|R|1]}\frac{A_{i,\phi}}{R_i^2(q_1+q_2+R_i)^2}\,. \tag{4.14}$$

For the case (B), first we consider which term gives nonzero contributions. Expanding the product of three $(q_1+R_i+z\lambda_2\widetilde{\lambda}_1)$ we find that following five terms:

$$\begin{aligned}&\langle\alpha|(q_1+R_1)(q_2+R_2)(q_3+R_3)|\beta]+\left\langle\alpha|(z\lambda_2\widetilde{\lambda}_1)(q_2+R_2)(q_3+R_3)|\beta\right]\\&\quad+\left\langle\alpha|(q_1+R_1)(q_2+R_2)(z\lambda_2\widetilde{\lambda}_1)|\beta\right]\\&\qquad+\left\langle\alpha|(q_1+R_1)(z\lambda_2\widetilde{\lambda}_1)(q_3+R_3)|\beta\right]+\left\langle\alpha|(z\lambda_2\widetilde{\lambda}_1)(q_2+R_2)(z\lambda_2\widetilde{\lambda}_1)|\beta\right]\end{aligned}$$

with $\alpha=\lambda_1+z\lambda_2, \beta=\widetilde{\lambda}_2-z\widetilde{\lambda}_1$. Remembering the $\frac{1}{z^3}$ factor from three propagators, we see that except the fourth term, all other terms will vanish under the limit $z\to\infty$. In another word, only the fourth term gives nonzero contribution.

Having identified the term, we write down its expression as

$$\begin{aligned}I_B = (-ig)^4\sum_{i_1+i_2+i_3+i_4=n}&\frac{\left\langle\lambda_1+z\lambda_2|(q_1+R_1)(q_1+R_2+z\lambda_2\widetilde{\lambda}_1)(q_1+R_3)|\widetilde{\lambda}_2-z\widetilde{\lambda}_1\right]}{(q_1+R_1+z\lambda_2\widetilde{\lambda}_1)^2(q_1+R_2+z\lambda_2\widetilde{\lambda}_1)^2(q_1+R_3+z\lambda_2\widetilde{\lambda}_1)^2}\\&\times\frac{A_{i_1i_2i_3i_4,\phi}}{R_1^2(R_2-R_1)^2(R_3-R_2)^2(R_4-R_3)^2}\,,\end{aligned} \tag{4.15}$$

where $R_1=\sum_{j=1}^{i_1}p_j$, $R_2=\sum_{j=1}^{i_1+i_2}p_j$, $R_3=\sum_{j=1}^{i_1+i_2+i_3}p_j$, $R_4=\sum_{j=1}^{n}p_j$, and

$$A_{i_1i_2i_3i_4,\phi} = A_{i_1+1}(p_1,\ldots,p_{i_1},p_{R_{i_1}})A_{i_2+1}A_{i_3+1}A_{i_4+1} \tag{4.16}$$

where A_{i_k+1}, $k=2,3,4$ have similar structure like A_{i_1+1} so we have written them briefly. Taking the residue we have

$$B[I_B] = (-ig)^4 \sum_{i_1+i_2+i_3+i_4=n} \frac{-1}{\langle 2|q_1+R_2|1]} \frac{A_{i_1 i_2 i_3 i_4,\phi}}{R_1^2(R_2-R_1)^2(R_3-R_2)^2(R_4-R_4)^2} \,. \tag{4.17}$$

Having these two boundary contributions (4.12) and(4.17) we can finally write down the boundary BCFW recursion relation as

$$\begin{aligned} A_{n+2}(q_1^-;p_1,\ldots,p_n;q_2^+) &= B[I_A]+B[I_B] \\ &\quad + \sum_{i=1,h=\pm}^{n-1} A_{i+2}(q_1^+(z_i);p_1,\ldots,p_i;q_i^h(z_i)) \frac{1}{(q_1+\sum_{j=1}^i p_j)^2} \times \\ &\quad \times A_{n-i+2}(-q_i^{-h}(z_i);p_{i+1},\ldots,p_n;q_2^-(z_i)) \,. \end{aligned} \tag{4.18}$$

There is one thing we want to remark for this helicity. Our above analysis is done with the $\langle 1|2]$-shifting. However, if we use the $[1|2\rangle$-shifting, it is easy to see that there is no boundary contribution. Thus we can calculate same amplitudes using two different methods: one with boundary contribution and one without. This will be a strong consistent check for our formula.

4.2 Explicit calculations for various helicity configurations

In this subsection we will use the BCFW recursion relations presented in previous subsection to calculate various amplitudes with all possible helicity configurations. All results are same as the one given by Feynman diagrams in appendix A.

4.2.1 The helicity $(h_{q_1}, h_{q_2}) = (+,-)$

This is the simplest case where boundary contributions are zero. However, as we have emphasized, the sub-amplitudes used in the recursion relation will involve other helicity configurations, thus the knowledge of boundary behavior is essential.

With two scalars, there is only one possible channel $\mathcal{I} = \{q_1^+, 1\}$ with $z = \frac{P_{q_1 1}^2}{\langle 2|P_{q_1 1}|1]}$ where we have defined $P_{q_1 i} = q_1 + p_1 + \ldots + p_i$. Putting this we have

$$A\Big(q_1^+;p_1,p_2;q_2^-\Big) = A\Big(\widehat{q}_1^+;p_1;-\widehat{P}_{q_1 1}^+\Big)\frac{1}{P_{q_1 1}^2} A\Big(\widehat{P}_{q_1 1}^-;p_2;\widehat{q}_1^-\Big) = (-ig)^2 \frac{[1|P_{q_1 1}|2\rangle}{P_{q_1 1}^2} \tag{4.19}$$

With four scalars there are three channels and we have

$$\begin{aligned} A\Big(q_1^+;p_1,p_2,p_3,p_4;q_2^-\Big) &= (-ig)^4 \sum_{i=1}^3 \frac{[1|(P_{q_1 1}+z_i q)(P_{q_1 2}+z_i q)(P_{q_1 3}+z_i q)|2\rangle}{\prod_{\substack{j=1\\ j\neq i}}^3 \left(1-\frac{z_i}{z_j}\right) P_{q_1 1}^2 P_{q_1 2}^2 P_{q_1 3}^2} \\ &\quad + (-ig)^2(-i\lambda)\left[\frac{[1|P_{q_1 1}|2\rangle}{P_{q_1 1}^2} + \frac{[1|P_{q_1 3}|2\rangle}{P_{q_1 1}^2}\right] \end{aligned} \tag{4.20}$$

where z_i are poles for these three channels. Using identities

$$\sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{1-\frac{z_i}{z_j}}=1,\quad \sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{z^m}{1-\frac{z_i}{z_j}}=1,(m\in\mathbb{Z},1\leq m<n),\quad \sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{z^n}{1-\frac{z_i}{z_j}}=\prod_{k=1}^{n}z_k \tag{4.21}$$

the first term can be reduced to

$$(-ig)^4\frac{[1|P_{q_11}P_{q_12}P_{q_13}|2\rangle}{P^2_{q_11}P^2_{q_12}P^2_{q_13}} \tag{4.22}$$

thus we get the same answer as in appendix.

