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Two Types of Mathematical Knowledge and Their Relation

Abstract. The distinction between procedural knowledge and conceptual knowledge seems to be possible at a terminological level. However, real problems begin when this distinction is to be operationalized by acceptable tasks, and the relation between the two knowledge types is to be clarified. This article tries to resolve some of these problems by using a constructivist approach.

Zusammenfassung. Die Unterscheidung des prozeduralen und begrifflichen Wissens scheint auf terminologischer Ebene evident zu sein. Die wirklichen Schwierigkeiten beginnen dann, wenn explizite Beziehungen zwischen diesen zwei Kenntnisstypen oder angepasste Aufgaben gesucht sind. Dieser Artikel stellt einen Versuch dar, einige von diesen Problemen im Sinne des Konstruktivismus zu analysieren und zu lösen.

Distinction by terminology

Because of different research frameworks and the fact that procedural and conceptual knowledge are not easy to define precisely (Carpenter, 1986), a number of views relating to procedural vs. conceptual knowledge can be found in the literature. For giving a framework for interpretations made by the reader as well as by ourselves, let us shortly summarize some of these views. The views are not identical, but most of them do more or less deal with the same kind of knowledge distinction.

- According to Ivic (1991), Piaget made a distinction between 'practical knowledge' (savoir-faire) and 'conceptual knowledge', whereas Vygotsky dealt with three levels of knowledge: 'manifest content' (facts, data and the like), 'instrumental knowledge' (methods, skills, procedures, etc.), and 'structural knowledge' (knowledge structures with underlying modes of thinking).
- Skemp (1979) examined the relationships between knowledge and actions by using three distinct yet closely connected constructs: 'knowing that', 'knowing how' and 'being able to'. The first denotes that an individual possesses an appropriate schema (conceptual structure), the second refers to his/her ability to come up with a particular task plan by using this schema, and the third denotes his/her ability to carry out such plan if available.
- As regards a computer program in Artificial Intelligence, 'declarative knowledge' is encoded as facts the program is "aware" of, whereas 'procedural knowledge' is encoded as procedures the program is typically "unconscious" of (Hofstadter, 1980).
- Papert (1980) pointed out that a common distinction between the two types of knowing is usually expressed as dualizations like 'knowing that' - 'knowing how', 'facts' - 'skills' or 'propositional knowledge' - 'procedural knowledge'. He underlined that such a distinction, which is typically reflected in memorizing facts vs. practicing skills, impoverishes the learning process.
- Procedural knowledge comprises productions (condition-action rules), whereas 'declarative knowledge' (this term was used instead of conceptual knowledge) is composed of tangled hierarchies of cognitive units (Anderson, 1983).

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- Having assumed that understanding refers to the individual's control over his/her process of knowing, Nesher (1986) made a distinction between learning algorithms and learning towards understanding, pointing out that 'algorithmic performance' and 'understanding' can only be examined separately after the learning has been completed.
- Gelman & Meck (1986) distinguished between 'procedural competence' and 'conceptual competence', assuming that the former relates to procedure performance, whereas the latter refers to knowledge of principles.
- Procedural knowledge is rich in algorithms for completing tasks but is lacking in relationships, whereas conceptual knowledge is rich in relationships but is lacking in algorithms for completing tasks (Hiebert & Wearne, 1986).
- VanLehn (1986) distinguished between 'schematic knowledge' comprising descriptions of actions, and 'theological knowledge', referring to information relating these descriptions and their parts to the intended purposes.
- By examining geometric knowledge, Schoenfeld (1986) contrasted 'empirical know-ledge' based on guess-and-test loops with 'deductive knowledge' comprising proofs.
- 'Mechanical knowledge' refers to factual data, and rules and algorithms (procedural knowledge), whereas 'meaningful knowledge' denotes knowledge (implicit of explicit) of concepts or principles (Baroody & Ginsburg, 1986).
- Skemp (1987) proposed three types of understanding: instrumental, relational and logical, each of which can be demonstrated in two modes of mental activity: intuitive and reflective. 'Instrumental understanding' refers to the ability to utilize certain rules without knowing why they work. 'Relational understanding' relates to the ability to infer particular rules or procedures by considering some general relationships. 'Logical understanding' denotes the ability to reason deductively by applying suitable patters of reasoning to relevant definitions, axioms and theorems.
- Procedures are modelled by productions (condition-action rules), whereas concepts refer to several kinds of relational representations such as A is kind of B, event A is caused by event B, object A is above object B, and event A happened after event B (Byrnes & Wasik, 1991).
- Procedural knowledge is viewed as sequences of actions, whereas conceptual knowledge refers to connected networks (Hiebert & Carpenter, 1992).
- Procedural knowledge denotes knowledge of procedures and mastery of computational skills, whereas conceptual knowledge relates to knowledge of various interconnections between conceptions that give meaning to mathematical procedures (Shimizu, 1996).
- While 'concept image' denotes "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes", 'concept definition' denotes "words used to specify that concept" (Tall & Vinner, 1981; p. 152).
- According to Tessmer et *al.* (1990), who examined concepts as context-dependent cognitive tools, each concept has as well declarative as procedural components, such as its definition, concept examples and non-example, connections to related knowledge, procedures for classification/identification, rules for use in different contexts and emotive connotations.
- Having defined 'procept' as "a combined mental object consisting of a process, a concept produced by that process, and a symbol which may be used to denote either of both" (e.g. 3/4), Gray & Tall made a distinction between 'procedural thinking' and 'proceptual thinking'. The former requires the use of procedures, whereas the latter

calls for "the use of procedures where appropriate and symbols as manipulable objects where appropriate." (Gray & Tall, 1993; pp. 6, 8)

- Sfard (1994) distinguished between 'operational thinking' and 'structural thinking'. While the former deals with processes in terms of operations on objects, the latter refers to objects made out of these processes.

