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The learning paradox and the learning miracle: thoughts on primary school mathematics education[1]

Zusammenfassung: Die Diskussionen über neue Formen des Lernens sind häufig vom Kern des Lernens weit entfernt. Im Gegensatz dazu wird in diesem Beitrag Lernen unter einer als solche erachteten Schlüsselfrage der Mathematikdidaktik überdacht: Wie können wir Lernende dazu veranlassen, auf der Basis ihres vorhandenen Wissens ein neues Verständnis auszubilden? Exemplarisch werden didaktische Bootstrapping-Strategien vorgestellt, um mit dieser "Paradoxie des Lernens" umzugehen. Ergänzend dazu wird auch dem "Wunder des Lernens" Aufmerksamkeit geschenkt. In Anbetracht der Qualität der Unterrichtskommunikation und der Erläuterungen, die Lernenden geboten werden, ist es oft erstaunlich, dass Lernen überhaupt stattfindet. Zur Entwicklung neuer didaktischer Einsichten müssen beide Seiten in Betracht gezogen werden.

Abstract: It is characteristic for the discussions about the new forms of learning that they are far removed from the core of the learning process. In contrast, in this contribution learning will be reconsidered from what is seen as a key question within the didactics of mathematics education: how can we prompt the students to construct new understanding based on what they already know. Examples are given of the didactical bootstrapping strategies that can be applied to handle this 'learning paradox'. In addition to this high-level didactics attention is also paid to the 'learning miracle'. In view of the quality of classroom communication and explanations given to students it is often remarkable that learning takes place. Taking both learning issues into account is necessary for developing new didactical insights.

1. Introduction

The answer to the question 'How do people learn?' is crucial for guiding our teaching of mathematics. Not for nothing was it one of Freudenthal's questions when he was asked what he saw as the major problems of mathematics education of the time[2] at the Fourth ICME in Berkeley in 1980. For Freudenthal, knowing how people learn was both the first step towards solving the everyday problems of practioners of how to teach learning, and the first step towards building a learning theory. I think that nothing has changed on that since 1980. Learning and our knowledge of it undoubtedly form the main pillars of education. At the same time, however, it is still being emphasized today that research in mathematics learning has not yet produced a successful and widely accepted theory of learning (see e.g. Bereiter, 1985; Uljens, 1992; Romberg, 1993; Roschelle, 1995; Schoenfeld, 1999).

This contribution will not change much about that. What I want to do as a developer and researcher of mathematics education is to look purposefully at the learning side. The underlying intention – or rather, the hope hiding underneath – is that reconsidering learning might give directions for further improvement of the theory and practice of mathematics education.

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[1] This article is an elaborated version of my plenary address at the 37th GDM Conference in Dortmund.

[2] This request was inspired by what Hilbert did in 1900 when he pronounced the major mathematical problems of his time at the Paris International Congress of Mathematicians. The list of thirteen problems Freudenthal came up with in 1980 is still used as a reference for state-of-art descriptions of mathematics education today. Furthermore, often the conclusion is drawn that none of these problems have been solved yet and are still of major interest; see for instance Adda (1998).

My review of learning will concentrate on two issues. The first one is related to what is called 'the learning paradox': the phenomenon that students construct a higher level of understanding based on their current lower level of knowledge. No matter how interesting this may be, I will not deal here with a philosophical or epistemological reflection, but will mainly pay attention to the didactical scaffolding or bootstrapping strategies that can be applied to make this shift in understanding happen.

The second issue is a little bit the opposite of the high-level didactics used to achieve the aforementioned shift in thinking, and takes us back down to earth: apparently many things sort themselves out. If there is one thing that classroom research brings to light over and over, it is that so much goes wrong in instruction. By this I refer specifically to the micro-level of teaching: the terrible quality of communication between teacher and student, and the many unclear and incorrect explanations children have to cope with when learning mathematics. The miracle is that, despite this, learning takes place. For that reason I have called this 'the learning miracle'.

Actually, both issues do not show a glimpse of the 'new forms of learning'[3] that one might expect in this age of booming technology. Indeed, this contribution is mainly about 'old' learning, but not completely.[4] I will start with discussing some ideas about 'new' learning. However, I will not spend long on this, because it concentrates too much on the outside of learning. I will not ignore it either, because once again it teaches us that we must look at the core of learning.

2. New forms of learning

I think that the best way to make clear that in learning the times are changing is to go to the internet. Using a search machine such as Google instead of Webster's Dictionary immediately gives you a wide scope of what new forms of learning are. The links mainly refer to learning with computer-based software, the use of new technical tools such as the graphic calculator, the use of hypertext and multimedia (including the use of video) and learning via the internet, e-learning, distance learning, networking, globalisation and computer-mediated conferences. Furthermore, there are also links to creativity, life-long learning, learning by constructing, cooperative learning, the joy of learning, story boards, and even the implications of LSD for human learning.[5][6]

An appropriate way to make clear what 'new forms of learning' are, is to contrast them

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[3] 'New forms of learning in education and teacher education' was the theme of the 37. Jahrestagung der Gesellschaft für Didaktik der Mathematik, Universität Dortmund, 3.-7. März 2003.
[4] See, for instance, Oliver (2000) who stressed that the recognition of alternative theories for learning coincided with the emergence of new learning technologies.
[5] When I used Google for the preparation of my lecture at the GDM 2003 in Dortmund, I was brought into contact with new learning in a very confronting manner. Searching for links about 'new forms of learning' the first hit was 'Hauptvortag Heuvel'. I must say, it feels a bit like looking into space with a telescope and watching the birth of a star or a black hole.
[6] A search on December 27, 2002 resulted in around 200 hits for 'new forms of learning' and similar English expressions. It will be not a surprise that the number of hits went down considerably when 'mathematics' was added to the search parameters. A bigger surprise resulted from performing the same search in German: the number of hits rose to around 3350.

with the traditional ways of learning. An example can be found in Fischer & Palen (1999) (see Fig. 1). Although their list is certainly not complete, it covers clearly the general purport of comparing traditional learning with new learning: the learning is more in the hands of the student and instead of being teacher-centered learning becomes more self-directed and collaborative. Fisher & Palen also mention lifelong learning and, of course, learning with computers.

Traditional forms of learning	New forms of learning
– Instructionism (teacher-centered) – Fixed curriculum – Decontextualized learning – Memorisation	– Self-directed learning – Learning on demand – Integration of working and learning – Collaborative knowledge construction – Organizational learning – Lifelong learning – Learning about computers _ learning with computers

Fig. 1: Traditional versus new forms of learning (adapted from Fisher & Palen, 1999)

This last characteristic, the use of computers, is often the first thing people think of in new learning. Yet this is a limited interpretation. As Riel (1998) elucidated the aspect of tools in relation to new forms of learning contains much more. In Fig. 3 we can see that this includes student created materials and student generated lessons, as well as simulations, virtual worlds interacting with reality and having many 'expert' voices in classroom.

