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Notes on the characterization of derivations

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Abstract. Although the characterization of ring derivations has an extensive literature, up to now, all of the characterizations have had the following form: additivity and another property imply that the function in question is a derivation. The aim of this note is to point out that derivations can be described via a single equation.

1. Introduction and preliminaries

The purpose of this paper is to provide new characterization theorems on derivations. At first, we will list some preliminary results that will be used in the sequel. All of these statements and definitions can be found in Kuczma [14] and also in Zariski–Samuel [17].

Let Q be a commutative ring and let P be a subring of Q. A function $f\colon P\to Q$ is called a *derivation* if it is additive, i.e.,

$$f(x+y)=f(x)+f(y)\quad (x,y\in P)$$

and also satisfies the equation

$$f(xy)=xf(y)+yf(x)\quad (x,y\in P).$$

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A fundamental example for derivations is the following. Let $\mathbb{F}$ be a field, and let in the above definition $P = Q = \mathbb{F}[x]$ be the ring of polynomials with coefficients from $\mathbb{F}$. For a polynomial $p \in \mathbb{F}[x]$, $p(x) = \sum_{k=0}^{n} a_k x^k$, define the function $f\colon \mathbb{F}[x] \to \mathbb{F}[x]$ as

$$f(p) = p',$$

where $p'(x) = \sum_{k=1}^{n} k a_k x^{k-1}$ is the derivative of the polynomial p. Then the function f clearly fulfills

$$f(p+q) = f(p) + f(q)$$

and

$$f(pq) = pf(q) + qf(p)$$

for all $p, q \in \mathbb{F}[x]$. Hence f is a derivation.

In $\mathbb{R}$ (the set of the real numbers) the identically zero function is obviously a (trivial) derivation. It is difficult to find however another example, because every real derivation has the following properties.

(1) If $f\colon \mathbb{R} \to \mathbb{R}$ is a derivation, then $f(x) = 0$ for all $x \in \mathrm{algcl}\mathbb{Q}$ (the algebraic closure of the set of the rational numbers).

(2) If $f\colon \mathbb{R} \to \mathbb{R}$ is a derivation, and f is measurable, or bounded above or below on a set which has positive Lebesgue measure, then f is identically zero.

In spite of the above properties, there exist non-trivial derivations in $\mathbb{R}$, see Theorem 14.2.2. in [14].

The characterization of derivations has an extensive literature, the reader should consult for instance Horinouchi–Kannappan [10], Jurkat [13], Kurepa [15], [16] and also the two monographs Kuczma [14] and Zariski–Samuel [17].

Nevertheless, to the best of the author's knowledge, all of the characterizations have the following form: additivity and another property imply that the function in question is a derivation. We intend to show that derivations can be characterized by one single functional equation.

More precisely, we would like to examine whether the equations occurring in the definition of derivations are independent in the following sense. Let $\lambda, \mu \in Q \setminus \{0\}$ be arbitrary, $f\colon P \to Q$ be a function and consider the equation

$$\lambda[f(x+y) - f(x) - f(y)] + \mu[f(xy) - xf(y) - yf(x)] = 0 \quad (x, y \in P).$$

Clearly, if the function f is a derivation, then this equation holds. In the next section we will investigate the opposite direction, and it will be proved that under some assumptions on the rings P and Q, derivations can be characterized through the above equation. This result will be proved as a consequence of the main theorem that will be devoted to the equation

$$f(x+y)-f(x)-f(y)=g(xy)-xg(y)-yg(x) \quad (x,y\in P),$$

where $f,g\colon P\to Q$ are unknown functions.

We remark that similar investigations were made by Dhombres [3], Ger [7], [8] and also by Ger–Reich [9] concerning ring homomorphisms. For instance, in Ger [7] the following theorem was proved.

Theorem 1.1. *Let X and Y be two rings, and assume that for all $x\in X$ there exists $e_x\in X$ such that $xe_x=x$, suppose further that Y has no elements of order 2 and does not admit zero divisors. If f is a solution of the equation*

$$f(x+y)+f(xy)=f(x)+f(y)+f(x)f(y) \qquad (x,y\in X)$$

such that $f(0)=0$, then either $3f$ is even and $3f(2x)=0$ for all $x\in X$, or f yields a homomorphism between X and Y.

