A REMARK ON CODAZZI TENSORS IN CONSTANT CURVATURE SPACES

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A Codazzi tensor on a riemannian manifold M with Levi-Civita covariant derivative ∇ is a symmetric (1,1) - tensor field A with

$$(\nabla_X A) Y = (\nabla_Y A) X \qquad \text{for all } X, Y.$$

Let M have constant sectional curvature k. Then we have the following

Example: If $f : M \to \mathbb{R}$ is a smooth function,

then $$A[f] := \text{Hess}\, f + k f \,\text{Id}$$

is a Codazzi tensor.

We claim the converse

Proposition. If A is a Codazzi tensor on a riemannian manifold of constant curvature k, then locally

$$A = A[f]$$

for some smooth function f.

Indication of a proof: For the euclidean case we simply apply the standard integrability condition twice. For the unit sphere or unit hyperbolic space we use the standard imbedding as a hypersurface into the euclidean or lorentzian vector space: $M^n \subset \mathbb{R}^{n+1}$. Let $\pi : \mathbb{R}^{n+1} \supset \{ tx \mid t > 0, \ x \in M \} =: \tilde{M} \to M$ be the orthogonal projection, and define a (1,1) - tensor $\tilde{A}$ on $\tilde{M}$ by

$$< \tilde{A} X, Y > = \| x \| < A\, d\pi (X), d\pi (Y) > \qquad \text{for } X, Y \in T_x \tilde{M}$$

where $\| x \| := \sqrt{| < x, x > |}$. Then $\tilde{A}$ turns out to be a Codazzi tensor, and the assertion follows easily from the euclidean case.