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Branch and Bound Algorithm for the Warehouse Location Problem with the Objective Function as Linear Fractional

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Abstract

In this paper, a branch and bound algorithm for a special class of warehouse location problems when the objective function is fractional, is developed. The branching decision rules help us to decide which warehouse has to be opened or closed from any node of branching tree. We propose the revised version of the algorithm suggested by Basheer M. Khumawala (1972). It is illustrated with the help of a numerical example.

Mathematical Subject Classification: Primary 90B80, Secondary 90C57, 90C32

Keywords

Fractional Programming, Branch and Bound, Open and Closed Warehouse

1. Introduction

Let there be m potential warehouses (with unlimited capacity) and n customers.

The warehouse location problem with fractional objective function is formulated as

$$\text{Minimize } Z = \frac{\sum C_{ij} X_{ij} + \sum F_i Y_i}{\sum P_{ij} X_{ij}}$$

$$\text{subject to} \qquad \sum X_{ij} = 1 \quad \forall \quad j = 1,2,\ldots,n$$

$$0 \le \sum_{j \in R_i} X_{ij} \le n_i Y_i \quad \forall \quad i = 1,2,\ldots,m$$

$$Y_i = 0 \quad \text{or} \quad 1 \quad \forall \quad i = 1,2,\ldots,m$$

$$P_{ij} \ge 0 \tag{1}$$

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where $C_{ij} = t_{ij} D_{ij}$

t_{ij} = the per unit cost which includes the FOB cost at the warehouse, the warehouse handling cost and transportation cost from warehouse i to customers j.

D_j = the demand of customer j

X_{ij} = the portion of D_j supplied by the warehouse i to customer j

F_i = the fixed cost associated with warehouse i

N_j = set of warehouses which can supply to customer j

R_i = set of those customers that can be supplied by the warehouse i

n_i = number of elements in R_i

P_{ij} = the profit incurred when product is supplied from warehouse i to customer j

The main difficulty to the problem arises when from m, the number of potential warehouses; an optimum subset of warehouses has to be selected which will be used in the system.

At each stage of the branch and bound algorithm a free warehouse has been selected. By a free warehouse, we mean a potential warehouse, which is not yet "open" or "closed". By an "open" warehouse, it is meant that it is assigned to be used. Similarly a "closed" warehouse is assigned not to be used. Initially, it is decided to keep all the warehouses open. However, all the warehouses need not to be kept open to fulfill the demands of all the customers. Out of all the warehouses that are available, an optimal subset has to be chosen. The warehouses of this subset will be kept open for use. The remaining warehouses will be closed. Let

$$Y_i = \begin{cases} 0 & \text{when the warehouse is closed} \\ 1 & \text{when the warehouse is open} \end{cases}$$

Basheer M. Khumawala [1] has given an algorithm which helps to decide which warehouses are to be kept open and which warehouses are to be kept closed. We have presented a modified version of this algorithm since our objective function is fractional.

It is observed that at each stage of the branch and bound algorithm, selection of a "free" warehouse has to be done from the set of free warehouses available at that point. The free warehouse that is selected has to be constrained open and closed that is, branching is done from the selected warehouse. This selection is done using the largest Ω rule, which is given at a later stage in this paper.

Kurt Spiel Berg, [2] refers to the "open" or "closed" plants as the 'side constraints'. The branch and bound algorithm for plant location was first developed by Effoymson and Ray [3] to yield an optimal solution.

A.S. Manne [4] gave an algorithm which is of interest both from the viewpoint of numerical analysis and also from the analogy with market mechanisms. Feldman E, F.A. Lehrer and T.R. Ray [5] have developed a heuristic for solving warehouse location problems when these economics are represented by continuous concave functions. G. Sa [6] gave exact branch and bound treatment for the capacitated plant- location problem.

2. Branch and Bound Algorithm for the Fractional Objective Function

In order to solve the problem (1) we remove the integral restriction on Y and solve the fractional programming problem with $Y > 0$.