Past tools for learning	Promising power tools for learning
– Textbooks and worksheet – Linear text student writing – Models and materials – Direct observation – Educational films broadcast reality – Teacher delivers lectures – Student reports to teacher on learning	– Primary sources and student created materials – Hypertext multimedia productions – Virtual creatures and simulations – Tools for remote observations – Virtual worlds interact with reality – Many 'expert' voices in classroom – Student generated lessons for others

Fig. 2: Past versus new tools for learning (Riel, 1998)

It is interesting to see how people regarded future tools for learning one hundred years ago. Asimov's (1986) book 'Futuredays', which is based on a series of cigarette cards designed in France to celebrate the beginning of the 20th century, shows us a nice example of this. According to one of the cards (see Fig. 3), learning in the year 2000 was going to be a case of grinding down textbooks, after which the subject matter could be transferred directly into the children's brains through wires. Not much has come out of this prediction – which by the way was not meant to be taken very seriously, however –,

elements of which can also be found in more serious literature.[7] The closest we come to this forecast is if we think of a classroom which has been furnished with head phones for a language lesson.
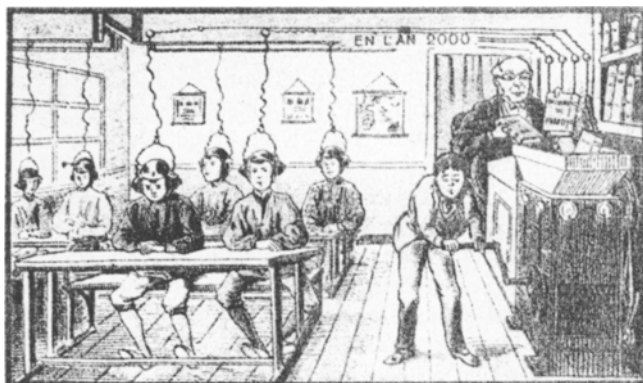

Fig.3: Picture from Asimov's 'Futuredays'

Alongside forms of learning and the tools used for them, there is a third approach to characterizing new learning: the underlying assumptions on which the new forms and tools are based. It was Grabinger who came up with the list in 1996 (see Oliver, 2000), of which a slightly adapted version is shown in Fig. 4.

Old assumptions about learning	New assumptions about learning
– Learners are receivers of knowledge	– Learners are active constructors of knowledge
– Learning is behavioristic and involves strenghening of stimulus and response	– Learning is cognitive and in a constant state of growth and evolution
– Skills and knowledge are best acquired independent of context	– Skills and knowledge are best required within realistic context
– Learners are blanc slates ready to be filled with knowledge	– Learners bring their own needs and experiences to learning situations
– People transfer learning with ease by learning abstract and decontextualised concepts	– People transfer learning with difficulty needing both content and context learning

Fig. 4: Old versus new assumptions about learning (Grabinger, 1996, p. 667; adapted from quotation by Oliver, 2000)

With some good will something can be found of the old assumptions in the 'Futuredays' way of learning: learners as receivers of knowledge, learning as reinforcing stimulus response connections, learning bare knowledge that is not linked to a context, learning that ignores students' own knowledge, transfer taking place automatically based

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[7] In 1910 Comenius was of the opinion that as soon as we have succeeded in finding the proper method "[k]nowledge can be impressed on the mind, in the same way that its concrete form can be printed on paper" (Comenius, 1910, p. 289; quoted by Wittmann, 2001, p. 7).

on abstract knowledge. The new assumptions about learning are the opposite of these, and can be summarized by the label 'social constructivist' (see also Romberg, 1993). Learning is seen as a process of personal understanding and meaning making which is active and interpretative and for which interaction in a social environment is seen to play a crucial role.

This view about learning can also be found in *the* standard work about learning published in 1999 by the U.S. National Research Council (NRC), titled 'How people learn' (Bransford et al., 1999)[8]. The intention of this book is to give a broad overview of research on learning and the implications for teaching. As shown in Fig. 5 the key findings on learning described in the later published expanded edition of the book (Bransford et al., 2000) have to do with: (1) pre-taught initial understanding and misconceptions of students; (2) the 'content' or 'range' of knowledge (facts, conceptual frameworks, and abilities for retrieval and application of knowledge); and (3) the active nature of the learning process, or the learner's control of learning.[9] According to Bransford et al. (2000) the following implications for teaching result from this: (1) instruction should take into account pre-existing understanding of students; (2) superficial coverage of a subject matter domain must be replaced by in-depth coverage of fewer topics, and instruction must include concepts 'at work' and a firm foundation of factual knowledge; and (3) metacognitive skills should be taught that give students opportunities to take control of their own learning.

Key findings research on learning	Implications for teaching
1. Students come to school with initial understanding	1. Instruction should engage with this pre-existing understanding
2. Developing competence requires (a) a deep foundation of factual knowledge, (b) understanding of facts in the context of conceptual frameworks, and (c) the organization of knowledge that facilitates retrieval and application	2. Subject matter must be taught in-depth, providing many examples of concepts 'at work' and a firm foundation of factual knowledge
3. Students can be actively involved in their learning and should have possibilities to control of their own learning	3. It is necessary to teach metacognitive skills that help students to take control of their own learning

Fig. 5: Key findings of research on learning and implications for teaching according to the NRC's publication 'How people learn' (Bransford et al., 2000)

Although the book also shows some cases of mathematics education[10] with the intention to show which (learning) knowledge base is fundamental to this education, the book in fact only shows at a general level what the teachers are doing and what decisions they are taking. The description does not touch on the core of the learning process. The cases do not

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[8] In 2000 an expanded edition was published: Brandsford et al., 2000.

[9] Although this sounds as if the book provides a very precise research base of the findings, this is not the case. Instead it remains fairly vague about which research these key findings are based on.

[10] This concerns work of respectively Lampert (1986) and Ball (1993), and work from the approach of Cognitively Guided Instruction (e.g. see Carpenter & Fennema, 1992).

reveal why the teachers behaved as they did and why their teaching strategies worked; that is to say, why their strategies resulted in learning. If the goal is to inform readers about how people learn, then these are the crucial questions that have to be answered.

The same distance from the heart of learning is reflected by a diagram that shows the key role that knowledge of learning plays in choosing a suitable teaching technique (see Fig. 6).
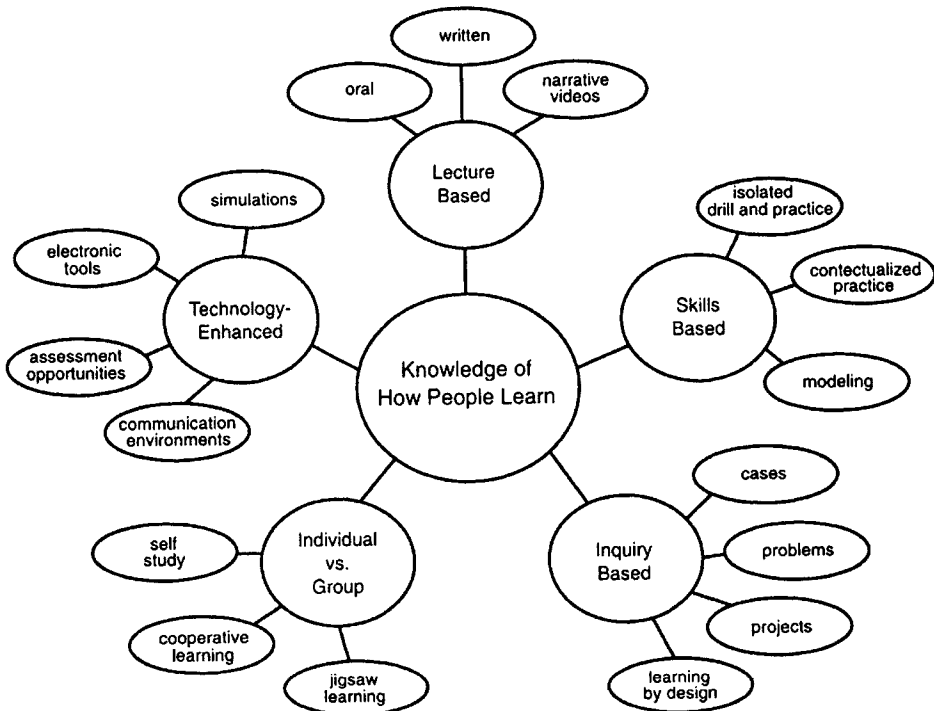