With six scalars there are five channels. Using identity (4.21) we can simplify the expression and get

$$\begin{aligned}A\Big(q_1^-;p_1,p_2,p_3,p_4;q_2^+\Big) &= (-ig)^6\frac{[1|P_{q_11}P_{q_12}P_{q_13}P_{q_14}P_{q_15}|2\rangle}{P^2_{q_11}P^2_{q_12}P^2_{q_13}P^2_{q_14}P^2_{q_15}} \\ &+(-ig)^4(-i\lambda)\Bigg[\frac{[1|P_{q_11}P_{q_12}P_{q_13}|2\rangle}{P^2_{q_11}P^2_{q_12}P^2_{q_13}P^2_{46}}+\frac{[1|P_{q_11}P_{q_12}P_{q_15}|2\rangle}{P^2_{q_11}P^2_{q_12}P^2_{q_15}P^2_{35}} \\ &\qquad+\frac{[1|P_{q_11}P_{q_14}P_{q_15}|2\rangle}{P^2_{q_11}P^2_{q_14}P^2_{q_15}P^2_{24}}+\frac{[1|P_{q_13}P_{q_14}P_{q_15}|2\rangle}{P^2_{q_13}P^2_{q_14}P^2_{q_15}P^2_{13}}\Bigg] \\ &+(-ig)^2(-i\lambda)^2\Bigg\{\frac{[1|P_{q_13}|2\rangle}{P^2_{q_13}P^2_{13}P^2_{46}}+\frac{[1|P_{q_11}|2\rangle}{P^2_{q_11}P^2_{26}}\left[\frac{1}{P^2_{24}}+\frac{1}{P^2_{35}}+\frac{1}{P^2_{46}}\right] \\ &\qquad+\frac{[1|P_{q_15}|2\rangle}{P^2_{q_15}P^2_{15}}\left[\frac{1}{P^2_{13}}+\frac{1}{P^2_{24}}+\frac{1}{P^2_{35}}\right]\Bigg\}\end{aligned} \tag{4.23}$$

which is again the right one.

4.2.2 The helicity $(h_{q_1},h_{q_2})=(-,+)$

This is the most complicated helicity configuration because there are two possible boundary contributions $B[I_A]$ and $B[I_B]$ with one or three fermion propagators respectively. Since for these two case (A) and (B), there are also corresponding pole contributions, we can combine pole part and boundary part together. Sometimes this combination gives simpler expression. Noticing this we move to the explicit calculations.

With only two scalars, there are only one $B[I_A]$ contribution and one pole contribution:

$$A\Big(q_1^-;p_1,p_2;q_2^+\Big)=A\Big(\widehat{q}_1^-;p_1;-\widehat{P}^-_{q_11}\Big)\frac{1}{P^2_{q_11}}A\Big(\widehat{P}^+_{q_11};p_2;\widehat{q}_2^+\Big)+B[I_A] \tag{4.24}$$

The boundary term of case (A) is given by

$$B[I_A]=(-ig)^2\frac{(R^2_{\mathcal{I}}+2R_{\mathcal{I}}\cdot q_2)}{\langle 2|R_{\mathcal{I}}|1]}\frac{A_{i,\phi}}{R^2_{\mathcal{I}}(q_1+q_2+R_{\mathcal{I}})^2} \tag{4.25}$$

where $R_{\mathcal{I}}=P_{\mathcal{I}}+q_1$. Adding them up we have

$$\begin{aligned}&-(ig)^2\frac{\Big\langle\lambda_1+z_{\mathcal{I}}\lambda_2|P_{\mathcal{I}}+z_{\mathcal{I}}q|\widetilde{\lambda}_2-z_{\mathcal{I}}\widetilde{\lambda}_1\Big]}{P^2_{\mathcal{I}}}\frac{A_{i,\phi}}{R^2_{\mathcal{I}}(q_1+q_2+R_{\mathcal{I}})^2}+B[I_A] \\ &\qquad=-(ig)^2\frac{\langle 1|P_{\mathcal{I}}|2]}{P^2_{\mathcal{I}}}\frac{A_{i,\phi}}{R^2_{\mathcal{I}}(q_1+q_2+R_{\mathcal{I}})^2}\end{aligned} \tag{4.26}$$

Using result (4.26) and and $z_1 = \frac{P_{q_1 1}^2}{\langle 2|P_{q_1 1}|1]}$ we can simplify to

$$A\Big(q_1^-; p_1, p_2; q_2^+\Big) = (-ig)^2 \frac{\langle 1|P_{q_1 1}|2]}{P_{q_1 1}^2} \tag{4.27}$$

With four scalars, both case (A) and case (B) give boundary contributions. For case (A) there are two nonzero contributions with splitting (1|234) and (123|4) (the splitting (12|34) will give zero). For case (B) there is only one contribution with splitting (1|2|3|4). For pole part we have three channels $\mathcal{I} = \{q_1, p_1\}, \{q_1, p_1, p_2\}, \{q_1, p_1, p_2, p_3\}$ with location of poles z_1, z_2, z_3 respectively. Putting all together we have expression

$$\begin{aligned}
&A\Big(q_1^-; p_1, p_2, p_3, p_4; q_2^+\Big) \\
&\quad= A\Big(\widehat{q_1}^-; p_1; -\widehat{P}_{q_1 1}^-\Big) \frac{1}{P_{q_1 1}^2} A\Big(\widehat{P}_{q_1 1}^+; p_2, p_3, p_4; \widehat{q_2}^+\Big) + B[I_A(1|234)] \\
&\quad\quad + A\Big(\widehat{q_1}^-; p_1, p_2; -\widehat{P}_{q_1 1}^+\Big) \frac{1}{P_{q_1 2}^2} A\Big(\widehat{P}_{q_1 1}^-; p_3, p_4; \widehat{q_2}^+\Big) \\
&\quad\quad + A\Big(\widehat{q_1}^-; p_1, p_2, p_3; -\widehat{P}_{q_1 1}^-\Big) \frac{1}{P_{q_1 3}^2} A\Big(\widehat{P}_{q_1 1}^+; p_4; \widehat{q_2}^+\Big) + B[I_A(123|4)] + B[I_B(1|2|3|4)] \\
&\quad= (-ig)^4 \sum_{i=1}^{3} \frac{\Big\langle \lambda_1 + z_i\lambda_2 | (P_{q_1 1} + z_i q)(P_{q_1 2} + z_i q)(P_{q_1 3} + z_i q) | \widetilde{\lambda}_2 - z_i \widetilde{\lambda}_1 \Big]}{\prod_{\substack{j=1 \\ j\neq i}}^3 \Big(1 - \frac{z_i}{z_j}\Big) P_{q_1 1}^2 P_{q_1 2}^2 P_{q_1 3}^2} \\
&\quad\quad + B[I_B(1|2|3|4)] + B[I_A(1|234)] \\
&\quad\quad + (-ig)^2(-i\lambda) \left[\frac{\Big\langle \lambda_1 + z_1\lambda_2 | P_{q_1 1} | \widetilde{\lambda}_2 - z_1 \widetilde{\lambda}_1 \Big]}{P_{q_1 1}^2} + \frac{\Big\langle \lambda_1 + z_3\lambda_2 | P_{q_1 3} | \widetilde{\lambda}_2 - z_3 \widetilde{\lambda}_1 \Big]}{P_{q_1 1}^2} \right] \\
&\quad\quad + B[I_A(123|4)]
\end{aligned}$$

It can be checked that terms with z_i^3 will cancel with the contribution from $B[I_B(1|2|3|4)]$. Terms with lower order of z_i sum to zero according to the identity in (4.21). z_i^3-independent terms equals $(-ig)^4 \frac{\langle 1|P_{q_1 1} P_{q_1 2} P_{q_1 3}|2]}{P_{q_1 1}^2 P_{q_1 2}^2 P_{q_1 3}^2}$ also by identity in (4.21). Finally we have

$$A\Big(q_1^-; p_1, p_2, p_3, p_4; q_2^+\Big) = (-ig)^4 \frac{\langle 1|P_{q_1 1} P_{q_1 2} P_{q_1 3}|2]}{P_{q_1 1}^2 P_{q_1 2}^2 P_{q_1 3}^2} + (-ig)^2(-i\lambda) \left[\frac{\langle 1|P_{q_1 1}|2]}{P_{q_1 1}^2} + \frac{\langle 1|P_{q_1 3}|2]}{P_{q_1 3}^2} \right] \tag{4.28}$$

With six scalars, there are five pole channels. For boundary contributions of case (A), there are three nonzero partitions (1|23456), (123|456) and (12345|6). For the case (B), there are four nonzero partitions (1|2|3|456), (123|4|5|6), (1|234|5|6) and (1|2|345|6).

