Acta Sci. Math. (*Szeged*)
76 (2010), 411–425

Semivariations of a vector measure

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Dedicated to Professor Heinz König on his 80th birthday

Communicated by L. Kérchy

Abstract. With a measure φ on a σ-algebra Σ of sets taking values in a Banach space two positive functions on Σ, called semivariations of φ, are associated. We characterize those functions as order continuous submeasures that are multiply subadditive in the sense of G. G. Lorentz (1952). In connection with some results of G. Curbera (1994) and the author (2003), we also discuss the special cases where φ is separable and nonatomic or has relatively compact range.

1. Introduction

Let Σ be a σ-algebra of sets, let X be a Banach space, and let $\varphi\colon \Sigma \to X$ be a (σ-additive vector) measure. Associated with φ are two positive functions $\tilde{\varphi}$ and $\bar{\varphi}$ on Σ, both called semivariations of φ in the literature (see the beginning of Section 3). It is well known that they are order continuous submeasures. Moreover, as noted in [8], they are multiply subadditive in the sense of Lorentz [9].

Theorem 1 of this paper[1)] shows that these properties characterize $\tilde{\varphi}$ and $\bar{\varphi}$ as set functions, which is a σ-additive counterpart of [8], Theorem 4. We also characterize $\tilde{\varphi}$ and $\bar{\varphi}$ in the case where φ has relatively compact range (Theorem 2).

Received March 23, 2009, and in revised form December 10, 2009.
AMS Subject Classification (2000): 28B05, 46G10, 28A12.
Key words and phrases: Banach space, vector measure, semivariation, relatively compact range, separable, nonatomic, submeasure, order continuous, multiply subadditive.

[1)] Some results of the paper were presented to the Third Meeting on Vector Measures, Integration and Applications (Eichstätt, Germany, September 2008) and to the 37th Winter School in Abstract Analysis (Kácov, Czech Republic, January 2009).

Other main results of the paper deal with $\tilde{\varphi}$ alone. They are inspired by a theorem of Curbera [1] (see also Theorem 5 in the final Section 7) on the space of φ-integrable scalar functions for φ separable and nonatomic. A corollary to that result is that for such φ there exists a c_0-valued measure ψ on Σ with $\tilde{\psi} = \tilde{\varphi}$. We give a new proof of this corollary and establish its counterpart for measures with relatively compact range (Theorem 3). We next show that c_0 does not have the analogous universality property for atomic measures (Theorem 4(b)). Finally, a development of the argument used to establish Theorem 3 allows us to give a proof for Curbera's theorem and its counterpart for measures with relatively compact range.

Our notation and terminology are mostly standard and follow [2] and [3]. For some explantions see Sections 2 and 3, the passage introducing Lemma 4 in Section 6, and the beginning of Section 7. Sections 2 and 3 also contain examples and auxiliary results. The main body of the paper consists of Sections 4–6. Sections 5–7 are independent of Section 4, as far as the proofs are concerned. Moreover, Sections 6 and 7 are independent in this sense, too.

2. Preliminaries on submeasures

Throughout the paper S stands for a nonempty set and Σ for a σ-algebra of subsets of S. Let $\eta\colon \Sigma \to [0,\infty)$ be a *submeasure*, i.e., η is increasing, subadditive and $\eta(\varnothing) = 0$. We call η *nonatomic* or *atomless* if for every $E \in \Sigma$ with $\eta(E) > 0$ there exist disjoint E_1, $E_2 \in \Sigma$ such that $E_1 \cup E_2 = E$ and $\eta(E_i) > 0$ for $i = 1, 2$.

We denote by $\operatorname{dens}\eta$ the density character of Σ equipped with the topology generated by the Fréchet–Nikodym semimetric

$$d_\eta(E,F) = \eta(E \triangle F) \qquad \text{for } E,\ F \in \Sigma.$$

We say that η is *separable* if $\operatorname{dens}\eta$ is countable.

We call η *order continuous* provided that $\eta(E_n) \to 0$ whenever (E_n) is a decreasing sequence in Σ with empty intersection.

Following [9], p. 455, we call η *multiply subadditive* (*m.s.* for short) if, given E, $E_1, \ldots, E_n$ in Σ and $k \in \mathbb{N}$ with

$$k 1_E = \sum_{i=1}^{n} 1_{E_i},$$

we have $k\eta(E) \le \sum_{i=1}^{n} \eta(E_i)$ (see also [8], Section 2).

We say that $\eta\colon\Sigma \to [0,\infty)$ is a (positive) *quasi-measure* if η is additive. Clearly, η is then a m.s. submeasure.

According to a result of Lorentz ([9], Theorem 4), a submeasure η on Σ is m.s. if and only if there exists a set Γ of quasi-measures on Σ such that[2)]

$$\sup\Gamma = \eta.$$

Following [8], Section 3, we call the smallest among the cardinalities of sets Γ as above the *degree* of η and denote it by $\deg\eta$.

Example 1 of [8] shows that the degree of a m.s. submeasure can be an arbitrary cardinal number ≥ 1. We shall now see that this is still so if we restrict attention to order continuous submeasures on σ-algebras of sets. The example just mentioned allows us to consider infinite cardinal numbers only.

