Recognition of Planar Point Configurations Using the Density of Affine Shape

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Abstract. In this paper, we study the statistical theory of shape for ordered finite point configurations, or otherwise stated, the uncertainty of geometric invariants. Such studies have been made for affine invariants in e.g. [GHJ92], [Wer93], where in the former case a bound on errors are used instead of errors described by density functions, and in the latter case a first order approximation gives an ellipsis as uncertainty region. Here, a general approach for defining shape and finding its density, expressed in the densities for the individual points, is developed. No approximations are made, resulting in an exact expression of the uncertainty region. Similar results have been obtained for the special case of the density of the cross ratio, see [May95,Åst96].

In particular, we will concentrate on the affine shape, where often analytical computations are possible. In this case confidence intervals for invariants can be obtained from a priori assumptions on the densities of the detected points in the images. However, the theory is completely general and can be used to compute the density of any invariant (Euclidean, similarity, projective etc.) from arbitrary densities of the individual points. These confidence intervals can be used in such applications as geometrical hashing, recognition of ordered point configurations and error analysis of reconstruction algorithms. Another approach towards this problem, in the case of similarity transformations, can be found in [Ken89]. For the special case of normally distributed feature points in a plane and similarity transformations, see [Boo86], [MD89].

Finally, an example will be given, illustrating an application of the theory for the problem of recognising planar point configurations from images taken by an affine camera. This case is of particular importance in applications, where details on a conveyor belt are captured by a camera, with image plane parallel to the conveyor belt and extracted feature points from the images are used to sort the objects.

keywords: shape, invariants, densities, error analysis, recognition, distribution of invariants

1 Introduction

In many disciplines (technology, science, biology, etc), one often encounters a need to describe geometrical shapes. This is usually done by extracting a configuration of feature points for the object at hand. Once these points have been extracted, they have to be treated collectively, and not as individual points. For example, the location and orientation in space is in general irrelevant for the shape, which means that one should not distinguish between configurations that can be transformed into each other by rotations and translations. To achieve this one often defines a function from the set of possible point configurations, which is constant or invariant under rotations and translations of a point configuration.

Often there are other immaterial properties at hand, like for example scaling. To treat this quantitatively, a parametrisation of point configurations is needed, that only bothers about essential properties, 'the shape', of the application at hand. What is meant by 'essential' can often be described in terms of a group of transformations, by identifying objects that can be transformed into each other by elements of the group.

When using such parametrisations of shape in practice, one soon encounters the problem of how uncertainties in the measurements of individual feature points affect the shape parameters. This is the subject of the present study, i.e. to device a general, method for parametrisation of shape, taking only relevant information into account, and to investigate the statistical density of the shape parameters given the statistical density for the individual points of the configurations.

As an example of a specific field of applications in computer vision we have the structure and motion problem. When deriving the structure of a 3D-scene from a sequence of 2D-images, one often has little or none calibration information about the cameras. Fortunately, cf [Spa96], it is still often possible to derive structure information, but only up to some class of transformations, i.e. a situation similar to the one described above. The algorithms for this only use some kind of shape data, with respect to some group of transformations. This means that the data needed by the algorithms consists of shape parameters, which are 'packages' of measurement parameters. To analyse the stability and robustness of such algorithms, the densities of these shape parameters are needed.

Another computer vision application of the result of this paper, is model based recognition. Here a data base is built containing e.g. affine or projective shapes of a number of model objects. The recognition problem consists of matching measured image features to the right model object. To do this, shapes are computed from image data. Knowing the densities of shape, this can be done in a firm way, using quantitative hypothesis testing. We give the necessary densities, which will allow us to determine confidence sets for shape below.

In this paper we will outline a general framework for shape. The result will be explicit parametrisations of shape spaces together with formulas for the exact density of shape, given as integrals. The general theory will be illustrated by a number of examples of affine shape. Furthermore, we will show how the theory can be used to solve the problem of recognising point configurations taken by affine cameras, using real images.

The pioneer in the study of shape along these lines is Kendall, cf [Ken89], who dealt with the case of positive similarity transformations. Bookstein, cf [Boo86], introduced variables for size and shape to treat this problem when all feature points belongs to a plane. The density for the shape variables was later computed in [MD89], where the feature points were assumed to be normally distributed.

In computer vision, the density of affine shape has been studied in [GHJ92], [Wer93] and [Hey95]. In [GHJ92], the uncertainty of the affine coordinates of a planar four point configuration is treated. However, instead of errors described by a density function, a bound on errors is derived. In [Hey95] a first order approximation is used to compute the density of the affine shape to a planar four point configuration, when the points are normally distributed. A similar method is used in [Wer93]. A first order approximation has also been applied for the density of similarity invariants in [RH95].

Densities of projective shape have been studied in [Åst96] and [May95], where exact densities for the cross ratio for four points on a line are computed.

The approach of the present paper uses an abstract setting, which makes it possible to cover all these situations simultaneously. For a more thorough treatment of the contents of this paper, we refer the reader to [Ber97].

2 Shape of finite ordered point configurations

In this section, we will define a general notion for the shape of finite point configurations. To begin with, some terminology is needed.