Fig. 6: Overview of teaching strategies that can be chosen based on knowledge about how people learn (from Bransford et al., 2000, p. 22)

Based on this knowledge it can be decided whether the teaching technique should be lecture-(or text-)based, skills-based, inquiry-based, be organized in an individual or group setting, or should be techology-enhanced. After this first decision for a main te-chique a more specific one can be chosen. The lecture can be given in oral or written form or by means of narrative videos. The skills-based method can be focused on drill and practice, conceptualized practice or modeling. The inquiries can be carried out on cases, problems, projects, and design activities. The individual or group setting can be filled in by self-study, cooperative learning, and jigsaw learning. Finally, the technology-enhanced method can be based on simulations and electronic tools, and can give oppor-tunities for assessment and can offer communication environments.

My problem with this scheme is that it does not deal with how learning works. This does not mean, however, that making use of these educational methods is not important – cooperative learning can elicit reflection and project work can make that students are confronted with the relations between different subdomains of mathematics. All of this is

important, but if we want our knowledge about learning to help us to improve our mathematics education, then our thinking about how students come to know must be related to what Wittmann (1998) calls 'the core of mathematics education'; it must be related to mathematical activity. In fact, we have to answer the question of how particular mathematics content – taken in a broader sense – can be made accessible to the students.

However, in most definitions of learning the content is left out. Take, for instance, the definition given by Greeno & Collins & Resnick (1996). They see learning as "the process by which knowledge is increased or modified" (p. 21). Descriptions like these are easy to give, but the difficulty only starts there. An immensely complex problem is hidden behind such a definition. What, for instance, is meant by 'knowledge' and what do 'increased' and 'modified' mean? Moreover, learning takes place on many levels. Every acquisition of a form of knowledge or ability through the use of experience can be called learning, but these superficial forms of learning are not what I want to focus on here. Within the context of learning mathematics I want to look especially at what Ohlsson (1995) calls 'deep learning', the acquisition of new insights. At that point we are dealing with shifts in understanding.

An example of that is the conceptual change that is necessary for understanding rational numbers. The number knowledge that children have developed for counting and operating with whole numbers is not sufficient for understanding fractions and must even be overturned completely. In contrast to natural numbers, rational numbers do not have unique successors; there is an infinite number of numbers between two rational numbers. Furthermore, the 'normal' view of the size of numbers is not valid anymore. The students have to understand that 1/5 is less than 1/3.

Not only in difficult domains do these shifts in understanding have to happen, they are also a necessary part of the learning process in, for instance, the area of early number. The step from 'counting all' to 'counting on' and the step to making use of 'known facts' imply a conceptual change as well.

Essential is how these new insights arise. Although interpretations vary, the current interpretation of learning can – as said before – be referred to as 'social constructivist', which means that "learners [are seen] as active constructors of knowledge in a social environment" (Romberg, 1993, p. 102). Moreover, when we are looking at the origins of new insights, the contemporary view is that "people construct new knowledge and understandings based on what they already know and believe" (Bransford et al. (1999, p. 10). This last view in particular gets us into trouble. Developing the new insight presupposes an understanding of a more sophisticated procedure in advance of discovering it. In other words, to obtain the new insight, the student has to possess prior knowledge which is at least as complex as the new insight itself.

A paradigmatic example of this, that is often quoted by Cobb (see e.g. 1987), is Holt's (1982) experience with the use of Cuisenaire rods: "Bill [a colleague of Holt] and I were excited about [Cuisenaire] rods because we could *see* strong connections between the world of rods and the world of numbers. We therefore assumed that children, looking at the rods and doing things with them, could see how the world of numbers and numerical operations worked. The trouble with this theory was that Bill and I *already* knew how the world of numbers worked. We could say: 'Oh, the rods behave just the way numbers do.' But if we *hadn't* known how numbers behaved, would looking at the rods have helped us to find out?" (Holt, 1982, p. 138-139).

3. The learning paradox

The previous example takes me to the central question of didactics: How can we trigger and guide the students' learning process to finally attain our goal of having the students develop new insights? Put bluntly, this is what it is all about: the moment at which the students make a conceptual shift and when insight is born. As said before, the problem with this shift is that the mathematical understanding, which has to be achieved, is at the same time needed to gain this understanding.

Since Fodor's contribution to the Piaget-Chomsky debate in Royaumont in 1975, this old epistemological problem[11] has been referred to as the 'learning paradox'. In the words of Fodor "[...] it is never possible to learn a richer logic on the basis of a weaker logic." (Fodor, 1980, p. 148)

A strong methaphor outside the field of mathematics education that explains what is meant by the learning paradox is Lionni's (1970) story 'Fish is fish'. It illustrates how people construct new knowledge based on their current knowledge with all its limitations. The story is about a fish who is keenly interested in learning about what happens on land, and who asks a frog to tell him about it. When the frog returns to the pond he describes all kinds of things like birds, cows, and people.
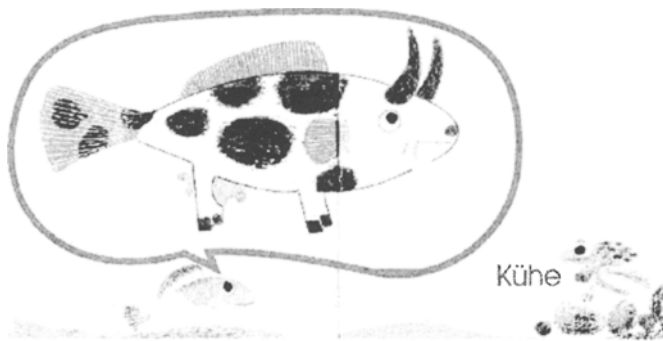

Fig. 7: Picture from Lionni's book 'Fisch ist Fisch'

The book shows pictures of the fish's representations of each of these descriptions: each is a fish-like form that is slightly adapted to accommodate the frog's descriptions – people are imagined to be fish who walk on their tailfins, birds are fish with wings, and cows are fish with udders (see Fig. 7).

One person who has given a very important contribution to our understanding of the learning paradox and who notably made the connection to its implications for education is Bereiter (1985). In his famous article 'Toward the solution of the learning paradox' he argues for taking the learning paradox seriously and pleads for developing educational strategies "for the 'bootstrapping' in cognitive growth" (ibid., p. 201) as he calls it. Drawing on the many sources, that according to him, the human cognitive system has available

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[11] This paradox has a very long history. According to Orton (1995) it might be stated in its earliest form by Plato in his dialogue 'The Meno' in which Menon and Socrates discuss the problem of how a concept might trigger its learning before it has been learned.

for promoting its own development, he comes to several theoretical principles that seem to hold promise for understanding how bootstrapping can occur. Based on these principles he describes examples of educational strategies that can be used for tackling the learning paradox. He mentions the following strategies: (a) 'chance plus selection', which implies providing students with a rich variety of solution strategies; (b) 'piggy backing', which implies that already available understanding or skills are used for other purposes; (c) 'affective boosting', which involves using problems in which students find something to be at stake for them; (d) 'field facilitation'; which means that relevant features of a problem are made perceptually prominent for the students; and (e) 'imitation'; which simply refers to something like copying what teachers and peers are thinking aloud.