Adding them up and simplifying with (4.21) we get

$$
\begin{aligned}
&A\Big(q_1^-;p_1,p_2,p_3,p_4,p_5,p_6;q_2^+\Big)\\
&\quad=(-ig)^6\frac{\langle 1|P_{q_11}P_{q_12}P_{q_13}P_{q_14}P_{q_15}|2]}{P_{q_11}^2P_{q_12}^2P_{q_13}^2P_{q_14}^2P_{q_15}^2}\\
&\qquad+(-ig)^4(-i\lambda)\left[\frac{\langle 1|P_{q_11}P_{q_12}P_{q_13}|2]}{P_{q_11}^2P_{q_12}^2P_{q_13}^2P_{46}^2}+\frac{\langle 1|P_{q_11}P_{q_12}P_{q_15}|2]}{P_{q_11}^2P_{q_12}^2P_{q_15}^2P_{35}^2}\right.\\
&\qquad\qquad\left.+\frac{\langle 1|P_{q_11}P_{q_14}P_{q_15}|2]}{P_{q_11}^2P_{q_14}^2P_{q_15}^2P_{24}^2}+\frac{\langle 1|P_{q_13}P_{q_14}P_{q_15}|2]}{P_{q_13}^2P_{q_14}^2P_{q_15}^2P_{13}^2}\right]\\
&\qquad+(-ig)^2(-i\lambda)^2\left\{\frac{\langle 1|P_{q_13}|2]}{P_{q_13}^2P_{13}^2P_{46}^2}+\frac{\langle 1|P_{q_11}|2]}{P_{q_11}^2P_{26}^2}\left[\frac{1}{P_{24}^2}+\frac{1}{P_{35}^2}+\frac{1}{P_{46}^2}\right]\right.\\
&\qquad\qquad\left.+\frac{\langle 1|P_{q_15}|2]}{P_{q_15}^2P_{15}^2}\left[\frac{1}{P_{13}^2}+\frac{1}{P_{24}^2}+\frac{1}{P_{35}^2}\right]\right\}
\end{aligned}
\tag{4.29}
$$

which can be checked with result in appendix A.

4.2.3 The helicity $(h_{q_1},h_{q_2})=(+,+)$

In this case, we have $n=odd$ for nonzero results. With $n=1$, there is no pole contribution, but there is one boundary contribution and we have

$$
A(q_1^+;p;q_2^+)=(-ig)\,[\lambda_1|\lambda_2-z\lambda_1]=(-ig)\,[1|2]
\tag{4.30}
$$

which can be checked to be right. With $n=3$ there are two pole contributions and one boundary contribution $B_{n=3}=(-ig)(-i\lambda)\frac{1}{P_{13}^2}\,[1|2]$ where $P_{ij}=p_i+p_{i+1}+\ldots+p_j$. Adding up we have

$$
\begin{aligned}
A(q_1^+;p_1,p_2,p_3;q_2^+)&=\widehat{A}(\widehat{p}_1^+;q_1;-\widehat{P}_{q_11}^+)\frac{1}{P_{q_11}^2}\widehat{A}(\widehat{P}_{q_11}^-;q_2,q_3;\widehat{p}_2^+)\\
&\qquad+\widehat{A}(\widehat{p}_1^+;q_1,q_2;-\widehat{P}_{q_12}^-)\frac{1}{P_{q_12}^2}\widehat{A}(\widehat{P}_{q_12}^+;q_3;\widehat{p}_2^+)+B_{n=3}\\
&=(-ig)^3\Bigg(\left[\widetilde{\lambda}_1\right|\frac{(P_{q_11}-z_1q)(P_{q_12}-z_1q)}{P_{q_11}^2(P_{q_12}^2-z_1P_2q)}\left|\widetilde{\lambda}_2-z_1\widetilde{\lambda}_1\right]\\
&\qquad+\left[\widetilde{\lambda}_1\right|\frac{(P_{q_11}-z_2q)(P_{q_12}-z_2q)}{(P_{q_11}^2-z_2P_{q_11}q)P_{q_12}^2}\left|\widetilde{\lambda}_2-z_2\widetilde{\lambda}_1\right]\Bigg)\\
&\quad+(-ig)(-i\lambda)\frac{1}{P_{13}^2}\,[1|2]\\
&=(-ig)^3\,[1|\,\frac{P_{q_11}}{P_{q_11}^2}\frac{P_{q_12}}{P_{q_12}^2}\,|2]+(-ig)(-i\lambda)\frac{1}{P_{13}^2}\,[1|2]
\end{aligned}
\tag{4.31}
$$

With $n=5$ the boundary part is given by $B_{n=5}=(-ig)(-i\lambda)^2\frac{[1|2]}{P_{15}^2}\left(\frac{1}{P_{13}^2}+\frac{1}{P_{24}^2}+\frac{1}{P_{35}^2}\right)$

Adding the pole part together we have

$$
\begin{aligned}
&A(q_1^+;p_1,p_2,p_3,p_4,p_5;q_2^+) \qquad (4.32)\\
&\quad = B_{n=5}+\widehat{A}((\widehat{q}_1^+;p_1;-\widehat{P}_{q_11}^+)\frac{1}{P_{q_11}^2}\widehat{A}(\widehat{P}_{q_11}^-;p_2,p_3,p_4,p_5;\widehat{q}_2^+)\\
&\qquad +\widehat{A}(\widehat{q}_1^+;p_1,p_2;-\widehat{P}_{q_12}^-)\frac{1}{P_{q_12}^2}\widehat{A}(\widehat{P}_{q_12}^+;p_3,p_4,p_5;\widehat{q}_2^+)\\
&\qquad +\widehat{A}(\widehat{q}_1^+;p_1,p_2,p_3;-\widehat{P}_{q_13}^+)\frac{1}{P_{q_13}^2}\widehat{A}(\widehat{P}_{q_13}^-;p_4,p_5;\widehat{q}_2^+)\\
&\qquad +\widehat{A}(\widehat{q}_1^+;p_1,p_2,p_3,p_4;-\widehat{P}_{q_14}^-)\frac{1}{P_{q_14}^2}\widehat{A}(\widehat{P}_{q_14}^+;p_5;\widehat{q}_2^+)\\
&\quad = (-ig)(-i\lambda)^2\frac{[1|2]}{P_{15}^2}\left(\frac{1}{P_{13}^2}+\frac{1}{P_{24}^2}+\frac{1}{P_{35}^2}\right)+(-ig)^5\left[\widetilde{\lambda}_1\right|\frac{P_{q_11}P_{q_12}P_{q_13}P_{q_14}}{P_{q_11}^2P_{q_12}^2P_{q_13}^2P_{q_14}^2}\left|\widetilde{\lambda}_2\right]\\
&\qquad +(-ig)^3(-i\lambda)\left(\left[\widetilde{\lambda}_1\right|\frac{P_{q_13}P_{q_14}}{P_{q_13}^2P_{q_14}^2}\left|\widetilde{\lambda}_2\right]+\left[\widetilde{\lambda}_1\right|\frac{P_{q_11}P_{q_14}}{P_{q_11}^2P_{q_14}^2}\left|\widetilde{\lambda}_2\right]+\left[\widetilde{\lambda}_1\right|\frac{P_{q_11}P_{q_12}}{P_{q_11}^2P_{q_12}^2}\left|\widetilde{\lambda}_2\right]\right)
\end{aligned}
$$

which is the right one.