Example 1. Let $\mathfrak{m}$ be a cardinal number $\geq \aleph_0$, and set $S = \{0,1\}^{\mathfrak{m}}$. Take for Σ the standard product σ-algebra of subsets of S and for λ the standard product measure on Σ. Fix $1 < p < \infty$, and set $\eta = \lambda^{1/p}$. By Minkowski's inequality η is a m.s. submeasure on Σ (cf. [9], Theorem 5; see also Example 2 and Proposition 1(b) in Section 3). The density character of (Σ, d_η) equals $\mathfrak{m}$, and so $\deg\eta \leq \mathfrak{m}$ (see [8], Remark 1). To establish the other inequality, consider a set Γ of quasi-measures on Σ with $\sup\Gamma = \eta$. Each $\mu \in \Gamma$ is then a λ-continuous positive measure. Denote by f_μ the Radon–Nikodym derivative of μ with respect to λ. Suppose, to get a contradiction, that $\operatorname{card}\Gamma < \mathfrak{m}$. We consider two cases.

Case 1: $\mathfrak{m} = \aleph_0$. Choose $\mu_0 \in \Gamma$ so that the set

$$E_0 = \{s \in S : f_{\mu_0}(s) \geq f_\mu(s) \text{ for all } \mu \in \Gamma\}$$

has positive λ-measure. Then $\mu_0(E) = \eta(E)$ for all $E \in \Sigma$ with $E \subset E_0$. Take $E_1 \in \Sigma$ such that $E_1 \subset E_0$ and $\lambda(E_1) = \frac{1}{2}\lambda(E_0)$. It follows that

$$\eta(E_0) = \eta(E_1) + \eta(E_0 \setminus E_1) = 2^{1-\frac{1}{p}}\lambda(E_0)^{\frac{1}{p}},$$

which is impossible.

Case 2: $\mathfrak{m} > \aleph_0$. In this case there exists $E_0 \in \Sigma$ such that $0 < \lambda(E_0) < 1$ and 1_{E_0} and f_μ are stochastically λ-independent for each $\mu \in \Gamma$. We then have

$$\mu(E_0) = \int 1_{E_0} f_\mu \, d\lambda = \lambda(E_0)\mu(S) \leq \lambda(E_0)$$

2) In what follows the symbols sup and max applied to a set of positive functions on Σ mean the pointwise supremum and maximum of that set, respectively.

(see [4], Theorem 45.A). Hence $\sup\{\mu(E_0) : \mu \in \Gamma\} \leq \lambda(E_0) < \eta(E_0)$, a contradiction.

3. Preliminaries on vector measures

Throughout this section X stands for a (real or complex) Banach space, with the norm denoted by $\|\cdot\|$. Let $\varphi\colon \Sigma \to X$ be a (vector) measure, i.e., φ is σ-additive. As is usual, we associate with φ three functions on Σ with values in $[0, \infty]$ defined by the formulas:

$$|\varphi|(E) = \sup\Big\{\sum_{i=1}^{n} \|\varphi(E_i)\| : E_i \in \Sigma \text{ are pairwise disjoint and } \bigcup_{i=1}^{n} E_i = E\Big\},$$

$$\tilde{\varphi}(E) = \sup\Big\{\Big\|\sum_{i=1}^{n} t_i\varphi(E_i)\Big\| : E_i \in \Sigma \text{ are pairwise disjoint and } \bigcup_{i=1}^{n} E_i = E, \text{ and } t_i \text{ are scalars with } |t_i| \leq 1\Big\},$$

$$\bar{\varphi}(E) = \sup\Big\{\|\varphi(F)\| : F \in \Sigma \text{ and } F \subset E\Big\},$$

for $E \in \Sigma$. The first one is a positive measure and is called the *variation* of φ. Both $\tilde{\varphi}$ and $\bar{\varphi}$ are called *semivariations* of φ. The former is also denoted by $\|\varphi\|$ or $|\varphi|_\infty$ in the literature. In the terminology of [10], p. 349, $\bar{\varphi}$ is the quasivariation of φ.

The following proposition collects some properties of $\tilde{\varphi}$ and $\bar{\varphi}$ needed in the sequel.

Proposition 1. *Let $\varphi\colon \Sigma \to X$ be a measure. Then*
(a) $\tilde{\varphi} = \sup\{|x^*\varphi| : x^* \in X^*$ *and* $\|x^*\| \leq 1\}$;
(b) $\tilde{\varphi}$ *and* $\bar{\varphi}$ *are order continuous m.s. submeasures on* Σ.

Part (a) is a special case of [2], Proposition I.1.11. That $\tilde{\varphi}$ and $\bar{\varphi}$ are submeasures is straightforward. In view of (a) and [9], Theorem 4, $\tilde{\varphi}$ is m.s., while $\bar{\varphi}$ is m.s., by [8], Lemma 2. The order continuity of $\tilde{\varphi}$ follows from [6], Lemma II.1.3. Clearly, $\bar{\varphi} \leq \tilde{\varphi}$, so that $\bar{\varphi}$ is also order continuous.

We note that a version of (a) also appears in [6], Lemma II.1.1, but the definition of the p-semivariation given there on p. 17 is incorrect; see an example due to S. Okada presented in [11], p. 205. (In [6], p stands for a seminorm on a linear space.) However, it is not this definition, but Lemma II.1.1 that is mostly applied in [6].

