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Explicit Solvability of Dual Pairs of Infinite Linear Programs

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Abstract

In this paper, we obtain explicit solutions of a pair of infinite linear programs in terms of generalized inverses. This is done by reducing one of the problems to an interval linear program. The main result is posed in the setting of a class of dual pairs of real Banach spaces and is illustrated with an example.

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1. Introduction

Let A be a real $m \times n$ matrix, $a, b \in \mathbb{R}^m$ with $a \leq b$, where '$\leq$' is the coordinate-wise ordering of vectors. For $c \in \mathbb{R}^n$, consider the following linear programming problem:

$$\text{Maximize } \; c^T x$$

$$\text{subject to } \; a \leq Ax \leq b.$$

Such problems, called interval linear programs (ILP) were first introduced and studied by Ben-Isreal and Charnes [2] who showed that if A is of full row rank then an explicit solution can be given in terms of generalized inverses of A. These were later extended in [14 & 15], where the restriction on A was removed. Later, iterative methods to solve interval linear programs were proposed by Robers, Ben-Israel and others. (See [3, 8, 9 & 10] and the references cited therein). Sposito [13] showed how, under certain assumptions the results of [14 & 15] can be extended to solve problems (that will be denoted by LP here) where the lower bound is removed. Sivakumar and Kulkarni extended the results of [2, 14 & 15] to infinite dimensional spaces in [6, 7 & 12].

The dual pair of linear programming problems that we study here are defined as follows. Let (X_1, Y_1) and (X_2, Y_2) be dual pairs of real Banach spaces, X_2 be a partially ordered real Banach space with P_2 as the positive cone. Let $A : X_1 \to X_2$ be a linear map, $\varphi \in Y_1$ and $a, b \in X_2$ with $a \leq b$. Consider the pair of (dual) linear programming problems given by

$$\text{LP:} \quad \text{Maximize} \quad \varphi(x)$$
$$\text{subject to} \quad Ax \leq b.$$

and

$$\text{LD:} \quad \text{Minimize} \quad w(b)$$
$$\text{subject to} \quad A^* w = \varphi, w \geq 0.$$

In this paper we give explicit solutions of these problems. This is done by using known results on interval linear programs. Here we take the approach similar to [13] and show how LP and LD can be solved explicitly. In this connection, it is pertinent to note that the strong duality theorem is used in [13] whereas we use only the weak duality theorem. This is significant in the present context as the strong duality theorem in general, does not hold in infinite dimensional spaces [1]. To summarize, our work reported here extends the results of explicit solvability of interval linear programs (both in finite dimensional and in infinite dimensional spaces) to include a larger class of problems.

The paper is organized as follows. In the next section we present the preliminaries that are required in the rest of the paper. Here we discuss the concept of a Ben-Israel-Charnes space and prove a result. Even though this idea was introduced in [12], the results therein have not been published so far. However, we do not intend to discuss this in detail and so only a brief introduction is given here. In the last section we consider the problem of explicit solvability of LP and LD. The main results in this section are Corollary 3.7 and Theorem 3.9. We conclude with an example illustrating the applicability of Theorem 3.9.

2. Preliminaries and Notations

Definition 2.1:

Let X be a real vector space. Then X is called partially ordered vector space if X has a partial order '$\leq$' defined on it satisfying the following: For $x, y \in X$ and $x \leq y$, $x + u \leq y + u$ for all $u \in X$ and $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.

Definition 2.2:

Let X be a partially ordered real vector space. Then the subset $C := \{x \in X / x \geq 0\}$ is called the positive cone of X. C is called a strictly positive cone if $x \leq y$ and $y \leq x$ imply $x = y$ for every $x, y \in X$.

Definition 2.3:

A vector space is said to be a partially ordered Banach space if it is a partially ordered vector space and also a Banach space with respect to some norm.