What is above all worth taking note of in this list – and which is a paradox in itself – is that lower forms of learning can be used to reach a higher level. I see it as a great achievement of Bereiter that he opened our eyes to this. Bereiter makes it clear that the learning that I would call 'surface learning' – and which has a negative connotation in our current understanding-oriented education – provides scaffolding for subsequent learning of higher cognitive strategies. In this respect I recognized a similar turn in didactical thinking in Wittmann's plenary lecture at ICMI 10 in Japan, where he argued "to integrate the practice of skills into substantial mathematical activities" (Wittmann, 2001, p. 12). According to Wittmann our focus on higher-order skills should not result in neglecting the importance and specific role of the basic skills.

4.   Shifts in understanding percentage — an example from RME

As the many reactions to Bereiter's article[12] have made clear, the learning paradox is – didactically speaking – a great challenge. Actually, one could say that every educational approach has its own ways to either avoid, resolve or overcome this paradox. In this contribution I cannot deal with the discussions that came up after Bereiter's article. Instead of doing this, I will address how in the Dutch instructional approach to mathematics – called Realistic Mathematics Education (RME)[13] – the students are prompted to make conceptual shifts in understanding mathematics. My focus will be on an example of the didactical use of models in teaching percentage. I will say in advance that I will not work out the example as a genetic epistemology, but that I will describe from a di-

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[12] See for instance the reaction by Newman & Griffin & Cole (1989) who discuss Bereiter's article from the perspective of the sociohistorical school or socio-cultural theory, the reaction of von Glasersfeld (1998) who sees in the scheme theory the key to the learning paradox, and especially the reaction by the socio-constructivists Cobb & Yackel & Wood (1992) and the discussion that resulted from this reaction (see Orton, 1995; Cobb, 1995). According to Cobb et al. the learning paradox results when one adopts a represential view of mind that requires learning processes in which the students have to modify their mental representations, and it disappears when students are offered opportunities to construct mathematical meanings based on the development of taken-as-shared ways of mathematical knowing that are compatible with those of the wider community. In other words, mathematical relationships are not only self-evident to the initiated but the students themselves have the possibility of apprehending them (see Cobb & Yackel & Wood, 1992, p. 11). Although this interpretation of learning mathematics obviously can be considered as a way to avoid the learning paradox, it is not very clear in explaining how the bootstrapping that results in a shift of understanding works.

[13] A concise overview of the philosophy and principles of RME can be found in Van den Heuvel-Panhuizen (2001).

dactical point of view how growth of understanding is elicited. And although I know that for some of you the interaction in the classroom is the heart of the matter (see e.g. Steinbring, 1991, 1998), I will also leave that out of consideration.

As an introduction to the example I will give a very short overview of RME and the use of models in this approach.

4.1 RME and the use of models

One of the basic concepts of RME is Freudenthal's (1971) idea of mathematics as a human activity. For him mathematics was not the body of mathematical knowledge, but the activity of solving problems and looking for problems, and, more in general, the activity of organizing all the information you have about a problem situation – which he called 'mathematizing' (Freudenthal, 1968).

It was Treffers (1978, 1987) who later explicitly distinguished 'horizontal mathematization' and 'vertical mathematization'. The first means that mathematical tools are brought forward and used to organize and solve a problem situated in daily life – Freudenthal (1991) called this *going from* the world of life to the world of symbols. Vertical mathematization stands for all kinds of re-organizations and operations done by the students within the mathematical system itself; it implies, for instance, the making of shortcuts – in Freudenthal's (ibid.) words this means *moving within* the world of symbols.

Another characteristic that is closely related to mathematization is what can be called the 'level principle' of RME (Van den Heuvel-Panhuizen, 2001). Students pass through different levels of understanding on which mathematization can take place: from devising informal context-connected solutions to reaching some level of schematization, and finally having insight into the general principles behind a problem and being able to see the overall picture. In short, this is meant by what is named 'progressive mathematization' (Treffers, 1987).

Models have a powerful role in achieving these rises in level. The person we have to thank for this insight is Streefland. About fifteen years ago, he elucidated in a Dutch article how models can fulfill the bridging function between the informal and the formal level, namely by shifting from a *'model of'* to a *'model for'* (Streefland, 1985; see also Streefland, 1993, 1996). In brief, this means that in the beginning of a particular learning process a model is constituted very closely connected to the problem situation at hand, and that later on, the context-specific model is generalized over situations and then becomes a model that can be used to organize related and new problem situations and to reason mathematically. In that second stage, the strategies that are applied to solve a problem are no longer related to that specific situation, but reflect a more general point of view.

In this process of schematization and generalization, again the roles of the designer and the teacher are very important. By designing a trajectory in which new problems prompt the students to arrive at adaptations of the initial 'concrete' model and by accentuating particular adaptations that the students come up with, the process of model development is guided. For a more extensive discussion about the didactical use of models in RME, see Van den Heuvel-Panhuizen (under review) and Gravemeijer (1999). In the following I will show you some snapshots from such a trajectory on percentage.

4.2 The didactical use of the bar model in learning percentage

This trajectory, in the development of which I was involved, is designed for 'Mathematics in context' (Romberg, 1997-1998), a mathematics curriculum for the U.S. middle school.[14] In this percentage trajectory, that is meant for 11 to 12 year-old students, the learning of percentage is embedded within the domain of rational number. This means that gaining knowledge of percentages is strongly entwined with learning fractions, decimals and ratios. However, since I mainly want to highlight the longitudinal character of the trajectory, as well as the shifts in understanding that occur within such a trajectory, I will limit myself to the percentage part of it. For the sake of clarity I will also limit myself to just one model: the bar model. I will describe how the bar model emerges and evolves, and supports the students' learning.

The learning-teaching trajectory on percentage is spread out over three teaching units.[15] Globally speaking, the trajectory starts with a qualitative way of working, with percentages as descriptors of so-many-out-of-so-many situations, and ends with a more quantitative way of working with percentages, by using them as operators.
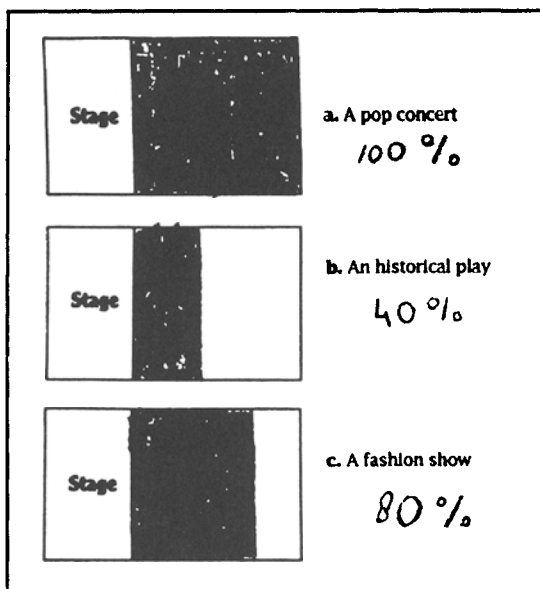

Fig. 8: Percentage of occupied seats in school theater

[15] The draft version of these units have been developed by staff members of the Freudenthal Institute. The draft of 'Per Sense' is developed by Van den Heuvel-Panhuizen and Streefland. This took place from 1991 to 1993. The draft version of 'Fraction Times' is developed by Keijzer, Van Galen and Gravemeijer. 'More or Less' was designed in draft by Keijzer, Van den Heuvel-Panhuizen, and Wijers.