We shall use the following standard notation. By $ca(\Sigma, X)$ we denote the Banach space of all measures $\varphi\colon \Sigma \to X$, the norm being defined by $\|\varphi\| = \tilde{\varphi}(S)$. If X is a scalar field, we abbreviate $ca(\Sigma, X)$ to $ca(\Sigma)$. Thus $ca_+(\Sigma)$ stands for the set of all positive finite measures on Σ. We set

$$cca(\Sigma, X) = \{\varphi \in ca(\Sigma, X) : \varphi(\Sigma) \text{ is relatively compact}\}^{3)}.$$

It is a closed subspace of $ca(\Sigma, X)$.

We say that $\varphi \in ca(\Sigma, X)$ is *nonatomic* or *atomless* [resp., *separable*] provided that so is the submeasure $\tilde{\varphi}$ (see Section 2).

We shall now give explicit formulas for $\tilde{\varphi}$ and $\bar{\varphi}$ in a well-known special case (cf. [2], Example I.1.16). Those formulas are related to Example 1 above.

Example 2. Let $\lambda \in ca_+(\Sigma)$ and let $1 \le p < \infty$. Define $\varphi\colon \Sigma \to L_p(\lambda)$ by $\varphi(E) = 1_E$. Clearly, φ is σ-additive. We have $\tilde{\varphi} = \bar{\varphi} = \lambda^{1/p}$. Only the inequality $\tilde{\varphi} \le \lambda^{1/p}$ needs checking. To this end, fix $E \in \Sigma$, and consider pairwise disjoint $E_1, \ldots, E_n \in \Sigma$ with $\bigcup_{i=1}^n E_i = E$ and scalars $t_1, \ldots, t_n$ with $|t_i| \le 1$. We then have

$$\Big\| \sum_{i=1}^n t_i \varphi(E_i) \Big\|_p^p = \sum_{i=1}^n |t_i|^p \lambda(E_i) \le \lambda(E).$$

Consequently, $\tilde{\varphi}(E) \le \lambda(E)^{1/p}$. Note that if λ is nonatomic and $1 < p < \infty$, then $|\varphi|$ is independent of p. In fact, $|\varphi| = \infty \cdot \lambda$ (cf. [2], Example I.1.16).

The following simple lemma will be used in the proofs of Theorems 1–3 and 5, Proposition 3 and Lemma 4.

Lemma 1. *Let $\alpha\colon \Gamma \to ca(\Sigma)$, where Γ is a nonempty set, be a mapping whose range is bounded and uniformly σ-additive. Define $\psi_\alpha\colon \Sigma \to l_\infty(\Gamma)$ by*

$$\psi_\alpha(E)(\mu) = \alpha(\mu)(E) \qquad \text{for all } E \in \Sigma \text{ and } \mu \in \Gamma.$$

We then have $\psi_\alpha \in ca(\Sigma, l_\infty(\Gamma))$,

$$\tilde{\psi}_\alpha = \sup\{|\alpha(\mu)| : \mu \in \Gamma\} \qquad \text{and} \qquad \bar{\psi}_\alpha = \sup\{\alpha(\mu)^- : \mu \in \Gamma\}.$$

If, moreover, α has relatively compact range, then so does ψ_α.

3) Here and in what follows the term *compact* always refers to the strong topology of the Banach space under consideration.

Proof. The formulas of the first part are seen, since we can interchange the order in which the corresponding suprema are taken and, for $\nu \in ca(\Sigma)$, we have $\tilde{\nu} = |\nu|$.

To establish the second part, it is enough to observe that, given α with relatively compact range and $\varepsilon > 0$, there exists $\alpha_0 \colon \Gamma \to ca(\Sigma)$ with finite-dimensional range such that

$$\|\psi_\alpha(E) - \psi_{\alpha_0}(E)\| < \varepsilon \qquad \text{for all } E \in \Sigma.$$

Fix $\varepsilon > 0$ and choose $\nu_1, \ldots, \nu_n \in \Gamma$ so that for each $\mu \in \Gamma$ there is an $i(\mu)$ with $1 \leq i(\mu) \leq n$ and

$$\|\alpha(\mu) - \alpha(\nu_{i(\mu)})\| < \varepsilon.$$

Set $\alpha_0(\mu) = \alpha(\nu_{i(\mu)})$ for $\mu \in \Gamma$. It is clear that α_0 is as desired.

■

The next lemma is a basic tool in the proofs of Theorem 3 in Section 6 and Theorem 5 in Section 7.

Lemma 2. *Let $\lambda \in ca_+(\Sigma)$ be separable and nonatomic and let $\lambda_1, \lambda_2, \ldots \in ca_+(\Sigma)$ be uniformly λ-continuous. Then there exist $\mu_1, \mu_2, \ldots \in ca(\Sigma)$ such that*

(α) $|\mu_n| = \lambda_n$ *for all* $n \in \mathbb{N}$;

(β) $\mu_n(E) \to 0$ *for all* $E \in \Sigma$.

Proof. Since λ_n are separable and nonatomic, we can find $\mu_{nm} \in ca(\Sigma)$, where n, $m \in \mathbb{N}$, with $|\mu_{nm}| = \lambda_n$ for all n, m and $\mu_{nm}(E) \to 0$ as $m \to \infty$ for all $E \in \Sigma$. Indeed, we can either mimic the standard Rademacher construction or make use of the Boolean isomorphism of λ_n to the Lebesgue measure on $[0, \lambda_n(S)]$ (see [4], Theorem 41.C).

