JEPE Vol 2, 2016, p. 87-104 *Orthogonal* $\perp$

EXISTENCE OF WEAK SOLUTIONS OF AN UNSTEADY THERMISTOR SYSTEM WITH p-LAPLACIAN TYPE EQUATION

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ABSTRACT. In this paper, we consider an unsteady thermistor system, where the usual Ohm law is replaced by a non-linear monotone constitutive relation between current and electric field. This relation is modeled by a p-Laplacian type equation for the electrostatic potential φ. We prove the existence of weak solutions of this system of PDEs under mixed boundary conditions for φ, and a Robin boundary condition and an initial condition for the temperature u.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ ($n = 2$ or $n = 3$) be a bounded domain with Lipschitz boundary $\partial\Omega$, and set $Q_T = \Omega \times]\,0, T\,[$ $(0 < T < +\infty)$.

Let $\boldsymbol{J}$ and $\boldsymbol{q}$ denote the electric current field density and the heat flux, respectively, of a thermistor occupying the domain Ω under unsteady operating conditions. Then the balance equations for the electric current and the heat flow within the thermistor material are the following two PDEs

$$\nabla \cdot \boldsymbol{J} = 0, \quad \frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{q} = f(x, t, u, \nabla\varphi) \quad \text{in } Q_T,$$

where $\varphi = \varphi(x, t)$ and $u = u(x, t)$ represent the electrostatic potential and the temperature, respectively (see, e.g., [29, Chap. 8]).

We make the following constitutive assumptions on $\boldsymbol{J}$ and $\boldsymbol{q}$

$$\boldsymbol{J} = \sigma\big(u, |\boldsymbol{E}|\big)\boldsymbol{E} \quad \text{Ohm's law}, \quad \boldsymbol{q} = -\kappa(u)\nabla u \quad \text{Fourier's law},$$

where

$$\begin{aligned} &\boldsymbol{E} = -\nabla\varphi \quad \text{density of the electric field}, \\ &\sigma = \sigma\big(u, |\boldsymbol{E}|\big) \quad \text{electrical conductivity}, \\ &\kappa = \kappa(u) \quad \text{thermal conductivity}. \end{aligned}$$

2010 *Mathematics Subject Classification.* 35J92, 35K20, 35Q79, 80A20.

Key words and phrases. Thermistor system, Robin boundary condition, p-Laplacian, saturation of current, self-heating.

Received 05/09/2016, accepted 27/10/2016.

With these notations the above system of PDEs takes the form

$$-\nabla\cdot\big(\sigma\big(u,|\nabla\varphi|\big)\nabla\varphi\big)=0 \quad \text{in } Q_T, \tag{1.1}$$

$$\frac{\partial u}{\partial t}-\nabla\cdot\big(\kappa(u)\nabla u\big)=f(x,t,u,\nabla\varphi) \quad \text{in } Q_T. \tag{1.2}$$

The function $f=f(x,t,u,\nabla\varphi)$ represents a heat source that will be specified below (see (1.13) and (H3), Section 2).

We supplement system (1.1)–(1.2) by boundary conditions for φ and u, and an initial condition for u. Without any further reference, throughout the paper we assume

$$\partial\Omega=\Gamma_D\cup\Gamma_N \ \text{ disjoint}, \quad \Gamma_D \ \text{ non-empty, open.}$$

Define

$$\Sigma_D=\Gamma_D\times\,]0,T\,[\,, \quad \Sigma_N=\Gamma_N\times\,]\,0,T\,[\,.$$

We then consider the conditions

$$\varphi=\varphi_D \ \text{ on } \Sigma_D, \quad \boldsymbol{J}\cdot\boldsymbol{n}=0 \ \text{ on } \Sigma_N, \tag{1.3}$$

$$\boldsymbol{q}\cdot\boldsymbol{n}=g(u-h) \ \text{ on } \partial\Omega\times\,]\,0,T\,[\,, \tag{1.4}$$

$$u=u_0 \ \text{ in } \Omega\times\{0\} \tag{1.5}$$

($\boldsymbol{n}$ = unit outward normal to $\partial\Omega$). The first condition in (1.3) means that there is an applied voltage φ_D along Σ_D, whereas the second condition characterizes electrical insulation of the thermistor along Σ_N. The Robin boundary condition (1.4)[1)] means that the flux of heat through $\partial\Omega\times\,]\,0,T\,[$ is proportional to the temperature difference $u-h$, where g denotes the thermal conductivity of the surface $\partial\Omega$ of the thermistor, and h represents the ambient temperature (cf. [10], [15], [22], [29, Chap. 8] and [32] (nonlinear boundary conditions)). □

We present two prototypes for the electrical conductivity σ. To this end, let $\sigma_0:\mathbb{R}\to\mathbb{R}_+$[2)] be a continuous function such that

$$0<\sigma_*\le\sigma(u)\le\sigma^*<\infty \quad \forall\, u\in\mathbb{R} \quad (\sigma_*,\sigma^*=\text{const}).$$

We then consider the following functions

$$\sigma(u,\tau)=\sigma_0(u)(\delta+\tau^2)^{(p-2)/2}, \quad (u,\tau)\in\mathbb{R}\times\mathbb{R}_+ \quad (\delta=\text{const}>0,\ \ 1<p<+\infty) \tag{1.6}$$

and

$$\sigma(u,\tau)=\sigma_0(u)\tau^{p-2}, \quad (u,\tau)\in\mathbb{R}\times\mathbb{R}_+ \quad (2\le p<+\infty). \tag{1.7}$$

The electrical conductivities which correspond to these functions $\sigma=\sigma(u,\tau)$ read

$$\sigma\big(u,|\boldsymbol{E}|\big)=\sigma_0(u)\big(\delta+|\boldsymbol{E}|^2\big)^{(p-2)/2} \tag{1.8}$$

and

$$\sigma\big(u,|\boldsymbol{E}|\big)=\sigma_0(u)|\boldsymbol{E}|^{p-2}, \tag{1.9}$$

[1)] This boundary condition is also called "Newton's cooling law" or "third boundary condition".

[2)] $\mathbb{R}_+=[0,+\infty\,[$.

respectively ($\boldsymbol{E}$ = electrical field density). Here, the factor $\sigma_0(u)$ characterizes the thermal dependence of the electrical conductivity of the thermistor material. Observing that $\boldsymbol{E} = -\nabla\varphi$, equ. (1.1) takes the form of p-Laplacian equations

$$-\nabla \cdot \big(\sigma_0(u)\big(\delta + |\nabla\varphi|^2\big)^{(p-2)/2}\nabla\varphi\big) = 0,$$

resp.

$$-\nabla \cdot \big(\sigma_0(u)|\nabla\varphi|^{p-2}\nabla\varphi\big) = 0.$$

Let $p = 2$. Then both (1.8) and (1.9) lead to $\boldsymbol{J} = \sigma_0(u)\boldsymbol{E}$. If the right hand side in (1.2) is of the form $f = \sigma_0(u)|\nabla\varphi|^2 = \boldsymbol{J} \cdot \boldsymbol{E}$ (Joule heat), (cf. (1.13) below), then (1.1)–(1.2) represents the "classical" thermistor system (see [1], [9], [15], [33]). This system has been studied in [18]–[20] with a degeneration of the coefficients $\sigma_0(u)$ and $\kappa(u)$ (cf. also [10] for a similar degeneration of $\sigma_0(u)$). □

Remark 1. (*The case* $1 < p \leq 2$.) Let be $\sigma = \sigma(u, \tau)$ as in (1.6). Then Ohm's law reads

(1.10) $$\boldsymbol{J} = \sigma_0(u)\big(\delta + |\boldsymbol{E}|^2\big)^{(p-2)/2}\boldsymbol{E}$$

(cf. (1.8)). To make things clearer, let $I = |\boldsymbol{J}|$ and $V = |\boldsymbol{E}|$ denote the current and voltage, respectively, in an electrical conductor. Equ. (1.10) then gives the current-voltage characteristic

(1.11) $$I = \sigma_0(u)(\delta + V^2)^{(p-2)/2}V.$$

If $p = 2$, then this current-voltage characteristic turns into the well-known linear (i.e., Ohmic) characteristic $I = \sigma_0(u)V$. If p is "sufficiently near to 1", then (1.11) can be used as an approximation of current-voltage characteristics for transistors (see, e.g., [23], [31, Chap. 6.2.2]).

The characteristic (1.11) continues to make sense if $p = 1$, i.e.,

(1.12) $$I = \frac{\sigma_0(u)}{(\delta + V^2)^{1/2}}\,V.$$

This current-voltage characteristic is widely used to describe the effect of saturation of current in certain transistors under high electric fields (see, e.g., [27, Chap. 2.5] for details). The following figure gives an illustration of the relationship between the limit case $p = 1$ and the effect of saturation of current.

