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DEVELOPMENT OF DOMINATING WAVES FROM SMALL DISTURBANCES IN FALLING VISCOUS-LIQUID FILMS

G. M. Sisoev and V. Ya. Shkadov

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For wavy liquid films, the principle of selection of the periodic solutions realized experimentally as regular waves is justified. By means of numerical methods, the bifurcations of the families of steady periodic waves and the attractors of the corresponding nonstationary problem are systematically studied. A comparison of the bifurcations and the attractors shows that, when several periodic solutions exist for a given wave number, the solution with the maximum wave amplitude and the maximum phase velocity develops from small initial disturbances (the dominating wave regime). With wave number variation, near the bifurcation points the attractor passes discontinuously from one family to another. This passage is accompanied by the appearance of two-periodic solutions in small neighborhoods of these points. The relations between the calculated parameters of the dominating waves are in a good agreement with all the available experimental data.

Capillary liquid-film flows demonstrate a variety of instabilities and developing wave structures. Researchers are attracted by the fact that there are technical means of experimentally registering the waves and measuring the wave parameters, which is not usually possible in other unstable fluid flows. Periodic waves of two kinds traveling with different phase velocities have been revealed: (i) slow waves with a single film thickness maximum per period and (ii) fast waves with a steep leading front preceded by a capillary ripple and a gradually sloping trailing front [1]. The surface profiles of the first kind resemble a harmonic wave while those of the second kind are more similar to a solitary wave. When small periodic disturbances are introduced in the entrance flow region, clearly registered regular waves develop.

The theoretical explanation of the experimental data on regular nonlinear waves in films with finite liquid flow rates is based of the equations derived in [2]. In [2], the first nonlinear periodic solution family was constructed and the parameters of waves of the first kind observed experimentally were calculated. In later studies [3–6], the class of nonlinear wave solutions was considerably extended. A periodic solution family of the second kind consistent with the data of [1] was found in [5]. In [4], solutions in the form of solitary waves were obtained. In [5, 6], two solitary wave sequences (fast and slow) were constructed and the existence of other nonlinear wave solutions of the equations of [2] was demonstrated.

While explaining the experimental results, the theoretical studies [3–6] raised a new fundamental problem. It turned out that, for given values of the governing parameters (the mean film thickness and the wavelength), several periodic solutions exist, each of which determines a regular wave regime. At the same time, in experiments under these conditions a perfectly definite regular regime develops from small disturbances and the results can be repeated in different experiments. The problem is to determine the principle of selection of these dominating solutions from the set of possible solutions.

In this paper, we give a detailed justification of the selection principle for dominating wave regimes. For this purpose, we performed extensive calculations of the periodic solution family bifurcations and the nonstationary problem attractors. Our purpose was to analyze the set of families of periodic and solitary wave solutions, to study the relations between the families and their conditions of formation from small disturbances. Below, universal relations between the dominating wave parameters (independent of the particular liquid properties) are obtained and used for comparison with experiment.

1. BASIC EQUATIONS AND THEIR SLIGHTLY NONLINEAR APPROXIMATIONS

For the mathematical description of the nonlinear waves developing due to hydrodynamic instability in thin liquid films falling along solid surfaces, we will use the equations derived in [2]:

$$\begin{gathered}\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}=0\\ \frac{\partial q}{\partial t}+\frac{6}{5}\frac{\partial}{\partial x}\left(\frac{q^2}{h}\right)=\frac{1}{5\delta}\left(h\frac{\partial^3 h}{\partial x^3}+h-\frac{q}{h^2}\right)\end{gathered}\tag{1.1}$$

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Here, $h(x, t)$ and $q(x, t)$ are the local values of the nondimensional layer thickness and the liquid flow rate. The mean layer thickness h_* and the velocity $U_* = gh_*^2/(3\nu)$ are taken as the scales; here, ν is the kinematic viscosity of the liquid, and g is the gravity force acceleration. The dimensional coordinate x', time t', flow rate q', and thickness h' are calculated using the formulas:

$$x' = xh_*\gamma^{1/3}\mathrm{Re}^{-2/9}, \quad t' = th_*\gamma^{1/3}U_*^{-1}\mathrm{Re}^{-2/9}, \quad q' = qh_*U_*, \quad h' = hh_*$$

$$\delta = 45^{-1}\gamma^{-1/3}\mathrm{Re}^{11/9}, \quad \gamma = \sigma\rho^{-1}\nu^{-4/3}g^{-1/3}, \quad \mathrm{Re} = 3U_*h_*\nu^{-1}$$

Here, ρ is the density and σ is the surface tension coefficient. The variable transformations containing the parameter δ were proposed in [4]. These transformations make it possible to apply the results of analyzing the solutions of (1.1) to flows of different liquids.