Let $\{E_i : i \in \mathbb{N}\}$ be a dense set in (Σ, d_λ). Choose m_n so that $|\mu_{nm_n}(E_i)| < 1/n$ for $n \in \mathbb{N}$ and $i = 1, \ldots, n$. Setting $\mu_n = \mu_{nm_n}$, we see that (α) holds and $\mu_n(E_i) \to 0$ as $n \to \infty$, where $i \in \mathbb{N}$ is arbitrary. This implies (β), by the uniform λ-continuity assumption.

■

4. General vector measures

The following theorem is, except for condition (ii), a σ-additive version of [8], Theorem 4, and has a similar proof.

Theorem 1. *For $\eta\colon \Sigma \to [0,\infty)$ the following five conditions are equivalent*:

(i) *η is an order continuous m.s. submeasure*;

(ii) *there exists a uniformly σ-additive $\Gamma \subset ca_+(\Sigma)$ such that* $\sup \Gamma = \eta$;

(iii) *there exist a Banach space X and $\varphi \in ca(\Sigma, X)$ such that $\tilde{\varphi} = \eta$*;

(iv) *there exist a Banach space X and $\varphi \in ca(\Sigma, X)$ such that $\bar{\varphi} = \eta$*;

(v) *there exist a Banach space X and $\varphi \in ca(\Sigma, X)$ such that $\tilde{\varphi} = \bar{\varphi} = \eta$.*

Proof. The equivalence of (i) and (ii) is a simple consequence of a result of Lorentz ([9], Theorem 4); see also [8], Theorem 1. Clearly, (v) implies (iii) and (iv). By Proposition 1(b), each of the conditions (iii) and (iv) implies (i). Finally, let (ii) hold, and apply Lemma 1 with α equal to the identity mapping. Setting $\varphi = \psi_\alpha$, we get (v).

∎

Remark 1. (cf. [8], Remark 5). In Theorem 1 we cannot restrict the size of X, keeping Σ arbitrary. Indeed, for every $\varphi \in ca(\Sigma, X)$ and every 1-norming subset M of X^* we have

$$\deg \tilde{\varphi} \le \operatorname{card} M \qquad \text{and} \qquad \deg \bar{\varphi} \le 2 \operatorname{card} M,$$

by [8], Propositions 2(c) and 3(a), respectively. On the other hand, $\deg \eta$, where η is an order continuous m.s. submeasure on a σ-algebra of sets, can be an arbitrary cardinal number ≥ 1 (see Example 1).

From Theorem 1 we immediately get the following corollary.

Corollary 1. *Let X be a Banach space and let $\varphi \in ca(\Sigma, X)$.*

(a) *There exist a Banach space Y and $\chi \in ca(\Sigma, Y)$ such that $\tilde{\chi} = \bar{\chi} = \tilde{\varphi}$.*

(b) *There exist a Banach space Z and $\chi \in ca(\Sigma, Z)$ such that $\tilde{\chi} = \bar{\chi} = \bar{\varphi}$.*

Theorem 1 suggests various questions of the following type. Specialize one of the conditions (i)–(v) and ask for the corresponding specializations of some of the remaining conditions. For instance, specialize the class of Banach spaces (e.g., consider only Hilbert spaces) or the class of vector measures (e.g., consider only those with relatively compact range) in conditions (iii)–(v). Some answers are contained in the next two sections.

5. Vector measures with relatively compact range

The following result is known; see [12], p. 104, for the implication (i)$\Rightarrow$(ii). We sketch a proof for the reader's convenience.

Proposition 2. *Let X be a Banach space and let $\varphi \in ca(\Sigma, X)$. Then the following two conditions are equivalent*:

(i) $\varphi \in cca(\Sigma, X)$;

(ii) $\{x^*\varphi : x^* \in X^*$ *and* $\|x^*\| \le 1\}$ *is relatively compact in* $ca(\Sigma)$.

Proof. Denote by $B(\Sigma)$ the Banach space of bounded Σ-measurable scalar functions on S equipped with the supremum norm. There exists a (unique) continuous linear operator $T_\varphi \colon B(\Sigma) \to X$ such that

$$T_\varphi(1_E) = \varphi(E) \qquad \text{for all } E \in \Sigma,$$

and we have $\|T_\varphi\| = \|\varphi\|$ (see [2], pp. 5–6). Given $x^* \in X^*$, we can then identify x^*T_φ with $x^*\varphi \in ca(\Sigma)$. On the other hand, condition (i) holds if and only if T_φ is compact; cf. [2], proof of Theorem VI.2.18. Since T_φ is compact if and only if so is $(T_\varphi)^*$, by Schauder's theorem (Theorem VI.5.2 of [3], where the term *completely continuous* is used), the assertion follows. ∎

Theorem 2. *For $\eta \colon \Sigma \to [0, \infty)$ the following four conditions are equivalent*:

(i) *there exists a relatively compact $\Gamma \subset ca_+(\Sigma)$ such that* $\sup\Gamma = \eta$;

(ii) *there exist a Banach space X and $\varphi \in cca(\Sigma, X)$ such that $\tilde{\varphi} = \eta$*;

(iii) *there exist a Banach space X and $\varphi \in cca(\Sigma, X)$ such that $\bar{\varphi} = \eta$*;

(iv) *there exist a Banach space X and $\varphi \in cca(\Sigma, X)$ such that $\tilde{\varphi} = \bar{\varphi} = \eta$.*

Proof. Lemma 1, applied to the identity mapping over Γ, shows that (i) implies (iv). Clearly, (iv) implies both (ii) and (iii).

