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Two embedding theorems pertaining to strong approximation of sine and cosine series

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Dedicated to Béla Csákány on his 80th birthday

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Abstract. In the present paper we prove two embedding theorems. Both give necessary and sufficient conditions, herewith improving and unifying some previous results.

1. Introduction

In [2] we proved two embedding theorems pertaining to sine series. These theorems were improved and unified in a joint paper A. S. Belov and L. Leindler [1]. The result of [1] is an essential generalization of the theorems of [2], namely, using the idea of Belov, it gives a necessary and sufficient condition for the embedding assertion considered in [2], and it holds for any positive p.

The theorems of [2] give only sufficient conditions, and show that under certain additional assumptions these conditions cannot be improved, furthermore they are proved only for $0 < p \le 1$ and $1 \le p < \infty$, separately.

Very recently D. Yu [4], among others, extended the theorems of [2] to new classes of functions having derivatives, too.

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In this paper we show that the theorems of Yu can be also improved. We shall establish two theorems, giving both necessary and sufficient conditions, the first one for the case if the order of the derivative is even, and the second one for the odd case.

Making the previous assertions a little be more perceivable we recall one theorem from each of the papers [2], [4] and [1].

As it has been mentioned, in [2] two theorems were proved, and their generalizations in [4]; but now we recall only one of them, namely we do believe it is sufficient to the comparison of the old and new theorems.

To establish the theorems we need numerous notions and notations, they are collected in Section 3.

Theorem L. ([2]). *Let $p \ge 1$ and $\omega(\delta)$ be a modulus of continuity. If $\boldsymbol{\lambda} := \{\lambda_n\}$ is a positive monotone sequence such that $\boldsymbol{\lambda}$ is quasi β-power-monotone increasing with some $\beta < 1$, then the condition*

$$(n\,\lambda_n)^{1/p}\omega\Big(\frac{1}{n}\Big) \ll 1 \tag{1.1}$$

implies

$$H_{S,\Omega_0} \subset S_p(\boldsymbol{\lambda}). \tag{1.2}$$

If there exists a positive nondecreasing sequence $\boldsymbol{\rho} := \{\rho_n\}$ tending to infinity such that the sequence $\{\lambda_n \rho_n^{-p}\}$ is simultaneously γ-power-monotone increasing with some $\gamma < 1$ and quasi α-power-monotone increasing with some $\alpha > 1-p$, and

$$(n\,\lambda_n)^{1/p}\omega\Big(\frac{1}{n}\Big) \gg \rho_n, \tag{1.3}$$

then (1.2) *does not hold; namely there exists a function f_0 such that*

$$f_0 \in H_{S,\Omega_0} \quad but \quad f_0 \notin S_p(\boldsymbol{\lambda}).$$

Theorem BL. ([1]). *Let $0 < p < \infty$ and $\omega(\delta)$ be a modulus of continuity. If $\boldsymbol{\lambda} := \{\lambda_n\}$ is an arbitrary sequence of nonnegative numbers, then the embedding relation* (1.2) *holds if and only if the condition*

$$\Lambda_n^{1/p}\omega\Big(\frac{1}{n}\Big) \ll 1, \quad with \quad \Lambda_n := \sum_{k=1}^{n} \lambda_k,$$

is fulfilled.

Theorem Y. ([4]). *Let $p \geq 1$, $\omega(\delta)$ be a modulus of continuity and $\{n_i\}$ satisfy the condition* (3.2). *If $\boldsymbol{\lambda} := \{\lambda_n\}$ is quasi η-power-monotone increasing with some $\eta < 1$, then condition* (1.1) *implies*

$$W^r H_{S,\Omega_r} \subset S_p^*(\boldsymbol{\lambda}, r), \tag{1.4}$$

for even r. If, in addition, $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma} := \{\gamma_n\}$ with $\sum_{n=1}^{\infty} (n\,\gamma_n)^{-1} < \infty$ satisfy the inequality