Let C_m^n be the set of ordered m-point configurations

$$\mathcal{X} = (p_1, p_2, \ldots, p_m)$$

in $\mathbb{R}^n$, where $p_i \in \mathbb{R}^n$ is the coordinate vector of point number i in $\mathcal{X}$. Thus, there is a natural isomorphism between C_m^n and $\mathbb{R}^{mn}$. As topology on C_m^n, we will use the one inherited from $\mathbb{R}^{mn}$.

Let G be a group of transformations $C_m^n \to C_m^n$. By the **G-orbit** of $\mathcal{X} \in C_m^n$ is meant the set

$$\{\mathcal{Y} \mid \mathcal{Y} = g(\mathcal{X}),\ g \in G\} \ .$$

We write $\mathcal{X} \sim \mathcal{Y}$, when $\mathcal{X}$ and $\mathcal{Y}$ are in the same orbit. For a group G of transformations $\mathbb{R}^n \to \mathbb{R}^n$, let G^m be the group of product transformations, $C_m^n \to C_m^n$, defined by

$$g(\mathcal{X}) = (g(p_1), \ldots, g(p_m)), \text{ when } \mathcal{X} = (p_1, \ldots, p_m) \in C_m^n \text{ and } g \in G \ .$$

For our applications, this is the most usual situation, i.e. the same transformation is applied to all points of the configuration. By abuse of language, we write G instead of G^m.

For the geometric applications we are interested in, the following terminology is convenient.

Definition 1. *Let G be a group of transformations $C_m^n \to C_m^n$. The **shape space** is defined as the set of orbits C_m^n/G. Let*

$$s : C_m^n \to C_m^n/G \ ,$$

*be the natural projection. For $\mathcal{X} \in C_m^n$, the orbit $s(\mathcal{X}) \in C_m^n/G$ is called the **shape** of $\mathcal{X}$ and each element of $s(\mathcal{X})$ is called a **representative** of $s(\mathcal{X})$. A function φ on C_m^n, which is constant on each orbit is called an **invariant**. The functions $\varphi = (\varphi_1, \ldots, \varphi_k)$ form a **complete set of invariants** if $\varphi(\mathcal{X}) = \varphi(\mathcal{Y})$ if and only if $\mathcal{X} \sim \mathcal{Y}$. As topology on C_m^n/G, we choose the strongest topology that makes s continuous.*

Since G forms a group, the orbits of C_m^n/G are disjoint. The space C_m^n can then be viewed as divided into disjoint strings, where each string is an orbit, see Figure 1.

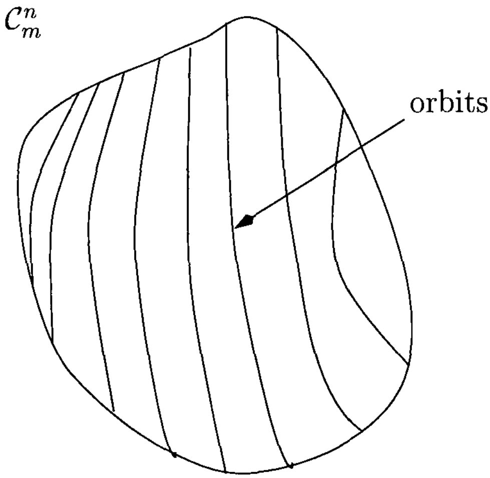

Fig. 1. Orbits of C_m^n.

According to Klein, in his Erlanger-program, geometry is the study of properties of geometric objects, which are invariant with respect to some class of transformation. This point of view fits with Definition 1, as is also illustrated by the following examples.

Example 1. Let G be the group of nonsingular Euclidean transformations, acting on C_m^n. Euclidean invariants are distances between points in the point configurations. The orbits consist of configurations that are congruent, in the sense of classical Euclidean geometry.

Example 2. When G is the group of affine transformations, $\mathcal{C}_m^n/G$ is isomorphic to a Grassman manifold, consisting of linear subspaces of some ambient linear space, cf [Spa92]. These subspaces have been called affine shape, and can be described explicitly as

$$s(\mathcal{X}) = \left\{ \xi \;\middle|\; \sum_{i=1}^{m} \xi_i x_i = 0,\ \sum_{i=1}^{m} \xi_i = 0,\ \mathcal{X} = (x_1, \ldots, x_m) \right\} .$$

3 Density of shape, general theory

We now introduce a parametrisation of point configurations $\mathcal{C}_m^n$, by taking the transformation group G into account, which will enable us to write down a formula for a density function on the shape space $\mathcal{C}_m^n/G$, when a density function on $\mathcal{C}_m^n$, i.e. the individual points of the configurations is known. This parametrisation will be done by pulling $\mathcal{C}_m^n$ back to two parameter sets A and B, such that B labels the orbits $\mathcal{C}_m^n/G$ and A provides parameters along each orbit. Recall that a function is called a **C^n-diffeomorphism** if it is invertible and $f, f^{-1} \in C^n$.