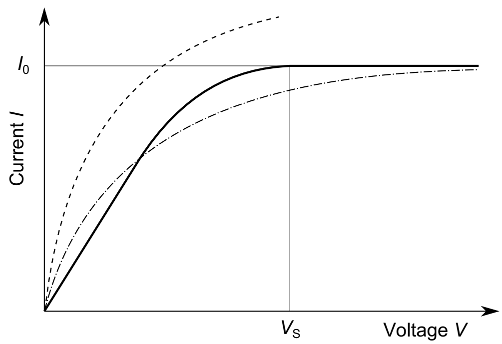

Fig. Current-voltage characteristic I vs. V ($I_0 = \sigma_0(u)$)

Broken line: $I = \frac{I_0}{(\delta+V^2)^{(2-p)/2}} V$, $1 < p < 2$ (cf. (1.11));
dotted line: $I = \frac{I_0}{(\delta+V^2)^{1/2}} V$ (cf. (1.12), i.e., asymptotic saturation of current $I \nearrow I_0$ when V increases;
bold-faced line: experimental data I vs. V of MOSFETs, i.e., linear slope $I = \sigma_0(u)V$ for voltages $V \ll V_S$ (cf. (1.11) with $p = 2$), and saturation of current $I = I_0$ for voltages $V \geq V_S$ (see, e.g., [23], [31, p. 304, fig 9]).

Finally, we notice that for the case $\delta = 0$ and $p = 1$, Ohm's law (1.10) and the current-voltage characteristic (1.11) have to be replaced by

$$\boldsymbol{J} \in \overline{B_{r_0}(0)} \quad \text{if} \quad \boldsymbol{E} = \boldsymbol{0}, \qquad \boldsymbol{J} = \frac{r_0}{|\boldsymbol{E}|}\boldsymbol{E} \quad \text{if} \quad \boldsymbol{E} \neq \boldsymbol{0},$$
$$0 \leq I \leq r_0 \quad \text{if} \quad V = 0, \qquad I = r_0 \quad \text{if} \quad V > 0,$$

respectively, where $\overline{B_{r_0}(0)} = \{\xi \in \mathbb{R}^n; |\xi| \leq r_0\}$, $r_0 = r_0(u)$ (cf. [21]).

Remark 2. (*The case* $2 \leq p < +\infty$.) In [11], the author considers the steady case of (1.1) with $\sigma = \sigma\big(|\nabla\varphi|\big)$, where

$$\lim_{\tau \to +\infty} \frac{\sigma(\tau)}{\tau^{p-2}} = a > 0, \quad p \geq 2$$

(cf. (1.7)). Electrical conductors obeying the constitutive law $\boldsymbol{J} = -\sigma\big(|\nabla\varphi|\big)\nabla\varphi$ are called *varistors* (= varying resistors).

Equ. (1.1) with this constitutive law is then studied under the boundary conditions

$$\varphi = 0 \quad \text{on } \Gamma'_D, \quad \varphi = \Phi \quad \text{on } \Gamma''_D, \quad \frac{\partial\varphi}{\partial\boldsymbol{n}} = 0 \quad \text{on } \Gamma_N \quad (\Gamma_D = \Gamma'_D \cup \Gamma''_D \text{ disjoint}),$$

where Φ is an unknown constant (cf. (1.3)). The constant Φ is related to $\nabla\varphi$ by a nonlocal boundary condition on Γ''_D which models a current limiting device (see, e.g., [15] for more details).

A second topic of [11] concerns the steady case of (1.1)–(1.2) with $\boldsymbol{J} = -\sigma(u)\nabla\varphi$ and $f = \sigma(u)|\nabla\varphi|^2$ under analogous boundary conditions as above.

Similar studies of the steady case of (1.1)–(1.2) with $\boldsymbol{J} = -\sigma(u,\varphi)\nabla\varphi$ and $f = \sigma(u,\varphi)|\nabla\varphi|^2$ can be found in [12]. □

Another type of non-Ohmic current-voltage characteristics is

$$I = \big(\sigma_0(x,u)V^{p(x)-2}\big)V, \quad 2 \leq p(x) < +\infty \quad (x \in \Omega),$$

where $p = p(x)$ is a jump function (cf. (1.7) and (1.9)). The experimental findings which lead to this characteristic, are presented in [14]. This characteristic is used to model both Ohmic and non-Ohmic behavior of the device material (i.e., $\{x \in \Omega;\ p(x) = 2\}$ and $\{x \in \Omega;\ 2 < p_i(x) < +\infty\}$, respectively, $(i = 1, \ldots, m)$) (see also [24] for more details). □

We present a prototype for the heat source term f in (1.2) which motivates hypotheses (H3) in Section 2.

Let be $\sigma = \sigma(u,\tau)$ as in (1.6) or (1.7). For $(x,u,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ we consider functions f such that

$$\text{(1.13)} \qquad \begin{cases} f(x,u,\xi) = \alpha(x,u,\xi)\sigma\big(u,|\xi|\big)|\xi|^2, \\ \alpha : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}_+ \quad \text{is Carathéodory}, \\ 0 \leq \alpha(x,u,\xi) \leq \alpha_0 = \text{const} \quad \forall\,(x,u,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \quad (\alpha_0 = \text{const}). \end{cases}$$

If $\alpha \equiv 1$, then

$$f(x,u,\nabla\varphi) = \sigma\big(u,|\nabla\varphi|\big)(-\nabla\varphi)\cdot(-\nabla\varphi) = \boldsymbol{J}\cdot\boldsymbol{E}.$$

Let be α of the form

$$\alpha(x,u,\xi) = \widehat{\alpha}(x,u,-\xi)$$

or

$$\alpha(x,u,\xi) = \widehat{\alpha}\big(x,u,-\sigma\big(u,|\xi|\big)\xi\big),$$

where $\widehat{\alpha} : \Omega\times\mathbb{R}\times\mathbb{R}^n$ is a Carathéodory function such that $0 \le \widehat{\alpha} \le 1$ everywhere. Then (1.2) models a self-heating process with source term

$$f = \alpha\,\boldsymbol{J}\cdot\boldsymbol{E},$$

where the factor

$$\alpha = \widehat{\alpha}(x,u,\boldsymbol{E}) \quad \text{or} \quad \alpha = \widehat{\alpha}(x,u,\boldsymbol{J})$$

characterizes a loss of Joule heat (cf. [24] for more details).

The existence of weak solutions to the *steady case* of (1.1)–(1.4) has been proved for the first time in [24] for $2 < p < +\infty$ and in [17] for $2 \le p(x) < +\infty$ ($n = 2$ in both papers). Extensions of these results for measurable exponents $p = p(x)$ such that $1 < p_1 \le p(x) \le p_2 < +\infty$ ($p_1, p_2 = \text{const}$), and any dimension n have been recently presented in [7], [8]. □

In [28], we proved the existence of a weak solution of (1.1)–(1.5) when the function $\tau \mapsto \sigma(u,\tau)$ is strictly monotone and f satisfies hypothesis (H3) below (see Section 2) which includes (1.13) as a special case. The aim of the present paper is to prove an analogous existence result when $\tau \mapsto \sigma(u,\tau)$ is merely monotone whereas the function f, however, has to satisfy a structure condition of type (1.13).

2. Weak formulation of (1.1)–(1.5)

We introduce the notations which will be used in what follows.

By $W^{1,p}(\Omega)$ $(1 \le p < +\infty)$ we denote the usual Sobolev space. Define

$$W^{1,p}_{\Gamma_D}(\Omega) = \big\{v \in W^{1,p}(\Omega);\, v = 0 \ \text{ a.e. on } \ \Gamma_D\big\}.$$

This space is a closed subspace of $W^{1,p}(\Omega)$. Throughout the paper, we consider $W^{1,p}_{\Gamma_D}(\Omega)$ equipped with the norm

$$|v|_{W^{1,p}} = \left(\int_\Omega |\nabla v|^p dx\right)^{1/p}.$$

Let X denote a real normed space with norm $|\cdot|_X$ and let X^* be its dual space. By $\langle x^*, x\rangle_X$ we denote the dual pairing between $x^* \in X^*$ and $x \in X$. The symbol $L^p(0,T,X)$ $(1 \le p \le +\infty)$ stands for the vector space of all strongly measurable mappings $u :]\,0,T\,[\to X$ such that the function $t \mapsto \big|u(t)\big|_X$ is in $L^p(0,T)$ (cf. [4, Chap. III, §3; Chap. IV, §3], [5, App.], [13, Chap. 1]). For $1 \le p < +\infty$, the spaces $L^p\big(0,T;L^p(\Omega)\big)$ and $L^p(Q_T)$ are linearly isometric. Therefore, in what follows we identify these spaces.