$$\omega\Big(\frac{1}{n}\Big) \ll (n\,\lambda_n\,\gamma_n)^{-1/p},$$

then (1.4) *also holds for odd r.*

If there exists a positive nondecreasing sequence $\boldsymbol{\rho} := \{\rho_n\}$ tending to infinity such that the sequence $\{\lambda_n\,\rho_n^{-p}\}$ is simultaneously γ-power-monotone increasing with some $\gamma < 1$ and α-power-monotone decreasing with some $\alpha > 1 - \min(1, \beta)p$, and (1.3) *holds, then* (1.4) *does not hold. In fact, there exists a function f_0^* such that*

$$f_0^* \in W^r H_{S,\Omega_r}, \quad \textit{but} \quad f_0^* \notin S_p(\boldsymbol{\lambda}, r).$$

The second theorems of [2] and [4] are analogous to Theorem L and Y with $0 < p \leq 1$, respectively.

2. New theorems

Our new theorems read as follows.

Theorem 1. *Let p be a positive number, r a nonnegative even integer and $\omega(\delta)$ be a modulus of continuity. If $\boldsymbol{\lambda} := \{\lambda_n\}$ is an arbitrary sequence of nonnegative numbers, then the embedding relation*

$$W^r H^*_{S,\Omega_r} \subset S_p^*(\boldsymbol{\lambda}, r) \tag{2.1}$$

holds if and only if the condition

$$\Lambda_n^{1/p} \omega\Big(\frac{1}{n}\Big) \ll 1 \tag{2.2}$$

is fulfilled.

Theorem 2. *Under the assumptions of Theorem* 1 *with odd* r *instead of an even one, and attaching a slight presumption on* $\omega(\delta)$*, that is, assuming that the sequence* $\{\omega(1/n)\}$ *is quasi* β*-power-monotone decreasing with some* $\beta > 0$*, then* (2.1) *holds if and only if*

$$\sum_{n=1}^{\infty} \frac{\Lambda_n}{n} \omega\Big(\frac{1}{n}\Big)^p < \infty. \tag{2.3}$$

Remarks. 1. It is worthy to comment the following embedding relations:

$$W^r H_{S,\Omega_r} \subset W^r H^*_{S,\Omega_r} \quad \text{and} \quad S^*_p(\boldsymbol{\lambda}, r) \subset S_p(\boldsymbol{\lambda}, r),$$

and thereafter to compare, e.g., the statements (1.2) and (1.4) if $r = 0$, furthermore (1.4) and (2.1).

2. It is clear that (2.3) claims more than (2.2). This follows from the fact that if r is odd then $f^{(r)}(x)$ has cosine series.

3. Notions and notations

Let $f(x)$ be a 2π-periodic function and let

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx \tag{3.1}$$

be its Fourier series. Let $s_n = s_n(x) = s_n(f, x)$ denote the n-th partial sum of (3.1). If $f(x)$ is continuous then it will be denoted by $f \in C(T)$.

We denote by $\|\cdot\|$ the usual supremum norm.

Let $\omega(\delta)$ be a modulus of continuity function.

We shall use the notation $L \ll R$ ($L \gg R$) at inequalities if there exists a positive constant K such that $L \le KR$ ($KL \ge R$) holds, not necessarily the same at each occurrence.

We also use the sequence $\Omega_r := \{n^{-r-1}\omega(1/n)\}$.

A sequence $\boldsymbol{\eta} := \{\eta_n\}$ of positive numbers is called *quasi* β*-power-monotone increasing* (*decreasing*) if there exists a constant $K := K(\beta, \boldsymbol{\eta}) \ge 1$ such that

$$Kn^\beta \eta_n \ge m^\beta \eta_m \quad (n^\beta \eta_n \le K m^\beta \eta_m)$$

holds for any $n \ge m$, in symbol $n^\beta \eta_n \uparrow$ ($n^\beta \eta_n \downarrow$).