Definition 2.4:

Let X_1 be a real Banach space and X_2 be a partially ordered real Banach space with P_2 as the positive cone. Let $A : X_1 \to X_2$ be linear, $\varphi \in X_1^*$ (the space of bounded linear functionals on X_1) and $a, b \in X_2$ with $a \leq b$. Consider the problem called Interval Linear Program denoted by $\text{ILP}(a, b, \varphi, A)$:

$$\text{Maximize } \varphi(x)$$

$$\text{subject to } a \leq Ax \leq b.$$

A vector $x^* \in X_1$ is said to be feasible for the problem $\text{ILP}(a, b, \varphi, A)$ if $a \leq Ax^* \leq b$. The problem $\text{ILP}(a, b, \varphi, A)$ is said to be feasible if there exists a feasible vector for it. A feasible vector x^* is said to be optimal if $\varphi(x^* - x) \geq 0$ for every feasible vector x. The problem $\text{ILP}(a, b, \varphi, A)$ is said to be bounded if $\text{sub}\{\varphi(x) : a \leq Ax \leq b\} < \infty$.

Definition 2.5:

Let X be a partially ordered real Banach space. Let I denote the identity map on X. We say that X is a Ben-Israel-Charnes space (B-C space, for short) if $\text{ILP}(a, b, \varphi, I)$ has an optimal solution for all $a, b \in X$, $a \leq b$ and $\varphi \in X^*$.

The notion of a B-C space, proposed in [12], is in recognition of the contributions of these authors to finite interval linear programs. It must be mentioned here that a class of production-planning and input-output problems have been shown to have explicit solutions in the setting of B-C spaces. This work appears in [5] and is the first instance of the use of the nomenclature B-C spaces in the literature.

In the next result we collect a family of B-C spaces.

Theorem 2.6: (Example 4.1.6 , [12])

Let μ be a σ- finite positive measure on a σ-algebra $\mathcal{M}$ in a nonempty set Y and $L^p(Y,\mu)$, $1 \le p < \infty$ be the space of (equivalent classes) of measurable p-integrable functions on Y. Then $L^p(Y,\mu)$ is a B-C space.

Proof:

Let $X = L^p(Y,\mu)$ and $C := \{f \in X : f \ge 0 \text{ a.e. } (\mu)\}$. Then X is a partially ordered Banach space with C as the positive cone. Let $\varphi \in X^*$. Then there exists a unique $h \in L^q(Y,\mu)$ (where q is the conjugate exponent of p) such that for all $f \in X$, $\varphi(f) = \int\limits_X h f \, d\mu$. Let $a, b \in X$ with $a \le b$ and ILP(a,b,φ,I) be feasible.

Define $Y_+ := \{y \in Y : h(y) \ge 0\}$, $Y_- := \{y \in Y : h(y) < 0\}$ and $\eta = b\chi_{Y_+} + b\chi_{Y_-}$, where χ_S denotes the characteristic function of a set S. It then follows that η is measurable. Further $\int |\eta|^p \, d\mu \le \|b\|^p + \|a\|^p$. This shows that $\eta \in X$. For any $u \in X$ satisfying $a \le u \le b$ a.e.(μ), we have

$$\varphi(\eta - u) = \int\limits_Y h\,(\eta - u)\,d\mu = \int\limits_{Y_+} h\,(\eta - u)\,d\mu + \int\limits_{Y_-} h\,(\eta - u)\,d\mu \ge 0.$$

Thus η is optimal for the problem ILP(a,b,φ,I).

Remark 2.7:

It follows from Theorem 2.6 that R^n, then n-dimensional real Euclidean space is a B-C space. Also ℓ^p, the space of real p-summable sequences for $1 \le p < \infty$ is a B-C space. This follows by taking $Y = N$ and μ to be the counting measure on N, in the above theorem.

Remark 2.8:

It can be verified that a Cartesian product of a finite number of B-C spaces is a B-C space with component-wise operations.

Remark 2.9:

$C([0,1])$, the space of all real valued continuous functions on $[0,1]$ with the norm $\|x\| = \sup\{|x(t)| : 0 \le t \le 1\}$ is not a B-C space. We prove this as follows. Consider the problem ILP($0,1,\varphi,I$), where

$$\varphi(f) = \int_0^{1/2} f(t)\,dt - \int_{1/2}^{1} f(t)\,dt$$

Let

$$f^0(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1/2 \\ 0 & \text{if } 1/2 \le t \le 1 \end{cases}$$

By approximating f^0 by continuous functions, it can be shown that $\|x\| = \sup\{\varphi(f) : f \in C([0,1]) : 0 \le f \le 1\} = 1/2$. But there is no continuous function g on $[0,1]$ such that $0 \le g \le 1$ and $\varphi(g) = 1/2$.