Mathematics educators, especially on the operational level of their enterprise, often make a distinction between procedural and conceptual knowledge by speaking just about "algorithmic performance" and "understanding" (e.g., Nesher, 1986). This easily leads to a polarization that procedural knowledge would be dynamic in nature, whereas conceptual knowledge would be static. Such an assumption would not, however, modern constructivist paradigms of teaching and learning mathematics. In the *MODEM project* (Haapasalo, 1993) for example, conceptual knowledge is viewed in dynamic way: it is constructed by using different kind of representation forms of the concept (especially verbal, graphic and symbolic), and active dynamic processing between these concept attributes. They may be of different characters, depending on what kind of *concept class* the particular concept represents (terms and examples are discussed later).

We see that it is the dynamic view of conceptual knowledge which should be highlighted more clearly. In our view, the two knowledge types can, in some cases, be distinguished only by the level of consciousness of the applied actions. We therefore assume the following proceduralo-conceptual knowledge distinction:

- *Procedural knowledge* denotes dynamic and successful utilization of particular rules, algorithms or procedures within relevant representation form(s). This usually requires not only the knowledge of the objects being utilized, but also the knowledge of format and syntax for the representational system(s) expressing them.

- *Conceptual knowledge* denotes knowledge of and a skilful "drive" along particular networks, the elements of which can be concepts, rules (algorithms, procedures, etc.), and even problems (a solved problem may introduce a new concept or rule) given in various representation forms.

Furthermore, procedural knowledge often calls for automated and unconscious steps, whereas conceptual knowledge typically requires conscious thinking. However, the former (like the above-mentioned instrumental understanding) may also be demonstrated in a reflective mode of thinking when, for example, the student skillfully combines two rules without knowing why they work.

We hope that our characterization not only emphasizes some important aspects that are missing in others' views, but also allows a radical departure from two traditional views on the relation between the two types of knowledge (i.e. procedural is based upon conceptual vs. conceptual is based upon procedural). At a higher level, one knowledge type can be based upon another (to be discussed later).

Distinction between procedural and conceptual tasks

Like the concept of 'problem', the proceduralo-conceptual knowledge distinction is at least person, content and context dependent. As regards educational context, it depends on the pedagogical theory guiding the teaching/learning process. This situation is immediately realized when suitable tasks are looked for. In the case of the percentage in Fig. 1, for example, we can ask which of the knowledge types it represents.[1] Keeping in mind our knowledge distinction, this percentage should be considered conceptual when we deal with its different representations and (conscious) transferring between them, and procedural when we execute a certain set of (automated) actions (within a particular representation).

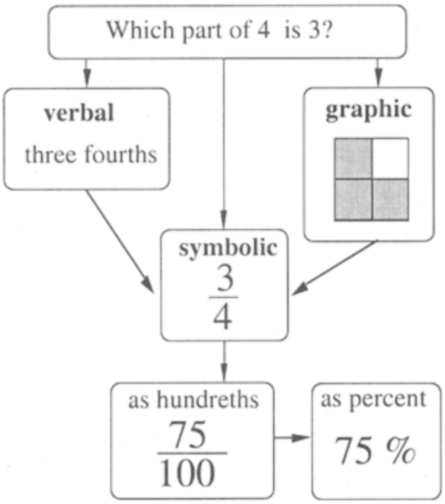

Fig. 1. Percentage as conceptual/procedural knowledge

According to Silver (1986), it is hard to develop conceptual (procedural) test items that are procedurally (conceptually) free as most items of knowledge have both conceptual and procedural features. Despite that, most empirical studies on procedural and conceptual knowledge to date have been based upon two sets of test items assessing the levels of these types of knowledge (e.g., Nesher, 1986; Byrnes & Wasik, 1991; Palmiter, 1991; cf. Shimizu, 1996). Beyond these studies seems to be an optimism that some (global) classification of procedural vs. conceptual test items is nevertheless attainable. (To our readings, no study, especially those with large samples, examined whether the applied classification is also reflected in the subjects' answers. It could be easily done by using a factor - or cluster analysis, for example). A sample of the applied tasks is given below.

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[1] Even this simple example (like also that in Fig. 3) can uncover some important issues relevant to the goals of this article. Furthermore, there are large empirical studies beyond these examples (Haapasalo 1992). We find that these and other examples examined in our paper can satisfactorily represent main features of secondary school mathematics.

- Zucker's study reported in Nesher (1986) dealt with decimals. Algorithmic performance was assessed by items dealing with the four arithmetic operations, whereas understanding was assessed by items calling for comparing decimals, estimating the outcomes of multiplication and division, reading and writing decimals and the concept of density in decimals (e.g., how many numbers are between 2.5 and 2.6?).
- In a study on fractions by Byrnes & Wasik (1991), procedural knowledge items required addition and multiplication with fractions, whereas conceptual knowledge items called for comparing fractions in a given context (eating pie), picture-symbol correspondence (e.g., write a fraction representing a shaded area), and simple morphi-sam items (given a picture, find another picture showing the same fraction shaded).

As regards problem solving, developing procedural test items requiring exac computation and conceptual test items requiring genuine understanding of the underlying domain is particularly complex enterprise that may be very person, content and context sensitive. It is our belief that in certain cases (for particular students, topics, etc.) these distinctive test items could nevertheless be created if the teacher has been from the very beginning actively involved in the planning and control of the learning process (i.e. assessment in a global sense; see "the sailing paradox" introduced later). As an example of a suitable task distinction, consider the following two problems on motion (note their identical underlying structure), bearing in mind that these problems are to be solved by students who have been previously familiarized with: (a) solving typical procedural tasks on meeting and overtaking when the speeds of both objects are given, by using arithmetic, algebraic and graphical means; and (b) the work with quantitative and qualitative graphs representing piecewise uniform motion of objects.