Equations (1.1) were derived in [2] from the complete boundary-value problem for the Navier-Stokes equations in two steps. First, the problem was reduced to the equations of the boundary layer with a self-induced pressure gradient. Then, the Galerkin method with a single approximating function along the normal to the surface was used. A generalization of the derivation of the equations for several basis functions is given in a paper by the same author[1].

For steady-state waves traveling with a constant phase velocity c, if the solution form is $h(\xi)$, $q(\xi)$, $\xi = x - ct$, the system (1.1) can be reduced to a single differential equation [2–6]:

$$h^3\frac{d^3h}{d\xi^3} + \delta\left[6(q_0 - c)^2 - c^2h^2\right]\frac{dh}{d\xi} + h^3 - q_0 - c(h - 1) = 0 \qquad (1.2)$$

$$q(\xi) = q_0 + c(h - 1)$$

The mean liquid flow rate q_0 and the phase velocity c are determined together with $h(\xi)$ in the course of solving the problem.

We note the limiting asymptotic slightly-nonlinear forms of Eqs. (1.1) for $\delta \to 0$ (infinitely small film thickness). Introducing the compressed coordinates $\tau = \varepsilon t$, $\eta = \varepsilon x$, $\varepsilon = \alpha_0 \equiv (15\delta)^{1/2}$, from (1.1) we subsequently obtain:

$$q = h^3 + O(\varepsilon^3)$$

$$\frac{\partial q}{\partial \tau} = 3h^2\frac{\partial h}{\partial \tau} + O(\varepsilon^3) = -3h^2\frac{\partial q}{\partial \eta} + O(\varepsilon^3) = -9h^4\frac{\partial h}{\partial \eta} + O(\varepsilon^3) \qquad (1.3)$$

Substituting (1.3) in the second of equations (1.1) gives:

$$q = h^3 + \varepsilon^3\left(h^3\frac{\partial^3 h}{\partial \eta^3} + h^6\frac{\partial h}{\partial \eta}\right) + O(\varepsilon^6)$$

Then, from the first of equations (1.1) we obtain:

$$\frac{\partial h}{\partial \tau} + 3h^2\frac{\partial h}{\partial \eta} + \varepsilon^3\frac{\partial}{\partial \eta}\left(h^3\frac{\partial^3 h}{\partial \eta^3} + h^6\frac{\partial h}{\partial \eta}\right) + O(\varepsilon^6) = 0 \qquad (1.4)$$

Changing variables in (1.4) gives:

$$\frac{\partial H}{\partial \omega} + \frac{\partial}{\partial \zeta}\left(CH + 3H^2 + \frac{\partial^3 H}{\partial \zeta^3} + \frac{\partial H}{\partial \zeta}\right) + O(\varepsilon^3) = 0 \qquad (1.5)$$

$$h = 1 + \varepsilon^3 H, \quad \zeta = \eta - (3 - C\varepsilon^3)\tau, \quad \omega = \varepsilon^3\tau$$

For the steady-state waves, from (1.5) we obtain the equation

$$\frac{d^3H}{d\zeta^3} + \frac{dH}{d\zeta} + 3H^2 + CH = \mathrm{const} \qquad (1.6)$$

which is an asymptotic limiting form of Eq. (1.2) [2].

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[1] V. Ya. Shkadov, V. E. Epikhin, E. A. Demekhin, A. V. Bunov, and L. V. Filyand, "Stability of the flows with contact surfaces (liquid layers, capillary jets) [in Russian]," *Report of Inst. Mech. MGU*, No. 2564, Moscow (1981).

The model equations (1.5), (1.6), first derived for a qualitative analysis of possible wave structures in films with finite flow rates[2], were later obtained in studying some problems of chemical kinetics and diffusion processes. This indicates that, apart from the narrow application to wave films, the basic system (1.1) is of more general interest as a system with dispersion and dissipation [7–9].

The detailed analysis of the bifurcations and the attractors of Eq. (1.5) given in [9] is used here to assign initial approximate values of the parameters c and q_0 and to check the calculation accuracy of the solutions of (1.1) and (1.2) for finite values of δ.

2. BIFURCATIONS OF THE STEADY-STATE WAVES

The trivial solution of (1.1) $h=1$, $q=1$ corresponding to a possible waveless flow is linearly unstable with respect to space-periodic disturbances $h_1\exp(i\alpha x)$, $q_1(t)\exp(i\alpha x)$ for the wave numbers $\alpha \in (0, \alpha_0)$. This allows us to consider a two-parameter family of solutions of (1.1) periodic in x and determined by the values of δ and α.