3. Explicit Solvability of LP and LD

In this section we study how LP and LD can be solved explicitly.

Definition 3.1:

Let X_1 and X_2 be real Banach spaces and A be a linear map with $D(T \& A) \subset X_1$ into X_2. A linear map T with $D(T) \subset X_2$ into X_1 is called a $\{1\}-$ inverse of A if, $R(A) \subset D(T)$, $R(T) \subset D(A)$ and $ATAx = Ax \;\; \forall x \in D(A)$.

Remark 3.2:

For $A \in BL(X_1, X_2)$, let $A(\{1\})$ denote the set of all bounded $\{1\}-$inverses of A. It is well known that if $R(A)$ is a closed subspace in X_2, then $A(\{1\})$ is nonempty [4].

Definition 3.3:

Let Y be a normal linear space. The annihilator of a subset E of Y denoted by E^0, is a subspace of Y^* defined by $E^0 = \{ \varphi \in Y^* : \varphi(x) = 0 \;\; \forall\, x \in E \}$.

The following two results have been proved in [6]. We give it here for completeness. The proofs have been omitted.

Lemma 3.4: (Lemma 2.7, [6])

Let X_1 and X_2 be real Banach spaces with X_2 partially ordered and $ILP(a, b, \varphi, A)$ be feasible. If ILP is bounded, then $\varphi \in N(A)^0$. The converse holds if A has a bounded $\{1\}-$inverse and $[a, b]$ is bounded in X_2. Here $[a,b] = \{x \in X_2 : a \le x \le b\}$.

Theorem 3.5: (Theorem 3.1, [6])

Let X_1 be a real normed linear space, X_2 be a B-C space, $a, b \in X_2$ such that $a \le b$, $[a, b]$ be bounded, $\varphi \in X_1^*$, $\varphi \in N(A^0)$ and η^0 be any optimal solution of $ILP(a, b, T^*\varphi, I)$. For $\gamma \in N(A)$ and for a bounded $T \in A(\{1\})$, let $x^0 = T\eta^0 + \gamma$. Then x^0 is optimal for $ILP(a, b, \varphi, A)$ iff x^0 is feasible for $ILP(a, b, \varphi, A)$.

Remark 3.6:

If $a = b = d$, then $\eta^0 = d$. Further, if P_2 is a strict cone, then $a \le Ax \le a$ is equivalent to $Ax = a$. Thus we have the following corollary.

Corollary 3.7:

Let P_2 be a strict cone and $ILP(d, d, \varphi, A)$ be feasible. Then $u^0 = Td + y$, $y \in N(A)$ is optimal for $ILP(d, d, \varphi, A)$.

Proof:

Since $ILP(d, d, \varphi, A)$ is feasible and P_2 is strict, $Ax = d$ has a solution. Thus $Au^0 = ATd = d$ since $T \in A(\{1\})$. So u^0 is feasible for $ILP(d, d, \varphi, A)$. By Theorem 3.5, u^0 is optimal for $ILP(d, d, \varphi, A)$.

We are now in a position to prove the main result. We need the following definition.

Definition 3.8:

Let X and Y be any two real vector spaces with a bilinear form defined from $X \times Y$ into $\mathbb{R}$, which we write as $\langle \cdot , \cdot \rangle$ with $\langle x , y \rangle$ a linear function of x for each fixed y in Y, and a linear function of y for each fixed x in X. If for each $x \ne 0$ in X there is some $y \in Y$ with $\langle x , y \rangle \ne 0$ and for each $y \ne 0$ in Y there is some $x \in X$ with $\langle x , y \rangle \ne 0$, then the pair of spaces (X, Y) is called a dual pair. Clearly (Y, X) is also a dual pair, if (X, Y) is a dual pair. For more details on dual pairs we refer to [11].