In the proof of the main result the celebrated *cocycle equation* will play a key role. About this equation one can read, e.g., in Aczél [1], Davison–Ebanks [2], Ebanks [5], Erdős [6], Hosszú [11] and also in Jessen–Karpf–Thorup [12]. In the next section we will however utilize only Theorem 3 of Ebanks [4], which reads as follows.

Theorem 1.2. *Let A be an integral domain and X a uniquely A-divisible unitary module over A. Then, $F,G\colon A^2\to X$ satisfy the equations*

$$(\beta)\qquad F(a+b,c)+F(a,b)=F(a,b+c)+F(b,c) \quad (a,b,c\in A),$$

$$(\gamma)\qquad G(a,b)=G(b,a) \quad (a,b\in A),$$

$$(\delta)\qquad cG(a,b)+G(ab,c)=aG(b,c)+G(a,bc) \quad (a,b,c\in A),$$

$$(\varepsilon)\qquad F(ac,bc)-cF(a,b)=G(a+b,c)-G(a,c)-G(b,c) \quad (a,b,c\in A),$$

$$(\zeta)\qquad \sum_{i=1}^{p}F(1,i1)=0, \quad p=\mathrm{char}A,$$

if and only if there is a map $f\colon A\to X$ representing F and G through the equations

$$(\mathrm{A})\qquad F(a,b)=f(a+b)-f(a)-f(b),$$

respectively,

$$\text{(B)} \qquad G(a,b) = f(ab) - af(b) - bf(a).$$

Moreover, if A is ordered, the same result holds with A replaced by

$$A_+ = \{a \in A \,|\, a > 0\}.$$

From this theorem with the choice $F \equiv 0$ (and interchanging the roles of f and g) the following statement can be obtained immediately.

Theorem 1.3. *Let A be a commutative ring, X be a module over A and $f\colon A \to X$ be a function such that*

$$f(ab) = af(b) + bf(a) \qquad (a, b \in A)\,.$$

Then the function $F\colon A \times A \to X$ defined by (A) fulfills

$$(\alpha) \qquad F(a,b) = F(b,a),$$

equation (β) is satisfied and also

$$(\eta) \qquad F(ac, bc) = cF(a,b)$$

holds for any $a, b, c \in A$.

Furthermore, in case A is an integral domain and X is a unitary module over A which is uniquely A-divisible, then the function F defined by the function f through equation (A) is the only function which satisfies equations (α), (β) and (η).

2. The main result

Our main result is contained in the following

Theorem 2.1. *Let* $\mathbb{F}$ *be a field,* X *a vector space over* $\mathbb{F}$ *and* $f, g\colon \mathbb{F} \to X$ *functions such that*

$$(\mathcal{E}) \qquad f(x+y) - f(x) - f(y) = g(xy) - xg(y) - yg(x)$$

holds for all $x, y \in \mathbb{F}$. *Then, and only then, there exist additive functions* $\alpha, \beta\colon \mathbb{F} \to X$ *and a function* $\varphi\colon \mathbb{F} \to X$ *with the property*

$$\varphi(xy) = x\varphi(y) + y\varphi(x) \qquad (x, y \in \mathbb{F}),$$

such that

$$f(x) = \beta(x) + \frac{1}{2}\alpha(x^2) - x\alpha(x) \qquad (x, y \in \mathbb{F})$$

and

$$g(x) = \varphi(x) + \alpha(x) \qquad (x, y \in \mathbb{F})$$

are satisfied.