Let H be a real Hilbert space with scalar product $(\cdot,\cdot)_H$ such that $X \subset H$ densely and continuously. Identifying H with its dual space H^* via Riesz' Representation Theorem, we obtain the continuous embedding $H \subset X^*$ and

$$\langle h, x\rangle_X = (h,x)_H \quad \forall\, h \in H,\ \forall\, x \in X. \tag{2.1}$$

Given any $u \in L^1(0,T;X)$ we identify this function with a function in $L^1(0,T;X^*)$ and denote it again by u. If there exists $U \in L^1(0,T;X^*)$ such that

$$\int_0^T u(t)\alpha'(t)dt \overset{\text{in}X^*}{=} -\int_0^T U(t)\alpha(t)dt \quad \forall\, \alpha \in C_c^\infty(\,]\,0,T\,[\,),$$

then U will be called derivative of u in the sense of distributions from $]\,0,T\,[$ into X^* and denoted by u' (see [5, App.], [13, Chap. 21]). □

Let $1 < p < +\infty$ be fixed. We make the following assumptions on the coefficients σ, κ and the right hand side f in (1.1)–(1.2):

$$\text{(H1)} \qquad \begin{cases} \sigma : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+ \ \text{ is continuous,} \\ c_1\tau^p - c_2 \le \sigma(u,\tau)\tau^2,\ 0 \le \sigma(u,\tau) \le c_3(1+\tau^2)^{(p-2)/2} \\ \forall\,(u,\tau) \in \mathbb{R}\times\mathbb{R}_+,\ \text{where } c_1, c_3 = \text{const} > 0 \text{ and } c_2 = \text{const} \ge 0; \end{cases}$$

$$\text{(H2)} \qquad \begin{cases} \kappa : \mathbb{R} \to \mathbb{R}_+ \ \text{ is continuous,} \\ 0 < \kappa_0 \le \kappa(u) \le \kappa_1 \quad \forall\, u \in \mathbb{R},\ \text{where } \ \kappa_0, \kappa_1 = \text{const}, \end{cases}$$

and

$$\text{(H3)} \qquad \begin{cases} f : Q_T \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+ \ \text{ is Carathéodory,} \\ 0 \le f(x,t,u,\xi) \le c_4\big(1+|\xi|^p\big) \\ \forall\,(x,t,u,\xi) \in Q_T\times\mathbb{R}\times\mathbb{R}^n,\ \text{where } \ c_4 = \text{const} > 0. \end{cases}$$

It is readily seen that (H1) and (H3) are satisfied by the prototypes for σ and f we have considered in Section 1. □

Definition. Assume (H1)–(H3) and suppose that the data in (1.3)–(1.5) satisfy

(2.2) $$\varphi_D \in L^p\big(0,T;W^{1,p}(\Omega)\big);$$

(2.3) $$g = \text{const}, \quad h = \text{const};$$

(2.4) $$u_0 \in L^1(\Omega).$$

The pair

$$(\varphi, u) \in L^p\big(0,T;W^{1,p}(\Omega)\big) \times L^q\big(0,T;W^{1,q}(\Omega)\big) \quad \Big(1 < q < \frac{n+2}{n+1}\Big)$$

is called *weak solution* of (1.1)–(1.5) if

$$\int_{Q_T} \sigma\big(u, |\nabla\varphi|\big)\nabla\varphi \cdot \nabla\zeta \, dxdt = 0 \quad \forall\, \zeta \in L^p\big(0,T; W^{1,p}_{\Gamma_D}(\Omega)\big); \tag{2.5}$$

$$\varphi = \varphi_D \quad \text{a.e. on } \Sigma_D; \tag{2.6}$$

$$\exists\, u' \in L^1\big(0,T; \big(W^{1,q'}(\Omega)\big)^*\big); \tag{2.7}$$

$$\begin{cases} \displaystyle\int_0^T \big\langle u'(t), v(t)\big\rangle_{W^{1,q'}} dt + \int_{Q_T} \kappa(u)\nabla u \cdot \nabla v \, dxdt + g\int_0^T\int_{\partial\Omega} (u-h)v \, d_xSdt \\ \displaystyle = \int_{Q_T} f(x,t,u,\nabla\varphi)v \, dxdt \quad \forall\, v \in L^\infty\big(0,T; W^{1,q'}(\Omega)\big); \end{cases} \tag{2.8}$$

$$u(0) = u_0 \quad \text{in } \big(W^{1,q'}(\Omega)\big)^*. \tag{2.9}$$

From (H1) and (H3) it follows that $f(\cdot,\cdot,u,\nabla\varphi)\in L^1(Q_T)$. Therefore, $u\in L^q\big(0,T;W^{1,q}(\Omega)\big)$ $\big(1 < q < \frac{n+2}{n+1}\big)$ is standard for weak solutions of parabolic equations with right hand side in L^1 (see, e.g., the papers cited in [28]).

We notice that $v \in L^\infty\big(0,T,W^{1,q'}(\Omega)\big)$ can be identified with a function in $L^\infty(Q_T)$ (cf. [28]). Hence, the integral on the right hand side of the variational identity in (2.8) is well-defined.

To make precise the meaning of (2.9), let $\frac{2n}{n+2} < q < \frac{n+2}{n+1}$. Then $\frac{nq}{n-q} > 2$ and $q' > n+2$. Identifying $L^2(\Omega)$ with its dual, we obtain

$$W^{1,q'}(\Omega) \underset{\text{continuously}}{\subset} W^{1,q}(\Omega) \underset{\text{compactly}}{\subset} L^2(\Omega) \underset{\text{continuously}}{\subset} \big(W^{1,q'}(\Omega)\big)^*. \tag{2.10}$$

Therefore, u can be identified with an element in $L^q\big(0,T;\big(W^{1,q'}(\Omega)\big)^*\big)$. Together with (2.7) this implies the existence of a function $\widetilde{u} \in C\big([0,T];\big(W^{1,q'}(\Omega)\big)^*\big)$ such that

$$\widetilde{u}(t) = u(t) \quad \text{for a.e. } t \in [0,T]$$

(see, e.g., [13, p. 45, Th. 2.2.1]).

On the other hand, there exists a uniquely determined $\widetilde{u}_0 \in \big(W^{1,q'}(\Omega)\big)^*$ such that

$$\langle \widetilde{u}_0, z\rangle_{W^{1,q'}} = \int_\Omega u_0 z \, dx \quad \forall\, z \in W^{1,q'}(\Omega). \tag{2.11}$$

Thus, (2.9) has to be understood in the sense

$$\widetilde{u}(0) = \widetilde{u}_0 \quad \text{in } \big(W^{1,q'}(\Omega)\big)^*.$$

Remark 3. Let (φ, u) be a sufficiently regular solution of (1.1)–(1.5). We multiply (1.1) and (1.2) by smooth test functions ζ and v, respectively, satisfying the conditions

$$\zeta = 0 \quad \text{on } \Sigma_D, \quad v(\cdot,T) = 0 \quad \text{in } \Omega.$$

Then we integrate the div-terms by parts over Ω and the term $\frac{\partial u}{\partial t}v$ by parts over the interval $[0,T]$. It follows

$$-\int_{Q_T} u\,\frac{\partial v}{\partial t}\,dxdt+\int_{Q_T}\kappa(u)\nabla u\cdot\nabla v\,dxdt+g\int_0^T\int_{\partial\Omega}(u-h)v\,d_xSdt$$

$$=\int_\Omega u_0 v(\cdot,0)dx+\int_{Q_T} f(x,t,u,\nabla\varphi)v\,dxdt. \tag{2.12}$$

This variational formulation of initial/boundary-value problems for parabolic equations is frequently used in the literature.

We notice that from a variational identity of type (2.12) it follows the existence of a distributional time derivative of u (see the arguments concerning (4.25) and (4.26) below).