- *Procedural task* - A car and a truck started simultaneously from towns that are 150 km apart. After what time did they meet each other if their speeds were 80 km/h and 60 km/h, respectively?
- *Conceptual task* - A mountaineer started his trip in the morning, arriving at a mountain house in the evening. Having spent the night there, the mountaineer started down the next morning by using the same trail. Is there a point on the trail where he was at the same place at the same time each day? Give a detailed explanation.

A recent study (Kadijevich 1999) examines a number of conceptual task types. It however does not deal with tasks on finding out different object representations and transferring between them, which are, according to our knowledge distinction, also conceptual. Figure 2 represents such a task. This task taken from the MODEM project (Haapasalo, 1993) deals with the conceptual field *Proportionality - Linear Dependence - Gradient of a Straight Line through the Origin.* The task can have two different kinds of function. It can measure concept understanding on the identification level between graphic and symbolic forms. However, in the process of concept building, organised according to MODEM philosophy, it helps pupil to construct links between concept attributes. What kind of thinking process a particular task is calling for from a pupil, depends very heavily on curricular context as well as on the quality and organization of the learning process. The same task can be purely mechanical for one pupil, whereas it can be a challenging problem for another pupil, requiring further development of conceptual knowledge (recall Fig. 1). This important issue will be highlighted by a sailing metaphor in our didactic considerations later.

Conventional textbooks mostly comprise procedural tasks promoting skills rather than understanding. Conceptual mathematics based upon conceptual tasks (like those in Dreyfus & Eisenberg (1990), Haapasalo (1993) and Kadijevich (1999)) is usually missing, which naturally result in poorly developed conceptual understanding. According to Dreyfus & Eisenberg (1990), mathematics education also needs to be based upon conceptual tasks as they, contrary to traditional procedural tasks, can fully assessed whether genuine understanding of the underlying domain is really achieved. Although these researchers only examine calculus requiring that this skill-oriented topic must turn into a concept oriented one, there is no doubt that their request is relevant to mathematics education in general. It is particularly true today when computer-based mathematics education is available. This is because computer can be used to introduce a new balance of instructional time by decreasing the time for procedural skills and increasing the time for conceptual understanding (Fey, 1989). Such a less proceduralized approach seems to promote better understanding (e.g., Schwarz, Dreyfus & Bruckheimer, 1990; Palmiter, 1991; cf. Simmons & Cope, 1997). Note that, according to Kaput (1992; p. 549), an important research question regarding technology and mathematics education is "How do different technologies affect the relation between procedural and conceptual knowledge, especially when the exercise of procedural knowledge is supplanted by (rather than supplemented by) machines?"

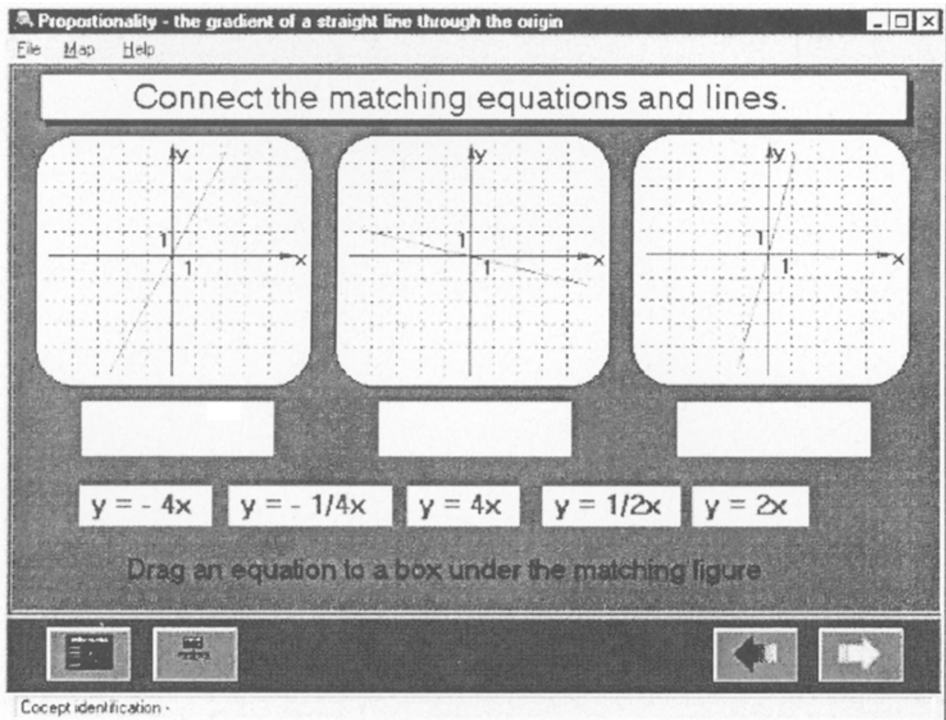

Fig. 2. A conceptual task in a CBL-program within MODEM: Identification between symbolic and graphic forms (downloadable at http://www.joensuu.fi/lenni/programs.html)

Searching the relation between the two knowledge types

Recalling Shimizu's (1996; p. 234) remark that "understanding how procedural knowledge and conceptual knowledge relate to one another is one of the major foci in mathematics education", let us try to clarify the relation between the two knowledge types. As procedural and conceptual knowledge cannot be measured directly, an analysis of the relation between conceptual and procedural knowledge may, before theoretical considerations, be based upon an analysis of the relation between success in procedural and conceptual tasks representing these knowledge types, which is, in itself, an important educational issue. Let us therefore assume students' knowledge types were successfully assessed by an acceptable set of procedural and conceptual test items, and that, for each knowledge type, each student's total score was classified low (L) or high (H) in respect to, for example, the total score mean. We find that only four of all potential outcomes have been empirically supported so far. These four outcomes are summarized in Table 1, the content of which *just* presents distinctive patterns realized through our readings. While the first outcome is evidenced by the absence of any correlation between the two knowledge total scores, the others are evidenced by a significant linear (or quadratic) correlation between these scores.