In view of Propositions 1(a) and 2, we deduce (i) from (ii) by taking

$$\Gamma = \{|x^*\varphi| : x^* \in X^* \text{ and } \|x^*\| \le 1\}.$$

A similar argument shows that (iii) implies (i) for X over $\mathbb{R}$. We only have to apply the formula

$$\bar{\varphi} = \sup\{(x^*\varphi)_+, (x^*\varphi)_- : x^* \in X^* \text{ and } \|x^*\| \le 1\}$$

(cf. [8], proof of Lemma 2). If the scalar field of X is $\mathbb{C}$, we consider X to be a Banach space over $\mathbb{R}$ (with the same norm) and note that this does not affect $\bar{\varphi}$. ∎

The following corollary of Theorem 2 is analogous to Corollary 1.

Corollary 2. *Let X be a Banach space and let $\varphi \in cca(\Sigma, X)$.*

(a) *There exist a Banach space Y and $\chi \in cca(\Sigma, Y)$ such that $\tilde{\chi} = \bar{\chi} = \tilde{\varphi}$.*

(b) *There exist a Banach space Z and $\psi \in cca(\Sigma, Z)$ such that $\tilde{\psi} = \bar{\psi} = \bar{\varphi}$.*

Remark 2. In general, it is not possible to decide whether $\varphi \in ca(\Sigma, X)$ has relatively compact range knowing only its semivariations $\tilde{\varphi}$ and $\bar{\varphi}$. Indeed, if Σ admits a nonatomic probability measure λ, then there exists $\varphi \in ca(\Sigma, L_1(\lambda))$ such that $\tilde{\varphi} = \bar{\varphi} = \lambda$ and $\varphi(\Sigma)$ is not relatively compact (see [2], Example III.1.2, and Example 2 in Section 3).

6. Vector measures and c_0

We start with the following known result; see [10], Proposition 1(ii), for a generalization. We give a simple proof for the reader's convenience.

Lemma 3. *Let $\psi \in cca(\Sigma, c_0)$, and denote by μ_n the n-th co-ordinate of ψ. We then have $\|\mu_n\| \to 0$.*

Proof. Clearly, $\mu_n(E) \to 0$ for $E \in \Sigma$. Moreover, by Proposition 2, (i)⇒(ii), $\{\mu_n : n \in \mathbb{N}\}$ is relatively compact. It follows that $\|\mu_n\| \to 0$.

■

We shall now characterize one of the semivariations of a c_0-valued measure in general and in the case where the range is relatively compact.

Proposition 3. *For $\eta\colon \Sigma \to [0, \infty)$ the following two conditions are equivalent:*

(i) *there exist $\mu_1, \mu_2, \ldots \in ca(\Sigma)$ such that $\mu_n(E) \to 0$ for $E \in \Sigma$ [resp., $\|\mu_n\| \to 0$] and $\sup\{|\mu_n| : n \in \mathbb{N}\} = \eta$;*

(ii) *there exists $\psi \in ca(\Sigma, c_0)$ [resp., $\psi \in cca(\Sigma, c_0)$] such that $\tilde{\psi} = \eta$.*

Proof. Suppose (ii) holds, and denote by μ_n the n-th co-ordinate of ψ. Set $\alpha(n) = \mu_n$ for $n \in \mathbb{N}$. Applying Lemma 1 and noting that $\psi_\alpha = \psi$, we get the first part of (i). The second part now follows, by Lemma 3.

Suppose (i) holds. By Nikodym's convergence theorem ([3], Corollary III.7.4), $\mu_1, \mu_2, \ldots$ are uniformly σ-additive. Applying Lemma 1 as before, we get (ii) with $\psi = \psi_\alpha$.

■

The first part of the forthcoming Theorem 3 is a consequence of a result of Curbera ([1], Theorem 1). We shall give a proof of that result based on Lemma 2 in Section 7. We also note that a version of Theorem 3, without any assumption on $\varphi \in ca(\Sigma, X)$ whatsoever, but with the weaker assertion that $|\psi| = |\varphi|$, is a special case of [7], Corollary 2. More comments on Theorem 3 are given after the proof of Theorem 4 below.

Theorem 3. *If X is a Banach space and $\varphi \in ca(\Sigma, X)$ [resp., $\varphi \in cca(\Sigma, X)$] is separable and nonatomic, then there exists $\psi \in ca(\Sigma, c_0)$ [resp., $\psi \in cca(\Sigma, c_0)$] such that $\tilde{\psi} = \tilde{\varphi}$.*

Proof. The separability of φ implies that $\varphi(\Sigma)$ is separable, and so X itself may be assumed separable. Under this assumption, X^* contains a 1-norming subset $\{x_n^* : n \in \mathbb{N}\}$. Set $\lambda_n = |x_n^* \varphi|$ for $n \in \mathbb{N}$. By [8], Proposition 2(c), we get

$$\sup\{\lambda_n : n \in \mathbb{N}\} = \tilde{\varphi}.$$

In view of Proposition 1(b), $\lambda_1, \lambda_2, \ldots$ are uniformly σ-additive. Moreover, $\{\lambda_n : n \in \mathbb{N}\}$ is relatively compact if so is $\varphi(\Sigma)$, by Proposition 2, (i)$\Rightarrow$(ii). Set $\lambda = \sum_{n=1}^{\infty} 2^{-n}\lambda_n$. We have $\lambda \in ca_+(\Sigma)$, $\lambda \leq \tilde{\varphi}$, and $\lambda(E) = 0$ implies $\tilde{\varphi}(E) = 0$ whenever $E \in \Sigma$. It follows that λ is separable and nonatomic. By [2], Theorem I.2.4, $\lambda_1, \lambda_2, \ldots$ are uniformly λ-continuous. Let $\mu_1, \mu_2, \ldots$ be given by Lemma 2. We then have

$$\sup\{|\mu_n| : n \in \mathbb{N}\} = \tilde{\varphi}.$$

An application of Proposition 3, (i)$\Rightarrow$(ii), completes the proof.