Remark 4. Let (φ,u) be a weak solution of (1.1)–(1.5). From (2.8) it follows that, for any $z\in W^{1,q'}(\Omega)$,

$$\big\langle u'(t),z\big\rangle_{W^{1,q'}}+\int_\Omega \kappa\big(u(x,t)\big)\nabla u(x,t)\cdot\nabla z(x)dx+g\int_{\partial\Omega}\big(u(x,t)-h\big)z(x)d_xS$$

$$=\int_\Omega f\big(x,t,u(x,t),\nabla\varphi(x,t)\big)z(x)dx \tag{2.13}$$

for a.e. $t\in[0,T]$, where the null set in $[0,T]$ of those t for which (2.13) fails, does not depend on z. We integrate (2.13) (with s in place of t) over the interval $[0,t]$ $(0\le t\le T)$ and integrate the first term on the left hand side by parts. Using the above notation $\widetilde{u}$ and (2.11), we obtain

$$\big\langle \widetilde{u}(t),z\big\rangle_{W^{1,q'}}+\int_0^t\int_\Omega \kappa\big(u(x,s)\big)\nabla u(x,s)\cdot\nabla z(x)dxds+g\int_0^t\int_{\partial\Omega}\big(u(x,s)-h\big)z(x)d_xSds$$

$$=\int_\Omega u_0(x)z(x)dx+\int_0^t\int_\Omega f\big(x,s,u(x,s),\nabla\varphi(x,s)\big)z(x)dxds. \tag{2.14}$$

Let be $p=2$ and let be $f(x,t,u,\xi)=\sigma_0(u)|\xi|^2$ $\big(\big((x,t),u,\xi\big)\in Q_T\times\mathbb{R}\times\mathbb{R}^n$; cf. (1.13)$\big)$. Taking $z\equiv 1$ in (2.14), we obtain

$$\big\langle \widetilde{u}(t),1\big\rangle_{W^{1,q'}}+g\int_0^t\int_{\partial\Omega}\big(u(x,s)-h\big)d_xSds=\int_\Omega u_0(x)dx+\int_0^t\int_\Omega \boldsymbol{J}\cdot\boldsymbol{E}\,dxds,\quad t\in]0,T].$$

3. Existence of weak solutions

Our existence result for weak solutions of (1.1)–(1.5) is the following

Theorem. *Assume* (H1) *and* (H2). *Suppose further that*

$$\big(\sigma\big(u,|\xi|\big)\xi-\sigma\big(u,|\eta|\big)\eta\big)\cdot(\xi-\eta)\ge 0\quad\forall\,u\in\mathbb{R},\ \forall\,\xi,\eta\in\mathbb{R}^n, \tag{3.1}$$

and

$$
(3.2)\qquad \begin{cases} f(x,t,u,\xi) = \alpha(x,t,u)\sigma\big(u,|\xi|\big)|\xi|^2 \quad \forall\, \big((x,t),u,\xi\big) \in Q_T \times \mathbb{R} \times \mathbb{R}^n, \\ \textit{where}\ \ \alpha : Q_T \times \mathbb{R} \to \mathbb{R}_+ \ \ \textit{is Carathéodory}, \\ 0 \le \alpha(x,t,u) \le \alpha_0 = \text{const} \quad \forall\, \big((x,t),u\big) \in Q_T \times \mathbb{R}, \\ \sigma = \sigma(u,\tau) \quad \textit{as in}\ \text{(H1)}. \end{cases}
$$

Let φ_D and u_0 satisfy (2.2) *and* (2.4)*, respectively, and suppose that*

$$
(3.3)\qquad g = \text{const} > 0, \quad h = \text{const}.
$$

Then there exists a pair

$$
(\varphi, u) \in L^p\big(0,T;W^{1,p}(\Omega)\big) \times \Big(\bigcap_{1<q<(n+2)/(n+1)} L^p\big(0,T;W^{1,q}(\Omega)\big)\Big)
$$

such that

$$
(3.4)\qquad \begin{gathered} (2.5)\ \textit{and}\ (2.6)\ \textit{are satisfied}, \\ \exists\, u' \in \bigcap_{n+2<r<+\infty} L^1\big(0,T;\big(W^{1,r}(\Omega)\big)^*\big), \end{gathered}
$$

and for any $n+2 < s < +\infty$ there holds

$$
(3.5)\qquad \begin{cases} \displaystyle\int_0^T \langle u', v\rangle_{W^{1,s}} dt + \int_{Q_T} \kappa(u)\nabla u \cdot \nabla v\, dxdt + g\int_0^T\int_{\partial\Omega} (u-h)v\, d_xSdt \\ \displaystyle = \int_{Q_T} f(x,t,u,\nabla\varphi)v\, dxdt \quad \forall\, v \in L^\infty\big(0,T;W^{1,s}(\Omega)\big), \end{cases}
$$

$$
(3.6)\qquad u(0) = u_0 \quad \textit{in}\ \ \big(W^{1,s}(\Omega)\big)^*.
$$

Moreover, u satisfies

$$
(3.7)\qquad \begin{cases} \displaystyle \|u\|_{L^\infty(L^1)} + \lambda \int_{Q_T} \frac{|\nabla u|^2}{\big(1+|u|\big)^{1+\lambda}}\, dxdt \\ \le c\big(1 + \|u_0\|_{L^1} + \big\|\,|\nabla\varphi_D|\,\big\|^p_{L^p}\big), \quad 0 < \lambda < 1\ ^{3)} \end{cases}
$$

$$
(3.8)\qquad u \in \bigcap_{1<r<(n+2)/n} L^r\big(0,T;L^r(\Omega)\big).
$$

The proof of this theorem is a further development of the approximation method we used in [28]. In this paper, the function $\tau \mapsto \sigma(u,\tau)$ is assumed to satisfy the condition of strict monotonicity

$$
\big(\sigma\big(u,|\xi|\big)\xi - \sigma\big(u|\eta|\big)\eta\big) \cdot (\xi - \eta) > 0 \quad \forall\, u \in \mathbb{R},\ \ \forall\, \xi, \eta \in \mathbb{R}^n,\ \ \xi \ne \eta.
$$

This condition allows to prove that the sequence $(\nabla\varphi_\varepsilon)_{\varepsilon>0}$ converges a.e. in Q_T as $\varepsilon \to 0$, where $(\varphi_\varepsilon, u_\varepsilon)_{\varepsilon>0}$ is an approximate solution of the problem under consideration. Therefore, the discussion in [28] includes the large class of source functions f characterized by (H3).

---

3) For notational simplicity, in what follows, for indexes we write $L^p(X)$ in place of $L^p(0,T;X)$. If there is no danger of confusion, we briefly write L^p in place of $L^p(E)$ $(E \subset \mathbb{R}^m)$.

However, due to (3.1), in the present paper we have to work only with the weak convergence of the sequence $(\varphi_\varepsilon)_{\varepsilon>0}$ in $L^q\big(0,T;W^{1,q}(\Omega)\big)$ as $\varepsilon\to 0$, which in turn makes the structure condition (3.2) necessary for the passage to the limit $\varepsilon\to 0$.

4. Proof of the theorem

We begin by introducing two notations. For $\varepsilon>0$, define

$$f_\varepsilon(x,t,u,\xi)=\frac{f(x,t,u,\xi)}{1+\varepsilon f(x,t,u,\xi)},\quad \big((x,t),u,\xi\big)\in Q_T\times\mathbb{R}\times\mathbb{R}^n.$$

To our knowledge, this approximation has been introduced for the first time by Bensoussan-Frehse [2] for the study of nonlinear elliptic systems in stochastic game theory. Detailed proofs of [2] are presented in [3]. Later on the above approximation has been widely used for the study of nonlinear elliptic and parabolic problems with right hand side in L^1.

The function f_ε is Carathéodory and satisfies the inequalities

$$0\le f_\varepsilon(x,t,u,\xi)\le\frac{1}{\varepsilon}\quad\forall\,\big((x,t),u,\xi\big)\in Q_T\times\mathbb{R}\times\mathbb{R}^n.$$

Let $(u_{0,\varepsilon})_{\varepsilon>0}$ be a sequence of functions in $L^2(\Omega)$ such that $u_{0,\varepsilon}\to u_0$ strongly in $L^1(\Omega)$ as $\varepsilon\to 0$. □

We divide the proof of the theorem into five steps.