4.2.4 The helicity $(h_{q_1},h_{q_2})=(-,-)$

This is, in fact, similar to previous one so we will be briefly. For $n=1$, similar to the case $(+,+)$, there is one boundary contribution and the result is $B_{n=1}=(-ig)\langle 1|2\rangle$. For $n=3$, we calculate one boundary contribution $B_{n=3}=(-ig)(-i\lambda)\frac{1}{Q_{13}^2}\langle 1|2\rangle$ and two pole contributions as following:

$$
\begin{aligned}
A(q_1^-;p_1,p_2,p_3;q_2^-) &= B_{n=3}+\widehat{A}(p_1^-;q_1;-P_{q_11}^-)\frac{1}{P_{q_11}^2}\widehat{A}(P_{q_11}^+;q_2,q_3;p_2^-)\\
&\quad +\widehat{A}(p_1^-;q_1,q_2;-P_{q_12}^+)\frac{1}{P_{q_12}^2}\widehat{A}(P_{q_12}^-;q_3;p_2^-)\\
&= (-ig)(-i\lambda)\frac{1}{P_{13}^2}\langle 1|2\rangle\\
&\quad +(-ig)^3\Bigg(\langle\lambda_1+z_1\lambda_2|\frac{(P_{q_11}-z_1q)(P_{q_12}-z_1q)}{P_{q_11}^2(P_{q_12}^2-z_1P_{q_12}q)}|\lambda_2\rangle\\
&\qquad +\langle\lambda_1+z_2\lambda_2|\frac{(P_{q_11}-z_2q)(P_{q_12}-z_2q)}{(P_{q_11}^2-z_2P_{q_11}q)P_{q_12}^2}|\lambda_2\rangle\Bigg)\\
&= (-ig)^3\langle 1|\frac{P_1}{P_1^2}\frac{P_2}{P_2^2}|2\rangle \qquad (4.33)
\end{aligned}
$$

With $n=5$, the boundary part is given by $B[\langle 5\rangle] = (-ig)(-i\lambda)^2 \frac{\langle 1|2\rangle}{P_{15}^2}\left(\frac{1}{P_{13}^2}+\frac{1}{P_{24}^2}+\frac{1}{P_{35}^2}\right)$, thus we have

$$
\begin{aligned}
&A(q_1^-;p_1,p_2,p_3,p_4,p_5;q_2^-) \qquad (4.34)\\
&= B_{n=5} + \widehat{A}(q_1^-;p_1;-P_{q_11}^-)\frac{1}{P_{q_11}^2}\widehat{A}(P_{q_11}^+;p_2,p_3,p_4,p_5;q_2^-)\\
&\quad +\widehat{A}(q_1^-;p_1,p_2;-P_{q_12}^+)\frac{1}{P_{q_12}^2}\widehat{A}(P_{q_12}^-;p_3,p_4,p_5;q_2^-)\\
&\quad +\widehat{A}(q_1^-;p_1,p_2,p_3;-P_{q_13}^-)\frac{1}{P_{q_13}^2}\widehat{A}(P_{q_13}^+;p_4,p_5;q_2^-)\\
&\quad +\widehat{A}(q_1^-;p_1,p_2,p_3,p_4;-P_{q_14}^+)\frac{1}{P_{q_14}^2}\widehat{A}(P_{q_14}^-;p_5;q_2^-)\\
&= (-ig)(-i\lambda)^2\frac{\langle 1|2\rangle}{P_{15}^2}\left(\frac{1}{P_{13}^2}+\frac{1}{P_{24}^2}+\frac{1}{P_{35}^2}\right) + (-ig)^5\langle\lambda_1|\frac{P_{q_11}P_{q_12}P_{q_13}P_{q_14}}{P_{q_11}^2P_{q_12}^2P_{q_13}^2P_{q_14}^2}|\lambda_2\rangle\\
&\quad +(-ig)^3(-i\lambda)\left(\langle\lambda_1|\frac{P_{q_13}P_{q_14}}{P_{q_13}^2P_{q_14}^2}|\lambda_2\rangle + \langle\lambda_1|\frac{P_{q_11}P_{q_14}}{P_{q_11}^2P_{q_14}^2}|\lambda_2\rangle + \langle\lambda_1|\frac{P_{q_11}P_{q_12}}{P_{q_11}^2P_{q_12}^2}|\lambda_2\rangle\right)
\end{aligned}
$$

is exactly what we get from Feynman diagrams.

5 Conclusion

In this paper, we have analyzed the on-shell constructibility more carefully. We showed that for some theories, although there is no any deformation which has vanishing boundary contribution, the boundary contributions can still be identified and calculated on-shell recursively. With the knowledge of boundary behavior we can write down the generalized BCFW recursion relation.

Our examples in this paper is simple in this sense that all boundary contributions can be directly analyzed by just a few Feynman diagrams. There are other examples where above direct analysis is not so straightforward, for example, the gauge theory with deformation $\langle 1^-|2^+]$. It will be interesting to find other methods to deal with these more complicated situations. Having the knowledge of boundary behavior we can have better idea for the off-shell and on-shell constructibility. Then we may have better understanding of S-matrix theory using, for example, the technique presented in [4].

There are many questions we can ask for ourselves. For example, the rational part of one-loop amplitudes can be calculated by BCFW recursion relation too (see, for example, [11]- [16]). It is found that sometimes there is nonzero boundary contribution. Using our new understanding, it maybe useful to recheck this problem.

Acknowledgments

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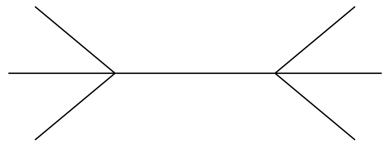


Figure 5. A 6-point Feynman diagram.

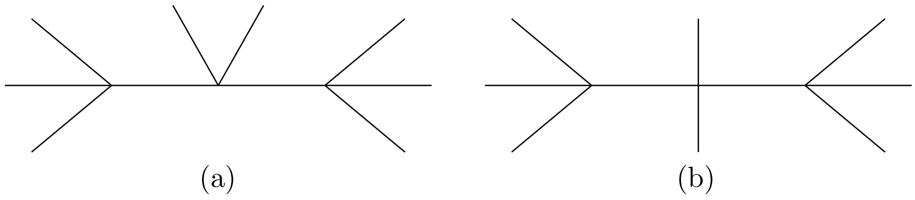


Figure 6. Three kinds of 8-point Feynman diagrams.

A Amplitudes from Feynman diagrams

In this appendix, we will calculate various amplitudes using Feynman diagrams directly to compare with results from boundary BCFW recursion relation. For simplicity we will focus on the ordered results.

A.1 Amplitude of pure scalar field

The Feynman rule for this theory is very simple: there is only one vertex with four scalar lines and coupling constant $-i\lambda$. Using this we get following results.

A.1.1 6-point amplitude

The typical Feynman diagrams for the 6-point amplitude are given in figure 5. There are six of them by cyclic ordering. Each one is given by $(-i\lambda)^2\frac{1}{P^2_{i(i+1)(i+2)}}$ where $P_{ijk}=p_i+p_j+p_k$. Adding them up we have

$$A_6^{\rm FD}(1,\ldots,6)=(-i\lambda)^2\left(\frac{1}{P^2_{123}}+\frac{1}{P^2_{126}}+\frac{1}{P^2_{156}}\right)\,, \tag{A.1}$$

where we have identified $i+6\equiv i$.