■

For the purpose of the next lemma and theorem, we recall that, given $\varphi \in ca(\Sigma, X)$ and $\lambda \in ca_+(\Sigma)$, we say that φ is λ-*continuous* and write $\varphi \ll \lambda$ if, for every $E \in \Sigma$, we have $\varphi(E) = 0$ whenever $\lambda(E) = 0$. This is equivalent to the usual ε–δ condition concerning $\tilde{\varphi}$ and λ (see, e.g., [6], Corollary II.1.2).

Lemma 4. *Let $\psi \in cca(\Sigma, c_0)$ and let $\lambda \in ca_+(\Sigma)$ have infinite range. If $\psi \ll \lambda$, then there exist E_1, $E_2 \in \Sigma$ such that*

$$\tilde{\psi}(E_1) + \tilde{\psi}(E_2) = \tilde{\psi}(E_1 \cup E_2) + \tilde{\psi}(E_1 \cap E_2)$$

and $\lambda(E_1 \setminus E_2)$, $\lambda(E_2 \setminus E_1) > 0$.

Proof. Let μ_n be as in Lemma 3. Set

$$\Sigma_n = \{E \in \Sigma : \tilde{\psi}(E) = |\mu_n|(E)\}.$$

It follows from Lemmas 1 and 3 that $\bigcup_{n=1}^{\infty} \Sigma_n = \Sigma$. Moreover, Σ_n is closed in (Σ, d_λ) for each $n \in \mathbb{N}$, since $\psi \ll \lambda$ and $|\mu_n| \ll \lambda$. With the usual identification of sets of λ-measure zero, we apply the Baire category theorem to conclude that Σ_{n_0} has nonempty interior for some n_0.

Fix an interior point E_0 of Σ_{n_0}. Let $F_1, F_2, \ldots$ be pairwise disjoint sets in Σ with $\lambda(F_i) > 0$ for each i. Then either

$$\lambda(E_0 \cap F_i) > 0 \quad \text{or} \quad \lambda((S \setminus E_0) \cap F_i) > 0$$

holds for infinitely many i. In the first case, choose $i_1 < i_2$ so that

$$E_0 \setminus F_{i_1},\ E_0 \setminus F_{i_2} \text{ and } E_0 \setminus (F_{i_1} \cup F_{i_2})$$

are in Σ_{n_0}, and set $E_k = E_0 \setminus F_{i_k}$, $k = 1, 2$. In the second case, choose $i_1 < i_2$ so that

$$E_0 \cup F_{i_1},\ E_0 \cup F_{i_2} \text{ and } E_0 \cup (F_{i_1} \cup F_{i_2})$$

are in Σ_{n_0}, and set $E_k = E_0 \cup F_{i_k}$, $k = 1, 2$. Clearly, in either case E_1 and E_2 have all the desired properties.

■

Theorem 4. *Let $1 < p < \infty$.*

(a) *If $\lambda \in ca_+(\Sigma)$ is nonatomic and nonzero, then there exists $\varphi \in ca(\Sigma, L_p(\lambda))$ such that $\varphi \ll \lambda$ and $\tilde{\varphi} \neq \tilde{\psi}$ whenever $\psi \in cca(\Sigma, c_0)$.*

(b) *There exists $\varphi \in ca(2^{\mathbb{N}}, l_p)$ such that $\tilde{\varphi} \neq \tilde{\psi}$ whenever $\psi \in ca(2^{\mathbb{N}}, c_0)$.*

Proof. We first observe that, in the situation of Lemma 4, we have

$$(*) \qquad \tilde{\psi}(E_1)^p + \tilde{\psi}(E_2)^p < \tilde{\psi}(E_1 \cup E_2)^p + \tilde{\psi}(E_1 \cap E_2)^p$$

provided that $\tilde{\psi}(E_1 \cup E_2) > \tilde{\psi}(E_1) > \tilde{\psi}(E_1 \cap E_2)$. Indeed, set

$$h = \tilde{\psi}(E_1 \cup E_2) - \tilde{\psi}(E_1) = \tilde{\psi}(E_2) - \tilde{\psi}(E_1 \cap E_2).$$

Clearly, $\tilde{\psi}(E_1) + h = \tilde{\psi}(E_1 \cup E_2)$ and $\tilde{\psi}(E_1 \cap E_2) + h = \tilde{\psi}(E_2)$, and $h > 0$. Since the function

$$[0, \infty) \ni x \mapsto (x + h)^p - x^p \in \mathbb{R}$$

is strictly increasing, $(*)$ follows.