1° *Existence of approximate solutions.* We have

Lemma 1. *For every $\varepsilon>0$ there exists a pair*

$$(\varphi_\varepsilon,u_\varepsilon)\in L^p\big(0,T;W^{1,p}(\Omega)\big)\times L^2\big(0,T;W^{1,2}(\Omega)\big)$$

such that

$$\text{(4.1)}\qquad\begin{cases}\varepsilon\displaystyle\int_{Q_T}|\nabla\varphi_\varepsilon|^{p-2}\nabla\varphi_\varepsilon\cdot\nabla\zeta\,dxdt+\int_{Q_T}\sigma\big(u_\varepsilon,|\nabla\varphi_\varepsilon|\big)\nabla\varphi_\varepsilon\cdot\nabla\zeta\,dxdt\\ =0\quad\forall\,\zeta\in L^p\big(0,T;W^{1,p}_{\Gamma_D}(\Omega)\big)^{\,4)}\,;\end{cases}$$

$$\text{(4.2)}\qquad \varphi_\varepsilon=\varphi_D\quad\textit{a.e. on }\ \Sigma_D;$$

$$\text{(4.3)}\qquad \exists\,u'_\varepsilon\in L^2\big(0,T;\big(W^{1,2}(\Omega)\big)^*\big);$$

$$\text{(4.4)}\qquad\begin{cases}\displaystyle\int_0^T\langle u'_\varepsilon,v\rangle_{W^{1,2}}dt+\int_{Q_T}\kappa(u_\varepsilon)\nabla u_\varepsilon\cdot\nabla v\,dxdt+g\int_0^T\int_{\partial\Omega}(u_\varepsilon-h)v\,d_xSdt\\ =\displaystyle\int_{Q_T}f_\varepsilon(x,t,u_\varepsilon,\nabla\varphi_\varepsilon)v\,dxdt\quad\forall\,v\in L^2\big(0,T;W^{1,2}(\Omega)\big);\end{cases}$$

$$\text{(4.5)}\qquad u_\varepsilon(0)=u_{0,\varepsilon}\quad\textit{in }\ L^2(\Omega).$$

---

4) If $1<p<2$, for $z\in W^{1,p}(\Omega)$ we define $\big|\nabla z(x)\big|^{p-2}\nabla z(x)=0$ a.e. in $\{x\in\Omega;\nabla z(x)=0\}$.

Proof. To begin with, we notice that, for all $\xi, \eta \in \mathbb{R}^n$,

$$\begin{aligned}&\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big)\cdot(\xi-\eta)\\ &\geq \begin{cases} \dfrac{p-1}{\big(1+|\xi|+|\eta|\big)^{2-p}}\,|\xi-\eta|^2 & \text{if } 1<p\leq 2,\\ \min\Big\{\dfrac{1}{2},\dfrac{1}{2^{p-2}}\Big\}\,|\xi-\eta|^p & \text{if } 2<p<+\infty\end{cases}\end{aligned} \tag{4.6}$$

(cf. [25, pp. 71, 74], [28]).

For $\varepsilon > 0$ and $(u,\tau) \in \mathbb{R}\times\mathbb{R}_+$, define

$$\begin{aligned}\sigma_\varepsilon(u,0) &= \sigma(u,0) && \text{if } \tau = 0,\\ \sigma_\varepsilon(u,\tau) &= \varepsilon\tau^{p-2}+\sigma(u,\tau) && \text{if } 0<\tau<+\infty.\end{aligned}$$

Thus, by (3.1) and (4.6),

$$\big(\sigma_\varepsilon\big(u,|\xi|\big)\xi - \sigma_\varepsilon\big(u,|\eta|\big)\eta\big)\cdot(\xi-\eta) \geq \varepsilon\big(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\big)\cdot(\xi-\eta) > 0$$

for all $u \in \mathbb{R}$ and all $\xi,\eta\in\mathbb{R}^n$, $\xi\neq\eta$.

The assertion of Lemma 1 now follows from [28, Lemma 1] with σ_ε in place of σ. □

2° *A-priori estimates.* We have

Lemma 2. *Let be $(\varphi_\varepsilon, u_\varepsilon)$ as in Lemma 1. Then, for all $0<\varepsilon\leq 1$,*

$$\varepsilon\big\|\,|\nabla\varphi_\varepsilon|\,\big\|^p_{L^p} + \|\varphi_\varepsilon\|^p_{L^p(W^{1,p})} \leq c\big(1+\big\|\,|\nabla\varphi_D|\,\big\|^p_{L^p}\big) \text{ [5)] }; \tag{4.7}$$

$$\begin{cases}\|u_\varepsilon\|_{L^\infty(L^1)} + \lambda\displaystyle\int_{Q_T}\frac{|\nabla u_\varepsilon|^2}{\big(1+|u_\varepsilon|\big)^{1+\lambda}}\,dxdt\\ \leq c\big(1+\|u_{0,\varepsilon}\|_{L^1}+\big\|\,|\nabla\varphi_D|\,\big\|^p_{L^p}\big), \quad 0<\lambda<1;\end{cases} \tag{4.8}$$

$$\|u_\varepsilon\|_{L^q(W^{1,q})} \ \leq\ c \qquad \forall\, 1<q<\frac{n+2}{n+1}, \tag{4.9}$$

$$\|u_\varepsilon\|_{L^r(L^r)} \ \leq\ c \qquad \forall\, 1<r<\frac{n+2}{n}, \tag{4.10}$$

$$\|u'_\varepsilon\|_{L^1((W^{1,q'})^*)} \ \leq\ c \qquad \forall\, 1<q<\frac{n+2}{n+1}. \tag{4.11}$$

Proof. By (4.2), the function $\varphi_\varepsilon - \varphi_D$ is in $L^p\big(0,T;W^{1,p}_{\Gamma_D}(\Omega)\big)$. Inserting this function into (4.1), we find

$$\begin{aligned}&\varepsilon\int_{Q_T}|\nabla\varphi_\varepsilon|^p dxdt + \int_{Q_T}\sigma\big(u_\varepsilon,|\nabla\varphi_\varepsilon|\big)|\nabla\varphi_\varepsilon|^2 dxdt\\ &= \varepsilon\int_{Q_T}|\nabla\varphi_\varepsilon|^{p-2}\nabla\varphi_\varepsilon\cdot\nabla\varphi_D\,dxdt + \int_{Q_T}\sigma\big(u_\varepsilon,|\nabla\varphi_\varepsilon|\big)\nabla\varphi_\varepsilon\cdot\nabla\varphi_D\,dxdt.\end{aligned}$$

[5)] Without any further reference, in what follows, by c we denote constants which may change their numerical value from line to line, but do not depend on ε.

From this, (4.7) easily follows by combining (H1) and Hölder's inequality.

Estimates (4.8)–(4.11) can be proved by following line by line the proof of [28, Lemma 2]. □

3° *Convergence of subsequences.* Let be $(\varphi_\varepsilon, u_\varepsilon)$ as in Lemma 1. From (4.7) and (4.9), (4.10) we conclude that there exists a subsequence of $(\varphi_\varepsilon, u_\varepsilon)_{\varepsilon>0}$ (not relabelled) such that

$$\varphi_\varepsilon \longrightarrow \varphi \quad \text{weakly in } L^p\big(0,T;W^{1,p}(\Omega)\big) \tag{4.12}$$

and

$$\begin{cases} u_\varepsilon \to u \ \text{ weakly in } L^q\big(0,T;W^{1,q}(\Omega)\big) & \Big(1<q<\dfrac{n+2}{n+1}\Big) \\ \text{and weakly in } L^r\big(0,T;L^r(\Omega)\big) & \Big(1<r<\dfrac{n+2}{n}\Big) \end{cases} \tag{4.13}$$

as $\varepsilon \to 0$. Then (4.2) and (4.12) yield $\varphi = \varphi_D$ a.e. on Σ_D, i.e., φ satisfies (2.6).

Next, fix any $1 < q < \frac{n+2}{n+1}$. Taking into account the embeddings (2.10), from (4.9) and (4.11) we obtain by the aid of a well-known compactness result [6, Prop. 1] or [30, Cor. 4] the existence of a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ (not relabelled) such that $u_\varepsilon \to u$ strongly in $L^q\big(0,T;L^2(\Omega)\big)$, and therefore

$$u_\varepsilon \longrightarrow u \quad \text{a.e. in } Q_T \ \text{ as } \varepsilon \longrightarrow 0. \tag{4.14}$$

We prove estimate (3.7). To begin with, we find an $0 < \varepsilon_0 \le 1$ such that

$$\|u_{0,\varepsilon}\|_{L^1} \le 1 + \|u_0\|_{L^1} \quad \forall\, 0<\varepsilon\le\varepsilon_0.$$

Then, given any $\psi \in L^\infty(0,T)$, $\psi \ge 0$ a.e. in $[0,T]$, from (4.8) it follows that

$$\int\limits_{Q_T} \big|u_\varepsilon(x,t)\psi(t)\big|\,dxdt \le C_0 \int\limits_0^T \psi(t)dt \quad \forall\, 0<\varepsilon\le\varepsilon_0 \tag{4.15}$$

where

$$C_0 := c\big(1 + \|u_0\|_{L^1} + \|\,|\nabla\varphi_D|\,\|^p_{L^p}\big).$$

Taking the $\liminf\limits_{\varepsilon\to 0}$ in (4.15), we find

$$\int\limits_{Q_T} \big|u(x,t)\psi(t)\big|\,dxdt \le C_0 \int\limits_0^T \psi(t)dt.$$

Hence,

$$\int\limits_{\Omega} |u(x,t)|\,dx \le C_0 \quad \text{for a.e. } t \in [0,T].$$

Next, from (4.8) and (4.14) we infer (by passing to a subsequence if necessary) that

$$\frac{\nabla u_\varepsilon}{\big(1+|u_\varepsilon|\big)^{(1+\lambda)/2}} \longrightarrow \frac{\nabla u}{\big(1+|u|\big)^{(1+\lambda)/2}} \quad \text{weakly in } \big[L^2(Q_T)\big]^n$$

as $\varepsilon \to 0$. Then taking the $\liminf\limits_{\varepsilon\to 0}$ in (4.8) gives

$$\lambda \int\limits_{Q_T} \frac{|\nabla u|^2}{\big(1+|u|\big)^{1+\lambda}}\,dxdt \le C_0.$$

□

Summarizing, from (4.12)–(4.14) we deduced the existence of a pair

$$(\varphi, u) \in L^p\big(0,T;W^{1,p}(\Omega)\big) \times \Big(\bigcap_{1<q<(n+2)/(n+1)} L^q\big(0,T;W^{1,q}(\Omega)\big)\Big)$$

which satisfies (2.6) and (3.7), (3.8). It remains to prove that (φ, u) satisfies the variational identity in (2.5) and that (3.4)–(3.6) hold true. This can be easily done by the aid of Lemma 3 and 4 we are going to prove next.