A.1.2 8-point amplitude

The diagrams for 8-point amplitude have two different topologies as given in figure 6. The expression of the first kind of diagrams has the form

$$(-i\lambda)^3\frac{1}{P^2_{i(i+1)(i+2)}P^2_{(i+5)(i+6)(i+7)}} \tag{A.2}$$

with eight cyclic orderings. The second kind of Feynman diagram has the form

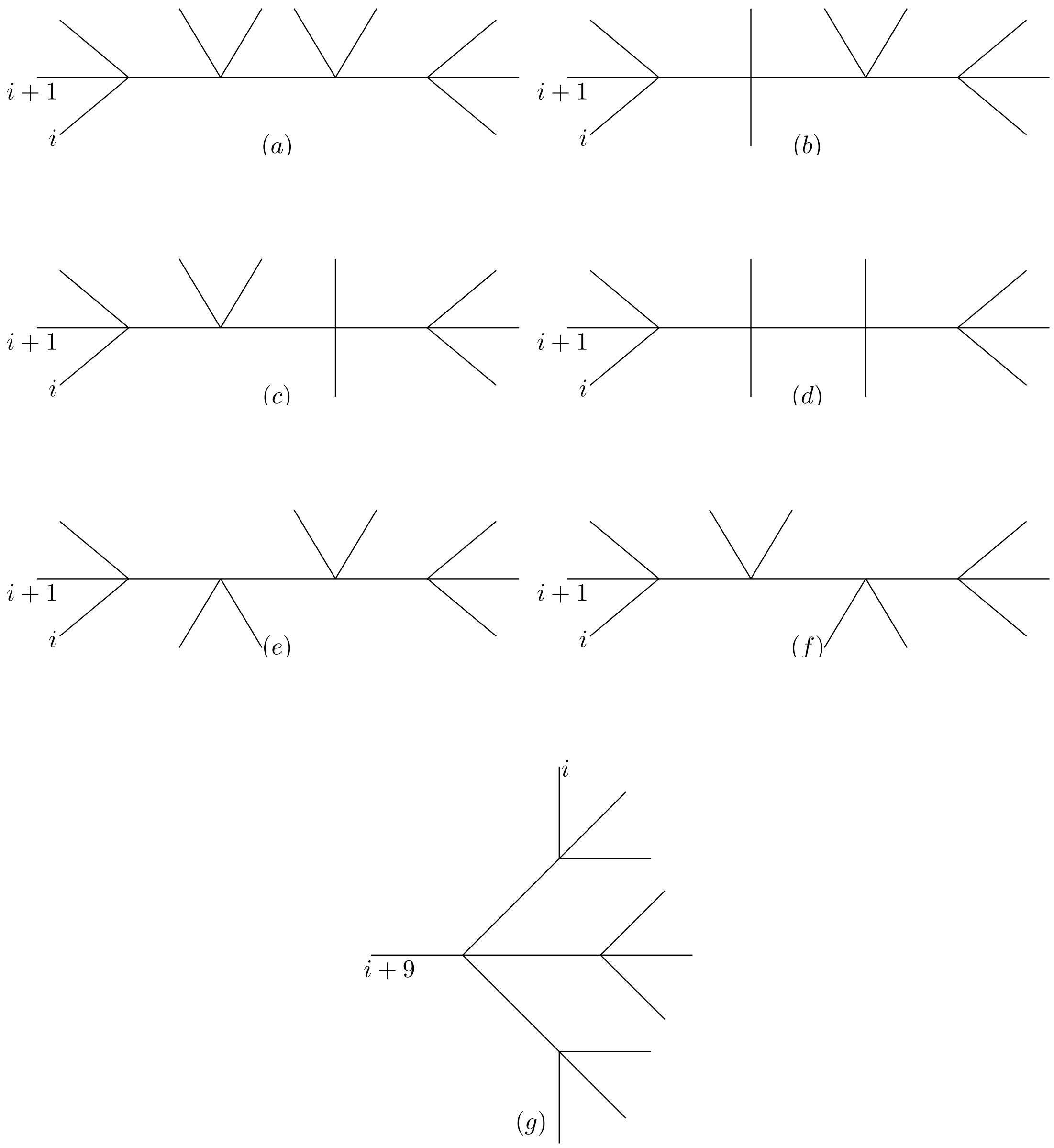
$$(-i\lambda)^3\frac{1}{P^2_{i(i+1)(i+2)}P^2_{(i+3)(i+4)(i+5)}} \tag{A.3}$$

with only four cyclic orderings, since it is obvious that this figure is same with shifting $i \to i+4$. Adding them up we have

$$\begin{aligned} A_8^{\mathrm{FD}}(1,\ldots,8) &= A_8^{FD(a)} + A_8^{FD(b)} \\ &= (-i\lambda)^3 \sum_{\sigma\in\mathbb{Z}_8} \left(\frac{1}{P^2_{\sigma(1)\sigma(2)\sigma(3)} P^2_{\sigma(6)\sigma(7)\sigma(8)}} + \frac{1}{2P^2_{\sigma(1)\sigma(2)\sigma(3)} P^2_{\sigma(5)\sigma(6)\sigma(7)}} \right) . \end{aligned} \tag{A.4}$$

A.1.3 10-point amplitude

The possible seven topologies of the diagrams are given in following:



and the corresponding expressions are given as

$$\mathcal{Q}_{(a)} = \frac{1}{P^2_{i(i+1)(i+2)} P^2_{i(i+1)(i+2)(i+3)(i+4)} P^2_{(i+7)(i+8)(i+9)}} \tag{A.5}$$

$$\mathcal{Q}_{(b)} = \frac{1}{P^2_{i(i+1)(i+2)} P^2_{i(i+1)(i+2)(i+3)(i+4)} P^2_{(i+6)(i+7)(i+8)}} \tag{A.6}$$

$$\mathcal{Q}_{(c)} = \frac{1}{P^2_{i(i+1)(i+2)} P^2_{(i-1)i(i+1)(i+2)(i+3)} P^2_{(i+6)(i+7)(i+8)}} \tag{A.7}$$

$$\mathcal{Q}_{(d)} = \frac{1}{P^2_{i(i+1)(i+2)} P^2_{(i-2)(i-1)i(i+1)(i+2)} P^2_{(i+5)(i+6)(i+7)}} \tag{A.8}$$

$$\mathcal{Q}_{(e)} = \frac{1}{P^2_{i(i+1)(i+2)} P^2_{i(i+1)(i+2)(i+3)(i+4)} P^2_{(i+5)(i+6)(i+7)}} \tag{A.9}$$

$$\mathcal{Q}_{(f)} = \frac{1}{P^2_{i(i+1)(i+2)} P^2_{(i-1)i(i+1)(i+2)(i+3)} P^2_{(i+5)(i+6)(i+7)}} \tag{A.10}$$

$$\mathcal{Q}_{(g)} = \frac{1}{P^2_{i(i+1)(i+2)} P^2_{(i+3)(i+4)(i+5)} P^2_{(i+6)(i+7)(i+8)}} \tag{A.11}$$

Among these seven topologies, three of them, i.e., (d), (e), (f), are intrinsic symmetric under $\sigma : [i] \mapsto [i+5]$, while remaining four are full Z_{10} ordering. Thus the final answer is given by

$$\begin{aligned}
&A^{\rm FD}_{10}(1,2,\ldots,10)\\
&= (-i\lambda)^4 \sum_{\sigma\in\mathbb{Z}_{10}} \left[\frac{1}{P^2_{\sigma(1)\sigma(2)\sigma(3)} P^2_{\sigma(1)\sigma(2)\sigma(3)\sigma(4)\sigma(5)} P^2_{\sigma(8)\sigma(9)\sigma(10)}} \right.\\
&\qquad + \frac{1}{P^2_{\sigma(1)\sigma(2)\sigma(3)} P^2_{\sigma(1)\sigma(2)\sigma(3)\sigma(4)\sigma(5)} P^2_{\sigma(7)\sigma(8)\sigma(9)}}\\
&\qquad + \frac{1}{2P^2_{\sigma(1)\sigma(2)\sigma(3)} P^2_{\sigma(10)\sigma(1)\sigma(2)\sigma(3)\sigma(4)} P^2_{\sigma(6)\sigma(7)\sigma(8)}}\\
&\qquad + \frac{1}{P^2_{\sigma(1)\sigma(2)\sigma(3)} P^2_{\sigma(10)\sigma(1)\sigma(2)\sigma(3)\sigma(4)} P^2_{\sigma(7)\sigma(8)\sigma(9)}}\\
&\qquad + \frac{1}{2P^2_{\sigma(1)\sigma(2)\sigma(3)} P^2_{\sigma(1)\sigma(2)\sigma(3)\sigma(4)\sigma(5)} P^2_{\sigma(6)\sigma(7)\sigma(8)}}\\
&\qquad + \frac{1}{2P^2_{\sigma(1)\sigma(2)\sigma(3)} P^2_{\sigma(1)\sigma(2)\sigma(3)\sigma(9)\sigma(10)} P^2_{\sigma(6)\sigma(7)\sigma(8)}}\\
&\qquad\qquad \left. + \frac{1}{P^2_{\sigma(1)\sigma(2)\sigma(3)} P^2_{\sigma(4)\sigma(5)\sigma(6)} P^2_{\sigma(7)\sigma(8)\sigma(9)}} \right]
\end{aligned} \tag{A.12}$$