Definition 2. *Let $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^{k'}$ be two open subsets for some k and k' and let G be a group of transformations $\mathcal{C}_m^n \to \mathcal{C}_m^n$. Suppose that there exist open sets $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^{k'}$, a bijection*

$$g : A \ni a \to g_a \in G \ ,$$

and a function

$$\zeta : B \to \mathcal{C}_m^n \ ,$$

such that the function

$$\gamma_\zeta : A \times B \to \mathcal{C}_m^n, \ \textit{defined by}\ \gamma_\zeta(a,b) = g_a \circ \zeta(b) \ ,$$

has a dense open range $U \subseteq \mathcal{C}_m^n$ and $\gamma_\zeta : A \times B \to U$, is a C^1-diffeomorphism. Then ζ is said to be a ***parametrisation of shape*** *and g a* ***parametrisation of*** *G.*

The reason for using a dense open subset of $\mathcal{C}_m^n$ instead of the whole set, is that it is in general not possible to parametrise the entire $\mathcal{C}_m^n$. However, the remaining set $\mathcal{C}_m^n \setminus U$ is of measure zero and are therefore irrelevant here.

The dimensions k, k' only depend on $\mathcal{C}_m^n$ and G, as γ_ζ is a C^1-diffeomorphism. The specific choices of parametrisations for shape and G merely represent a choice of coordinates on $U \subseteq \mathcal{C}_m^n$, such that ζ labels the orbits and g describes the location within each orbit, see Figure 2.

It can be shown that B and $s(U)$ are homeomorphic, see [Ber97]. Thus, $s(U)$ is also metrizable, since B is a metric space. We call $s(U)$ the **non-degenerate shape space**.

We now turn to the main question, to define densities on shape spaces. To this end assume that $\mathcal{C}_m^n$ has a density function $0 \le \phi \in L^1(\mathcal{C}_m^n)$, where $\int \phi d\mathcal{X} =$

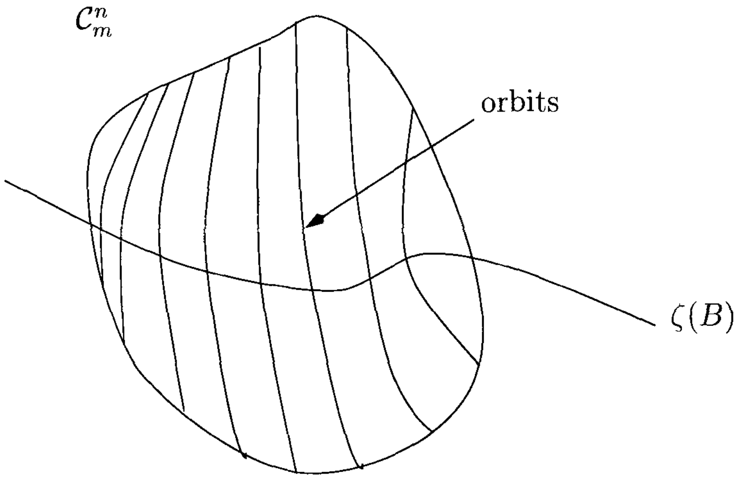

Fig. 2. Parametrisation of $\mathcal{C}_m^n$. Each orbit has a unique intersection with $\zeta(B)$.

1. Often it is obtained from density functions for the individual points of the configurations. We want to define a density function on $\mathcal{C}_m^n/G$, i.e. a density function for shapes. Let γ_ζ be as in Definition 2, then

$$\int_{\mathcal{C}_m^n} \phi(\mathcal{X})d\mathcal{X} = \int_B \left(\int_A \phi \circ \gamma_\zeta(a,b) \left|\det(\gamma'_\zeta)\right| da\right) db \ ,$$

where $\det(\gamma'_\zeta)$ is the functional determinant of $\gamma_\zeta(a,b)$. It is thus a determinant of order $k+k'$. Here the function

$$B \ni b \to \int_A \phi \circ \gamma_\zeta(a,b) \left|\det(\gamma'_\zeta)\right| da \in L^1$$

is defined almost everywhere. As B and the non-degenerate shape space $s(U)$ are homeomorphic, it is natural to use this function when defining a density of shape.

Definition 3. *Let $\phi \in L^1(\mathcal{C}_m^n)$ be a density function, $\zeta : B \to \mathcal{C}_m^n$ a parametrisation of shape, $g : A \ni a \to g_a \in G$ a parametrisation of the group G and set $\gamma_\zeta(a,b) = g_a \circ \zeta(b)$. The function $\mathfrak{D}(\phi,\zeta) : B \to \mathbb{R}$, defined by*

$$\mathfrak{D}(\phi,\zeta) : b \to \int_A \phi \circ \gamma_\zeta(a,b) \left|\det \gamma'_\zeta\right| da \qquad (1)$$

is called the ***density of shape*** *on $\mathcal{C}_m^n$, with respect to G.*

Observe that the density of shape depends on the parametrisation of shape, which can be chosen in many ways. This seems to bring in an ambiguity. Also the parametrisation of G can be done in many ways. However, it can be shown, see [Ber97] that the density of shape is independent of the parametrisation of G and that there exist canonical transformations between the density of shape for different parametrisations of shape. This canonical transformation implies that if we integrate over a set in $s(U)$ the result will be independent of the specific choice of parametrisation of shape ζ.