These solutions of (1.1) with the wave number α can be represented as Fourier expansions:

$$h = \sum_{k=-N}^{N} h_k(t) \exp(i\alpha kx), \qquad q = \sum_{k=-N}^{N} q_k(t) \exp(i\alpha kx) \tag{2.1}$$
$$h_k = h^*_{-k}, \qquad q_k = q^*_{-k}, \qquad h_0 = 1$$

Here, the asterisk signifies complex conjugacy. The expansion coefficients $h_k(t)$, $q_k(t)$ are determined by numerical integration of the dynamic system resulting from applying the Galerkin method [1, 2] to (1.1). At the initial instant, we assign the values of $h_k(0)$ and $q_k(0)$. The numerical solution of this Cauchy problem describes the development of the initial disturbances in time and, for $t \to \infty$, the formation of the limiting wave structures (the attractors of system (1.1)) for various δ and α.

The study of the bifurcations of steady-state waves periodic in ξ reduces to seeking nontrivial solutions of the nonlinear algebraic system resulting from the application of the Galerkin method to (1.2). If for some δ and α one such solution is known, continuation in the parameter α makes it possible to construct the solution family originating at the bifurcation point [5]. The solutions for one family of waves is calculated by passing to a fictitious system of differential equations in accordance with the invariant imbedding method. These differential system solutions were corrected using the Newton-Kantorovich method (see footnote[1]).

The calculations, made with carefully controlled accuracy, showed a multiform pattern of bifurcations of the periodic solutions of Eq. (1.2) which with increase in δ becomes considerably more complex.

We will characterize each periodic solution by the wave profile $h(\xi)$ and two parameters: the phase velocity c and the mean flow rate q_0. We will also use the maximum and minimum film thicknesses h_{max} and h_{min} and the wave amplitude $a=h_{max} - h_{min}$.

The first family of periodic solutions of (1.2) branches smoothly from the trivial solution on the neutral curve at $s\equiv\alpha/\alpha_0=1$ [2]. First family solutions exist for any $s \in (0, 1]$ and for $s \to 0$ are transformed into concave solitary waves with the phase velocity $c < 3$. In [5], a second family of periodic solutions of (1.2) branching discontinuously from the first family in the neighborhood of $s_2 \approx 0.5$ was obtained. Second family solutions exist for any $s \in (0, s_2(\delta)]$ and, for $s \to 0$, transform into convex solitary waves with $c > 3$. In [5], the existence of other solutions in the form of periodic steady-state waves with bifurcation points in neighborhoods of the values $s=1/3$, $1/4$, ... was predicted. Convincing confirmation was obtained by studying the solitary waves. The first solutions of (1.2) in solitary wave form were obtained in [4]. Then, in [6] for any $\delta \in [0; 0.2]$ two sequences of solutions of (1.2) corresponding to solitary waves (for brevity, they will be called solitons) were constructed. The phase velocity of the fast waves is $c > 3$, while that of the slow waves is $c < 3$. It was shown that the slow solitons with minimum phase velocity c are related by the passage to the limit $s \to 0$ with the first family, while the fast solitons with maximum c are related with the second family of periodic solutions. The values of c for both families approach $c=3$. The existence of a spectrum of solitons, into which the periodic waves can transform at $s \to 0$, allows us to assume the existence of bifurcation sequences for the periodic steady-state solution families. Systematic thorough calculations make it possible to construct the ordered bifurcation set as a whole. Below, we present some of our results.

For any δ there exist m ($m \geq 2$) families of periodic steady-state solutions $\gamma_k\{h(\xi), c, q_0\}$ of Eq. (1.2) with m independent of δ.

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[2]V. Ya. Shkadov, *Problems of Nonlinear Hydrodynamic Stability of Viscous Liquid Layers, Capillary Jets, and Internal Flows* [in Russian], Diss. D. Sci., Moscow State Univ., Mech-Math. Dep., Moscow (1973).

TABLE 1

	$j=2$				$j=3$			
	s	c	q_0	a	s	c	q_0	a
s_j	0.4763	2.914	1.040	0.3401	0.3208	2.932	1.032	0.2975
s_j'	0.5056	2.976	1.123	0.6996	0.3839	3.193	1.174	0.8561

We will call an arbitrary γ_k the family of the k-th bifurcation. As in [5], the families γ_l and γ_m will be called "the first and second families", respectively. This emphasizes that these families exist at any δ and represent two fundamentally different wave types. The families $\gamma_2,...,\gamma_{(m-1)}$ will be called "additional" because they appear at finite values of δ and their number increases with increase in this parameter. In the series of particular calculations for $\delta=0.04$, 0.1, 0.15, and 0.247, we obtained the bifurcation numbers $m=2$, 3, 4, and 7, respectively.