Let $\boldsymbol{\gamma} := \{\gamma_n\}$ be a given positive sequence. A null-sequence $\mathbf{c} := \{c_n\}$ ($c_n \to 0$) of real numbers satisfying the inequalities

$$\sum_{n=m}^{\infty} |\Delta c_n| \le K(\mathbf{c})\gamma_m \quad (\Delta c_n := c_n - c_{n+1}),\ m = 1, 2, \dots,$$

with a positive constant $K(\mathbf{c})$ is said to be *a sequence of $\boldsymbol{\gamma}$ rest bounded variation*, in symbol: $\boldsymbol{c} \in \boldsymbol{\gamma}RBVS$.

If $\boldsymbol{\gamma} \equiv \mathbf{c}$, then $\boldsymbol{\gamma}RBVS \equiv RBVS$. We emphasize that if $\mathbf{c} \in \boldsymbol{\gamma}RBVS$ it may have infinitely many zeros and negative terms, but this is not the case if $\mathbf{c} \in RBVS$.

Next we enroll the definitions of the classes of functions to be considered:

$$W^r H_{S,\Omega_r} := \Big\{ f : f = \sum_{n=1}^{\infty} b_n \sin nx,\ f^{(r)} \in C(T),\ \{b_n\} \in \Omega_r RBVS \Big\},$$

$$W^r H^*_{S,\Omega_r} := \Big\{ f : f = \sum_{n=1}^{\infty} b_n \sin nx,\ \{b_n\} \in \Omega_r RBVS \Big\},$$

$$S_p(\boldsymbol{\lambda}, r) := \Big\{ f : \Big\| \sum_{n=1}^{\infty} \lambda_n |s_n^{(r)} - f^{(r)}|^p \Big\| < \infty \Big\}.$$

Let $\{n_i\}$ be a positive sequence satisfying

$$n_0 = 1, \quad 1 < Q_1 \le \frac{n_{i+1}}{n_i} \le Q_2, \quad i = 0, 1, \dots, \tag{3.2}$$

where Q_1 and Q_2 are constants depending only on $\{n_i\}$.

If $n_{i-1} \le n < n_i, i = 1, 2, \dots$, then denote

$$T_n^{(r)}(x) := \Big| \sum_{k=n+1}^{n_i - 1} b_k \sin^{(r)} kx \Big| + \sum_{j=i}^{\infty} \Big| \sum_{k=n_i}^{n_{i+1}-1} b_k \sin^{(r)} kx \Big|.$$

Finally define

$$S_p^*(\boldsymbol{\lambda}, r) := \Big\{ f : \Big\| \sum_{n=1}^{\infty} \lambda_n (T_n^{(r)})^p \Big\| < \infty \Big\}.$$

If $r = 0$, we simply write H_{S,Ω_0} and $S_p(\boldsymbol{\lambda})$ instead of $W^0 H_{S,\Omega_0}$ and $S_p(\boldsymbol{\lambda}, r)$, respectively, and so on.

4. Auxiliary results

We require the following lemmas.

Lemma 1. ([3]). *If $\{b_n\} \in \Omega_r RBVS$, then for any $0 \le \alpha \le r$, $\{n^\alpha b_n\} \in \Omega_{r-\alpha} RBVS$.*

Lemma 2. ([4]). *If $\{b_n\} \in \Omega_r RBVS$ then for any $x > 0$*

$$T_n^{(r)}(x) \ll x^{-1} n^{-1} \omega\Big(\frac{1}{n}\Big), \quad r = 0, 1, \ldots, \; n = 1, 2, \ldots, \tag{4.1}$$

furthermore if $n < N$ then we also have

$$T_n^{(r)}(x) \ll \Big(x \sum_{k=n+1}^{N} k^{r+1} |b_k| + x^{-1} N^{-1} \omega\Big(\frac{1}{N}\Big)\Big), \quad r = 2\ell, \; \ell = 0, 1, \ldots, \tag{4.2}$$

and

$$T_n^{(r)}(x) \ll \Big(\sum_{k=n+1}^{N} k^{r} |b_k| + \omega\Big(\frac{1}{N}\Big)\Big), \quad r = 2\ell + 1, \; \ell = 0, 1, \ldots. \tag{4.3}$$