Example 3. As an illustration of Definition 2 and 3, let G be the group of nonsingular affine transformations $\mathbb{R}^2 \to \mathbb{R}^2$. Extend G to the space $\mathcal{C}_4^2$, which consist of all ordered four point configurations in $\mathbb{R}^2$. Using matrix notation, we write

$$\mathcal{X} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \in \mathcal{C}_4^2,$$

where each column represents a point in $\mathbb{R}^2$. Let

$$\zeta : \mathbb{R}^2 \ni b \to \begin{pmatrix} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \end{pmatrix} \in \mathcal{C}_4^2$$

be a parametrisation of shape and

$$g : A \ni a \to g_a(\cdot) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} (\cdot) + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in G, \quad \det(\{a_{ij}\}) \neq 0 \ ,$$

be a parametrisation of G, where $A \subset \mathbb{R}^6$ is open and dense. Then γ_ζ in Definition 2 is given by

$$\gamma_\zeta : A \times \mathbb{R}^2 \ni (a, b) \to \begin{pmatrix} a_{11} & a_{12} & a_1 \\ a_{21} & a_{22} & a_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \in \mathcal{C}_4^2 \ .$$

Here γ_ζ has a dense open range in $\mathcal{C}_4^2$ and $\gamma_\zeta : A \times \mathbb{R}^2 \to \gamma_\zeta(A \times \mathbb{R}^2)$ is a C^1-diffeomorphism.

If $\phi \in L^1(\mathcal{C}_4^2)$ is a density function, inserting γ_ζ in (1) in Definition 3, gives us a density function of affine shapes for $\mathcal{C}_4^2$. Examples will be given below.

The following theorem shows how the density of $\phi \in \mathcal{C}_m^n$ can be transformed without changing the the density of shape.

Theorem 1. *Let $\phi \in L^1(\mathcal{C}_m^n)$, be a density function, $g : A \to g_a \in G$ a parametrisation of G and ζ a parametrisation of shape. If $f \in G$ is a C^1-diffeomorphism, then*

$$\mathfrak{D}\left(\phi, \zeta\right) = \mathfrak{D}\left(|\det f'|\, \phi \circ f, \zeta\right).$$

As a consequence of Theorem 1, neither scaling nor translation of the density $\phi \in L^1(\mathcal{C}_m^n)$, affect the density of affine or positive similarity shape.

4 Density of affine shape

We will now focus on the affine group of transformations. First the theory of densities of affine shape will be outlined, then these densities will be computed and the special case of affine shape of four planar points will be treated in detail.

4.1 Theory

Let us start with a parametrisation of C_m^n as in (1) in Definition 2, when $m \geq n+2$ and G is the group of affine transformations.

Proposition 1. *Let $\phi \in L^1(C_m^n)$ be a density function for C_m^n, where $m \geq n+2$, and let G be the group of nonsingular affine transformations $\mathbb{R}^n \to \mathbb{R}^n$ with the parametrisation*

$$g : A \ni a = (\{y_{ij}\}, \{t_k\}) \to g_a(\cdot) = \{y_{ij}\}(\cdot) + \{t_k\} \in G, \quad \det\{y_{ij}\} \neq 0 \ ,$$

where $\{y_{ij}\}$ is an $n \times n$ matrix, $\{t_k\}$ is an $n \times 1$ matrix and $A \subset \mathbb{R}^{n^2} \times \mathbb{R}^n$. Furthermore, let

$$\zeta : \mathbb{R}^{n(m-n-1)} \ni \{b_{ij}\} \to \left(I \ 0 \ \{b_{ij}\}\right) \in C_m^n$$

be a parametrisation of affine shape, where I is the $(n \times n)$ identity matrix and $\{b_{ij}\}$ is a $(n \times (m-n-1))$ matrix. Then the density of affine shape is given by

$$\mathfrak{D}(\phi, \zeta)(b) = \int |\det\{y_{ij}\}|^{m-n-1} \phi \circ \gamma_\zeta(a, b) da,$$

and

$$\mathfrak{D}(\phi, \zeta) = \mathfrak{D}(|\det C|^m \phi \circ f, \zeta)$$

for every nonsingular affine transformation $f(b) = Cb + d$.

It is easily seen that γ_ζ fulfills the conditions in Definition 2. All that has to been done then is to compute the functional determinant of γ'_ζ. The details are left to the reader.

4.2 Density of affine shape for four normally distributed planar points

In this subsection, we compute the density of affine shape for four points in a plane, when the points are distributed normally and independent. The densities will provide a useful tool for determining search areas in shape space, when comparing an objects shape with some shapes in a data base.

When the four points are distributed with the same mean, it is possible to obtain a closed form solution, and when the means are different, the density of shape is given as a one dimensional integral, which generally has to be evaluated numerically.

Let

$$\phi(x, y) = \prod_{i=1}^{4} \psi(x_i)\psi(y_i) \ ,$$

where $\psi(x) = \pi^{-1/2} e^{-x^2}$, is the density for four points in a plane, i.e. $\phi \in L^1(C_4^2)$ and $\int \phi \, dxdy = 1$. Let

$$g : A \ni \alpha = (\{y_{ij}\}, \{t_k\}) \to g_\alpha(\cdot) = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} (\cdot) + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \quad \det\{y_{ij}\} \neq 0 \ , \tag{2}$$

be a parametrisation of the group G of nonsingular affine transformations $\mathbb{R}^2 \to \mathbb{R}^2$, where $A \subset \mathbb{R}^6$ is open and dense. Choose

$$\zeta : \mathbb{R}^2 \ni (x_1, x_2) \to \begin{pmatrix} 1\,0\,0\,x_1 \\ 0\,1\,0\,x_2 \end{pmatrix} ,$$

as a parametrisation of shape. Set

$$\gamma_\zeta : A \times \mathbb{R}^2 \ni (\alpha, x) \to g_\alpha \circ \zeta(x) \in \mathcal{C}_4^2 .$$