Let (X_0, Y_0) be the solution to the given problem with objective function value Z_0. If some Y_k is fractional then

(i) the restriction $Y_k = 0$ (warehouse is fixed closed) is added to the problem and problem is again solved. Let Z_1 be the objective function values with $Z_1 \geq Z_0$

(ii) The restriction $Y_k = 1$ is added to the problem and the problem is again solved. Let Z_2 be the objective function value and $Z_2 \geq Z_0$. Then $\tilde{Z} = \min(Z_1, Z_2)$ is the new lower bound on Z.

The procedure results in the construction of a tree whose nodes are represented by the Z's and the corresponding values of the fixed Y's (optimum solution is given by Z and Y's indicate whether a warehouse is fixed open or closed). If a node is reached where all the components of Y's are integers it will be called a terminal node as opposed to a non-terminal node where at least one Y is fractional. When a terminal node is reached all the warehouses are fixed open or closed. When there are free warehouses left then it is a non-terminal node.

The solution at a terminal node will be referred to as the terminal solution. In the branch and bound tree then the nodes are observed to be such that the optimal value of Z is less than the current upper bound. At the non-terminal nodes we get two additional nodes- one where we fix the warehouse 'open' and the other where we fix the warehouse 'closed'. The fractional program is solved at these nodes. In this manner we continue with the branch and bound algorithm till we reach a terminal node. If we come across a node where at least one customer's demand cannot be satisfied by a non-closed warehouse, we do not branch that particular node further.

If we reach a stage where the objective function value less than the current upper bound cannot be found, we stop. The optimum solution is the current upper bound.

Because of the assumption of the unlimited warehouse capacity the optimal solution to the fractional programs at any node can be easily found.

Let K_0 -denote the index set of warehouses that are fixed open

K_1 -denote the index set of warehouses that are fixed closed

K_2 -denote the index set of warehouses that are fixed free

$$X_{ij} = 1 \quad if \quad C_{ij} + \frac{g_i}{n_i} - P_{ij} = \min_{k \in K_0 \cup K_2} \left\{ C_{kj} - P_{kj} + \frac{g_k}{n_k} \right\}$$

$$= 0 \quad otherwise$$

where n_i is the number of customers that can be supplied by warehouse i.

$$Y_i = 1 \quad if \quad i \in K_0 \quad \text{(Open warehouse)}$$

$$= \sum \frac{X_{ij}}{n_i} \quad i \in K_2$$

$$= 0 \quad \text{(Closed warehouse)}$$

and where $g_k = F_k \quad k \in K_2$

$= 0 \quad k \in K_1$

The first simplification determines a minimum bound for opening a warehouse for $i \in K_2, \quad j \in R_i$

$$\nabla_{ij} = \min_{k \in N_j \cap (K_0 \cup K_2);\, k \neq i} \left[\max\left\{ \left(C_{kj} - P_{kj}\right) - \left(C_{ij} - P_{ij}\right) \right\}, 0 \right]$$

$$\Delta_i = \sum_{j \in R_i} \nabla_{ij} - F_i$$

Δ_i indicate the difference of the "the sum of the minimum saving for the warehouse i over all the customers it can supply when the profit incurred is taken into account" and "the fixed open cost F_i" Clearly if it has a positive value the corresponding warehouse must be opened.

∇_{ij} measures the minimum cost saving (when the profit incurred is taken into account) for customer j^{th} if warehouse i was opened.

Using the second simplification n_i can be reduced

For $i \in K_2$, $j \in R_i$

$$\min_{k \in N_j \cap K_0} \left[\max\{(C_{kj} - P_{kj}) - (C_{ij} - P_{ij})\} < 0\right]$$

The n_i reduce by 1. If any already opened warehouse is able to supply a customer j cheaper than any of the free warehouses at the node (profit incurred included in the savings) then such a customer should be supplied by open warehouse.

The third simplification determines a maximum bound on the cost reduction for opening a warehouse. If this bound is negative the warehouse will be fixed closed.