A.2 Amplitude of two fermions and n scalars

The Feynman rules for ordered scalar QCD is given by figure 7. Using this we can calculate various amplitudes $A_{2,n}(q_1; p_1, \ldots, p_n; q_2)$ where q_1, q_2 for two fermions and p_i for scalars. For these amplitudes, we need to notice that since the scalar part has only $\lambda\phi^4$ vertex, $A_{2,n}$ with helicities $(h_{q_1}, h_{q_2}) = (+,+)/(-,-)$ is not zero only when $n = odd$ while with

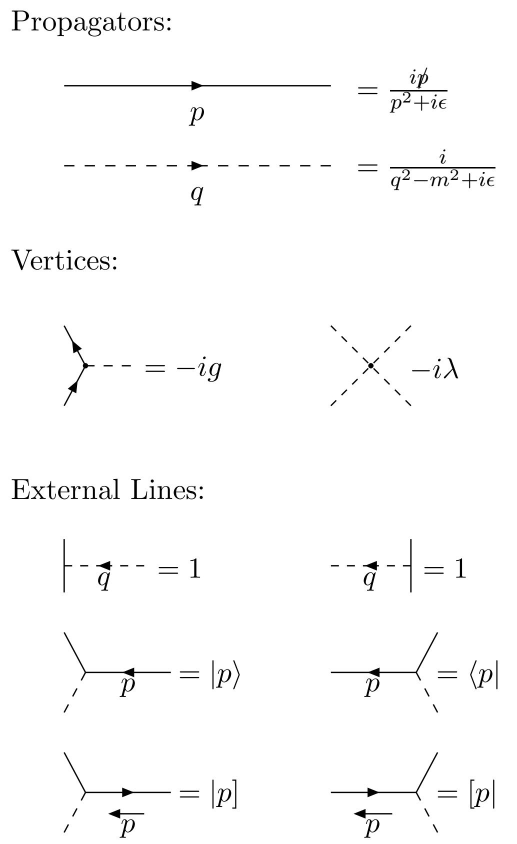


Figure 7. Feynman rules for fermion-scalar field.

$(h_{q_1}, h_{q_2}) = (+,-)/(-,+)$ it is not zero only when $n = even$. It is also important to notice that when we write $|1\rangle, |2\rangle$ they mean $|\lambda_{q_1}\rangle, |\lambda_{q_2}\rangle$.

A.2.1 Amplitude of $A(q_1^+, q_2^+; p_1, \ldots, p_n)$

For this case we have $n = odd$. For this case we have $n = odd$. With $n = 1$ we have

$$A(q_1^+; p; q_2^+) = (-ig)\,[1|2] \tag{A.13}$$

With $n = 3$ we have

$$A(q_1^+; p_1, p_2, p_3; q_2^+) = (-ig)^3 \frac{[1|\, P_{p_1 1} P_{p_1 2}\, |2]}{P^2_{p_1 1} P^2_{p_1 2}} + (-ig)(-i\lambda) \frac{1}{P^2_{13}}\,[1|2] \tag{A.14}$$

where we have defined $P_{q_1 i} = q_1 + p_1 + \ldots + p_i$ and $P_{ij} = p_i + p_{i+1} + \ldots + p_j$ With $n = 5$ we have

$$\begin{aligned} A(q_1^+; p_1, p_2, p_3, p_4, p_5; q_2^+) \;=\; & (-ig)^5 \frac{[1|\, P_{p_1 1} P_{p_1 2} P_{p_1 3} P_{p_1 4}\, |2]}{P^2_{p_1 1} P^2_{p_1 2} P^2_{p_1 3} P^2_{p_1 4}} \\ & + (-ig)^3 (-i\lambda) \left(\frac{[1|\, P_{p_1 3} P_{p_1 4}\, |2]}{P^2_{p_1 3} P^2_{p_1 4}} + [1|\, \frac{P_{p_1 1} P_{p_1 4}\, |2]}{P^2_{p_1 1} P^2_{p_1 4}} + \frac{[1|\, P_{p_1 1} P_{p_1 2}\, |2]}{P^2_{p_1 1} P^2_{p_1 2}} \right) \\ & + (-ig)(-i\lambda)^2 \frac{[1|2]}{P^2_{15}} \left(\frac{1}{P^2_{13}} + \frac{1}{P^2_{24}} + \frac{1}{P^2_{35}} \right) \end{aligned} \tag{A.15}$$

A.2.2 Amplitude of $A(q_1^-, q_2^-; p_1, \ldots, p_n)$

With $n = 1$ we have

$$A(q_1^-; p; q_2^-) = (-ig)\langle 1|2\rangle \tag{A.16}$$

With $n = 3$ we have

$$A^{\rm FD}(q_1^-; p_1, p_2, p_3; q_2^-) = (-ig)^3 \frac{\langle 1|\, P_{p_1 1} P_{p_1 2}\, |2\rangle}{P_{p_1 1}^2 P_{p_1 2}^2} + (-ig)(-i\lambda)\frac{1}{P_{13}^2}\langle 1|2\rangle \tag{A.17}$$

With $n = 5$ we have

$$\begin{aligned}
&A^{\rm FD}(q_1^-; p_1, p_2, p_3, p_4, p_5; q_2^-) \\
&\quad = (-ig)^5 \frac{\langle 1|\, P_{p_1 1} P_{p_1 2} P_{p_1 3} P_{p_1 4}\, |2\rangle}{P_{p_1 1}^2 P_{p_1 2}^2 P_{p_1 3}^2 P_{p_1 4}^2} \\
&\qquad + (-ig)^3(-i\lambda)\left(\frac{\langle 1|\, P_{p_1 3} P_{p_1 4}\, |2\rangle}{P_{p_1 3}^2 P_{p_1 4}^2} + \frac{\langle 1|\, P_{p_1 1} P_{p_1 4}\, |2\rangle}{P_{p_1 1}^2 P_{p_1 4}^2} + \frac{\langle 1|\, P_{p_1 1} P_{p_1 2}\, |2\rangle}{P_{p_1 1}^2 P_{p_1 2}^2}\right) \\
&\qquad + (-ig)(-i\lambda)^2 \frac{\langle 1|2\rangle}{P_{15}^2}\left(\frac{1}{P_{13}^2} + \frac{1}{P_{24}^2} + \frac{1}{P_{35}^2}\right)
\end{aligned} \tag{A.18}$$

A.2.3 Amplitude of $A(q_1^+, q_2^-; p_1, \ldots, p_n)$

In this case we need to have even number of n. With $n = 2$ we have

$$A\Big(q_1^+; p_1, p_2; q_2^-\Big) = (-ig)^2 \frac{[1|P_{q_1 1}|2\rangle}{P_{q_1 1}^2}\ . \tag{A.19}$$

where because the color ordering we have defined $P_{q_1 i} = q_1 + p_1 + p_2 + \ldots + p_i$. With $n = 4$ we have

$$A\Big(q_1^+; p_1, p_2, p_3, p_4; q_2^-\Big) = (-ig)^4 \frac{[1|P_{q_1 1} P_{q_1 2} P_{q_1 3}|2\rangle}{P_{q_1 1}^2 P_{q_1 2}^2 P_{q_1 3}^2} + (-ig)^2(-i\lambda)\left[\frac{[1|P_{q_1 1}|2\rangle}{P_{q_1 1}^2} + \frac{[1|P_{q_1 3}|2\rangle}{P_{q_1 1}^2}\right]. \tag{A.20}$$