Then the density of shape is

$$\mathfrak{D}(\phi, \zeta)(x) = \int_A |\det \{y_{ij}\}| \, \phi \circ \gamma_\zeta d\alpha . \tag{3}$$

After tedious symbolic computations, we obtain

$$\mathfrak{D}(\phi, \zeta)(x) = \frac{1}{2\pi\sqrt{2}} \left(1 - x_1 - x_2 + x_2^2 + x_1^2 + x_1 x_2\right)^{-3/2} .$$

Observe that the level curves of this density function are ellipses. A simulation was performed, where a sequence of point configurations were drawn, each with the density function ϕ. From the map γ_ζ, a unique $x = (x_1, x_2)$ was obtained and the plot for that variable is shown to the right. $\mathfrak{D}(\phi, \zeta)(x)$ is shown to the left. The results are shown in Figure 3, where the simulated data are shown to the right.

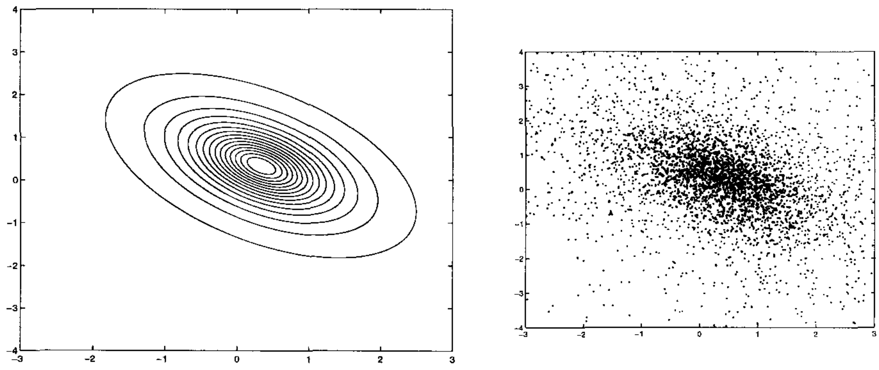

Fig. 3. Computed density function and simulated.

Now assume that the points are normally distributed and independent, with respective means $m_j \in \mathbb{R}^2$, $j = 1, 2, 3, 4$ but with the same standard deviation, i.e.

$$\phi(x, y) = \prod_{j=1}^{4} \psi(x_j - m_{x_j}) \psi(y_j - m_{y_j}) , \tag{4}$$

where $\psi(x) = (2\pi\sigma^2)^{-1/2}e^{-x^2/(2\sigma^2)}$. By Theorem 1, with

$$g_{a_0} : \mathbb{R}^2 \ni v \mapsto \sigma R v - m \ ,$$

where R is a rotation matrix and $m \in \mathbb{R}^2$ is a vector, it is no restriction to assume that $\psi(x) = \pi^{-1/2}e^{-x^2}$ and that $m_{j_1} = (0,0)$, $m_{j_2} = (0, m_{y_{j_2}})$, for some $j_1 \neq j_2$.

It is possible to write the density of shape (3) as an integral in one dimension, which generally has to be evaluated numerically. As an example, assume that $m_1 = (0,0)$, $m_2 = (0,1)$ $m_3 = (1,1)$ and $m_4 = (1,0)$. The density of shape is then given by

$$\mathfrak{D}(\phi,\zeta) = \frac{1}{\pi^2}\int_0^\infty \frac{u(x_1,x_2,\omega)}{\omega} d\omega \ ,$$

where

$$u(x_1,x_2,\omega) = \frac{4\omega(3a+\omega^2-b)\cos(\frac{c\omega}{a+\omega^2}) + 2c(a-\omega^2)\sin(\frac{c\omega}{a+\omega^2})}{(a^3+3a^2\omega+3a\omega^4+\omega^6)} e^{-\frac{b+2\omega^2}{a+\omega^2}} \ ,$$

$$a(x_1,x_2,\omega) = (x_1+x_2)^2 + (x_1-1)^2 + (x_2-1)^2 \ ,$$

$$b(x_1,x_2,\omega) = (x_1+x_2)^2 + (x_1-1)^2 \ ,$$

and

$$c(x_1,x_2,\omega) = 2(1-x_2) \ .$$

The density function is illustrated in Figure 4. Note that the level curves are not ellipses in this case, indicating a deviation from the first order approximation.

In order to compute (3), with $\phi(x,y)$ given by (4), we use the Fourier transformation $\widehat{|t|}$ of $t \mapsto |t|$, where the Fourier transformation is taken in distribution sense according to Schwartz. It can be shown, c.f. [Ber97], that

$$\mathfrak{D}(\phi,\zeta) = -\frac{1}{\pi}\int_0^\infty \frac{\psi'(\omega)-\psi'(-\omega)}{\omega} d\omega, \tag{5}$$

where

$$\psi(x_1,x_2,\omega,m) = \int_A \phi \circ \gamma_\zeta e^{i\omega \det\{y_{ij}\}} d\alpha.$$

For general means m_j, $j = 1,\ldots,4$, is straight forward, to compute ψ from the formula

$$\int e^{-(z_1t^2+z_2t)}dt = e^{z_2^2/(4z_1)}\sqrt{\frac{\pi}{z_1}}, \qquad \Re(z_1) > 0,$$

which is obtained by changing the path of integration to the complex plane. As the left hand side is continuous in z_1, the principal branch is chosen for the square root.