4° *Passage to the limit $\varepsilon \to 0$.* We have

Lemma 3. *Let be $(\varphi_\varepsilon, u_\varepsilon)$ as in Lemma 1, and let be (φ, u) as in (4.12), (4.13). Then*

$$\int_{Q_T} \sigma\big(u, |\nabla\varphi|\big)\nabla\varphi\cdot\nabla\zeta\, dxdt = 0 \quad \forall\, \zeta \in L^p\big(0,T;W^{1,p}_{\Gamma_D}(\Omega)\big) \tag{4.16}$$

i.e., (φ, u) satisfies (2.5);

$$\sigma\big(u_\varepsilon, |\nabla\varphi_\varepsilon|\big)\nabla\varphi_\varepsilon \longrightarrow \sigma\big(u, |\nabla\varphi|\big)\nabla\varphi \;\; \text{weakly in} \;\; \big[L^{p'}(Q_T)\big]^n \;\; \text{as} \;\; \varepsilon \longrightarrow 0; \tag{4.17}$$

$$\sigma\big(u_\varepsilon, |\nabla\varphi_\varepsilon|\big)|\nabla\varphi_\varepsilon|^2 \longrightarrow \sigma\big(u, |\nabla\varphi|\big)|\nabla\varphi|^2 \;\; \text{weakly in} \;\; L^1(Q_T) \;\; \text{as} \;\; \varepsilon \longrightarrow 0. \tag{4.18}$$

Proof of (4.16) (cf. the "monotonicity trick" in [26, pp. 161, 172], [34, p. 474]). The function $\varphi_\varepsilon - \varphi_D$ is in $L^p\big(0,T;W^{1,p}_{\Gamma_D}(\Omega)\big)$ (see (4.2)). Thus, given any $\psi \in L^p\big(0,T;W^{1,p}_{\Gamma_D}(\Omega)\big)$, the function $\zeta = \varphi_\varepsilon - \varphi_D - \psi$ is admissible in (4.1). By the monotonicity condition (3.1) ($\xi = \nabla\varphi_\varepsilon$ and $\eta = \nabla(\psi + \varphi_D)$),

$$\begin{aligned}
0 &= \varepsilon \int_{Q_T} |\nabla\varphi_\varepsilon|^{p-2}\nabla\varphi_\varepsilon\cdot\nabla\big(\varphi_\varepsilon - (\psi+\varphi_D)\big)dxdt \\
&\quad + \int_{Q_T} \sigma\big(u_\varepsilon, |\nabla\varphi_\varepsilon|\big)\nabla\varphi_\varepsilon\cdot\nabla\big(\varphi_\varepsilon - (\psi+\varphi_D)\big)dxdt \\
&\geq -\varepsilon \int_{Q_T} |\nabla\varphi_\varepsilon|^{p-2}\nabla\varphi_\varepsilon\cdot\nabla(\psi+\varphi_D)dxdt \\
&\quad + \int_{Q_T} \sigma\big(u_\varepsilon, \big|\nabla(\psi+\varphi_D)\big|\big)\nabla(\psi+\varphi_D)\cdot\nabla\big(\varphi_\varepsilon - (\psi+\varphi_D)\big)dxdt.
\end{aligned}$$

The passage to the limit $\varepsilon \to 0$ gives

$$0 \geq \int_{Q_T} \sigma\big(u, \big|\nabla(\psi+\varphi_D)\big|\big)\nabla(\psi+\varphi_D)\cdot\nabla\big(\varphi - (\psi+\varphi_D)\big)dxdt \tag{4.19}$$

(cf. (4.7), (4.12) and (4.14)).

Let $\zeta \in L^p\big(0,T;W^{1,p}_{\Gamma_D}(\Omega)\big)$. For any $\lambda > 0$, we insert $\psi = \varphi - \varphi_D \mp \lambda\zeta$ into (4.19), divide then by λ and carry through the passage to the limit $\lambda \to 0$. It follows

$$\int_{Q_T} \sigma\big(u, |\nabla\varphi|\big)\nabla\varphi\cdot\nabla\zeta\, dxdt = 0.$$

Proof of (4.17). From (H1) and (4.7) it follows that there exists a subsequence of $(\nabla\varphi_\varepsilon)_{\varepsilon>0}$ (not relabelled) such that

$$\sigma\big(u_\varepsilon, |\nabla\varphi_\varepsilon|\big)\nabla\varphi_\varepsilon \longrightarrow \boldsymbol{F} \quad \text{weakly in } \big[L^{p'}(Q_T)\big]^n \text{ as } \varepsilon \longrightarrow 0.$$

The function $\zeta = \varphi - \varphi_D$ being admissible in (4.1), we find

$$\int\limits_{Q_T} \boldsymbol{F}\cdot\nabla(\varphi-\varphi_D)dxdt = \lim_{\varepsilon\to 0}\int\limits_{Q_T}\sigma\big(u_\varepsilon, |\nabla\varphi_\varepsilon|\big)\nabla\varphi_\varepsilon\cdot\nabla(\varphi-\varphi_D)dxdt = 0.$$

Thus, using (4.1) with $\zeta = \varphi_\varepsilon - \varphi_D$, it follows

$$\begin{aligned} \int\limits_{Q_T} \boldsymbol{F}\cdot\nabla\varphi\, dxdt &= \int\limits_{Q_T}\boldsymbol{F}\cdot\nabla\varphi_D\, dxdt \\ &= \lim_{\varepsilon\to 0}\int\limits_{Q_T}\sigma\big(u_\varepsilon, |\nabla\varphi_\varepsilon|\big)\nabla\varphi_\varepsilon\cdot\nabla\varphi_D\, dxdt \\ &\geq \liminf_{\varepsilon\to 0}\int\limits_{Q_T}\sigma\big(u_\varepsilon, |\nabla\varphi_\varepsilon|\big)|\nabla\varphi_\varepsilon|^2 dxdt. \end{aligned} \tag{4.20}$$

Claim (4.17) is now easily seen by the aid of the "monotonicity trick" with respect to the dual pairing $\big(\big[L^p(Q_T)\big]^n, \big[L^{p'}(Q_T)\big]^n\big)$. Indeed, let $\boldsymbol{G}\in\big[L^p(Q_T)\big]^n$. Using (3.1) with $\xi = \boldsymbol{G}$, $\eta = \nabla\varphi_\varepsilon$, we find by the aid of (4.12), (4.20) and Lebesgue's Dominated Convergence Theorem

$$\int\limits_{Q_T}\sigma\big(u, |\boldsymbol{G}|\big)\boldsymbol{G}\cdot(\boldsymbol{G}-\nabla\varphi)dxdt \geq \int\limits_{Q_T}\boldsymbol{F}\cdot(\boldsymbol{G}-\nabla\varphi)dxdt.$$

Hence, given $\boldsymbol{H}\in\big[L^p(Q_T)\big]^n$ and $\lambda>0$, we take $\boldsymbol{G} = \nabla\varphi\pm\lambda\boldsymbol{H}$, divide by $\lambda>0$ and carry through the passage to the limit $\lambda\to 0$ to obtain

$$\int\limits_{Q_T}\sigma\big(u, |\nabla\varphi|\big)\nabla\varphi\cdot\boldsymbol{H}\, dxdt = \int\limits_{Q_T}\boldsymbol{F}\cdot\boldsymbol{H}\, dxdt.$$

Whence (4.17).