Lemma 3. *If $\{b_n\} \in \Omega_r RBVS$ and r is odd, then for any $x \ge 0$*

$$T_n^{(r)}(x) \ll \sum_{k=n+1}^{\infty} \frac{1}{k} \omega\Big(\frac{1}{k}\Big). \tag{4.4}$$

Proof of Lemma 3. Inequality (4.4) clearly follows from (4.3) and the definition of $T_n^{(r)}(0)$.

■

Lemma 4. ([1]). *Let $p > 0$ and*

$$f(x) := f_{\Omega_0}(x) := \sum_{n=1}^{\infty} n^{-1} \omega\Big(\frac{1}{n}\Big) \sin nx.$$

If

$$\Big\| \sum_{n=1}^{\infty} \lambda_n |f - s_n|^p \Big\| < \infty$$

then

$$\Lambda_n \omega\Big(\frac{1}{n}\Big)^p \ll 1.$$

The proof of Lemma 4 can be found in [1] as the proof of necessity of Theorem BL.

5. Proofs of the theorems

Proof of Theorem 1. First we prove the *Sufficiency.* We show that if (2.2) holds and the function

$$f(x) := \sum_{n=1}^{\infty} b_n \sin nx \tag{5.1}$$

belongs to $W^r H^*_{S,\Omega_r}$, then it also belongs to $S^*_p(\boldsymbol{\lambda}, r)$.

Since $T_n^{(r)}(x)$ is even and 2π-periodic function, and $T_n^{(r)}(0) = 0$, it is enough to consider the case $0 < x < \pi$. Let

$$\frac{\pi}{N} \le x < \frac{\pi}{N-1}, \quad 2^m \le N < 2^{m+1}, \quad m \ge 1,$$

then

$$\sum_{n=1}^{\infty} \lambda_n |T_n^{(r)}(x)|^p = \Big(\sum_{n=1}^{N-1} + \sum_{n=N}^{\infty}\Big) \lambda_n |T_n^{(r)}(x)|^p =: B_1(x) + B_2(x). \tag{5.2}$$

By (4.1) and (2.2) we get

$$\begin{aligned} B_2(x) &\ll x^{-p} \sum_{n=N}^{\infty} \lambda_n \Big(n^{-1}\omega\Big(\frac{1}{n}\Big)\Big)^p \ll N^p \sum_{k=m}^{\infty} \sum_{n=2^k}^{2^{k+1}} \lambda_n n^{-p} \omega\Big(\frac{1}{n}\Big)^p \\ &\ll N^p \sum_{k=m}^{\infty} 2^{-kp} \omega\Big(\frac{1}{2^k}\Big)^p \sum_{n=2^k}^{2^{k+1}} \lambda_n \ll 1. \end{aligned} \tag{5.3}$$

Moreover, by (4.2),

$$B_1(x) \ll \Big\{ x^p \sum_{n=1}^{N-1} \lambda_n \Big(\sum_{k=n+1}^{N} k^{r+1} |b_k| \Big)^p + \sum_{n=1}^{N-1} \lambda_n \omega\Big(\frac{1}{N}\Big)^p \Big\}. \tag{5.4}$$

Here the second sum is clearly $O(1)$, thus it is enough to estimate the first sum.