In the first teaching unit 'Per Sense', the instruction begins with building on the informal, preschool and outside school knowledge of students. The worksheet in Fig. 8 shows student work related to an assignment connected to the school theater. The students are asked to indicate for different performances how busy the theater will be. They can do this by coloring in the part of the hall that is occupied and then writing down the percentage of the seats that is occupied. This is asked before the students have been taught about percentages.

It was remarkable how easily the children got to work on this assignment. Observations during the try-outs of the teaching unit showed that the scenario to elicit the use of bars through the school theater activity worked. For the students, this coloring in of theater halls also became a way to express other kinds of so-many-out-of-so-many situations. Here, in other words, a first shift from a 'model of' to a 'model for' is made. Another interesting finding was that the students spontaneously used fractions to 'explain' percentages (see Fig. 9).
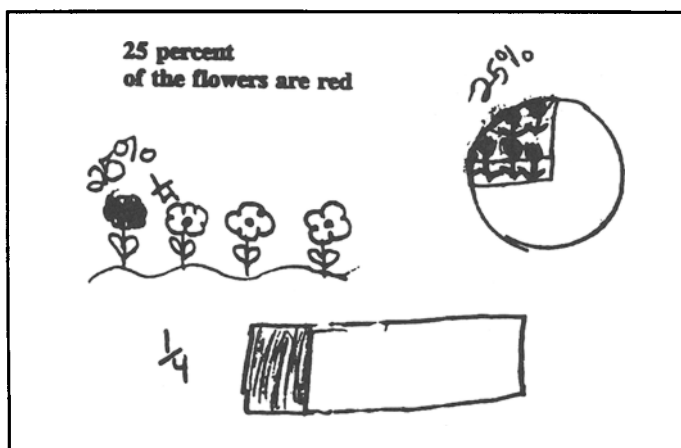

Fig. 9: The use of drawings to expresss percentages

This is used in the next chapter of the 'Per Sense' unit that includes a set of problems in the context of parking. The students are asked to compare parking lots with respect to their fullness. Again, the students are asked to indicate the degree of occupation for each parking lot by coloring in the frame that represents the parking lot. Based on this, it can be determined which parking lot is the fullest (see Fig. 10).

The next step is that the rectangular frame that represents the 'real' parking lot is replaced by an 'occupation meter' which is similar to, for instance, a display to check the amount of dust in a vacuum cleaner or a charge indicator for batteries. Like these, the occupation meter offers the students a way to represent the parking lot's fullness. They can again color in the occupied part (see Fig. 11).
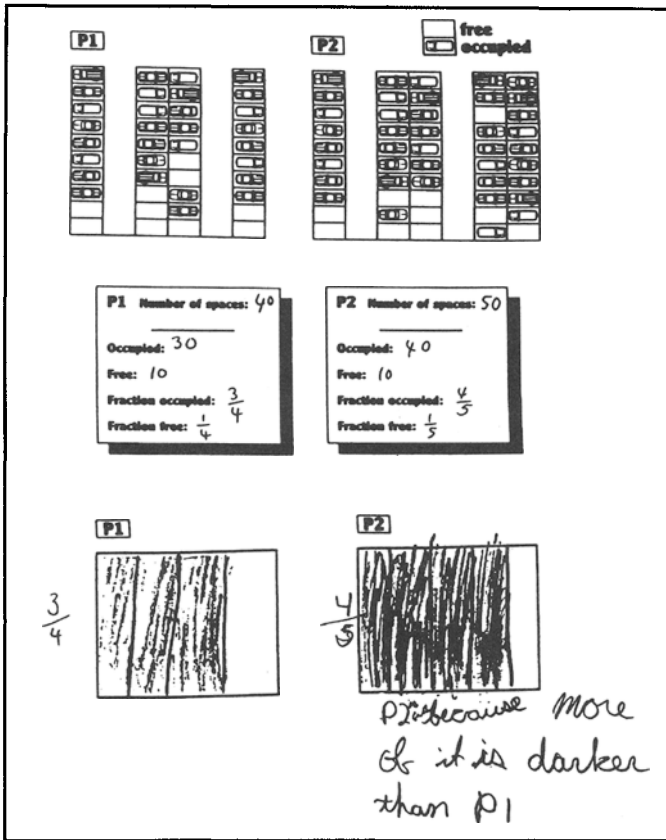
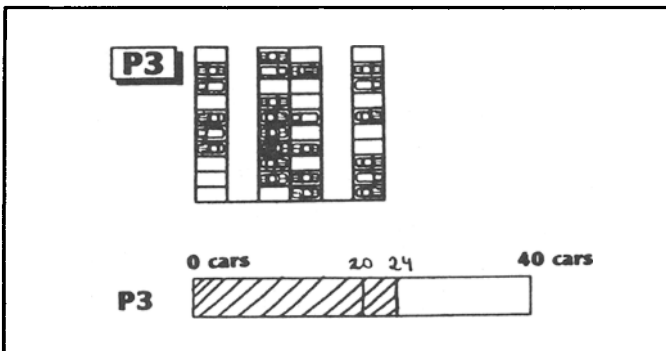Fig. 10: Comparing the fullness of parking lots

Fig. 11: The 'occupation meter' shows the fullness of the parking lot

Moreover, after doing this, the 'occupation meter' visualizes the percentage of occupied spaces. If the meter is completely colored in, it means that the parking lot is 100% full. If 24 out of 40 spaces are occupied the parking lot is filled for, let us say as a pre-

liminary first answer, a little bit over 50%. But after indicating 75% as the middle between 50% and 100%, and using 75% as a reference, 60% might come up as 'a good guess' (see Fig. 12).
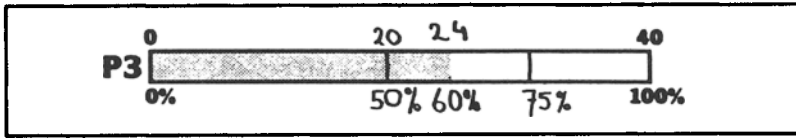

Fig. 12: The 'occupation meter' reveals the percentage of fullness

At this moment I should give some attention to the often neglected function of choosing specific numbers that prompt students to discover particular relations between numbers and to make use of clever strategies.

Because of time restraints I will continue instead with the next chapter of the 'Per Sense' unit, in which the 'occupation meter' gradually changes into a plain bar model. In other words, again a shift is made from a 'model of' to a 'model for'. This means that the model is no longer exclusively connected to the parking lot context, but can help, for instance, to compare the different preferences of fans for particular baseball souvenirs. Moreover, the shift gives access to a higher level of understanding, in which the bar is used to reason about so-many-out-of-so-many situations. Especially in cases where the problems concern numbers that cannot be simply converted to an easy fraction or percentage, the bar gives a good hold for estimating an approximate percentage. An example of this is shown in the following worksheet (Fig. 13).
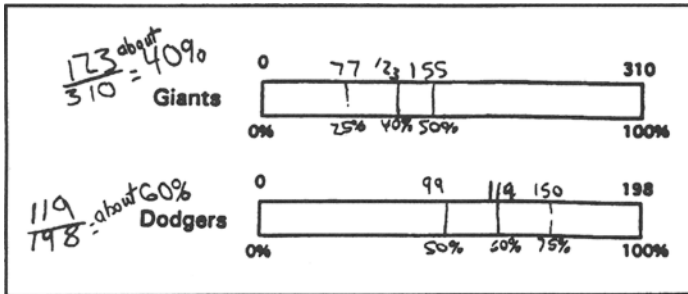

Fig. 13: Using the bar as an estimation model

The problem is about two groups of fans, Giants fans and Dodgers fans, who have been interviewed about their favorite baseball souvenir. In total, 310 Giant fans have been interviewed and 123 of them chose the cap as their favorite souvenir. In the case of the Dodgers fans, 119 out of 198 fans chose the cap. The students are asked which fans like the cap the most? On the worksheet you can see how this student found the answer by means of an estimation strategy.