For $i \in K_2$, $j \in R_i$ $\varpi_{ij} = \min_{k \in N_j \cap K_0} \left[\max\{(C_{kj} - P_{kj}) - (C_{ij} - P_{ij})\}, 0\right]$

Let $\Omega_i = \sum_{j \in R_i} \varpi_{ij} - F_i$

This expression is similar to ∇_{ij} except that in this case comparisons are made over fixed open warehouses.

If $\Omega_i < 0$ then $Y_i = 0$ for all branches emanating from this node. Basheer M. Khumawala [1] has given different branching decision rules. However, the largest Ω rule was found to be most efficient. According to this rule the free warehouse having the largest Ω from the set of free warehouses at the node having the positive Ω must be selected for the branching. Also as mentioned by Basheer M. Khumawala [1] customer j should be supplied from the warehouse i if $\nabla_{i'j} \geq \frac{F_i}{n_i}$ if $i' \in K_2$ and $\nabla_{i'j} > 0$ if $i' \in K_0$ where i' is that value of I that minimizes C_{ij} over all i in $N_j \cap (K_0 \cup K_2)$.

Note:- At every node one cycles through these simplification until no further opening or closing of warehouses is to be obtained.

2.1. Numerical Example

Consider the 5 × 8 warehouse location problem. The demands of all customers are identical.

Initialization:- At the first node $K_0 = K_1 = \phi$ the empty set and $K_2 = \{1, 2, 3, 4, 5\}$, the set R_i ($i = 1,\ldots,5$) and $N_j = (j = 1,\ldots,8)$ are initialized. Initial Lower bound = 0, Upper bound = + ∞

Fixed Warehouse Costs F_i	Ware-house	Customers C_{ij} | Profits P_{ij}							
		1	2	3	4	5	6	7	8
100	1	120|6	180|9	100|5	L	60|3	L	180|9	L
70	2	210|12	L	150|7	240|15	55|2	210|10	110|6	165|8
60	3	180|9	190|10	110|6	195|11	50|1	L	L	195|11
90	4	210|12	190|10	150|7	180|9	65|4	120|6	160|8	120|6
80	5	170|8	150|7	110|6	150|7	70|5	195|11	200|1	L

where L indicates a forbidden route. For example warehouse1 cannot supply to 4^{th}, 6^{th} and 8^{th} customers as can be observed from the above table.

(1) $\nabla_{11} = (170-8)-(120-6) = 162-114 = 48$

$\nabla_{13} = (110-6)-(100-5) = 104-95 = 9$

$\Delta_1 = (48+9)-100 = 57-100 = -43 < 0$

(2) $\nabla_{27} = (160-8)-(110-6) = 152-104 = 48$

$\Delta_2 = 48-70 = -22 < 0$

(3) $\nabla_{35} = (55-2)-(50-1) = 53-49 = 4$

$\Delta_3 = 4-60 = -56 < 0$

(4) $\nabla_{46} = (195-11)-(120-6) = 184-114 = 70$

$\nabla_{48} = (165-8)-(120-6) = 157-114 = 43$

$\Delta_4 = 113 - 90 = 23 > 0$

(5) $\nabla_{52} = (180-9)-(150-7) = 171-143 = 28$

$\nabla_{54} = 28$

$\Delta_5 = 56 - 80 < 0$

The fourth warehouse is fixed open since $\Delta_4 > 0$

Here $K_0 = \{4\}$, $K_2 = \{1,2,3,5\}$

Customers 6 and 8 are best supplied from open warehouse 4. They are eliminated from consideration of free warehouse.