In the case $p > 1$, we shall use in due course the assumption $\{b_n\} \in \Omega_r RBVS$, the Hölder inequality, Abel rearrangement and (2.2), thus we get

$$
\begin{aligned}
\sum\nolimits_1 &:= \sum_{n=1}^{N-1} \lambda_n \Big(\sum_{k=n+1}^{N} k^{r+1}|b_k| \Big)^p \ll \sum_{n=1}^{N-1} \lambda_n \Big(\sum_{k=n+1}^{N} \omega\Big(\frac{1}{k}\Big)\Big)^p \\
&\ll \sum_{n=1}^{N-1} \lambda_n N^{p-1} \sum_{k=n+1}^{N} \omega\Big(\frac{1}{k}\Big)^p \ll N^{p-1} \sum_{k=1}^{N} \omega\Big(\frac{1}{k}\Big)^p \sum_{n=1}^{k} \lambda_k \ll N^p.
\end{aligned}
\tag{5.5}
$$

If $0 < p \le 1$ then instead of the Hölder inequality we use the inequality $(\sum a_n)^p \le \sum a_n^p$, and utilize the "blocking method". Let $m_n := [\log_2(n+1)]$, where $[y]$ denotes the integer part of y. Thus

$$
\begin{aligned}
\sum\nolimits_1 &\ll \sum_{n=1}^{N} \lambda_n \Big(\sum_{\ell=m_n}^{m} \sum_{k=2^\ell}^{2^{\ell+1}} \omega\Big(\frac{1}{k}\Big)\Big)^p \ll \sum_{n=1}^{N} \lambda_n \sum_{\ell=m_n}^{m} \Big(\sum_{k=2^\ell}^{2^{\ell+1}} \omega\Big(\frac{1}{k}\Big)\Big)^p \\
&\ll \sum_{n=1}^{N} \lambda_n \sum_{\ell=m_n}^{m} 2^{\ell p} \omega\Big(\frac{1}{2^\ell}\Big)^p \ll \sum_{\ell=1}^{m} 2^{\ell p} \omega\Big(\frac{1}{2^\ell}\Big)^p \sum_{m_n \le \ell} \lambda_n \\
&\ll \sum_{\ell=1}^{m} 2^{\ell p} \omega\Big(\frac{1}{2^\ell}\Big)^p \sum_{n=1}^{2^{\ell+1}} \lambda_n \ll N^p.
\end{aligned}
$$

This, (5.4) and (5.5) imply that

$$B_1(x) \ll 1$$

holds for any positive p.

Collecting these inequalities and (5.2), the assertion $f \in S_p^*(\boldsymbol{\lambda}, r)$ is verified.

In order to prove the *Necessity* we show that if (2.1) holds, then (2.2) also upholds.

Let us consider the following function:

$$
f(x) := f_{\Omega_r}(x) := \sum_{n=1}^{\infty} n^{-r-1} \omega\Big(\frac{1}{n}\Big) \sin nx.
\tag{5.6}
$$

This function clearly belongs to $W^r H_{S,\Omega_r}^*$, indeed $f \in W^r H_{S,\Omega_r}$ ($\subset W^r H_{S,\Omega_r}^*$) also holds, see in [1], consequently by (2.1) $f \in S_p^*(\boldsymbol{\lambda}, r)$ also holds, whence

$$
C := \Big\| \sum_{n=1}^{\infty} \lambda_n |T_n^{(r)}(x)|^p \Big\| \ll 1
\tag{5.7}
$$

follows. Since $T_n^{(r)}(x) \geq |f^{(r)}(x) - s_n^{(r)}(x)|$, thus by (5.7)

$$\Big\| \sum_{n=1}^{\infty} \lambda_n |f^{(r)}(x) - s_n^{(r)}(x)|^p \Big\| \ll 1$$

is also valid. But $f^{(r)}(x) \equiv f_{\Omega_0}(x)$ given in Lemma 4, consequently we can apply Lemma 4, and it shows that (2.2) maintains, as stated.

Herewith the proof is complete.