Since $\lim_{\omega\to 0}\{\psi'(\omega)-\psi'(-\omega)\}\omega^{-1}$ is bounded, it follows, by partial integration of (5), that

$$\mathfrak{D}(\phi,\zeta) = -\frac{1}{\pi}\int_0^\infty \frac{\psi(\omega)+\psi(-\omega)-2\psi(0)}{\omega^2} d\omega.$$

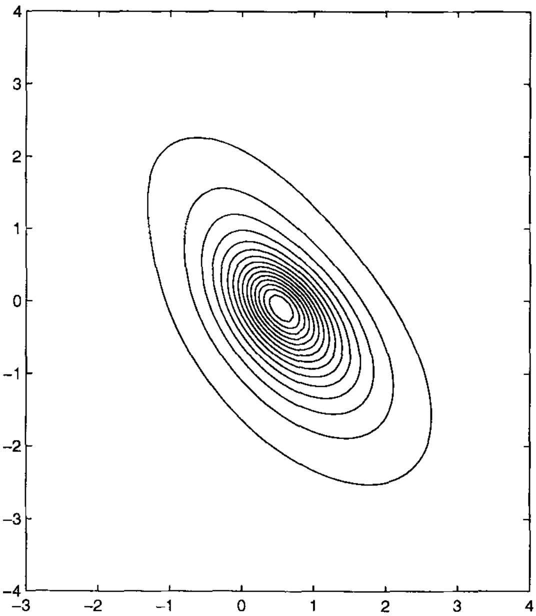

Fig. 4. Computed density function.

We will use this formula to compute a number of densities of shape below in Section 6. For general means, it is possible to obtain a closed form solution of the integrand, but the solution is very complicated. Instead we compute the integrand for each mean $m = (m_1, m_2, m_3, m_4)$, $m_j \in \mathbb{R}^2$.

4.3 Density of affine shape for four uniformly distributed planar points

An interesting situation is when we have four points in plane, which are uniformly and independently distributed in $[0, 1] \times [0, 1]$. This case corresponds to four randomly chosen points in an image. The density is important to know, when calculating false alarm rates, i.e. the possibility that four randomly chosen points $\mathcal{X}$ have shape $s(\mathcal{X})$ inside the confidence area for some of the model points.

Let

$$\phi(x, y) = \prod_{i=1}^{4} \psi(x_i)\psi(y_i),$$

where ψ is defined as

$$\psi(x) = \begin{cases} 0 & |x - 1/2| > 1/2, \\ 1 & |x - 1/2| \le 1/2 \end{cases},$$

be the uniform joint density in $\mathcal{C}_4^2$ With the parametrisation of G in (2) and

$$\zeta : \mathbb{R}^2 \ni (x_1, x_2) \to \begin{pmatrix} 1 & 0 & x_1 & 0 \\ 0 & 1 & 0 & x_2 \end{pmatrix} \in \mathcal{C}_4^2$$

as a parametrisation of shape, it is possible to compute the density of shape in closed form, though some effort is needed.

That ζ is in fact a parametrisation of shape can be seen if, given a point configuration $\mathcal{X} = (p_1, p_2, p_3, p_4) \in \mathcal{C}_4^2$, we let $t^T = (t_1, t_2)$ be the intersection of the lines obtained by adjoining the points p_1 with p_3 and p_2 with p_4. Thus

$$\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} p_1 - t & p_2 - t \end{pmatrix}$$

and

$$\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}^{-1} \begin{pmatrix} p_1 - t & p_2 - t \end{pmatrix} \quad .$$

This computation is possible for almost all $\mathcal{X} \in \mathcal{C}_4^2$.

The density of affine shape, given as an integral, is

$$\mathfrak{D}(\phi, \zeta)(x_1, x_2) = |1 - x_1|\,|1 - x_2| \int |\det(A)|\, \phi \circ \gamma_\zeta(y, x) dy$$

and when computed, it turns out to be a piecewise rational function in $\mathbb{R}^2$. By Theorem 1, the density $\mathfrak{D}(\phi, \zeta)$ is independent of any non-singular affine transformation of ϕ.

5 Application of affine shape to recognition

The densities $\mathfrak{D}(\phi, \zeta)$ in Section 4.2 can be used to limit the search area in a data base, that is used for recognition. The problem is to identify an unknown object with some object in a reference set, by using affine shape of extracted feature points.

For a set of planar objects, four feature points are extracted from each to give a set of point configurations $\{\mathcal{X}_j\}_{j=1}^n$. The affine shape $s(\mathcal{X}_j) = (x_{1j}, x_{2j}) \in \mathbb{R}^2$ is computed and stored in a data base $\{s(\mathcal{X}_j)\}_{j=1}^n$. If we assume that the feature points $\mathcal{X}_j$ have a normal independent density function ϕ_j, it is possible to compute the density of shape $\mathfrak{D}(\phi_j, \zeta)$, by the method described in Section 4.2.