Proof of (4.18). Define

$$g_\varepsilon = \big(\sigma\big(u_\varepsilon, |\nabla\varphi_\varepsilon|\big)\nabla\varphi_\varepsilon - \sigma\big(u_\varepsilon, |\nabla\varphi|\big)\nabla\varphi\big)\cdot\nabla(\varphi_\varepsilon-\varphi) \quad \text{a.e. in } Q_T.$$

By the aid of (4.17), (4.16) and $u_\varepsilon\to u$ a.e. in Q_T (see (4.14)) one easily obtains

$$\lim_{\varepsilon\to 0}\int\limits_{Q_T} g_\varepsilon\, dxdt = 0.$$

By (3.1), $g_\varepsilon\geq 0$ a.e. in Q_T. Thus

$$\lim_{\varepsilon\to 0}\int\limits_{Q_T} g_\varepsilon z\, dxdt = 0 \quad \forall\, z\in L^\infty(Q_T). \tag{4.21}$$

We next multiply each term of the equation

$$\sigma\big(u_\varepsilon, |\nabla\varphi_\varepsilon|\big)|\nabla\varphi_\varepsilon|^2 = g_\varepsilon + \sigma\big(u_\varepsilon, |\nabla\varphi_\varepsilon|\big)\nabla\varphi_\varepsilon\cdot\nabla\varphi + \sigma\big(u_\varepsilon, |\nabla\varphi|\big)\nabla\varphi\cdot\nabla(\varphi_\varepsilon-\varphi)$$

by $z \in L^\infty(Q_T)$ and integrate over Q_T. Then (4.18) follows from (4.21), (4.17) and (4.14), (4.12). □

The next lemma is fundamental to the passage to the limit $\varepsilon \to 0$ in (4.4).

Lemma 4. *Let be* $(\varphi_\varepsilon, u_\varepsilon)$ *as in Lemma* 1, *and let be* (φ, u) *as in* (4.12), (4.13). *Then, for any* $z \in L^\infty(Q_T)$,

$$\lim_{\varepsilon\to 0} \int_{Q_T} f_\varepsilon(x,t,u_\varepsilon,\nabla\varphi_\varepsilon) z\,dxdt = \int_{Q_T} f(x,t,u,\nabla\varphi) z\,dxdt. \tag{4.22}$$

Proof. For notational simplicity, we write $(\cdot,\cdot)$ in place of the variables (x,t).

The structure condition (3.2) and the definition of f_ε yield

$$\int_{Q_T} \frac{f(\cdot,\cdot,u_\varepsilon,\nabla\varphi_\varepsilon)}{1+\varepsilon f(\cdot,\cdot,u_\varepsilon,\nabla\varphi_\varepsilon)}\, z\,dxdt - \int_{Q_T} f(\cdot,\cdot,u,\nabla\varphi) z\,dxdt = J_{1,\varepsilon} + J_{2,\varepsilon} + J_{3,\varepsilon}$$

where

$$\begin{aligned} J_{1,\varepsilon} &= \int_{Q_T} A_\varepsilon B_\varepsilon dxdt, \\ A_\varepsilon &= z\alpha(\cdot,\cdot,u_\varepsilon)\Big(\frac{1}{1+\varepsilon\alpha(\cdot,\cdot,u_\varepsilon)\sigma\big(u_\varepsilon,|\nabla\varphi_\varepsilon|\big)|\nabla\varphi_\varepsilon|^2} - 1\Big) \\ B_\varepsilon &= \sigma\big(u_\varepsilon,|\nabla\varphi_\varepsilon|\big)|\nabla\varphi_\varepsilon|^2, \end{aligned}$$

and

$$\begin{aligned} J_{2,\varepsilon} &= \int_{Q_T} z\big(\alpha(\cdot,\cdot,u_\varepsilon) - \alpha(\cdot,\cdot,u)\big)B_\varepsilon\,dxdt, \\ J_{3,\varepsilon} &= \int_{Q_T} z\alpha(\cdot,\cdot,u)\big(B_\varepsilon - \sigma\big(u,|\nabla\varphi|\big)|\nabla\varphi|^2\big)dxdt. \end{aligned}$$

Observing that $0 \le \alpha \le \alpha_0 = \text{const}$ a.e. in Q_T (see (3.2)), we find

$$|A_\varepsilon| \le \alpha_0 \|z\|_{L^\infty} \quad \text{a.e. in } Q_T, \quad \forall\,\varepsilon > 0. \tag{4.23}$$

On the other hand, from

$$\int_{Q_T} \alpha(\cdot,\cdot,u_\varepsilon)\sigma\big(u_\varepsilon,|\nabla\varphi_\varepsilon|\big)|\nabla\varphi_\varepsilon|^2 dxdt \le c \quad \forall\,\varepsilon > 0$$

it follows (by going to a subsequence if necessary) that

$$\varepsilon\alpha(\cdot,\cdot,u_\varepsilon)\sigma\big(u_\varepsilon,|\nabla\varphi_\varepsilon|\big)|\nabla\varphi_\varepsilon|^2 \longrightarrow 0 \quad \text{a.e. in } Q_T \text{ as } \varepsilon \longrightarrow 0.$$

Hence,

$$A_\varepsilon \longrightarrow 0 \quad \text{a.e. in } Q_T \text{ as } \varepsilon \longrightarrow 0. \tag{4.24}$$

From (4.23), (4.24) and $B_\varepsilon \to \sigma\big(u,|\nabla\varphi|\big)|\nabla\varphi|^2$ weakly in $L^1(Q_T)$ (see (4.18)) we conclude with the help of Egorov's theorem and the absolute continuity of the integral that

$$J_{1,\varepsilon} = \int_{Q_T} A_\varepsilon B_\varepsilon\,dxdt \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0$$

(see, e.g., [16, p. 54, Prop. 1 (i)]). Analogously,

$$J_{k,\varepsilon} \longrightarrow 0 \quad \text{as} \ \ \varepsilon \longrightarrow 0 \quad (k = 2, 3).$$

Whence (4.22). □

5° *Proof of* (3.4)–(3.6). Let $n+2 < r < +\infty$ (i.e., setting $q = r'$, then $1 < q < \frac{n+2}{n+1}$, $q' = r$, and vice versa).

Let be $z \in W^{1,r}(\Omega)$ and $\psi \in C^1\big([0,T]\big)$, $\psi(T) = 0$. We set $v(x,t) = z(x)\psi(t)$ for a.e. $(x,t) \in Q_T$. An integration by parts gives

$$\begin{aligned}\int_0^T \langle u_\varepsilon', v\rangle_{W^{1,2}} dt &= -\big\langle u_\varepsilon(0), z\big\rangle_{W^{1,2}}\psi(0) - \int_0^T \langle z\psi', u_\varepsilon\rangle_{W^{1,2}} dt \\ &= -\int_\Omega u_\varepsilon(\cdot,0) z\, dx\, \psi(0) - \int_{Q_T} u_\varepsilon z\psi' dx dt \qquad \text{[by (2.1)]}\end{aligned}$$

(see [13, p. 54, Prop. 2.5.2 with $p = q = 2$, $r = 1$ therein]).

With the help of (4.13), (4.14) and (4.22) the passage to the limit $\varepsilon \to 0$ in (4.4) (with $v = z\psi$ therein) is easily done. We find

$$\begin{aligned}&-\int_{Q_T} uz\psi' dx dt + \int_{Q_T} \kappa(u)\nabla u \cdot \nabla z\psi\, dx dt + g\int_0^T\int_{\partial\Omega} (u-h)z\psi\, d_xS dt \\ &= \int_\Omega u_0 z\, dx \psi(0) + \int_{Q_T} f(x,t,u,\nabla\varphi) z\psi\, dx dt\end{aligned} \tag{4.25}$$

(recall $u_\varepsilon(\cdot,0) = u_{0,\varepsilon} \to u_\varepsilon$ strongly in $L^1(\Omega)$). Following line by line the arguments in [28], from (4.25) we deduce the existence of the distributional derivative

$$u' \in L^1\big(0,T; \big(W^{1,r}(\Omega)\big)^*\big)$$

(cf. [5, p. 154, Prop. A6]), i.e., (3.4) holds. Moreover, we have

$$\int_0^T \big\langle u'(t), z\psi(t)\big\rangle_{W^{1,r}} dt + \big\langle \widetilde{u}(0), z\big\rangle_{W^{1,r}}\psi(0) = -\int_{Q_T} uz\psi' dx dt \qquad \text{[by (2.1)]}, \tag{4.26}$$

where $\widetilde{u} \in C\big([0,T]; \big(W^{1,r}(\Omega)\big)^*\big)$ is as in Section 2 (see [13, p. 54, Prop. 2.5.2 with $p = 1$, $q = +\infty$, $r = 1$ therein]). We insert (4.26) into (4.25) and obtain

$$\begin{aligned}&\int_0^T \big\langle u'(t), z\psi(t)\big\rangle_{W^{1,r}} dt + \big\langle \widetilde{u}(0), z\big\rangle_{W^{1,r}}\psi(0) \\ &+ \int_{Q_T} \kappa(u)\nabla u \cdot \nabla z\psi\, dx dt + g\int_0^T\int_{\partial\Omega} (u-h)z\psi\, d_xS dt \\ &= \int_\Omega u_0 z\, dx\, \psi(0) + \int_{Q_T} f(x,t,u,\nabla\varphi) z\psi\, dx dt\end{aligned} \tag{4.27}$$

for all $z \in W^{1,r}(\Omega)$ and all $\psi \in C^1\big([0,T]\big)$, $\psi(T)=0$.