With $n = 6$ we have

$$\begin{aligned}
&A\Big(q_1^+; p_1, p_2, p_3, p_4, p_5, p_6; q_2^-\Big) \\
&\quad = (-ig)^6 \frac{[1|P_{q_1 1} P_{q_1 2} P_{q_1 3} P_{q_1 4} P_{q_1 5}|2\rangle}{P_{q_1 1}^2 P_{q_1 2}^2 P_{q_1 3}^2 P_{q_1 4}^2 P_{q_1 5}^2} \\
&\qquad + (-ig)^4(-i\lambda)\Bigg[\frac{[1|P_{q_1 1} P_{q_1 2} P_{q_1 3}|2\rangle}{P_{q_1 1}^2 P_{q_1 2}^2 P_{q_1 3}^2 P_{46}^2} + \frac{[1|P_{q_1 1} P_{q_1 2} P_{q_1 5}|2\rangle}{P_{q_1 1}^2 P_{q_1 2}^2 P_{q_1 5}^2 P_{35}^2} \\
&\qquad\qquad + \frac{[1|P_{q_1 1} P_{q_1 4} P_{q_1 5}|2\rangle}{P_{q_1 1}^2 P_{q_1 4}^2 P_{q_1 5}^2 P_{24}^2} + \frac{[1|P_{q_1 3} P_{q_1 4} P_{q_1 5}|2\rangle}{P_{q_1 3}^2 P_{q_1 4}^2 P_{q_1 5}^2 P_{13}^2}\Bigg] \\
&\qquad + (-ig)^2(-i\lambda)^2\Bigg\{\frac{[1|P_{q_1 3}|2\rangle}{P_{q_1 3}^2 P_{13}^2 P_{46}^2} + \frac{[1|P_{q_1 1}|2\rangle}{P_{q_1 1}^2 P_{26}^2}\left[\frac{1}{P_{24}^2} + \frac{1}{P_{35}^2} + \frac{1}{P_{46}^2}\right] \\
&\qquad\qquad + \frac{[1|P_{q_1 5}|2\rangle}{P_{q_1 5}^2 P_{15}^2}\left[\frac{1}{P_{13}^2} + \frac{1}{P_{24}^2} + \frac{1}{P_{35}^2}\right]\Bigg\}\ .
\end{aligned} \tag{A.21}$$

A.2.4 Amplitude of $A(q_1^-, q_2^+; p_1, \ldots, p_n)$

With $n = 2$ we have

$$A\Big(q_1^-; p_1, p_2; q_2^+\Big) = (-ig)^2 \frac{\langle 1|P_{q_1 1}|2]}{P_{q_1 1}^2} \ . \tag{A.22}$$

With $n = 4$ we have

$$A\Big(q_1^-; p_1, p_2, p_3, p_4; q_2^+\Big) = (-ig)^4 \frac{\langle 1|P_{q_1 1} P_{q_1 2} P_{q_1 3}|2]}{P_{q_1 1}^2 P_{q_1 2}^2 P_{q_1 3}^2} + (-ig)^2(-i\lambda)\left[\frac{\langle 1|P_{q_1 1}|2]}{P_{q_1 1}^2} + \frac{\langle 1|P_{q_1 3}|2]}{P_{q_1 1}^2}\right] . \tag{A.23}$$

With $n = 6$ we have

$$\begin{aligned} A\Big(&q_1^-; p_1, p_2, p_3, p_4, p_5, p_6; q_2^+\Big) \\ &= (-ig)^6 \frac{\langle 1|P_{q_1 1} P_{q_1 2} P_{q_1 3} P_{q_1 4} P_{q_1 5}|2]}{P_{q_1 1}^2 P_{q_1 2}^2 P_{q_1 3}^2 P_{q_1 4}^2 P_{q_1 5}^2} \\ &\quad + (-ig)^4(-i\lambda)\left[\frac{\langle 1|P_{q_1 1} P_{q_1 2} P_{q_1 3}|2]}{P_{q_1 1}^2 P_{q_1 2}^2 P_{q_1 3}^2 P_{46}^2} + \frac{\langle 1|P_{q_1 1} P_{q_1 2} P_{q_1 5}|2]}{P_{q_1 1}^2 P_{q_1 2}^2 P_{q_1 5}^2 P_{35}^2}\right. \\ &\qquad\qquad \left. + \frac{\langle 1|P_{q_1 1} P_{q_1 4} P_{q_1 5}|2]}{P_{q_1 1}^2 P_{q_1 4}^2 P_{q_1 5}^2 P_{24}^2} + \frac{\langle 1|P_{q_1 3} P_{q_1 4} P_{q_1 5}|2]}{P_{q_1 3}^2 P_{q_1 4}^2 P_{q_1 5}^2 P_{13}^2}\right] \\ &\quad + (-ig)^2(-i\lambda)^2\left\{\frac{\langle 1|P_{q_1 3}|2]}{P_{q_1 3}^2 P_{13}^2 P_{46}^2} + \frac{\langle 1|P_{q_1 1}|2]}{P_{q_1 1}^2 P_{26}^2}\left[\frac{1}{P_{24}^2} + \frac{1}{P_{35}^2} + \frac{1}{P_{46}^2}\right]\right. \\ &\qquad\qquad \left. + \frac{\langle 1|P_{q_1 5}|2]}{P_{q_1 5}^2 P_{15}^2}\left[\frac{1}{P_{13}^2} + \frac{1}{P_{24}^2} + \frac{1}{P_{35}^2}\right]\right\} . \end{aligned} \tag{A.24}$$

B Amplitudes with more external fermions

In section four, we have focused on the case with only two fermions. In this appendix, we will discuss the general case with multiple pair of fermions. For amplitudes with $2l$ fermions and n scalars, there are l fermion lines, which are connected to each other by scalar lines. The amplitudes will be the form

$$A = \sum_i \mathcal{S}_i \prod_{j=1}^{l} \mathcal{Q}_{ij} \tag{B.1}$$

where $\mathcal{S}_i$ are scalar parts and each $\mathcal{Q}_{i,j}$ is following form

$$\mathcal{Q}_{i,j}(q_i^-, q_j^+; R_1, \ldots, R_m) \sim i^m \frac{\Big\langle \lambda_i|R_1|R_2|\ldots|R_m|\widetilde{\lambda}_j\Big]}{R_1^2 R_2^2 \ldots . R_m^2} \tag{B.2}$$

where depending on the helicities of i, j, we may need to change $\lambda \to \widetilde{\lambda}$.

Now we choose two fermions, for example, q_1, q_2 to make the $\langle 1|2]$-deformation. There are two categories of Feynman diagrams: (A) two fermions q_1, q_2 are connected by same fermion line; (B) two fermions q_1, q_2 locate at different fermion line and are connected through scalar propagators.

For the category (A), the boundary behavior is exactly the same one as we have discussed in section four with only two fermions. Thus we can write down similar boundary contributions and add them to the boundary BCFW recursion relation.

For the category (B) things are different. First there is at least one scalar propagator connecting fermion lines and having $\frac{1}{z}$ dependence. Second, when there are two nearby fermion propagators along same fermion line, because

$$\frac{\left\langle \alpha|(R_1+z\lambda_2\widetilde{\lambda}_1)(R_2+z\lambda_2\widetilde{\lambda}_1)|\beta\right\rangle}{(R_1+z\lambda_2\widetilde{\lambda}_1)^2(R_2+z\lambda_2\widetilde{\lambda}_1)^2} = \frac{\left\langle \alpha|(R_1+z\lambda_2\widetilde{\lambda}_1)R_2|\beta\right\rangle}{(R_1+z\lambda_2\widetilde{\lambda}_1)^2(z\lambda_2\widetilde{\lambda}_1)^2} + \frac{\left\langle \alpha|R_1(z\lambda_2\widetilde{\lambda}_1)|\beta\right\rangle}{(R_1+z\lambda_2\widetilde{\lambda}_1)^2(R_2+z\lambda_2\widetilde{\lambda}_1)^2}, \tag{B.3}$$

we have another $\frac{1}{z}$ dependence instead of naive z^0-dependence. Using above two observations, we can discuss case by case:

- (B-1) *Helicity* $(h_{q_1}, h_{q_2}) = (+,+)$*:*

  For this case, along the line with z-dependence (this line will be constituted by fermion propagators and scalar propagators), we have one z from external wave-function of q_2, i.e., $\left|\widetilde{\lambda}_2 - z\widetilde{\lambda}_1\right]$, $\frac{1}{z^s}$ from $s \geq 1$ scalar propagators and f fermion propagators with naive z^0-dependence. However, when there are m pair nearby propagators as discussed above, we have another $\frac{1}{z^m}$-dependence. Putting all together, we found that to have nonzero boundary contributions, we need to satisfy following conditions: (a) there is only one scalar propagator depending on z; (b) there is no any nearby z-depending fermion propagator pair.