Set

$$\Omega_b(j) = \left\{ (x, y) \;\middle|\; \mathfrak{D}(\phi_j, \zeta) > a, \int_{\mathfrak{D}(\phi_j, \zeta) > a} \mathfrak{D}(\phi_j, \zeta) dx dy = b \right\} \quad ,$$

where $0 \leq b \leq 1$ is set by a human operator. In the example illustrated in Figure 4, Ω_b is the set, such that when integrating $\mathfrak{D}(\phi_j, \zeta)$ over the set, which is bounded by a level curve, the integral should be equal to b. b gives the level of significance.

For an unknown object, four feature points are extracted, forming a point configuration $\mathcal{Y}$, and the shape $s(\mathcal{Y})$ is computed. We identify $\mathcal{Y}$ as object j in the data base if $\mathcal{Y} \in \Omega_b(j)$.

Moreover, let ϕ_u be the uniform and independent joint density for four points in a rectangular subset of $\mathbb{R}^2$. If the density of shape $\mathfrak{D}(\phi_u, \zeta)$ is integrated over the union of all $\Omega_b(j)$, $j = 1, 2, \ldots$, this will give the false alarm rate. The false alarm rate measures the probability that four randomly chosen points is decided to be an object in the database. Letting P_{FAR} denote the false alarm rate, we have

$$P_{FAR} = \int_{\cup \Omega_b(j)} \mathfrak{D}(\phi_u, \zeta) dx dy \ . \tag{6}$$

6 Experiments

In order to simulate objects arriving at a conveyor belt, five different quadrangles were cut out of a piece of black paper. These five quadrangles were then put on a larger piece of white paper in six different positions. One image for each position were captured, with image plane parallel to the papers, see Figure 5.

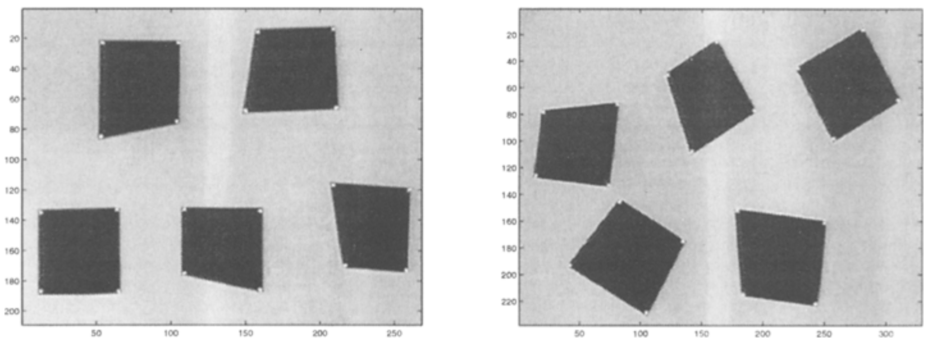

Fig. 5. Two of the images of quadrangles with detected corners.

The images were taken using a standard S-VHS video camera, giving images of size 500×700 pixels approximately. The sizes of the quadrangles were about 100 pixels. Each of these images were then subsampled giving images of half the size, with quadrangles of size about 50 pixels. Some of the images were then subsampled once more, giving quadrangles of size about 25 pixels.

The corners in the images were detected using Harris' corner detector, see Figure 5 again. The correspondence problem was solved by hand, but could easily be solved automatically, by identifying the corners belonging to each object and then walking around the contour. Since this is outside the scope of the present paper we chose to solve this problem manually.

The shape of each configuration in each image are plotted in Figure 6, where an asterisk is used for the original resolution, a circle for the subsampled resolution and a plus for the subsubsampled resolution. The 90% confidence areas for the five corresponding different densities of shape are shown in the same figure, when the points are assumed to be distributed as in Section 4.2 with $\sigma = 1/1.96$ and means

$$m_1 = \begin{pmatrix} 20 & 0 & 0 & 20 \\ 0 & 20 & 0 & 20 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 20 & 0 & 0 & 16 \\ 0 & 20 & 0 & 20 \end{pmatrix}, \quad m_3 = \begin{pmatrix} 20 & 0 & 0 & 20 \\ 0 & 20 & 0 & 24 \end{pmatrix},$$

$$m_4 = \begin{pmatrix} 20 & 0 & 0 & 24 \\ 0 & 20 & 0 & 20 \end{pmatrix}, \quad m_5 = \begin{pmatrix} 20 & 0 & 0 & 20 \\ 0 & 20 & 0 & 16 \end{pmatrix},$$

respectively. The reason for using $\sigma = 1/1.96$ is that we assume that the errors in the measurements of the corners are normally distributed, with 95% of the measurements giving an error smaller than one pixel. The 90% confidence areas corresponds to an uncertainty of the measured points that is about $1/20$ of the side length of the quadrangle.