Proof. Define the functions $\mathcal{C}_f, \mathcal{D}_f, \mathcal{C}_g$ and $\mathcal{D}_g$ on $\mathbb{F} \times \mathbb{F}$ by

$$\begin{aligned} \mathcal{C}_f(x,y) &= f(x+y) - f(x) - f(y), \\ \mathcal{D}_f(x,y) &= f(xy) - xf(y) - yf(x), \\ \mathcal{C}_g(x,y) &= g(x+y) - g(x) - g(y), \\ \mathcal{D}_g(x,y) &= g(xy) - xg(y) - yg(x), \end{aligned}$$

respectively. In view of Theorem 1.2., we immediately get that the pairs $(\mathcal{C}_f, \mathcal{D}_f)$ and $(\mathcal{C}_g, \mathcal{D}_g)$ fulfill the system of equations (α)–(ε). Furthermore, equation $(\mathcal{E})$ yields that

$$(\mathcal{E}^*) \qquad \mathcal{C}_f(x,y) = \mathcal{D}_g(x,y)$$

for all $x, y \in \mathbb{F}$. Due to equation (ε),

$$(2.1) \qquad \mathcal{C}_g(xz, yz) - z\mathcal{C}_g(x,y) = \mathcal{D}_g(x+y, z) - \mathcal{D}_g(x,z) - \mathcal{D}_g(y,z)$$

holds for all $x, y, z \in \mathbb{F}$. Interchanging the role of x and z in the previous equation, we obtain that

$$(2.2) \qquad \mathcal{C}_g(xz, xy) - x\mathcal{C}_g(z,y) = \mathcal{D}_g(y+z, x) - \mathcal{D}_g(z,x) - \mathcal{D}_g(y,x)$$

for any $x, y, z \in \mathbb{F}$. Let us subtract equation (2.2) from (2.1) to obtain

$$\begin{aligned}\mathcal{C}_g(xz,yz) - z\mathcal{C}_g(x, y) - \mathcal{C}_g(xz, xy) + x\mathcal{C}_g(z, y) \\ = \mathcal{D}_g(x + y, z) - \mathcal{D}_g(x, z) - \mathcal{D}_g(y, z) - \mathcal{D}_g(z + y, x) + \mathcal{D}_g(z, x) + \mathcal{D}_g(y, x).\end{aligned}$$

Because of $(\mathcal{E}^*)$, the function $\mathcal{D}_g$ can be replaced by $\mathcal{C}_f$. This implies however that

$$\begin{aligned}\mathcal{C}_g(xz, yz) - z\mathcal{C}_g(x, y) - \mathcal{C}_g(xz, xy) + x\mathcal{C}_g(z, y) \\ = \mathcal{C}_f(x + y, z) + \mathcal{C}_f(x, y) - \mathcal{C}_f(x, y + z) - \mathcal{C}_f(y, z) = 0,\end{aligned}$$

for all $x, y, z \in \mathbb{F}$, where we used that the function $\mathcal{C}_f$ fulfills (α) and (β). This equation with $z = 1$ yields that

$$\mathcal{C}_g(x, xy) = x\mathcal{C}_g(1, y),$$

or if we replace y by y/x $(x \neq 0)$,

$$\mathcal{C}_g(x, y) = x\mathcal{C}_g\Big(1, \frac{y}{x}\Big) \quad (x, y \in \mathbb{F}, x \neq 0).$$

We will show that from this identity the homogeneity of $\mathcal{C}_g$ follows. Indeed, let $t, x, y \in \mathbb{F}$, $t, x \neq 0$, be arbitrary, then

$$\mathcal{C}_g(tx, ty) = tx\mathcal{C}_g\Big(1, \frac{ty}{tx}\Big) = tx\mathcal{C}_g\Big(1, \frac{y}{x}\Big) = t\mathcal{C}_g(x, y).$$