  The condition (a) implies that the line connecting q_1, q_2 involves only two fermion lines. Furthermore, condition (b) tells us that there is at most one fermion propagator along each fermion line (remembering that we have two fermion lines here). Using $(R+z\lambda_2\widetilde{\lambda}_1)\left|\widetilde{\lambda}_2 - z\widetilde{\lambda}_1\right]$ is only order of z and $[\widetilde{\lambda}_1|(R+z\lambda_2\widetilde{\lambda}_1| = [\widetilde{\lambda}_1|R|$, we see that no any z-depending fermion propagator can be nearby the external particle q_1, q_2, thus the only boundary contribution is the one without any fermionic propagator depending on z as shown by figure 8. The figure can also be represented as

  $$C\left[\widetilde{\lambda}_1|\alpha\right]\frac{1}{(R+z\widetilde{\lambda}_2\lambda_1)^2}\left[\beta|\widetilde{\lambda}_2 - z\widetilde{\lambda}_1\right] \tag{B.4}$$

  where α, β, C are z-independent part from other components of diagrams.

  The helicity $(-,-)$ will be similar and we will not discuss it further.

- (B-2) *Helicity* $(h_{q_1}, h_{q_2}) = (+,-)$*:*

  For this case, along the line with z-dependence, we have $\frac{1}{z^s}$ from $s \geq 1$ scalar propagators and f fermion propagators with naive z^0-dependence. Thus there is no boundary contribution.

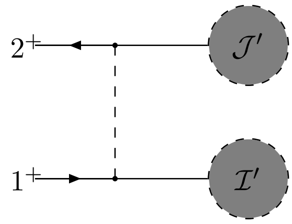


Figure 8. Special case.

- (B-3) *Helicity* $(h_{q_1}, h_{q_2}) = (-,+)$*:*

  For this case, along the line with z-dependence, we have z^2 from external wave-functions $\left|\widetilde{\lambda}_2 - z\widetilde{\lambda}\right]$ and $|\lambda_1 + z\lambda_2\rangle$, $\frac{1}{z^s}$ from $s \geq 1$ scalar propagators and f fermion propagators with naive z^0-dependence. This is the most complicated case with many possibilities, so we list them one-by one.

  With only one scalar propagator (i.e., $s = 1$), we have several diagrams which can be represented as following:

$$\begin{aligned}
(I-1):&\quad C\,\langle\lambda_1 + z\lambda_2|\alpha]\,\frac{1}{(Q + z\widetilde{\lambda}_2\lambda_1)^2}\left[\beta|\widetilde{\lambda}_2 - z\widetilde{\lambda}_1\right] \sim z^1\\
(I-2):&\quad C\left\langle\lambda_1{+}z\lambda_2|(R_1{+}z\lambda_2\widetilde{\lambda}_1)|\alpha\right]\frac{1}{(Q{+}z\widetilde{\lambda}_2\lambda_1)^2(R_1{+}z\lambda_2\widetilde{\lambda}_1)^2}\left[\beta|\widetilde{\lambda}_2{-}z\widetilde{\lambda}_1\right] \sim z^0\\
(I-3):&\quad C\,\langle\lambda_1{+}z\lambda_2|\alpha]\,\frac{1}{(Q{+}z\widetilde{\lambda}_2\lambda_1)^2(R_1{+}z\lambda_2\widetilde{\lambda}_1)^2}\left[\beta|(R_1{+}z\lambda_2\widetilde{\lambda}_1)|\widetilde{\lambda}_2 - z\widetilde{\lambda}_1\right] \sim z^0\\
(I-4):&\quad C\frac{\left\langle\lambda_1 + z\lambda_2|(R_1 + z\lambda_2\widetilde{\lambda}_1)(R_2 + z\lambda_2\widetilde{\lambda}_1)|\alpha\right]}{(R_1 + z\lambda_2\widetilde{\lambda}_1)^2(R_2 + z\lambda_2\widetilde{\lambda}_1)^2(Q + z\widetilde{\lambda}_2\lambda_1)^2}\left[\beta|\widetilde{\lambda}_2 - z\widetilde{\lambda}_1\right] \sim z^0\\
(I-5):&\quad C\,\langle\lambda_1 + z\lambda_2|\alpha]\,\frac{\left[\beta|(R_1 + z\lambda_2\widetilde{\lambda}_1)(R_1 + z\lambda_2\widetilde{\lambda}_1)|\widetilde{\lambda}_2 - z\widetilde{\lambda}_1\right]}{(R_1 + z\lambda_2\widetilde{\lambda}_1)^2(R_1 + z\lambda_2\widetilde{\lambda}_1)^2(Q + z\widetilde{\lambda}_2\lambda_1)^2} \sim z^0
\end{aligned} \tag{B.5}$$

  The case (I-1) tells us that there is no any z-depending fermion propagator. Case (I-2) and (I-3) represent the diagrams with one z-depending fermion propagator nearby external particles q_1, q_2 respectively. Case (I-4) and (I-5) represent the diagrams with two z-depending fermion propagators nearby external particles q_1, q_2 respectively. It is worth to notice that we have not included the case where we have two z-depending fermion propagators and each external q_i has one propagator nearby, since this one has vanished boundary contribution.

  With two scalar propagators, there are again two situations we need to consider. The first one is that there are only two fermion lines involved. The expression for this one is

$$(II-1):\qquad C\,\langle\lambda_1 + z\lambda_2|\alpha]\,\frac{1}{(Q_1 + z\widetilde{\lambda}_2\lambda_1)^2(Q_2 + z\widetilde{\lambda}_2\lambda_1)^2}\left[\beta|\widetilde{\lambda}_2 - z\widetilde{\lambda}_1\right]\ . \tag{B.6}$$

The second one is that there are three fermion lines involved. For this one, there are two different representations. The first one is

$$C\left\langle \lambda_1 + z\lambda_2|\alpha\right] \frac{1}{(Q_1 + z\widetilde{\lambda}_2\lambda_1)^2} [\gamma|\delta] \frac{1}{(Q_2 + z\widetilde{\lambda}_2\lambda_1)^2} \left[\beta|\widetilde{\lambda}_2 - z\widetilde{\lambda}_1\right], \tag{B.7}$$

where there is no fermion propagator in middle fermion line depending on z. However, since we have only triple-vertex $\overline{\psi}\phi\psi$, this case can not be realized. The second one is

$$(II-2): \; C\left\langle \lambda_1 + z\lambda_2|\alpha\right] \frac{1}{(Q_1 + z\widetilde{\lambda}_2\lambda_1)^2} \times \\ \times \frac{\left[\gamma|(R + z\lambda_2\widetilde{\lambda})|\delta\right\rangle}{(R + z\lambda_2\widetilde{\lambda})^2} \frac{1}{(Q_2 + z\widetilde{\lambda}_2\lambda_1)^2} \left[\beta|\widetilde{\lambda}_2 - z\widetilde{\lambda}_1\right] \tag{B.8}$$

where there is one fermion propagator in middle fermion line depending on z.

Adding all seven cases together, we get total boundary contribution when q_1, q_2 are not in same fermion line.

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