To prove (3.5), we take $\psi \in C_c^1\big(\,]\,0,T\,[\,\big)$ in (4.27). A routine argument yields

$$\begin{aligned}\big\langle u'(t), z\big\rangle_{W^{1,r}} &+ \int_\Omega \kappa(u)\nabla u\cdot\nabla z\,dx + g\int_{\partial\Omega}(u-h)z\,d_xS\\ &= \int_\Omega f(x,t,u,\nabla\varphi)z\,dx \end{aligned}\tag{4.28}$$

for all $z \in W^{1,r}(\Omega)$ and a.e. $t \in [0,T]$, where the null set in $[0,T]$ of those t for which (4.28) fails, does not depend on z. Now, given $v \in L^\infty\big(0,T;W^{1,s}(\Omega)\big)$ $(n+2<s<+\infty)$, we insert $z = v(\cdot,t)$ into (4.28) (with $r=s$ therein) and integrate over the interval $[0,T]$. Whence (3.5).

Equ. (3.6) in $\big(W^{1,s}(\Omega)\big)^*$ is now easily seen. Indeed, let $z \in W^{1,s}(\Omega)$ $(n+2<s<+\infty)$, and let $\psi \in C^1\big([0,T]\big)$, $\psi(0)=1$ and $\psi(T)=0$. We multiply (4.28) by $\psi(t)$ and integrate over $[0,T]$. Combining (4.27) and (4.28), we obtain

$$\big\langle \widetilde{u}(0), z\big\rangle_{W^{1,s}} = \int_\Omega u_0 z\,dx,$$

i.e., (3.6) holds (cf. (2.11) with $q'=s$ therein).

The proof of the theorem is complete.

References

[1] S. N. Antontsev and M. Chipot, *The thermistor problem: existence, smoothness, uniqueness, blowup*, SIAM J. Math. Anal. **25** (1994), 1128–1156.

[2] A. Bensoussan and J. Frehse, *Jeux différentiels stochastique et systèmes d'équations aux dérivées partielles non linéaires*, C. R. Acad. Sc. Paris, sér. I **293** (1981), 187–190.

[3] A. Bensoussan and J. Frehse, *Nonlinear elliptic systems in stochastic game theory*, J. Reine Angew. Math. **350** (1984), 23–67.

[4] N. Bourbaki, *Éléments de Mathématique*, Livre VI, Intégration 1–4, Hermann, Paris, 1965.

[5] H. Brezis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Publ. Comp., Amsterdam, 1973.

[6] F. E. Browder, *Strongly nonlinear parabolic equation of higher order*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Natur., Rend. Lincei (9) Mat Appl. **77** (1986), 159–172.

[7] M. Bulíček, A. Glitzky and M. Liero, *Systems describing electrothermal effects with $p(x)$-Laplacian-like structure for discontinuous variable exponents*, SIAM J. Math. Anal. **48** (2016), 3496-3514.

[8] M. Bulíček, A. Glitzky and M. Liero, *Thermistor systems of $p(x)$-Laplace-type with discontinuous exponents via entropy solutions*, `www.wias-berlin.de/publications/wias-publ./no.2247`.

[9] G. Cimatti, *Existence of weak solutions for the nonstationary problem of the Joule heating of a conductor*, Ann. Mat. Pura Appl. **162** (1992), 33–42.

[10] G. Cimatti, *The thermistor problem with Robin boundary condition*, Rend. Semin. Mat. Univ. Padova **135** (2016), 175–199.

[11] G. Cimatti, *Voltage-current characteristics of varistors and thermistors*, Boll. Unione Mat. Ital. (9) **2** (2009), 635–650.

[12] G. Cimatti, *The mathematics of varistors*, Euro. Jnl of Applied Mathematics (to appear).

[13] J. Droniou, *Intégration et espaces de Sobolev à valeurs vectorielles*, `www-gm3.univ-mrs.fr/polys/`.

[14] A. Fischer, P. Pahner, B. Lüssem, K. Leo, R. Scholz, T. Koprucki, J. Fuhrmann, K. Gärtner and A. Glitzky, *Self-heating, bistability, and thermal switching in organic semiconductors*, Phys. Rev. Letters **110** (2013), 126601.

[15] A. C. Fowler, I. Frigaard and S. D. Howison, *Temperature surges in current-limiting circuit devices*, SIAM J. Appl. Math. **52** (1992), 998–1011.

[16] M. Giaquinta, J. Modica and J. Souček, *Cartesian currents in the calculus of variations*, vol. I, Springer-Verlag, Berlin 1998.

[17] A. Glitzky and M. Liero, *Analysis of $p(x)$-Laplace thermistor models describing the electrothermal behavior of organic semiconductor devices*, www.wias-berlin.de/publications/wias-publ./no.2143.

[18] M. T. Gonzáles and F. Ortegón Gallego, *The evolution thermistor problem with degenerate thermal conductivity*, Comm. Pure Appl. Anal. **1** (2002), 313–325.

[19] M. T. Gonzáles Montesinos and F. Ortegón Gallego, *The evolution thermistor problem under the Wiedemann-Franz law with metallic conduction*, Discrete Cont. Dyn. Systems, Series B **8** (2007), 901–923.

[20] M. T. Gonzáles Montesinos and F. Ortegón Gallego, *The thermistor problem with degenerate thermal conductivity and metallic conduction*, Discrete Cont. Dyn. Systems, Suppl. 2007, 446–455.

[21] J. Haskovec and C. Schmeiser, *Transport in semiconductors at saturated velocities*, Comm. Math. Sci. **3** (2005), 219–233.

[22] S. D. Howison, J. F. Rodrigues and M. Shillor, *Stationary solutions to the thermistor problem*, J. Math. Anal. Appl. **174** (1993), 573–588.

[23] K. A. Jenkins and K. Rim, *Measurements of the effect of self-heating in strained-silicon MOSFETs*, IEEE Electr. Device Lett. **23** (2002), 360–362.

[24] M. Liero, T. Koprucki, A. Fischer, R. Scholz and A. Glitzky, *p-Laplace thermistor modeling of electrothermal feedback in organic semiconductor devices*, Z. Angew. Math. Phys. **66** (2015), 2957–2977.

[25] P. Lindqvist, *Notes on the p-Laplace equation*, Report. University of Jyväskylä, Department of Mathematics and Statistics, 102. Univ. Jyväskylä 2006, 80 pp.

[26] J.-L. Lions, *Quelques méthodes de résolution de problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[27] P. Markowich, C. Ringhofer and C. Schmeißer, *Semiconductor equations*, Springer-Verlag, Wien, New York, 1990.

[28] J. Naumann, *On the existence of weak solutions of a thermistor system with p-Laplacian type equation: the unsteady case*, Discrete Cont. Dyn. Systems, Series S (to appear).

[29] M. P. Shaw, V. V. Mitin, E. Schöll and H. L. Gubin, *The physics of instabilities in solid state electron devices*, Plenum Press, New York, 1992.

[30] J. Simon, *Compact sets in the spaces $L^p(0,T;B)$*, Annali Mat. Pura Appl. **146** (1987), 65–96.

[31] S. M. Sze and K. Ng Kwok, *Physics of semiconductor devices*, 3rd ed., J. Wiley, New Jersey, 2007.

[32] X. Xu, *A p-Laplacian problem in L^1 with nonlinear boundary conditions*, Comm. Part. Diff. Equs. **19** (1994), 143–176.

[33] G. Yuan and Z. Liu, *Existence and uniqueness of the C^α solution for the thermistor problem with mixed boundary value*, SIAM J. Math. Anal. **25** (1994), 1157–1166.

[34] E. Zeidler, *Nonlinear functional analysis and its applications. II/B: Nonlinear monotone operators*, New York, Berlin, Springer-Verlag, 1990.

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