Each family γ_k ($k=1, 2,..., m$) originates at the bifurcation point s_k and exists for $s \in (0, s_k']$, where $s_k \le s_k'$. The first main family smoothly branching from the trivial solution takes the maximum value $s_1=s_1'=1$. The minimum value s_m belongs to the second main family, and with increase in δ the value s_m is displaced toward the point $s=0$. The intermediate bifurcation points $s_2 > s_3 > ... > s_{m-1}$ belong to additional families.

The first main family γ_1 appears as a 2π-periodic harmonic wave with an infinitely small amplitude at $s_1=1$. Each subsequent family γ_k ($k=2,..., m$) appears at the bifurcation point s_k as a $2\pi/k$-periodic and almost harmonic solution with a finite amplitude. In the direction of smaller s from the bifurcation point all the families are continued up to $s=0$, with the families $\gamma_1,..., \gamma_{m-1}$ transforming into negative (slow) solitary waves and the second main family into a positive (fast) solitary wave. In the direction of increasing values of s from the bifurcation point each family $\gamma_2,..., \gamma_m$ exists on a small interval $s \in [s_k, s_k']$. On this interval there exist at least two solutions of a given family for any s. In particular, this means that at the bifurcation point two solutions appear.

For instance, for $\delta=0.1$ there are three bifurcations. The values of s_k, s_k' are given in Tab. 1. The above-mentioned features of the new family bifurcations are illustrated in Figs. 1 and 2. In Fig. 1, the continuous lines correspond to three families of periodic solutions in the plane (s, c). In Fig. 2*a*, an additional family γ_2 is plotted, while Fig. 2*b* shows the second main family γ_3. These two families appear near the points $s=1/2$ and $s=1/3$ as $2\pi/2$- and $2\pi/3$-periodic solutions with a finite amplitude. With decrease in s, the evolution of the profiles of the waves of each family from harmonic to soliton becomes clear.

As is clear from Fig. 1, for a given $\delta=0.1$ there may exist from one to five steady-state wave regimes which differ with respect to the values of a, q_0, c depending on the value of $s \in (0, 1]$. For the case $\delta=0.15$, for which in Fig. 4 we have plotted four families of periodic solutions in the plane (h_{max}, c), the number of solutions varies from one to seven. This figure clearly shows that the second, third, and fourth families branch off the first basic family.

3. ATTRACTORS AND DOMINATING WAVES

The presence of several solutions for traveling stationary nonlinear waves at fixed δ and s raises the question of the possibility of their realizability, i.e. which of several solutions will predominate in the development of small disturbances. For a fixed δ, the notion of "optimality" introduced in [2] classifies the waves according to their amplitude (or flow rate) on the interval of s. In [3], the problem of solution classification according to stability with respect to small disturbances was considered and the wave regimes for finite δ were demonstrated to be unstable, with the amplification factors being strong functions of δ and s. The optimum regimes proved to be least unstable and hence the assumption of their realizability in experiments was, in a some sense, confirmed. In subsequent studies of the instability of the wave solutions of Eqs. (1.1), rare cases of stability were also observed (small zones in the plane (δ, s)). However, both these approaches concern the first family and do not lead to firm conclusions about the advantages of each of several solutions for fixed δ and s.

A more constructive approach is to obtain a direct numerical time-dependent solution of Eqs. (1.1), to determine the limiting regimes attracting the solutions developing from small initial disturbances, and to compare them with the periodic steady-state wave bifurcations. The numerical solution of this problem begun in [3] and continued in [10, 11] made it possible to establish the following: (i) the self-development of the initial disturbances terminates in either the formation of a traveling steady-state wave or passage to a two-periodic (oscillating in space and time) wave regime, (ii) complex transition processes with the formation of intermediate wave structures occur. Further studies of the unsteady problem, including more extensive numerical experiments and a comparison of their results with the bifurcations of the traveling steady-state waves, led to the idea of dominating waves.