■

Proof of Theorem 2. *Sufficiency.* We prove that if (2.3) holds and the function given in (5.1) belongs to $W^r H^*_{S,\Omega_r}$, then it also belongs to $S_p^*(\boldsymbol{\lambda}, r)$. Let $f \in W^r H^*_{S,\Omega_r}$, then $\{b_n\} \in \Omega_r RBVS$. Thus, by (4.3) and the definition of $T_n^{(r)}(0)$, we get that for any $x \geq 0$

$$|T_n^{(r)}(x)|^p \ll \Big(\sum_{k=n+1}^{\infty} \frac{1}{k} \omega\Big(\frac{1}{k}\Big)\Big)^p =: \sum\nolimits_2 . \tag{5.8}$$

Since $\omega(1/n)n^{\beta} \downarrow$, we have

$$\sum_{k=n+1}^{\infty} \frac{1}{k^{1+\beta}} \omega\Big(\frac{1}{k}\Big) k^{\beta} \ll \omega\Big(\frac{1}{n}\Big),$$

thus

$$\sum\nolimits_2 \ll \omega\Big(\frac{1}{n}\Big)^p \ll \sum_{k=n}^{\infty} \frac{1}{k} \omega\Big(\frac{1}{k}\Big)^p.$$

This and (5.8) imply that

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_n |T_n^{(r)}(x)|^p &\ll \sum_{n=1}^{\infty} \lambda_n \sum_{k=n}^{\infty} \frac{1}{k} \omega\Big(\frac{1}{k}\Big)^p \\ &\ll \sum_{k=1}^{\infty} \frac{1}{k} \omega\Big(\frac{1}{k}\Big)^p \sum_{n=1}^{k} \lambda_n = \sum_{k=1}^{\infty} \frac{\Lambda_k}{k} \omega\Big(\frac{1}{k}\Big)^p \ll 1, \end{aligned}$$

and this proves that $f \in S_p^*(\boldsymbol{\lambda}, r)$ holds.

Necessity. If (2.1) holds, we show that (2.3) also maintains. Let us consider the function given in (5.6). As we have seen in the proof of Theorem 1, this function $f = f_{\Omega_r} \in S_p^*(\boldsymbol{\lambda}, r)$ and satisfies (5.7). Since

$$T_n^{(r)}(0) = \sum_{k=n+1}^{\infty} k^{-1} \omega\Big(\frac{1}{k}\Big),$$

thus

$$C \gg \sum_{n=1}^{\infty} \lambda_n \Big(\sum_{k=n+1}^{\infty} k^{-1} \omega\Big(\frac{1}{k}\Big)\Big)^p \gg \sum_{n=1}^{\infty} \lambda_n \omega\Big(\frac{1}{n}\Big)^p =: \sum\nolimits_3 . \tag{5.9}$$

Since $\omega\left(\frac{1}{n}\right) n^{\beta} \downarrow$, thus

$$\sum_{k=n}^{\infty} \frac{1}{k^{1+\beta p}} \Big(\omega\Big(\frac{1}{k}\Big) k^{\beta}\Big)^p \le \Big(\omega\Big(\frac{1}{n}\Big) n^{\beta}\Big)^p \sum_{k=n}^{\infty} \frac{1}{k^{1+\beta p}} \ll \omega\Big(\frac{1}{n}\Big)^p .$$

Using this we get

$$\sum\nolimits_3 \gg \sum_{n=1}^{\infty} \lambda_n \sum_{k=n}^{\infty} \frac{1}{k} \omega\Big(\frac{1}{k}\Big)^p = \sum_{k=1}^{\infty} \frac{1}{k} \omega\Big(\frac{1}{k}\Big)^p \sum_{n=1}^{k} \lambda_n = \sum_{k=1}^{\infty} \frac{\Lambda_k}{k} \omega\Big(\frac{1}{k}\Big)^p .$$

This and (5.9) imply that (2.3) holds.

The proof is complete. ∎

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