Conceptual knowledge scores	Inactivation view		Simultaneous activation view		Dynamic interaction view		Genetic view	
high	X	X		X	X	X		X
low	X	X	X		X		X	X
Procedural knowledge scores	low	high	low	high	low	high	low	high

Tab. 1. Possible relations between scores in conceptual and procedural knowledge (X denotes "many students", empty means "none or few")

These outcomes are characterized as follow.

- *Inactivation view:* Procedural and conceptual knowledge are not related. This view has been evidenced by Nesher (1986) and Resnick & Omanson (1987), for example.
- *Simultaneous activation view:* Procedural knowledge is a necessary and sufficient condition for conceptual knowledge. This view can be recogonized in Hiebert (1986), Byrnes & Wasik (1991) and Haapasalo (1993).
- *Dynamic interaction view:* Conceptual knowledge is a necessary but not sufficient condition for procedural knowledge. This view was thoughtfully examined in Byrnes & Wasik (1991).

- *Genetic view:* Procedural knowledge is a necessary but not sufficient condition for conceptual knowledge. This view can be recognized in, for example, Kline (1980), Kitcher (1983), Vergnaud (1990), Gray & Tall (1993), and Sfard (1994). It is important to note that, to our readings, no study has reported a pattern similar to that presented in Tab. 1 (although such a pattern might in fact be present in some undertaken studies). However, it is Shimizu (1996) who documented that, because of a separation of students' procedural and conceptual knowledge, good procedural knowledge can be demonstrated with missing or very limited conceptual knowledge, which provides some implicit evidence for the genetic view.

Before looking for a suitable pedagogical interpretation of the views represented above, we would like to underline that these four views evidence no general conclusion regarding the relation between procedural and conceptual knowledge (or success in procedural and conceptual tasks representing these knowledge types), but we should ask ourselves if such a conclusion can be eventually reached in a constructivist sense at all, having in mind various teaching approaches, different student abilities, various mathematical topics and associated problems, etc. Undoubtedly, the answer is negative. Thus, instead of searching for this conclusion, we should: (a) make ourselves aware of various empirical outcomes and probable underlying P-C links constructed through, for example, categories classification, productions utilization, microworlds coordination or proceptual thinking (to be discussed latter); and (b) try to examine them in terms of some relevant educational variables (which are, to our knowledge, not clearly articulated in perhaps most research studies). To achieve this end, we will use two different pedagogical approaches called *developmental* and *educational*, being aware that some oversimplifications or somewhat free generalizations will occasionally be made.

Before examining these approaches, we would like to underline that procedural and conceptual knowledge must be somehow related when the learning *process* is our focus. However, it is the variables in the assessment of learning processes that promote or obstruct possible qualitative and quantitative links between the two types of knowledge. By assessment in the global sense we mean the planning, realizing and control of the learning processes, made not only by the teacher, but especially by the learners and learner teams themselves. So, if we happen to enter a classroom after that pupils have worked two weeks with fractions, for example, it would be unsound to declare that these children - or especially which of them - would be able to learn percentage represented in figure 1. Paradoxally, the relevant persons who could make this prognosis are the people involved in this process. Maybe they are making some important investments, which will be payed back in the form of dynamic knowledge elements represented in figure 3, for example.

In order to highlight the relevance of these variables, let us consider the planning of learning processes for some important topics of mathematics (e.g. for a concept field) by using a sailing metaphor: If You can choose the starting harbour freely in a round-world-sailing competition, it would be unsound to declare after few days or weeks whether Your tactics will be successful, even less to give any prognosis about who is going to win. If You have relevant meteo-geographical knowledge (You are as teacher conscious about dynamic and skilful drives along concept networks) it is sometimes reasonable to start against the wind (to invest in conceptual knowledge) instead of choosing the most tempting alternative to just go with the wind (go for immediate procedures without much understanding what to do and/or why to do), hoping that things would be fine as time passes. This metaphor seems appropriate for the issue under consideration, and its meaning will be clarified in several parts of our text. We don't want to consider it in fuller detail as it may then lose its constructivist spirit. Although it is not easy to put constructivist elements in an article regarding constructivism, we would like to do so for the benefit of the reader interested in a refinement of the examined topic. We therefore would like to invite the reader to bear this metaphor in mind when examining any effort regarding:

- a global classification of procedural and conceptual tasks,
- a general relation between the two knowledge types,
- the most recommendable order for sequencing procedural and conceptual knowledge within a particular mathematical topics.

Actually the teacher, and the student (or student team) involved in planning and utilization of the learning process are the only relevant subjects who could do this classification through continuous dynamic assessment.

It is important to underline that clarifying the relation between the two knowledge types is still too complicated to be expressed within any single paper in fuller detail. We will therefore highlight here just some of the most important aspects by using all the way 'fractions' as an example. Why has this example been chosen? It is one of the most fascinating *conceptual fields*[2] of school mathematics (see Hiebert & Behr, 1988). This conceptual field has therefore been a challenge for the above-mentioned MODEM-project (Haapasalo, 1993; see also www.joensuu.fi/lenni/modemfin.html), yielding a large empirical study MODEM2 reported in Haapasalo (1992). Although an examination of the relation between conceptual and procedural knowledge was not so essential in the project's paradigm as finding effective collaborative learning environments, the project utilization does offer evidence in relation to the issues under scrutiny.