If $x = 0$, we get from the above identity that $\mathcal{C}_g(0, 0) = 0$, thus for arbitrary $t \in \mathbb{F}$,

$$\mathcal{C}_g(t0, t0) = 0 = t\mathcal{C}_g(0, 0).$$

Furthermore, in case $t = 0$, then for any $x, y \in \mathbb{F}$

$$\mathcal{C}_g(tx, tx) = \mathcal{C}_g(0, 0) = 0 = t\mathcal{C}_g(x, y).$$

This means that the function $\mathcal{C}_g$ is homogeneous and fulfills equations (α) and (β). In view of Theorem 1.3, there exists a function $\varphi\colon \mathbb{F} \to X$ such that

$$\varphi(xy) = x\varphi(y) + y\varphi(x) \qquad (x, y \in \mathbb{F})$$

and

$$\mathcal{C}_g(x, y) = \varphi(xy) - x\varphi(y) - y\varphi(x) \qquad (x, y \in \mathbb{F})$$

hold. Due to the definition of the function $\mathcal{C}_g$, this yields that

$$g(x) = \varphi(x) + \alpha(x) \qquad (x \in \mathbb{F}),$$

where the function φ fulfils the above identity and $\alpha: \mathbb{F} \to X$ is additive. Writing this representation of the function g into equation $(\mathcal{E})$, we have that

$$f(x+y) - f(x) - f(y) = \alpha(xy) - x\alpha(y) - y\alpha(x) \qquad (x, y \in \mathbb{F}). \tag{2.3}$$

Since the function α is additive, the two-place function

$$\mathcal{D}_\alpha(x,y) = \alpha(xy) - x\alpha(y) - y\alpha(x) \qquad (x, y \in \mathbb{F})$$

is a symmetric, biadditive function. Therefore, $\mathcal{D}_\alpha$ can be written as the Cauchy difference of its trace, that is

$$\mathcal{D}_\alpha(x,y) = \frac{1}{2}\alpha((x+y)^2) - (x+y)\alpha(x+y) - \left(\frac{1}{2}\alpha(x^2) - x\alpha(x)\right) - \left(\frac{1}{2}\alpha(y^2) - y\alpha(y)\right)$$

$(x, y \in \mathbb{F})$. In view of equation (2.3), this yields that the function

$$x \mapsto f(x) - \left(\frac{1}{2}\alpha(x^2) - x\alpha(x)\right) \qquad (x \in \mathbb{F})$$

is additive. Thus there exists an additive function $\beta: \mathbb{F} \to X$ such that

$$f(x) = \beta(x) + \frac{1}{2}\alpha(x^2) - x\alpha(x) \qquad (x \in \mathbb{F})\,.$$

■

According to Theorem 1.2, with the aid of this theorem the following corollary can be immediately obtained.

Corollary 2.2. *Let $\mathbb{F}$ be an ordered field, X a vector space over $\mathbb{F}$, $\mathbb{F}_+ = \{x \in \mathbb{F} | x > 0\}$ and $f, g: \mathbb{F}_+ \to X$ functions such that*

$$f(x+y) - f(x) - f(y) = g(xy) - xg(y) - yg(x)$$

holds for all $x, y \in \mathbb{F}_+$. Then the functions f and g can be extended to functions $\widetilde{f}, \widetilde{g}: \mathbb{F} \to X$ such that

$$\widetilde{f}(x) = \beta(x) + \frac{1}{2}\alpha(x^2) - x\alpha(x) \qquad (x, y \in \mathbb{F})$$

and

$$\widetilde{g}(x) = \varphi(x) + \alpha(x) \qquad (x, y \in \mathbb{F}),$$

where $\alpha, \beta: \mathbb{F} \to X$ are additive function and $\varphi: \mathbb{F} \to X$ fulfils

$$\varphi(xy) = x\varphi(y) + y\varphi(x) \qquad (x, y \in \mathbb{F}).$$

From Theorem 2.1. with $g(x) = -\frac{\mu}{\lambda} f(x)$ the following corollary can be derived easily.

Corollary 2.3. *Let $\mathbb{F}$ be a field and X a vector space over $\mathbb{F}$, $\lambda, \mu \in \mathbb{F}\setminus\{0\}$ arbitrarily fixed. Then the function $f\colon \mathbb{F} \to X$ is a derivation if and only if*

$$\lambda[f(x+y) - f(x) - f(y)] + \mu[f(xy) - xf(y) - yf(x)] = 0$$

holds for all $x, y \in \mathbb{F}$.

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