In order to provide the students with a more precise strategy, later on in this chapter their attention is also drawn to the 1%-benchmark. This is done more or less casually through a headline in the newspaper, which is about a dramatically low attendance by Tigers fans (see Fig. 14). This 1%-benchmark is introduced to open the way to calculate

percentages, but the approach chosen in this trajectory is different from the usual way of precise calculation in which first the 1% is calculated precisely. In contrast, here the 1% benchmark is only used as a rough approximation by which the total amount is divided.

TIGERS' SUPPORT WANES
TUESDAY'S GAME VISITED BY 1,598 — ONLY 16 TIGERS FANS

Fig. 14: Introduction of 1% as a benchmark

The bar also has a different role here than it had before. It is no longer a tool used to operate on, but is used to guide the students in calculating the percentage. The bar tells them what calculation they have to carry out to find the answer. To know the percentage that belongs to 91 in the Marathon problem we have to determine how many steps of 1% we have to make. So we can divide 91 by 16 to find the approximate percentage of drop outs (Fig. 15).
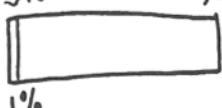
Year of marathon	Total number of competitors	Number of drop outs	Percent of drop outs	Describe your strategy
1991	1,603	91	$\approx 5\%$	≈ 16 1,603 $16\overline{)91}$ $\frac{-80}{11}$

Fig. 15: Using the bar as a calculation model with 1% as a benchmark

At this point I will skip several steps in the trajectory; among them the activities in which the students see, via a differently scaled bar (from 0 tot 1), that they can also write a percentage as a decimal – which is clearly a form of vertical mathematization.

Later, in grade 6, in the unit called 'More or Less', the students are confronted with situations of change. Then they learn to express – both in an additive (+ or –25%) and in a multiplicative way (× 0.75 or × 1.25) – new situations as percentages of the old ones. This part of the trajectory starts with a situation of price reduction. The example that is shown in the following worksheet (Fig. 16) is about a supermarket that introduced new price tags. The stu-

dents are asked to check the sale prices by making only one calculation on their calculator. In this case this means that they have to multiply 3.20 by 0.75.
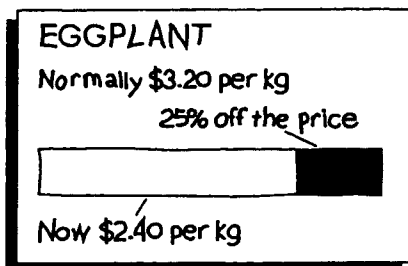

Fig. 16: Checking the sale price by one multiplication

Here again I will skip some activities, such as using the bar as an elastic strip to find a double reduction with a particular percentage.

Another example that shows how helpful the bar can be in understanding complex situations is when the bar is used in circumstances that ask for backward reasoning. This is the case in the problem in which the sale price and the discount percentage are given and the students have to find the original price.
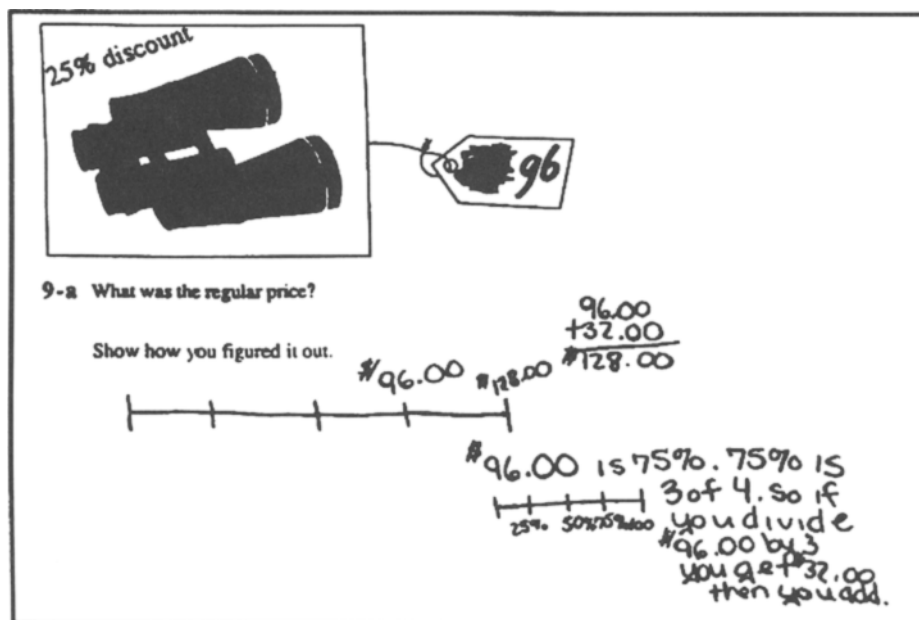

Fig. 17: The double number line as a support for backwards reasoning

The student work (see Fig. 17) shows that, instead of the bar, a simple double number line is used to support the backwards reasoning. In a way it confirms the natural switch

from one version of the model to another. Crucial for both versions is that they help the students to understand that the sale price equals 75% of the original price and that they can find the original price, for instance, by dividing the sale price by 3 and then adding that part to the sale price.
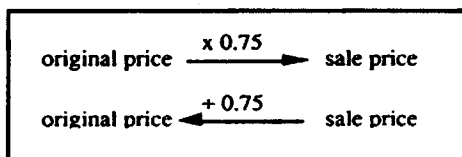

Fig. 18: Finding the original price as the reverse of finding the sale price

On a higher level, however, the original price can be found by means of a one-step division by dividing the sale price by seventy-five hundredths, which is the opposite of finding the sale price when the original price and the percentage of discount have been given (see Fig. 18). Actually, this latter solution is again an example of vertical mathematization. It is based on a shortcut within the mathematical system.

The previous snapshots from the learning-teaching trajectory show how the bar model can be used didactically to elicit shifts of understanding of percentage. During this process of growing understanding the bar gradually changes from a picture of a so-many-out-of-so-many situation to an occupation meter and later to a double number line. At the same time, it changes from a concrete context-connected representation to a more abstract *representational model* that moreover is going to function as an *estimation model*, and to a model that guides the students in choosing the calculations that have to be made; this means that the model then becomes a *calculation model*. At the end of the trajectory, when the problems become more complex, it can also be used as a *thought model* for getting a grip on these problem situations. In fact, the modeling activities do not produce one single model, but a chain of models.
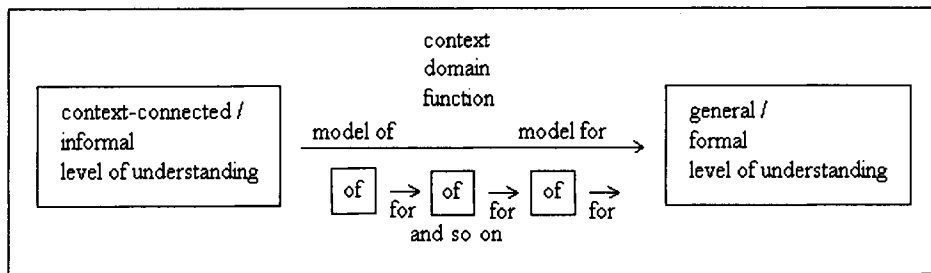

Fig. 19: Different levels of understanding and the shifts from 'model of' to 'model for'

Just as it is not a case of one bar model, but of a chain of models that together form the conceptual model that incorporates the relevant aspects of the rational number concept, there also is not just one shift from 'model of' to 'model for'. In fact, there is a series of continuous local shifts, which implies that a model, which on a context-connected level symbolizes informal solutions, in the end becomes a model for formal solutions on a more general level.