(a): Set $\varphi(E) = 1_E$ for all $E \in \Sigma$; see Example 2, where it is shown that $\tilde{\varphi}^p = \lambda$. Hence $\tilde{\varphi}^p$ is additive. Moreover, $\tilde{\varphi}(E \cup F) > \tilde{\varphi}(E)$ whenever $E, F \in \Sigma$ and $\lambda(F \setminus E) > 0$. An application of Lemma 4 and $(*)$ shows that φ is as desired.

(b): Denote by (e_n) the standard basis of l_p, and set

$$\varphi(E) = \sum_{n=1}^{\infty} \frac{1}{n} 1_E(n) e_n \qquad \text{for all } E \in 2^{\mathbb{N}}.$$

Clearly, $\varphi \in ca_+(2^{\mathbb{N}}, l_p)$ and we have

$$\tilde{\varphi}(E) = \|\varphi(E)\| \qquad \text{for all } E \in 2^{\mathbb{N}}.$$

Consequently, $\tilde{\varphi}^p \in ca_+(2^{\mathbb{N}})$ and $\tilde{\varphi}$ is strictly increasing. Since the range of a vector measure on $2^{\mathbb{N}}$ is compact, according to a well-known result (see, e.g., [5], Theorem 10), we can apply Lemma 4, with $\lambda = \tilde{\varphi}^p$, and $(*)$ to complete the proof. ■

Theorem 4(a), with λ separable, shows that a simultaneous strengthening of both parts of Theorem 3 is not possible. It follows from Theorem 4(b) that the nonatomicity assumption of Theorem 3 cannot be dispensed with.

The author does not know whether Theorem 4 subsists for $p = 1$.

Combining Theorem 4(a) and Theorem 3, we get the following corollary.

Corollary 3. *If $\lambda \in ca_+(\Sigma)$ is separable, nonatomic and nonzero, then there exists $\varphi \in ca(\Sigma, c_0)$ such that $\varphi \ll \lambda$ and $\tilde{\varphi} \neq \tilde{\psi}$ whenever $\psi \in cca(\Sigma, c_0)$.*

7. Appendix

The purpose of this section is to give a new proof of a result of Curbera ([1], Theorem 1) and to establish a parallel result for vector measures with relatively compact range. Our proof seems more transparent than the original one. It is similar to that of Theorem 3, with a key role played by Lemma 2, but it is more involved.

As in the previous sections, Σ stands for a σ-algebra of subsets of a nonempty set S. We denote by $s(\Sigma)$ the linear space of Σ-measurable simple functions on S.

We start by recalling one of the two well-known equivalent definitions of the integral of a Σ-measurable scalar function f on S with respect to $\varphi \in ca(\Sigma, X)$, where X is a Banach space (see also [6], Section II.2). We say that f is *φ-integrable* provided it is $x^*\varphi$-integrable for each $x^* \in X^*$ and there exists a mapping $\Sigma \ni E \mapsto \int_E f\,d\varphi \in X$ such that

$$x^* \int_E f\,d\varphi = \int_E f\,d(x^*\varphi) \qquad \text{for all } E \in \Sigma \text{ and } x^* \in X^*.$$

This mapping is then an X-valued measure on Σ, by the Orlicz–Pettis theorem ([2], Corollary I.4.4). We denote it by $f\varphi$. In view of Proposition 1(a), we have

$$(f\varphi)^{\sim}(E) = \sup\Big\{ \int_E |f|\,d|x^*\varphi| : x^* \in X^* \text{ and } \|x^*\| \le 1 \Big\}$$

for $E \in \Sigma$. We denote by $L(\varphi)$ the linear space of φ-integrable functions on S equipped with the seminorm $\|\cdot\|_\varphi$, defined by

$$\|f\|_\varphi = (f\varphi)^{\sim}(S) \qquad \text{for } f \in L(\varphi).$$

In particular, we have $\|1_E\|_\varphi = \tilde{\varphi}(E)$ for $E \in \Sigma$.

Lemma 5. *Let X be a Banach space and let $\varphi \in ca(\Sigma, X)$ be separable. Then there exist $x_1^*, x_2^*, \ldots$ in the unit ball of X^* such that*

$$\|h\|_\varphi = \sup\{\|h\|_{x_n^*\varphi} : n \in \mathbb{N}\} \qquad \textit{for all } h \in s(\Sigma).$$

Proof. As is easily seen, $s(\Sigma)$ is separable with respect to the seminorm $\|\cdot\|_\varphi$. Fix a dense subset $\{h_m : m \in \mathbb{N}\}$ of $s(\Sigma)$. Choose $x_n^* \in X^*$ with $\|x_n^*\| \le 1$ so that

$$\|h_m\|_\varphi = \sup\{\|h_m\|_{x_n^*\varphi} : n \in \mathbb{N}\} \qquad \text{for } m \in \mathbb{N}.$$

Set

$$p(h) = \sup\{\|h\|_{x_n^*\varphi} : n \in \mathbb{N}\} \qquad \text{for } h \in s(\Sigma).$$

Clearly, p is a seminorm on $s(\Sigma)$ and $p(h) \le \|h\|_\varphi$ for all $h \in s(\Sigma)$. Hence p is continuous on $s(\Sigma)$. Since $p(h_m) = \|h_m\|_\varphi$ for $m \in \mathbb{N}$, the assertion follows. ∎