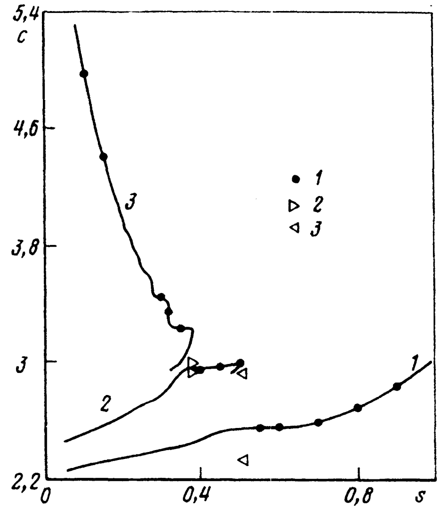

Fig. 1. Periodic solutions for $\delta = 0.1$. The symbols denote the calculations for the unsteady problem.

We will now analyze the systematic numerical results for the unsteady solutions of Eqs. (1.1).

The attractors of Eqs. (1.1) for finite δ were studied by direct numerical solution of the dynamic system resulting from the use of the Galerkin method for various initial conditions. For each specified relative wave number s, integration over t was performed until the solution reached a clearly identified limiting state. For calculating the nonlinear operations in Fourier coefficient space, we used the fast Fourier transform. In the course of the calculations we eliminated the spurious representation errors which can result in nonlinear numerical instability. The number of harmonics was varied from 32 to 256. For large times, controlled accuracy was ensured by increasing the parameter N in (2.1) with increase in the wave profile complexity. It was required that, at any instant, $|h_N| < \varepsilon_1 |h_1|$ ($\varepsilon_1 \ll 1$).

We studied initial ($t=0$) conditions of two kinds: (i) one non-zero Fourier harmonic was specified, (ii) the first $N/2$ non-zero small-amplitude harmonics calculated using a pseudorandom number generator were specified. The first case corresponds to the artificial excitation of a wave of a certain frequency, while the second case models the natural wave development due to hydrodynamic instability.

Some results of calculating the limiting values of c, a are given in the plane (s, c) (Fig. 1.) and in the plane (h_{max}, c) (Fig. 3). Points *1* show the quantities corresponding to unsteady problem attractors. Their values and the limiting solution as a whole are completely determined by the values of δ and s and are independent of the type of initial conditions and the non-zero harmonic amplitude. The analysis of the results presented in Figs. 1 and 3 indicates that the attractors of Eqs. (1.1) coincide with those steady-state wave solutions of Eq. (1.2) which, for given δ and s, have the largest amplitude and phase velocity. The passage of the attractor from one periodic wave family to the other with variation of s occurs jumpwise (the two discontinuities corresponding to the number of branching families are clearly visible in Fig. 1). Figure 4*a* shows the stages of wave profile evolution in one calculation. The passage from an initial stochastic distribution of the harmonic amplitudes to an almost harmonic wave and its subsequent transformation into a wave of the second main family are clearly apparent. It is this wave, shown by the broken line in Fig. 4*a*, which has the maximum values of c and a for the given s. For small s ($s < s_m'$), the atractors were second family waves with the maximum phase velocity and amplitude. In Fig. 1, the left upward branch corresponds to these solutions. We note that, in the experiments [1, 12], for fairly small frequencies fast solitary waves typical of the second family were also observed. Limiting waves formed as $t \to \infty$ and almost independent of the boundary condition kind predominate over the other possible wave structures.

Near the values of s at which a discontinuous passage from one family to the other occurs, the onset of oscillating limiting solutions is also possible. Table 2 contains examples of the maximum and minimum values of the amplitude and the flow rate for the oscillating regimes when $\delta = 0.15$.

The oscillating two-periodic solutions may have their own development dynamics. In calculating the variants, we obtained both slow monotonous variations of the oscillation amplitude and subsequent fast passages to zero amplitude.

For the oscillating regimes, it is not possible to introduce a single wave velocity; accordingly, in Fig. 1, for $\delta = 0.1$ the numbers *2*, *3* refer to the minimum and maximum values of the first harmonic velocity for $s = 0.384$ and 0.507, respectively, the first regime being slightly damped.