As shown in the diagram in Fig. 19, these local shifts refer to (a) shifts in context (when the students become aware that what is used for the occupation of the theater can als be used for the part of flowers[part of flowers?]), (b) shifts in the (sub)domain (when the students become aware of the relationship between percentages and fractions and decimals) and (c) shifts in function (when the students use the model in different ways).

The trajectory depicted here should in no way be seen as a fixed recipe. It is meant as a scenario that offers a learning environment for learning percentage, including the means needed to make these shifts in understanding, while they at the same time provoke the shifts, because of the fact that there is a certain necessity for making them. The multi-level quality of the models makes it possible that the students can see a given problem with different eyes. In this way, the didactical use of models in RME could be considered as a form of what Bereiter (1985) calls 'field facilitation'. It is a way of directing the students' attention, prompting the acquisition of new and more advanced strategies. However, this does not mean that RME has solved the learning paradox. As for Bereiter, within RME bootstrapping is accepted as a mechanism that happens and we are continuously looking for teaching methods that can make it happen.

But there is more happening in classrooms than the scenarios that are developed for teaching mathematics. This takes me to the second learning issue that I would like to address in this contribution: the learning miracle.

5.   The learning miracle

Let me first explain what I do *not* mean by a learning miracle. It is not – to return to my previous topic – the miracle that so many students wait for when solving a problem (see Fig. 20).
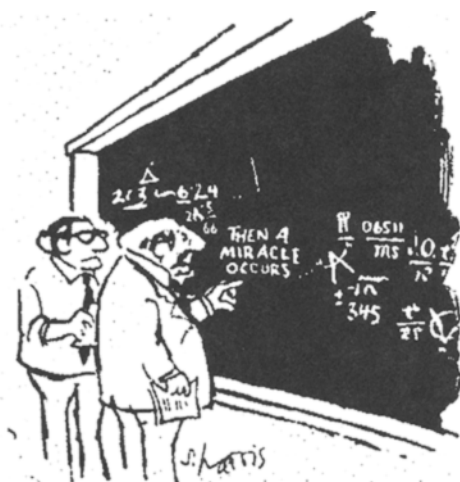
"I think you should be more explicit here in step two."

Fig. 20: Solution strategy with miracle[16]

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[16] I found this cartoon on a web site dedicated to Brenda H. Loyd, the former president of the U.S. National Council on Measurement in Education.

Nor do I mean the success stories about, for example, learning a foreign language in ten days, or the existence of mathematical prodigies that seem to violate all rules of learning. And I certainly do not mean the rise in state test scores that occurred at the price of an increase in drop-out rates in the Texas of then Governor Bush, which is wrongly referred to as the 'Texas Miracle' (see e.g. Haney, 2000).

No, what I am thinking of when I talk about the learning miracle, are the many experiences of flawed instruction students undergo during their school years, which apparently do not stop most of them from learning mathematics in the end. Students are clearly resistant to the not very good teaching that educational practice is so full of. Let me show some examples of such teaching.

5.1 A classroom vignette about learning fractions

The first example I found in an article by Davis (1997, p. 357-358). The teacher is considered highly competent. The lesson is about addition of fractions and is being given in an eight-grade classroom. It is one of the first lessons of a unit on the subject.

"[...] Wendy [who is the teacher] wrote a series of addition statements on the chalkboard,

$$\frac{1}{5}+\frac{1}{5} \quad \frac{1}{8}+\frac{5}{8} \quad \frac{1}{4}+\frac{1}{2} \quad \frac{1}{2}+\frac{2}{5}_.$$

[...] A few moments later she asked for students to volunteer their answers. Tim offered the first one (i.e., to 1/5 + 1/5): 'One tenth.'

'Now, let's think about that one,' Wendy suggested. 'Pretend we have a chocolate bar and that we cut it into fifths.' On the chalkboard she drew a rectangle and sliced it into five equal-sized pieces [...] (see Fig. 21). 'If you take one fifth,' she said as she shaded in the first of the five sections, 'and I take one fifth,' shading in the second section. 'how much is gone altogether?' 'Two fifth,' Tim responded correctly. 'So,' Wendy continued, pointing to the shaded parts of the diagram, 'what is one fifth plus one fifth?'
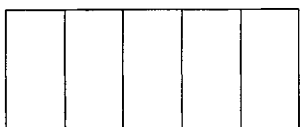

Fig. 21: Chocolate bar example

'Two fifths.' 'Two fifth. That's right. How many of you got that?' Most of the students raised their hands. 'Good. Adding fractions is just like adding anything else. One horse plus one horse is two horses; one tree plus one tree is two trees; one fifth plus one fifth is two fifths.' "

5.2 A classroom vignette about learning ratio

The next example is taken from the Dutch MOOJ study (Van den Heuvel-Panhuizen & Vermeer, 1999). The fragment is from a sixth-grade classroom. The children have been

working on the following problem: "A car covers a distance from 20.7 km to 21.7 km[17] in exactly 1 minute. What is the average speed?" This problem is discussed in a whole-class setting. The teacher suggests the ratio table and finally it is found that the average speed is 60 km/hr (see Fig. 22).

1 km	10 km	60 km
1 min	10 min	60 min

Fig. 22: Ratio table

Mariska comments that the jump from 1 minute to 60 minutes can also be made in one go, but the teacher does not think that is a different approach and reacts by saying: "That doesn't change anything." This remark ends the discussion and the teacher continues with the next problem, which is the following one: "The distance from 21.7 km to 22.7 km is covered in 2 minutes. What is the average speed now?"

The teacher wants the students to look at the table again. But Felix, a fairly good student, immediately says: "Thirty kilometers, because it's half." The teacher reacts to this with: "That's hocus pocus." It looks like the teacher does not understand what Felix means, and that therefore the turn is given to another student. This student does fill in the whole ratio table and the session is finished without making any connection to the previous problem; which is surely rather strange in a lesson about ratio.

5.3 A classroom vignette about learning subtraction

The protocols we made of observed lessons within the Dutch MORE research (Gravemeijer et al., 1993), also painfully exposed just how anti-didactical classroom communication often is.[18] See, for instance, the following fragment of a lesson in grade 1, recorded halfway through the school year.[19]

Teacher: "I have ten cents and I buy apples for 8 cents. What problem goes with that?" [...] Teacher: "Well, you have to tell me how to do that, 10 minus 8. Who knows? How do you do 10 minus 8?" [There is hardly any reaction from the classroom.] Teacher: "10 minus 8, who knows how you should do that? John?" John: "2 cents." Teacher: "How did you calculate that? That's what I want to know." John: "Then I set out the 8 and then I add 2 and then I know." Teacher: "You add 2. That's pretty smart. But do you remember how we agreed to do those difficult subtraction problems before holiday [the teacher means the Christmas holiday]? ... Wobke?" Wobke: "Count back." Teacher: "Yes, but counting back 8 steps is a lot. I can't do that. That is much too difficult." Other student: "Say the list." Teacher: "Which list?" [More questions and answers follow.] Teacher: "Of course you have to do the list of ten. Let's do it all together." Students: "10 is 10 and 0, 10 is 9 and 1." Teacher: "There it comes." Students: "10 is 8 and 2." Teacher: "And 2! So what is 10 minus 8?"