Developmental approach

Many researchers find that procedural knowledge enables conceptual knowledge development. In the sense of four views represented above, this is connected to the genetic view or the simultaneous activation view. An instructional interpretation might be: Utilize procedural knowledge and reflect on the outcome. Let us call this position *developmental approach* as it reflects the philogenesis of mathematical knowledge (Kline, 1980; Kitcher, 1983) as well as its ontogenesis (Vergnaud, 1990; Grey & Tall, 1993), especially in early years of mathematical education. Although this approach is not clearly articulated in research studies, it seems that among its advocates are Papert (1980), Vergnaud (1990), Gray & Tall (1993) and Sfard (1994), at least in the sense that the teaching/learning process should utilize this philogentico-ontogenetical pattern.

Before explaining what this approach would mean in the case of fractions, let us take an "evolutionary cavalcade" as follow.

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[2] "...large sets of situations whose analysis and treatment require several kinds of concepts, procedures, and symbolic representations that are connected with one another." (Vergnaud, 1990; p. 23)

- As regards the philogenesis of mathematical knowledge, procedural knowledge has still developed faster than conceptual knowledge since makers of mathematics were primarily guided by the pragmatic aspects of their discipline (e.g., Kline, 1980; Kitcher, 1983). In other words, procedures were devised first; conceptual clarifications have been undertaken latter[3] - or are still waiting to be done[4]. Is this the reason why most ancient, medieval and even new-age mathematical textbooks look like "do this, do that" recipe books?
- As regards the individual development of mathematical knowledge, it seems that again procedural knowledge develop faster than conceptual knowledge. Students "often choose the right thing to do without being able to mention the reason for it. Such behaviours very often reveals the existence of powerful implicit concepts and theorems" that may be called "concepts-in-action and theorems-in-action. Such knowledge cannot be properly called "conceptual", as conceptual knowledge is necessarily explicit." (Vergnaud, 1990; p. 20) However, "objects-in-actions" may eventually conceptualize procedural knowledge.
- The dominance of procedural over conceptual knowledge seems quite natural in general. As an example, recall that the first child's answer to a question "What is a chair?" is not "A chair is an item of furniture" but "A chair is for sitting". Other supportive examples are provided, for example, by Nesher (1986). By examining the process of counting, she concluded that its invariants are likely to be conceptualized only after some exercises on counting have been performed. She empirically found that the same holds true for the conception of the mean.
- The developmental approach is supported by some aspects of the theories of mental development proposed by Piaget and Vygotsky. A recent analysis of these theories undertaken by Ivic (1991) underlines that: (a) for Vygotsky, scientific knowledge is based upon instrumental (procedural) and structural (conceptual) components, (b) for Piaget, "conceptual knowledge comes developmentally later and is based on practical achievements [procedural knowledge]", and (c) for both Piaget and Vygotsky, metacognition (an awareness of one's own thinking processes) is related to "mature stages of development and to the appearance of conceptual systems of knowledge." (p. 24) This analysis clearly emphasises that it is the presence of metacognition that is crucial to conceptual knowledge development and that for this reason procedural knowledge acquisition is generally more accessible to human beings than conceptual knowledge one.
- The developmental approach is also in accord with the theory of reification, especially with its component assuming that knowledge develops through the transition from process to object- based thinking (Sfard, 1994), e.g., from operational to structural

[3] The development of programming languages also illustrates the dominance of procedural over conceptual knowledge. Imperative (or procedural) languages such as BASIC and FORTRAN, which instruct the computer what to do in a step-by-step manner, were developed first, whereas declarative languages like PROLOG, which describe the logical structure of the problems leaving the task of deriving the result to the computer, have been developed later.

[4] Instead of appearing as separate synonymous (conceptual) words for tenths, hundredths, ..., the terms 'desi', 'centi', 'milli', ... are used as (procedural) prefix for units, purely. *MODEM* took the opposite position resulting in the following outcome: pupils learned units, and accuracy into the bargain, and showed significantly better understanding of decimals than the comparison group (see Haapasalo 1993, pp. 20-21).

thinking. Although this theory (inspired by the work of Piaget) may not be relevant to knowledge development in general (Confrey & Costa, 1996), it is clear that we should question "the possibility of teaching for understanding in mathematics without attending to the algorithmic and procedural aspects" (Nesher, 1986; p. 8).

We find that many aspects of the considerations included in the cavalcade above could benefit pedagogical planning of learning environments. The basic underlying idea of the developmental approach may be expressed in our sailing world in the following way: "Choose the starting harbour so that you have a nice wind behind you. Get used to your boat, climate and have a touch for sailing. After learning to manage your boat in different conditions, try to find principles how the wind is acting on the sail, to optimize the sail positions, to learn to understand weather reports, as well."

What kinds of starting harbours would be available for fraction-sailing? The MODEM approach benefits from classifying concepts into three classes[5] like Steinhöfel et al. (1976):

- As an *object concept*, a fraction (e.g., 1/2) relates to a real or mental thing that can be characterized through its representatives. It is just a way of expressing a (real) situation in verbal, graphic or symbolic form, like "one of the two pupils in our group is girl" and "each of two hemispheres represent the same fraction 1/2 in the graphic form".
- As an *operation concept*, a fraction refers to one or more operations to be carried out by manipulating objects. For example, "Taking one half of a tennis ball is something else than taking one half of a football - or of my daddy's salary."
- As a *relation concept*, a fraction such as 1/2 denotes the relationship between the numbers 1 and 2, linking other procedural-conceptual knowledge like dividing and proportionality.