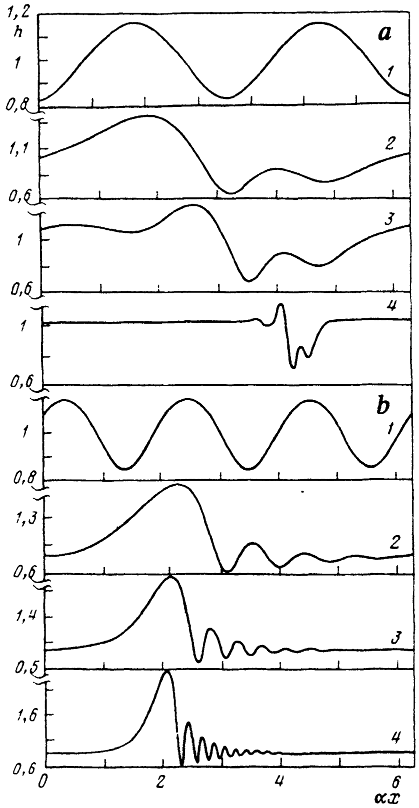

Fig. 2

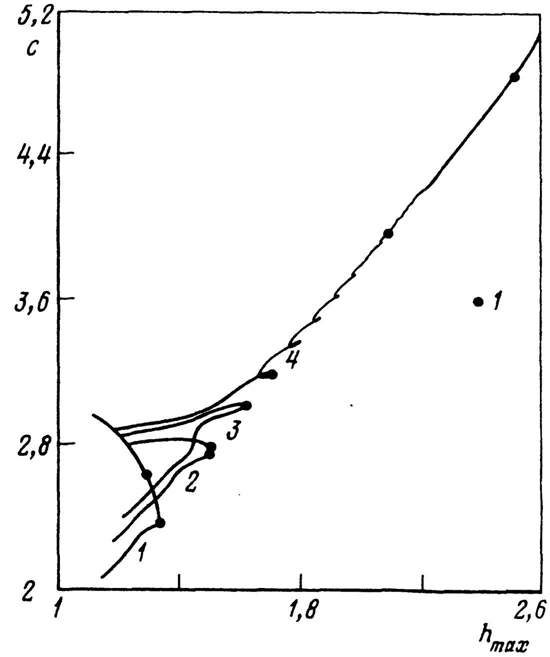

Fig. 3

Fig. 2. Periodic wave families for $\delta = 0.1$: (*a*) γ_2, $s = 0.4763$, 0.4, 0.25, 0.05 (curves *1–4*); (*b*) γ_3, $s = 0.3208$, 0.26, 0.15, 0.08 (curves *1–4*).

Fig. 3. Periodic solutions for $\delta = 0.15$. Points *1* refer to the calculations for the unsteady problem.

TABLE 2

s	a_{max}	a_{min}	q_{max}	q_{min}
0.3	1.102	0.995	1.272	1.221
0.35	1.083	0.604	1.260	1.103
0.466	0.920	0.539	1.191	1.074
0.5	0.817	0.792	1.155	1.149

The time variation of the spatial wave structure is particular apparent from considering the spectrum development

$$\left\langle\left(\frac{dh}{dx}\right)^2\right\rangle = \frac{\alpha}{2\pi}\int_0^{2\pi/\alpha}\left(\frac{dh}{dx}\right)^2 dx = \alpha^2\sum_{k=1}^{N} b_k, \quad b_k = 2k^2|h_k^2|.$$

Figure 4*b* shows the time variation of the spectrum components b_k for the waves presented in Fig. 4*a*. Connecting the points (k, b_k) with a broken line, we obtain a spectral image of the nonlinear wave, which is particularly convenient for tracing the wave evolution due to hydrodynamic instability.

With respect to the calculations presented in Figs. 4*a*, *b*, as for the other calculations, it is possible to distinguish the following stages: (i) nonlinear excitation of the multiple harmonics, (ii) fast harmonic growth from the neighborhood of the fastest-growing harmonics in accordance with the linear theory, (iii) the formation of intermediate nonlinear structures and their transformation due to instability into subharmonics and short waves, and (iv) the formation of the