Lemma 6. *Let $\varphi \in ca(\Sigma, X)$ and $\psi \in ca(\Sigma, Y)$, where X and Y are Banach spaces. If $\|h\|_\varphi = \|h\|_\psi$ for all $h \in s(\Sigma)$, then*

$$L(\varphi) = L(\psi) \quad \textit{and} \quad \|f\|_\varphi = \|f\|_\psi \qquad \textit{for all } f \in L(\varphi).$$

Proof. Let $f \in L(\varphi)$, and choose $h_n \in s(\Sigma)$ with $h_n \to f$ pointwise and $|h_n| \le |f|$ for all $n \in \mathbb{N}$. By [6], Theorem II.4.2, $\|h_n - f\|_\varphi \to 0$ holds. Since (h_n) is a Cauchy sequence with respect to $\|\cdot\|_\psi$, there exists $f' \in L(\psi)$ with $\|h_n - f'\|_\psi \to 0$ (see [6], Theorems IV.4.1 and IV.7.1). Clearly, $\|f\|_\varphi = \|f'\|_\psi$. We claim that $f = f'$ a.e. with respect to $\tilde{\psi}$. Indeed, let $y_0^* \in Y^*$ be a Rybakov functional for ψ, i.e., for $E \in \Sigma$ we have $\tilde{\psi}(E) = 0$ if (and only if) $|y_0^*\psi|(E) = 0$ (see [2], Theorem IX.2.2). Since $\|h_n - f'\|_{y_0^*\psi} \to 0$, we may assume that $h_n \to f'$ a.e. with respect to $\tilde{\psi}$. This establishes our claim and shows that $f \in L(\psi)$ and $\|f\|_\varphi = \|f\|_\psi$. Thus, $L(\varphi) \subset L(\psi)$. The other inclusion follows by symmetry. ∎

Theorem 5. *If X is a Banach space and $\varphi \in ca(\Sigma, X)$ [resp., $\varphi \in cca(\Sigma, X)$] is separable and nonatomic, then there exists $\psi \in ca(\Sigma, c_0)$ [resp., $\psi \in cca(\Sigma, c_0)$] such that*

$$L(\psi) = L(\varphi) \quad \text{and} \quad \|f\|_\psi = \|f\|_\varphi \qquad \text{for all } f \in L(\psi).$$

Proof. Let $x_1^*, x_2^*, \dots$ be chosen according to Lemma 5. Set $\lambda_n = |x_n^*\varphi|$ for $n \in \mathbb{N}$. In view of Proposition 1(b), $\lambda_1, \lambda_2, \dots$ are uniformly σ-additive. Moreover, $\{\lambda_n : n \in \mathbb{N}\}$ is relatively compact if so is $\varphi(\Sigma)$, by Proposition 2, (i)⇒(ii). Set $\lambda = \sum_{n=1}^{\infty} 2^{-n}\lambda_n$. We have $\lambda \in ca_+(\Sigma)$, $\lambda \le \tilde{\varphi}$ and $\lambda(E) = 0$ implies $\tilde{\varphi}(E) = 0$ whenever $E \in \Sigma$. It follows that λ is separable and nonatomic. By [2], Theorem I.2.4, $\lambda_1, \lambda_2, \dots$ are uniformly λ-continuous. Let $\mu_1, \mu_2, \dots$ be given by Lemma 2, and set $\psi = (\mu_n)$. By the Nikodym convergence theorem (see [3], Corollary III.7.4), we have $\psi \in ca(\Sigma, c_0)$. In view of Lemma 1, $\psi(\Sigma)$ is relatively compact if so is $\varphi(\Sigma)$. Moreover, the formula

$$(f\psi)(E) = \Big(\int_E f \, d\mu_n \Big)$$

holds for all $E \in \Sigma$ and $f \in L(\psi)$. In view of Lemma 1, this implies

$$\|f\|_\psi = \sup\{\|f\|_{x_n^*\varphi} : n \in \mathbb{N}\} \qquad \text{for } f \in L(\psi).$$

Hence $\|h\|_\psi = \|h\|_\varphi$ for $h \in s(\Sigma)$. An application of Lemma 6 completes the proof. ∎

It is not clear to the author whether, in the situation of Theorem 3, the assertion of Theorem 5 always holds, in which case Lemma 5 would be superfluous. A simple example shows, however, that, in general, for $\varphi, \psi \in ca(\Sigma, X)$ with $\tilde{\varphi} = \tilde{\psi}$ we do not have $\|h\|_\varphi = \|h\|_\psi$ for arbitrary $h \in s(\Sigma)$.

Example 3. Let Σ be the σ-algebra of all subsets of the set $\{1,2\}$. Consider φ, $\psi \in ca(\Sigma, l_\infty^{(2)})$ defined by

$$\varphi = (\delta_1, 2\delta_2) \quad \text{and} \quad \psi = (\delta_1 + \delta_2, 2\delta_2),$$

where δ_i stand for the Dirac measure on Σ concentrated at i. By Lemma 1, we have

$$\tilde{\varphi} = \max(\delta_1, 2\delta_2) = \max(\delta_1 + \delta_2, 2\delta_2) = \tilde{\psi}.$$

On the other hand, setting $h(1) = 2$ and $h(2) = 1$, we have

$$\int h\, d\delta_1 = \int h\, d(2\delta_2) = 2 \qquad \text{and} \qquad \int h\, d(\delta_1 + \delta_2) = 3.$$

Hence $\|h\|_\varphi < \|h\|_\psi$.

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