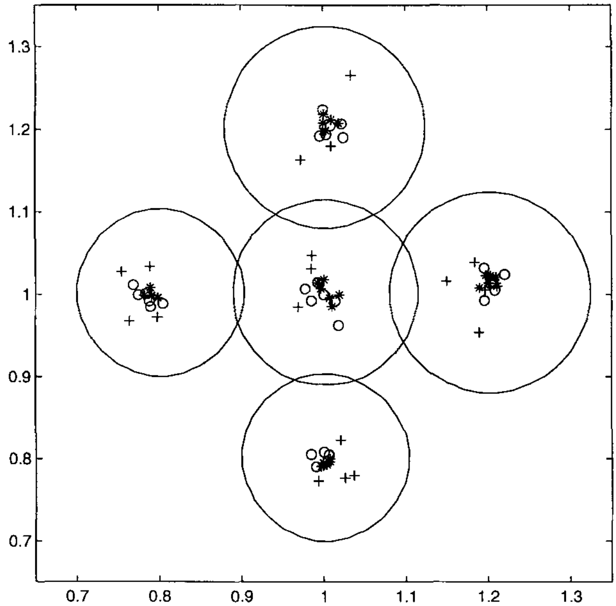

Fig. 6. 90% confidence areas for the shape of four point configurations with different means but with the same standard deviation, together with measured values.

In Figure 7, the points from the middle of Figure 6 are shown together with 90% confidence areas, when the means of the points are given by

$$\begin{pmatrix} 20 & 0 & 0 & 20 \\ 0 & 20 & 0 & 20 \end{pmatrix}, \quad \begin{pmatrix} 40 & 0 & 0 & 40 \\ 0 & 40 & 0 & 40 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 80 & 0 & 0 & 80 \\ 0 & 80 & 0 & 80 \end{pmatrix},$$

and the standard deviation is $\sigma = 1/1.96$ for all points. This corresponds to an uncertainty of about 1/20, 1/40 and 1/80 of the side length of the quadrangle respectively, which is in appliance with the actual sizes of the objects.

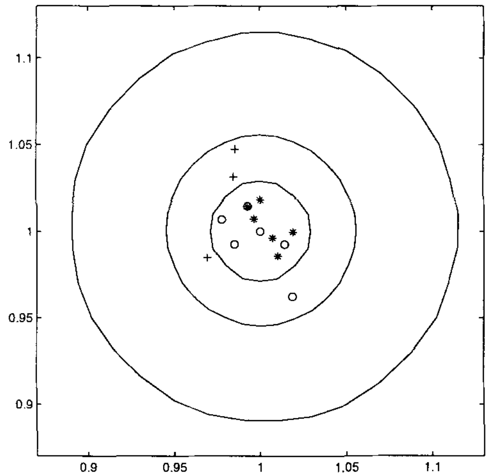

Fig. 7. 90% confidence areas for shape of four point configurations with different standard deviation but with the same mean.

Figure 7 indicates that the measured shapes are more accurate than the predicted, since all points lie well inside the 90% confidence area. There can be several reasons for this. Firstly, the measured corner positions, from Harris' corner detector, may be more accurate than one pixel. Secondly, the errors in the different corner positions may not be independent. It is well known that the position of a corner tends to move towards the interior of an object when the resolution diminishes. This bias in the measurements, gives highly correlated errors, when they are not modeled as a bias but rather as an error. Moreover, moving all points a little towards the center of the object does not affect the affine shape significantly. The false alarm rate given by (6), can be computed from results in [Ber97]. A computation gives that $P_{FAR} = 0.0420$.

In order to show that the confidence areas need not always be ellipses, Figure 8 shows the density of shape when

$$m = \begin{pmatrix} 0 & 10 & -7 & -16 \\ 0 & 0 & -2 & 2 \end{pmatrix} \tag{7}$$

and $\sigma = 1/\sqrt{2}$.

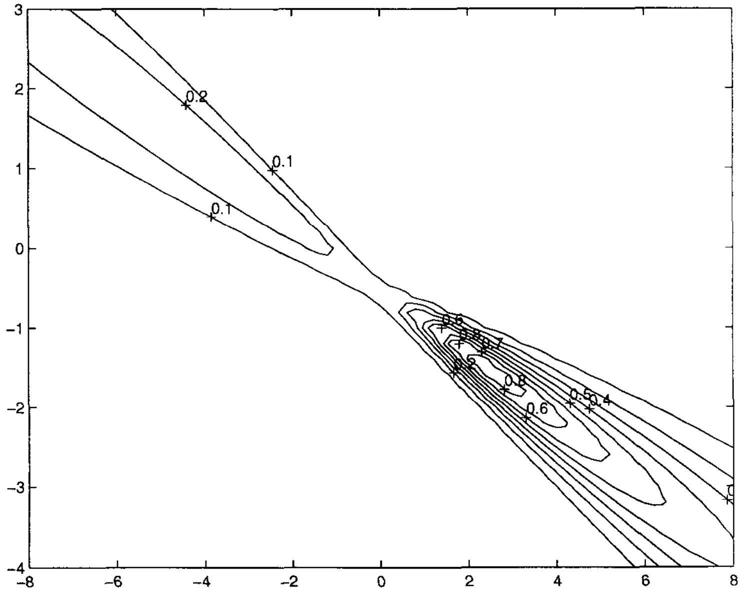

Fig. 8. Contour plot of $\mathfrak{D}(\phi, \zeta)^{1/4}$, when $\sigma = 1/\sqrt{2}$ and m is given by (7).

7 Conclusions

In this paper we have presented a theoretical framework for computing the density of different types of invariants, given the density of the underlying points. This framework is applied to affine invariants for planar four point configurations, especially for the case of normally distributed points and uniformly distributed points. From the density of shape, different kinds of levels of significance, can be calculated, as well as false alarm rates. The results are verified on real images of five different quadrangles, showing the applicability of the theory.

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