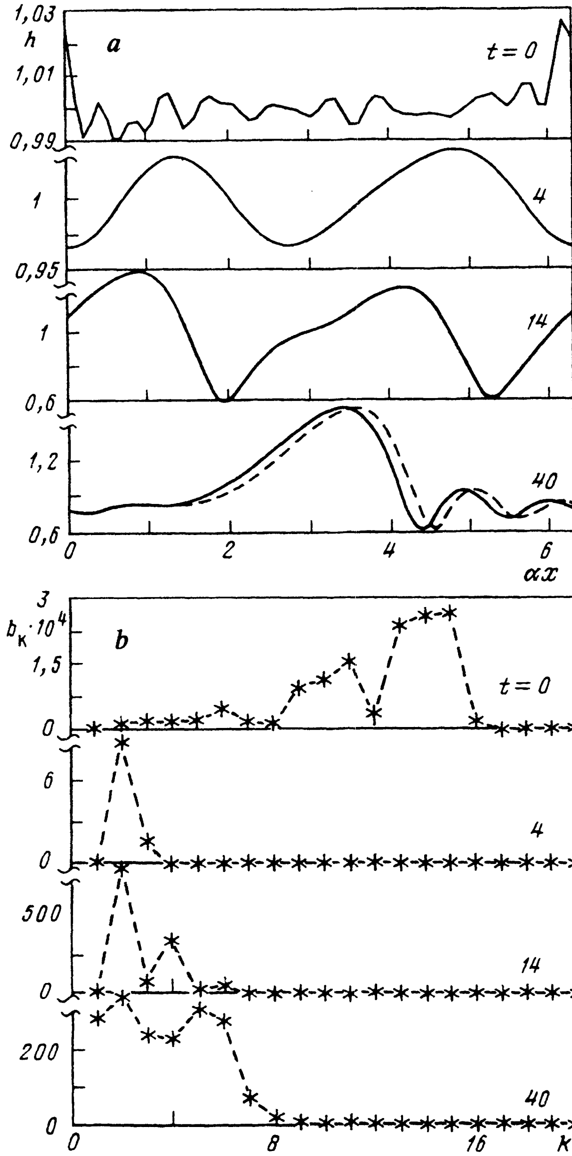

Fig. 4. Wave regime evolution for d = 0.1, $s = 0.3$.

equilibrium amplitude distribution over the spectrum due to the development of the subharmonics and short waves.

The spectrum fills up progressively with increase in the basic harmonic wavelength, i. e. with approach to the point $s = 0$. The maximum energy belongs to a harmonic with a fairly high number k_{max}, excited by the nonlinear interaction, rather than to the basic harmonic. For example, for $\delta = 0.1$ we obtained $k_{max} = 3$; 5; and 9 for $s = 0.3$; 0.2; and 0.1, respectively. Thus, with approach to the soliton (as $s \to 0$) along the second main family branch the pattern of the limiting nonlinear wave becomes more and more complex and the calculations become more and more difficult. The fact is that the second family has a mean flow rate maximum at small $s \leq 0.1$. Although, with increase in s, the phase velocity and the stationary wave amplitude still increase, the mean flow rate decreases and the attractors are no longer dominating. In particular, in the calculation for $s = 0.05$ and $\delta = 0.04$ we used the limiting wave corresponding to $s = 0.1$.

4. DOMINATING WAVES IN EXPERIMENTS

Dominating waves appear when small initial disturbances self-develop to the steady-state wave regime. For a given parameter δ, the corresponding solutions of (1.1) are completely determined by the value of s.

We will use the notion of dominating waves to interpret some general experimental facts, concerning, in particular, the region of existence of steady-state periodic waves and the dependence of the phase velocity and wavelength on the wave amplitude or maximum height [1, 12].

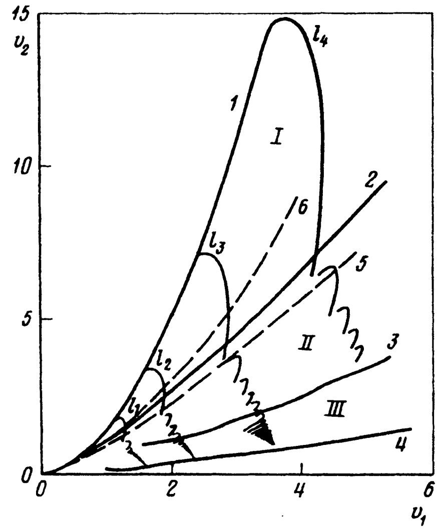

Fig. 5

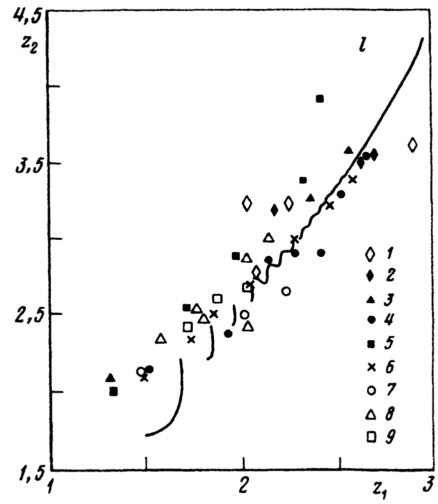

Fig. 6

Fig. 5. Dominating wave solutions: curves l_1–l_4 correspond to $\delta = 0.1$, 0.15, 0.247, 0.4.

Fig. 6. Comparison of the dominating waves for $\delta = 0.15$ with the experimental data: $\gamma = 3298$, 1129, 491, 465, 195, 105, 722, 1147, 1116 (points *1*–*9*).