Many mathematical concepts belong to all these types. For example, a 'function' is usually defined as an object concept by identifying it with its graph consisting of points, each of them represents a pair of numbers determined by that function. On the other hand, this definition characterizes it as a relation concept. Furthermore, if the function is defined via a certain law, it can be seen as an operator f directed to an object x producing y.

It is clear that each of these concept types offers a very different starting harbour. As well the content as the context impact decisively what kind of experiences a particular child connects to each instructions. We see from the cavalcade above that counting would offer a nice starting harbour for whole numbers. This would mean accelerating conceptual knowledge by counting procedures (and usually vice versa). The counting skills are implicit in the sense of Vergnaud (1990, p. 20): "A person often chooses the right thing to do without being able to mention the reasons for it". Having assumed implicitly that these reasons will be articulated in one or another way as time passes, textbooks writers and teachers seem to approach most topics procedurally. As regards fractions, they like to serve them as numbers, and get pupils, with great hurry, to do calculations with them, without carrying too much what to do and why the particular procedures work. This might be the basic reason why several studies evidenced poor understanding of fractions (e.g. Haapasalo 1992; 1993, p. 19). Being familiar with whole numbers, children tend to work with fractions by using procedures relating to whole numbers, like 2/3 + 3/4 = 5/7, for example.

[5] These classes are integrated in the proceptual thinking by Gray & Tall (1993).

Another developmental starting harbour would be to utilize children's spontaneous dividing procedures in certain "microworlds" that have relevance to their every-day life. A symbolic form like 2/7 + 3/7 could be connected to a situation: "I got today two dollars to be divided for next week so that each day I could spent the same amount of money. Tomorrow I am going to get three dollars more. If I estimate how much I would have to spend each day next week, it would be reasonable to divide five dollars by seven. Could I gather two dollars more, it would be easy!" These kinds of starting harbours, which allow producing a right answer without knowing much about fraction as a concept, can lead teacher to over-expectations concerning child's ability to get out of context-oriented situations.

The same yields for the often-used starting point to visualize symbolic expressions by paper folding. A mismatch can occur because of data overflow: The fraction 1/2, for example, means simultaneously dividing by two, dividing in equal parts, the product of a dividing process, a denotation of the division, a take-a-half-operator, or even the relation between the numbers 1 and 2 (proportionality). Before seeing the situation in a holistic way - which means having sufficient conceptual knowledge - children should be able to process many data chunks simultaneously. This strongly contradicts what we know about human limits in serial processing. Of cause, many beautiful elements of the simultaneous activation view could basically be involved by this approach. Real problems begin, when the teacher wants to offer prototypes of a certain concept type, but a pupil is trying to interpret, even receive them as prototypes from another concept type.

In spite of these critical considerations above we still see that conceptual knowledge does not need, especially in the beginning of a learning period, to be explicit. A good teacher and good learner can together produce a link from procedural to conceptual knowledge.

By making somewhat free generalizations, we summarize the possible solutions below.

- Assume the learner's innate ability to divide the world (not a problem) into several (conceptual) *microworlds* enabling different fraction procedures to be applied within each of them, (compare: adding numbers in little worlds based upon finger manipulation, money facts and LOGO turtle geometry facts). We can share Papert's (1987) conjecture that it is basically the elaboration and coordination of these microworlds that enables conceptual knowledge to develop out of such fractured procedural knowledge.
- Suppose the existence of "a combined mental object consisting of a process, a concept produced by that process, and a symbol which may be used to denote either of both" such as 'one half', 'fifty-fifty', 1/2. We can adopt the idea of *proceptual thinking* by Gray & Tall (1993; p. 8): Procedural and conceptual knowledge are related through utilizing "procedures where appropriate and symbols as manipulable objects where appropriate".

Educational approach

Most, perhaps the majority of, researchers/educators assume the dependence of procedural on conceptual knowledge. Among these are Anderson (1983), Carpenter (1986), Byrnes & Wasik (1991), Hiebert & Carpenter (1992) and Haapasalo (1993). What they assume may briefly be summarized as: it is conceptual knowledge that enables procedural knowledge development. An instructional implication is: Build meaning for procedural knowledge before mastering it (Hiebert & Carpenter, 1992). Let us call this position *educational approach* since it seems to fulfil educational needs typically requiring a large body of knowledge to be understood and to have supposed transfer effect.

The educational approach may be supported by the dynamic interaction (DI) view or the simultaneous activation (SA) view, which seem to come out in many works made by the mathematics educators in 80's and 90's. However, several researchers including Silver (1986), Nesher (1986), Schoenfeld (1986) and Shimizu (1996), who seem to prefer the educational approach, have underlined that the developmental approach may be relevant as well. Silver (1986), for example, gave two straightforward examples. The first stresses that one's knowledge of a concept depends on a procedure that distinguishes the concept's examples from its non-examples. The second, which deals with related problems, points out that conceptual categorizing of similar problems is primarily based upon procedural knowledge used in (needed for) solving them.

Instead of having a cavalcade of researchers' views again, we find that it is the didactical relevance of the DI and SA views that we should try to highlight in this part. Because the MODEM project philosophy combined both these views in a natural way, we come back to our fraction example again in terms of the two views. Let us first put the main idea in the language of our sailing metaphor: "Before just starting to sail (resp. going for a round-world race), learn the basic idea for the relation of the sail and the wind. Test[6] the basic functions in simplified situations, and learn to understand weather reports (resp. the impact of geographical facts on the climate, as well). Estimate your skills for choosing the conditions (a starting harbour) according to that." The question is about nothing but regulating the balance between the theoretical knowledge of the physics beyond sailing and the practical experiences. For this, certain simplifications for the *construction space* (learning environment) are often necessary, still allowing a room for radical constructions from the learner's side.