$R_1 = \{1, 2, 3, 5, 7\}$ $n_1 = 5$

$R_2 = \{1, 3, 4, 5, 7\}$ $n_2 = 5$

$R_3 = \{1, 2, 3, 4, 5\}$ $n_3 = 5$

$R_4 = \{1, 2, 3, 4, 5, 6, 7, 8\}$ $n_4 = 8$

$R_5 = \{1, 2, 3, 4, 5, 7\}$ $n_5 = 6$

We continue the procedure

$\varpi_{11} = (210-12)-(120-6) = 198-114 = 84$

$\varpi_{12} = (190-10)-(180-9) = 180-171 = 9$

(1) $\varpi_{13} = (150-7)-(100-5) = 143-95 = 48$

$\varpi_{15} = (65-4)-(60-3) = 61-57 = 4$

$\Omega_1 = (84+9+48+4)-100 = 145-100 = 45 > 0$

$\varpi_{25} = (65-4)-(55-2) = 61-53 = 8$

(2) $\varpi_{27} = (160-8)-(110-6) = 152-104 = 48$

$\Omega_2 = 56-70 = -14 < 0$

$\varpi_{31} = (210-12)-(180-9) = 198-171 = 27$

(3) $\varpi_{33} = (150-7)-(110-6) = 143-104 = 39$

$\varpi_{35} = (65-4)-(50-1) = 61-49 = 12$

$\Omega_3 = (27+39+12)-60 = 78-60 = 18 > 0$

$$\varpi_{51} = (210-12)-(170-8) = 198-162 = 36$$
$$\varpi_{52} = (190-10)-(150-7) = 180-143 = 37$$
(4) $$\varpi_{53} = (150-7)-(110-6) = 143-104 = 39$$
$$\varpi_{54} = (180-9)-(150-7) = 171-143 = 28$$
$$\Omega_5 = (36+37+39+28)-80 = 140-80 = 60 > 0$$

ϖ_{ij} is similar to the ∇_{ij} of first simplification except that here the comparison is made only over all the fixed open warehouses *i.e.* ϖ_{ij} is minimum saving for supplying to customer j that can be made if warehouse i was opened at that node. If sum of savings for warehouse i over all customers it can supply fails to exceed F_i then such a warehouse should be closed.

Now $K_0 = \{4\}, K_1 = \{2\}, K_2 = \{1, 3, 5\}$

At this stage warehouse 2 is fixed closed and warehouse 4 is fixed open.

The free warehouses are 1, 3, 5.

Customer 7 who was best supplied from warehouse 2 now can be best supplied by open warehouse 4.

$$R_1 = \{1,2,3,5\} \qquad n_1 = 4$$
$$R_3 = \{1,2,3,4,5\} \qquad n_3 = 5$$
$$R_5 = \{1,2,3,4,5\} \qquad n_5 = 5$$

We continue the simplification cycle

(1) $$\nabla_{11} = (170-8)-(120-6) = 162-114 = 48$$

$$\nabla_{13} = (110-6)-(100-5) = 104-95 = 9$$
$$\Delta_1 = 57-100 = -43 < 0$$

(2) $$\nabla_{35} = (60-3)-(50-1) = 57-49 = 8$$

$$\Delta_3 = 8-60 = -52 < 0$$

(3) $$\nabla_{52} = (180-9)-(150-7) = 171-143 = 28$$

$$\nabla_{54} = 28$$
$$\Delta_5 = 56-80 < 0$$

No decision could be made in the above cycle. One cycles through these simplifications until no further opening or closing can be made. Whenever we are not in a position to make a decision regarding the opening and closing of a free warehouse by simplification as in the above case, branching is done where a free warehouse is constrained open and closed respectively.

2.2. Programming Solution

Customers 6, 7 and 8 are best supplied from open warehouse 4, because $\nabla_{46}, \nabla_{47}, \nabla_{48}$ are positive. These values will be positive when customer is getting cheaper from warehouse 4 after it has gained its profit.

Now X_{ij} = the portion of D_j supplied from warehouse i

$$\sum X_{ij} = 1$$

$X_{46} = X_{47} = X_{48} = 1 \Rightarrow$ Complete demand supplied by warehouse 4.