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[17] These data are derived from the hectometer posts alongside the Dutch highways.
[18] More on this can be found in Streefland & Te Woerd (1991a, 1991b) and Gravemeijer & Van den Heuvel-Panhuizen & Streefland (1990).
[19] MORE Observation, school 24, NZR, grade 1, 7-1-1988.

5.4 A classroom vignette about learning estimation

The next example, from a third grade class, is also from the MORE-research.[20] The problem that is being worked on is estimating the length of an illustration of a pencil (see Fig. 23).

Teacher: "Now, there you see [...] a pencil. Now you are not supposed to, eh, measure the pencil! [...] first you must estimate how many centimeters of millimeters the pencil is. Get your scratchpad and write down for yourself, the pencil is about that many centimeters. You're not allowed to measure it. Take a good look. How much do you think it is? Write down a figure. Write down a figure. How many centimeters. [...]

Right, you wrote that down. Now you must write down another figure, for the same pencil. Imagine that the pencil isn't exactly 5 centimeters. Like that broken ruler, that wasn't 10 centimeters like Jeroen said, but 10 centimeters and 5.6 millimeter [*sic*]. Now maybe the pencil isn't 5 centimeter at all, but there are some millimeters more. Now write down that figure. How many centimeters and how many millimeters do you think the pencil is long? And you are definitely not allowed to measure it."
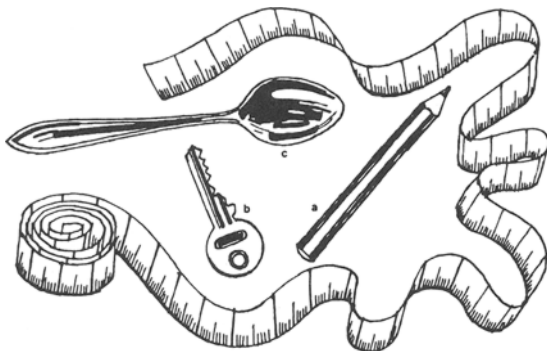

Fig. 23: Textbook sheet

Later on in the lesson rounding off is discussed. The teacher asks the students whether they know what that is. He finds the answers the students give unsatisfactory.

Teacher: "No. Do you know what rounding off is? Very understandable that you didn't know. Imagine you have 7 centimeter. Okay, a 7 centimeter pencil. That is fine. You

[20] MORE Observation, school 9, WIG, grade 3, 9-3-1990.

don't need to round off anything. But now you have a pencil that is 7 centimeter and 2 millimeter long, because you just sharpened it, and my pencil is 7 centimeter and 2 millimeter. Now I'm going to round it off. What is rounding off? I'm going to tell you. I'm going to round off this number. Those millimeters have to go. I don't want to see them anymore. Too much trouble. Imagine having to ask in a shop: 'Do you have 7 pencils that are 7 centimeter and 2 millimeter long?' The man would have to measure first if that is possible. So what are we going to do now? We agreed that if it is under 5 centimeter [the teacher means millimeter!], the millimeters disappear and we just say: 'This pencil is 7 centimeter long. Look, the 2 is lower than 5.' If you don't look here, you certainly won't understand [said by the teacher to a student who isn't paying attention]. And then we would for example get the same pencil that is lying next to it, because that is 7 centimeter and 6 millimeter... ."

And this lesson goes on like that for a while. I do not think I need to give any comment here.

5.5 Imperfect teaching

The type of teaching I have shown you is no exception. Each of us could easily add to the list of examples, either based on observing colleagues or on what we know from our own teaching practice. Characteristic for many of these examples is that they come from classroom situations where teachers – who are often very experienced – do their best, but where nevertheless things are going wrong, where students go too fast or express themselves so clumsily that the teacher can make heads nor tails of what they mean, and where teachers give explanations that mathematically and didactically speaking leave much to be desired.

That education often is imperfect and messy even in the best circumstances is not a newly discovered phenomenon. It has been pointed out by a number of authors (see e.g. Duckworth, 1987; Desforges & Cockburn, 1987; Sizer, 1997; Greeno, 1998). Sizer, a famous school reformer and former dean of Harvard's graduate school for education has the following to say about this: "Good teaching and learning are rarely linear, neat, predictable. The serendipities and distractions and fascinations that crowd into every classroom [...] conspire against that. Learning – and therefore teaching – is messy, but messy does not mean bad any more than orderly means good." (Sizer, 1997)

Another aspect of the imperfection of teaching has been exposed in a revealing manner by the research of Desforges & Cockburn (1987). They observed and video-taped the lessons of seven experienced teachers in classes from Kindergarten to grade 3, each for a period of three weeks. These obervational data were compared with data from tests made by the students before and after the teaching period, and diagnostic interviews that were held during the teaching.

It emerged from the analysis that in about 50% of the observed teaching events the children knew what to do before they were told. Another remarkable fact was that only very occasionally did the students point this out to the teacher. Furthermore, even when the students did, it did not stop the teacher from explaining again. Another outcome of this research was that of all the children who could not do a task before the teacher started her instruction 60% still could not do so twenty minutes afterwards (which amounts to 30% of all cases, because 50% of the children were already able to do the task before instruction).

6. Conclusion

To conclude, when we look at classroom practice we cannot do otherwise than draw the conclusion that quite often a 'learning miracle' occurs. More often than we would like, the quality of the classroom communication and the explanations given by teachers leave much to be desired. Despite this, the majority of children does learn mathematics. They may not all learn it at the highest achievable level, but it cannot be denied that with this occasionally flawed teaching we can at least maintain the knowledge of mankind, and can even enlarge it.

Please understand that the foregoing cannot be taken as a license for bad teaching. What I want to draw attention to is that children can apparently 'take' a lot where learning is concerned, as is the case for physical and mental resilience. Not every flu or cold virus that is going round will infect everybody that comes into contact with it, and not every bad experience will be traumatic. In mathematics education, it seems almost as if they have a repair or compensation capacity; a kind of 'mathematical buffer' that protects them for instance from faulty explanations (see Fig. 24).
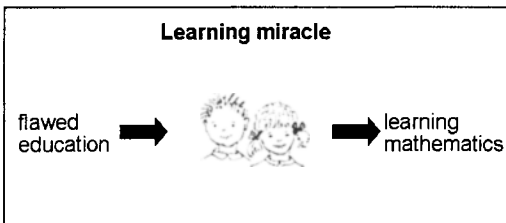

Fig. 24: 'Mathematical buffer' of children

The odd thing is that this mechanism is hardly taken into account when researching and developing mathematics education. In examples as given previously we only see what goes wrong in teaching and we desperately try to find ways to improve it. I would like to argue in favor of taking a closer look at the learning that takes place in spite of flawed teaching. In other words, we should not just look at the high-level didactics required to achieve shifts in understanding: the learning miracle is just as important as the learning paradox. Maybe taking both into account would even lead us to new forms of learning in education.

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