Keeping in mind that *fractions* is an enormous conceptual field, the starting harbour was to let children construct fraction as an object concept (i.e. not any number, yet). The basic question was: "Does your team think that it would be possible to express verbally which part of the pupils in the team are girls?" Pupils could easily find 10-20 different kinds of expressions, which include the relevant attributes for a fraction as an object concept. The most relevant idea is that "A thing which has a *Gestalt* of 1 (i.e. type of a cake or a group which must be identified as 1, at first), can be divided in equal parts, and from these parts a particular amount can be taken". Some examples of the outcome are: "two of seven", "two sevenths", "a group is divided in seven equal parts and two is taken", "the amount of sevenths is two", etc. Beyond each of these expressions there is, of course, different kind of experience-based procedural knowledge (and maybe naive concepts). This means that at the beginning, the developmental approach is automatically involved, and we should not try to avoid it. It could be compared with a wake-up-voltage, which is needed as a trigger for another higher voltage by an amplifier. Or coming back to our metaphor: It would not be sound to go sailing when the water is as smooth as a glass.

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[6] This might be quite appropriate as a basic philosophy for a sailing academy, but not in a competition.

We find that the DI view means organizing the (social) learning environment so that pupils' radical constructions could converge towards viable, context-free expressions, to be used as attributes for conceptual understanding. For that, the following kinds of problems were used: "Does your team think that it would be possible to express the same thing by using only numbers?" The teams represented (very proudly!) expressions like 2 (7), 2o7, 7[2]. After arguing and testing own denotations (e.g., by asking someone else to explain them) pupils chose the viable symbolic expression 2/7, easily understood by every-man on the street. Now the pupils were ready to define a fraction as an object concept in symbolic *(S)*, graphic *(G)* and verbal *(V)* forms, and process between these forms[7] (attributes) on the identification (**I**) and production (**P**) levels (see **P**XY in Fig. 3). This conceptual knowledge can be used not only for constructing (calculation) procedures, but also for widening the conceptual understanding toward fraction as an operational and relational concept types. SA method[8] gives excellent possibilities to let conceptual and procedural knowledge accelerate each other. Clarifying our metaphor of using electric circuits: "Use naive procedural knowledge as a trigger for amplifying it to conceptual understanding with the DI method. Then You have enough (conceptual) power to be used in different kind of proceduralo-conceptual circuits (educational tasks), like represented in the figure 3."

In the previous chapter we warned against teacher's eagerness to decide in advance how particular knowledge units are to be sequenced. The most beautiful idea beyond the SA method is to let children plan and test their own constructions. The process does not need to begin from concrete to abstract, but between abstract and concrete things, and even between abstract things, especially when children have an opportunity to quickly manipulate representatives of these things. For example, a real situation from a child's every-day-life (like in Fig. 3) can offer opportunities to connect spontaneous procedural knowledge (and pre-concepts) with (even abstract) conceptual knowledge. In which order a particular child processes each of the sub-tasks in this kind of learning environment, depends on pedagogical variables. Before reaching the level of *production* (**P**) (assessed of course before the leaflet was given), children tested verbal *(V)*, symbolic *(S)* and graphic *(G)* concept attributes on the level of *identification* (**I**). In all empirical studies of MODEM, this sub-phase played the most important role in concept building. As we underlined that any procedural knowledge can be interpreted as an operation concept, the identification types *(***I***VV,* **I***VS,* **I***VG,* **I***GG,* **I***GS,* **I***SS)* could be assumed as useful tasks for constructing the basic attributes of procedures, or procepts in terms of Gray & Tall (1993), as well. Note that in Fig. 3, many kind of thinking activities may be utilized: a production of conceptual knowledge and an identification of procedural knowledge, for example. On the other hand, for some children who have not reached the production level of the object concept, this kind of SA environment can help to find links between different kind of representation forms.

For highlighting some advantages of SA over that of DI, we would like to tell the following true-story example. In spite of reaching via SA and DI automatic processions between V, G and S forms of fractions 2/7 and 3/7, some pupils could not give any reasonable (verbal) expression for the symbolic form 2/7 + 3/7. In this case, the problem was not in the

[7] For avoiding data overflow, it is appropriate to take only one symbolic notation, at first. A successful processing between *(S)*, *(V)* and *(G)* is a symptom of a holistic conceptual structure, and children can easily link new denotations to this. Here slash ("/") is used (e.g., 3/7) to easier word-processing.
[8] Although we have defined SA and DI as views, it seems appropriate here to speak about *methods*.

conceptual understanding of fraction, but in that of the operation concept +. This symbol made suddenly the very rich verbalizations to be degenerated to "two sevenths plus three sevenths", which didn't seem to trigger any cognitive resources. Those who could easily construct solution(s) for the addition task above used processing via very rich verbalizations (first **P**SV, then **P**VV, and finally **P**VS), like:

- "Two of seven and three of seven make altogether five of seven.
  I can express this in symbols, if I want: 5/7"
- "If the amount of sevenths is two, and I increase it by three, it makes five in all.
  I make 5/7, shortly."

The above mentioned example shows that a link between conceptual and procedural knowledge can be missed in any approach. However, it is more essential to ask how this link can be construct in each pedagogical approach. In this MODEM example just an encouragement was needed, and the cognitive link appeared: "Let's take a race between teams for finding as many verbalizations as possible for this dead symbol + !" This activity would have been evidently unnecessary by using a leaflet like in Fig. 3.