The dominating waves can be combined into three groups. The first group includes only first family waves and exists for $s \in [s_2', 1]$. The second group includes all the families of additional bifurcations $s \in [s_m', s_2']$. Finally, the third group includes only second family waves and exists for $s \in [s_m'', s_m']$. Here, s_m'' is the point at which the mean flow rate is maximum. In the numerical experiments, on choosing s to the left of this point steady-state wave regimes were not attained. In a certain neighborhood of s_m'' we obtained undamped oscillations with the wave numbers s and $2s$.

Figure 5 shows the dominating wave solutions for four values of δ. In the last case, we have reproduced the solutions only for the first five bifurcations. For convenience of comparison with experiment, we chose the coordinates v_1 and v_2 from [12] which are related with the universal parameters of Eqs. (1.1), (1.2) by the formulas

$$v_1 = k\,\mathrm{Re} = 15\delta q_0^{4/3}\alpha, \qquad v_2 = \mathrm{Re}\,\mathrm{Fi}^{-1/11} = 7.508\delta^{9/11} q_0$$

Here, k and Re are the wave number and the Reynolds number from [12], and $\mathrm{Fi} = \gamma^3$. In Fig. 5, we have also plotted the neutral curve *1* and the boundary curves *2*, *3*, *4* from [12] showing the boundaries of the regions I, II, and III in which the waves were observed in the experiments. There is almost complete coincidence between the boundary points dividing the three groups of dominating waves and the experimental boundaries of the wave regimes. In regions II and III, clearly registered steady-state waves developing from small time-periodic disturbances introduced in the initial region of film flow are formed. In region II, less regular waves are formed, also due to the development of external disturbances (naturally developing waves). The fastest-growing waves in the linear approximation (curve *5*), which have an advantage in the case of natural onset, also belong to this region. However, due to the existence of two-periodic oscillating regimes near the bifurcation points of the intermediate families the wave pattern can be more complex, even for artificial excitation. In the experiments [1], besides waves of the second and third groups first group waves were also observed for $\delta \in [0.04; 0.2]$. The corresponding points are located near the optimum regime curve *6* [2].

Usually, in experiments with wave films the parameters c and a are recorded and the linear dependence $c(a)$ is noted. This dependence is also obtained in the numerical calculations. The linear relation between c and $h_{\max}$ ai more clearly traceable. As is clear from Fig. 3, the phase velocities and the maximum wave height are related by a complex nonmonotonic dependence. However, for the dominating waves the points in the plane $(h_{\max}, c)$ are grouped along a single straight line, which is particularly clear for the second main family solutions. With increase in δ, the behavior of the continuous curves becomes more complex but the main linear dependence remains.

In [12], the experimental data for the wavelengths λ are presented in the nondimensional complexes

$$z_1' = \left(\frac{gA^3}{\nu^2}\right)^{1/3} \mathrm{Re}^{-0.46}\,\mathrm{Fi}^{-0.02}, \qquad z_2' = \left(\frac{\lambda}{\sigma/\rho g}\right)^{1/3}$$

Here, A is the maximum wave height. On the basis of Eqs. (1.1), (1.2) and the associated transformations, it may be concluded that the following parameters are universal and independent of the particular liquid properties:

$$z_1 = \gamma^{1.04/11} z_1' = (45\delta)^{-1.14/11} \left(\frac{q_0}{3}\right)^{-0.46} h_{\max}$$

$$z_2 = \gamma^{1/22} z_2' = (2\pi)^{1/3} (45\delta)^{1/33} \alpha^{-1/3}$$

In Fig. 6, the broken curve l represents the dominating waves for $\delta = 0.15$ (the three breaks correspond to the bifurcation points). The corresponding curves for $\delta = 0.1$ and 0.247 lie close to the curve l and are not reproduced. Clearly, the calculated values of z_1, z_2 and the experimental points [12] for different liquids fall within the same region in the plane (z_1, z_2).

Summary. The notion of dominating waves naturally orders the families of periodic and solitary wave solutions of Eqs. (1.1), (1.2) and relates them with the wave structures which develop under experimental conditions. For each δ, the set of dominating waves consists of all the solutions of the branching families γ_k which, for a fixed s, have the maximum amplitude and phase velocity. It is precisely these solutions which develop from small initial disturbances with time. This set is defined on the segments $\Delta_k s$, $k = 1, \ldots, m$ in the region of variation of $s \in (0; 1]$. These segments are separated by narrow intervals near the points of passage from one family to another. The parameter δ is crucial for the complication of the nonlinear dynamics of the wave solutions. With increase in δ, the number of branching families of solutions γ_k quickly increases and the role of random factors in the excitation of the intermediate oscillating regimes should also increase. This agrees with the fact that, at large δ, regular wave solutions were not observed experimentally.

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