$Y_2 = 0$ [Warehouse 2 is fixed closed]

$Y_4 = 1$ [Because warehouse 4 is fixed open]

For customers 1, 2, 4 we have

$$\begin{cases} \nabla_{11} > F_1/n_1 \left[\nabla_{11} = 48 > \dfrac{100}{4} = 25\right] \\ \nabla_{52} > F_5/n_5 \left[\nabla_{52} = 28 > \dfrac{80}{5} = 16\right] \\ \nabla_{54} > F_5/n_5 \left[\nabla_{54} = 28 > \dfrac{80}{5} = 16\right] \\ X_{11} = X_{52} = X_{54} = 1 \end{cases}$$

For Customer 3, we have

$$\begin{cases} C_{13} - P_{13} + \frac{g_1}{n_1} = 100 - 5 + 25 = 120 \\ C_{33} - P_{33} + \frac{g_3}{n_3} = 110 - 6 + 12 = 116 \\ C_{43} - P_{43} + \frac{g_4}{n_4} = 150 - 7 + 11 = 154 \\ C_{53} - P_{53} + \frac{g_5}{n_5} = 110 - 6 + 16 = 120 \end{cases}$$

$X_{33} = 1$ Similarly $X_{35} = 1$

Thus $X_{11} = X_{32} = X_{33} = X_{54} = X_{35} = X_{46} = X_{47} = X_{48} = 1$ (all other $X_{ij} = 0$)

$Y_2 = 0$ [warehouse 2 is fixed closed]

$Y_4 = 1$ [Because warehouse 4 is fixed open]

Minimize $Z = \dfrac{\sum C_{ij} X_{ij} + \sum F_i Y_i}{\sum P_{ij} X_{ij}}$

$$\sum P_{ij} X_{ij} = 6 + 7 + 6 + 7 + 1 + 6 + 8 + 6 = 47$$

$$\begin{Bmatrix} Y_1 = \frac{X_{11} + X_{12} + X_{13} + X_{14} + X_{15}}{n_1} = \frac{1}{4} \\ Y_3 = X_{31} + X_{32} + X_{33} + X_{34} + X_{35} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5} \\ Y_5 = \frac{2}{5} \end{Bmatrix}$$

$$Z = \frac{(C_{11} + C_{52} + C_{53} + C_{54} + C_{35} + C_{46} + C_{47} + C_{48}) + (1 \times 90 + \frac{1}{4} \times 100 + \frac{2}{5} \times 60 + \frac{2}{5} \times 80)}{47}$$

$$Z = \frac{(120 + 150 + 110 + 150 + 50 + 120 + 160 + 120) + (90 + 25 + 24 + 32)}{47}$$

$$= \frac{1151}{47} = 24.4$$

The solution is feasible and non-terminal. The new lower bound is 24.4. Using the largest Ω rule, warehouse 5 is selected for branching. Let warehouse 5 be opened. Warehouses 1 and 3 are fixed closed during the simplification cycle. Therefore

$$K_0 = \{4,5\}$$
$$K_1 = \{1,2,3\}$$

Warehouses 4 and 5 now provide all the customers

$$X_{46} = X_{47} = X_{48} = 1$$

$$X_{51} = X_{52} = X_{53} = X_{54} = X_{45} = 1$$

Here $Z = \dfrac{1215}{52} = 23.3.$

This node is terminal and fractional programming solution is 23.3. This is the new upper bound.

Suppose warehouse 5 is fixed closed. Warehouse 2 was already closed and warehouse 4 was already open. Hence $Y_2 = 0, Y_4 = 1, Y_5 = 0$ also it is seen after performing calculations that

$$X_{11} = X_{32} = X_{33} = X_{44} = X_{35} = X_{46} = X_{47} = X_{48} = 1$$

$$Y_1 = \tfrac{1}{4} \text{ and } Y_3 = \tfrac{3}{5}$$

Hence $Z_1 = \dfrac{1211}{52} = 23.2$

No warehouse could be fixed open or closed during the simplification cycle. The fractional programming solution was 23.2. Branching continues and results are illustrated in the figure. The calculations are as follow.