Theorem 3.9:

Let (X_1, Y_1) and (X_2, Y_2) be two dual pairs of real Banach spaces and X_2 be a partially ordered real Banach space. Let P_1 and P_2 be the positive cones in X_1 and X_2, respectively with P_1 strict. Let Y_1 and X_2 both be B-C spaces. Let $A: X_1 \to Y_1$ be a bounded linear map with closed range, LP and LD be feasible, $b \in R(A)$ and let $w^0 = T^* \varphi + \psi$, $\psi \in N(A^*)$ be feasible for LD where $T \in A(\{1\})$. Then for $v \in N(A^*)$, $x^0 = Tb + v$ is optimal for LP and w^0 is optimal for LD.

Proof:

From the hypotheses, if follows that LD is equivalent to

$$\text{Maximize } w(-b)$$

$$\text{subject to } \varphi \le A^* w \le \varphi, \; w \ge 0.$$

The above problem is $ILP(\varphi, \varphi, -b, A^*)$ with the additional constraint $w \ge 0$. Since $w^0 = T^*\varphi + \psi$, $\psi \in N(A^*)$, by Corollary 3.7 w^0 is optimal for $ILP(\varphi, \varphi, -b, A^*)$. Since $w^0 \ge 0$ (by hypothesis), we conclude that w^0 is optimal for LD. Also $Ax^0 = ATb = b$ as $b \in R(A)$. Thus x^0 is feasible for LP. Finally

$$-w(-b) = w^0(b) = (T^*\varphi + \psi)b = (T^*\varphi)(b) = \varphi T(b) = \varphi(x^0 - \nu) = \varphi(x^0).$$

Hence by the (weak) duality theorem (see [1], for instance) x^0 is optimal for LP and w^0 is optimal for LD.

We illustrate the main result in the following example.

Example 3.10:

Let $X_1 = X_2 = L^2([0,1])$. Let $P_2 = P_1 = \{f \in L^2([0,1]) : f \ge 0 \text{ a.e.}(\mu)\}$. Define the operator $A: X_1 \to X_2$ by

$$A\,x(s) = s\int_0^1 t^2 x(t)\,dt + s^2 \int_0^1 t\,x(t)\,dt, \quad s \in [0,1],$$

Let $\varphi(s) = \frac{1}{12}s^2 + \frac{1}{20}s$ and $b(s) = s^2$.

Then the primal is

LP: $$\text{Maximize} \int_0^1 \left(\frac{1}{12}t^2 + \frac{1}{20}t\right) x(t)\, dt$$

$$\text{subject to } s\int_0^1 t^2 x(t)\, dt + s^2 \int_0^1 t\, x(t)\, dt \le s^2, \quad \forall s \in [0,1].$$

and the dual is

LD: $$\text{Minimize} \int_0^1 t^2 w(t)\, dt$$

$$\text{subject to } s\int_0^1 t^2 w(t)\, dt + s^2 \int_0^1 t\, w(t)\, dt = \frac{1}{12}s^2 + \frac{1}{20}s, \ \forall s[0,1],$$

$$w(s) \ge 0, \text{a.e.in } s \in [0,1].$$

Here $A = A^*$ and $\varphi(s) \in R(A) = R(A^*) = N(A)^{\perp}$. Hence LP is bounded. We note that $R(A)$ is finite dimensional and hence closed. Thus A has a bounded $\{1\}$- inverse. The matrix of $A^2_{|R(A)}$ with respect to the basis $\{s, s^2\}$ (of $R(A)$) is $\frac{1}{240}\begin{pmatrix} 31 & 24 \\ 40 & 31 \end{pmatrix}$. It follows that the Moore Penrose inverse of A is given by $A^+ = DA$, where $D = \frac{1}{240}\begin{pmatrix} 31 & -24 \\ -40 & 31 \end{pmatrix}$. We then have

$$w^0 = A^+\varphi(s) = s - s^2.$$

It can be verified that w^0 is feasible for LD. By Theorem 3.9, w^0 is optimal for LD and $x^0 = A^+b(s) = 48s - 60s^2$ is optimal for LP. The optimal value is $\frac{1}{20}$.

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