When warehouses 1, 2 and 5 are fixed closed and warehouse 3 and 4 are fixed open.

$$X_{35} = X_{44} = X_{33} = X_{32} = X_{31} = 1$$

Then $Y_1 = Y_2 = Y_5 = 0$

$$Y_3 = Y_4 = 1$$

$$\min Z = \frac{(C_{32}+C_{32}+C_{33}+C_{44}+C_{35}+C_{46}+60+90)}{7+10+6+9+1+6+8+6}$$
$$= \frac{180+190+110+180+50+120+160+120+150}{53}$$
$$= \frac{1260}{53} = 23.77$$

When warehouses 2, 3 and 5 are fixed closed and warehouse 1 and 4 are fixed open.

$$\text{Then} \quad \begin{aligned} X_{46} &= X_{47} = X_{48} = 1 \\ X_{11} &= X_{12} = X_{13} = X_{44} = X_{15} = 1 \end{aligned}$$

$$\min Z = \frac{(C_{11}+C_{12}+C_{13}+C_{44}+C_{15}+C_{46}+C_{47}+C_{48})+100+90}{6+9+5+9+3+6+8+6}$$
$$= 23.6$$

Thus the new node with values 23.7 and 23.6 are terminal but their solutions exceed the upper bound. The objective function Z = 23.3 is the optimum solution since the objective function value is higher in the other two cases.

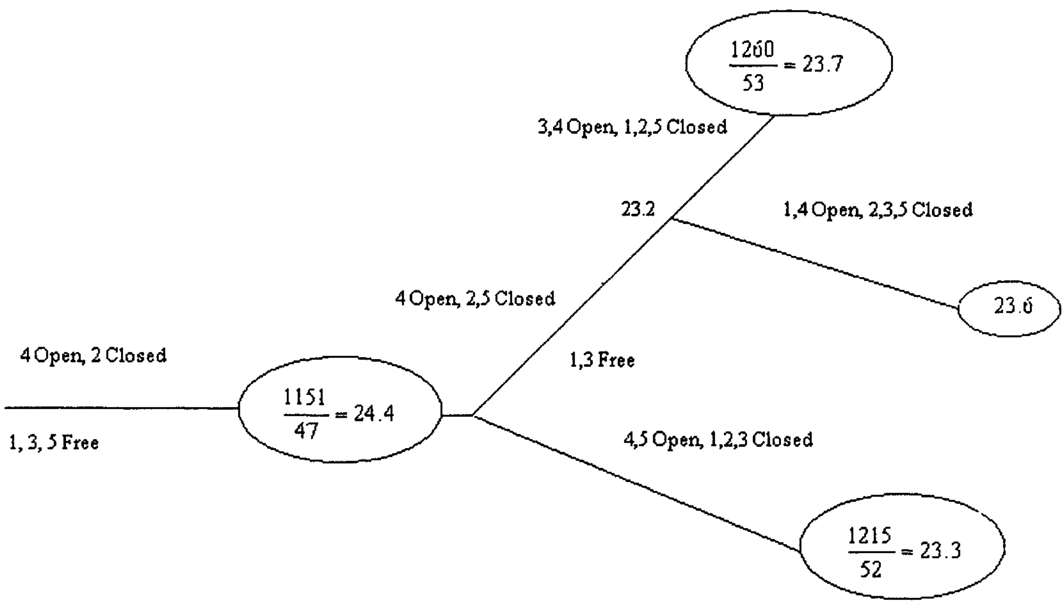

Figure 1: Branch and Bound tree for illustrated example.

3. Conclusion

To conclude in this paper we have observed that the branch and bound method employed by Basheer M. Khumawala [1] for linear programming problem is equally effective when applied to a fractional programming problem.

Various other methods can be employed to solve the warehouse location problem. A modified Benders decomposition method and a dual ascent and adjustment method within a branch and bound framework are efficient solution approaches. These methods have been discussed for linear integer programming problems [7]. It would be interesting to discuss these methods for fractional programming problems.

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