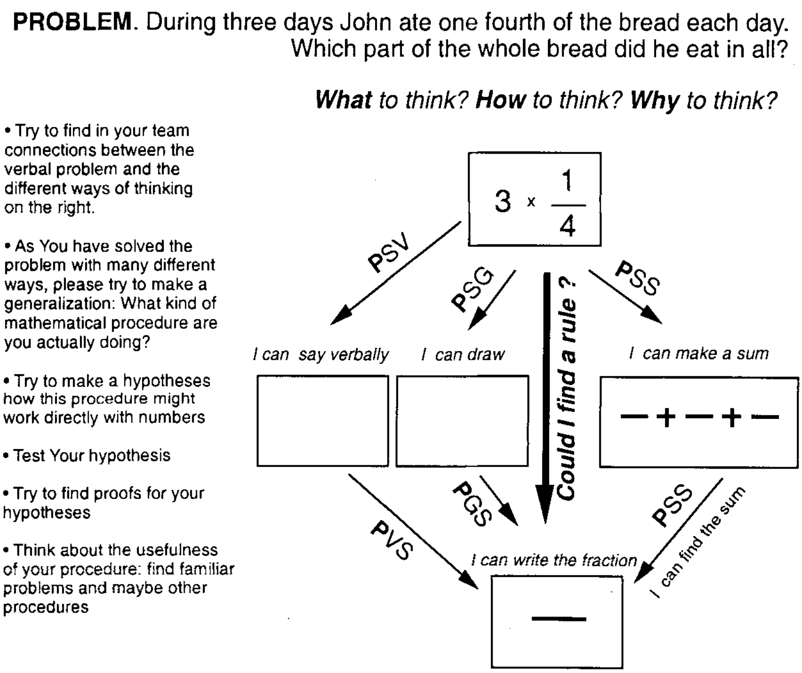

Fig. 3. An example of combining the DI and SA views in the MODEM project (P*SG*, for example, denotes **p**roduction from *s*ymbolic form to *g*raphic form)

We are closing this chapter on the educational approach by summarizing three proposals regarding a link between conceptual and procedural knowledge within this approach. Before analyzing them, the reader may again examine distinctions of Byrnes & Wasik, Gelman & Meck, and Anderson presented in the first chapter of this paper.

- Procedures are linked to concepts by means of categories, which are partly defined in procedural terms by functions of objects comprising them. If an object A is classified as a member of category X, then rules appropriate to this category are put into practice. Of course, if a misclassification occurs, inappropriate procedures are applied. For example, categorizing fractions as whole numbers would generate unsuitable additions like 2/3 + 3/4 = 5/7. (Byrnes & Wasik, 1991).
- "Conceptual competence or 'principled knowledge' is coordinated with the planning- and procedure-generation system that makes up procedural competence and thus helps determine the actual procedures used" (Gelman & Meck, 1986; p. 30). In other words, it is *utilization competence* that links conceptual and procedural knowledge, enabling conceptual knowledge to be skilfully proceduralized. For example, in geometric tasks calling for locus constructions (e.g., construct a triangle, being given $a + b$, c, and b), an *enabling condition* (an utilization competence item) is typically the following rule: "To determine a point that lies on a line with certain properties, construct this line obtaining a locus for that point". Such rule, the existence of which is usually overlooked by an expert, may not be obvious to many students, especially novices. (Schoenfeld, 1986). As regards our fraction example (2/3 + 3/4), an enabling condition may be the following rule: "To determine a sum by applying a common measurement unit, come up with such a unit enabling re-measuring of both addends."
- New task-specific productions (condition-action rules) have been initially developed through applying the available conceptual knowledge interpretively by means of some general problem solving productions. It is therefore some set of general productions that provide a link between the available declarative knowledge (i.e. conceptual knowledge in our terms) and procedural knowledge being developed. For example, initial two-column-proof skills, for a task on writing reasons for given statements, are generated through interpreting the declarative data from the task (i.e., given, wanted and statements) and a previously examined proof sample (e.g., given, wanted, statements and reasons) by means of some general problem solving productions, such as "IF the goal is to name a particular relation (e.g., why is AB = CD) *and* that relation is not yet known (e.g., underlying reason) THEN a sub-goal is to find out the relation". As regards our example (2/3 + 3/4), a relevant general production may be: "IF the goal is to find out a relation (e.g., the requested sum) *and* a set of methods for doing that is known (e.g., verbally, graphically and arithmetically) *and* there is a not yet tried method (e.g., graphic) THEN a sub-goal is to try out that method".

  Newly generated skills (i.e., task-specific productions) comprise procedural knowledge after knowledge *compilation* has been taken place.[9] This compilation is based upon *composition* collapsing of a sequence of productions into a single product-ion and *proceduralization* building versions of productions not requiring declara-tive knowledge retrieval (i.e. automating productions). The compilation enables a transition from an interpretative to a direct declarative knowledge utilization, which deals with specific productions such as "IF the goal is to give a reason for a statement and that statement is among the given THEN POP the goal with success". (Anderson, 1983)

[9] Having completed knowledge compilation, new productions need to be properly selected in the course of problem solving. Proper selections are achieved through *tuning productions* based upon the processes of: *generalization* extending their range of applicability, *discrimination* restricting their range of applicability, and *strengthening* regulating their amount of activation in competition with other productions that seem relevant to the task at hand. For details, see Anderson (1983).

Closing remarks

According to Vygotsky, procedural knowledge does precede conceptual knowledge ontogenetically, but it is school learning that precedes intellectual development, enabling learning to takes place in the learner's zone of proximal development[10]. The underlying theoretical position for such learning is: A higher mental activity appears first dependently through inter-personal social activities, re-appearing latter independently through intellectual actions. This mechanism for transformation of inter-psychic phenomena into intra-psychic phenomena is activated through imposing on the learner developed knowledge and cultural tools, e.g., systems of scientific concepts, built-in modes of thinking; methods, algorithms, procedures; computers and other tools that aid thinking. (Vygotsky, 1978; Ivic, 1989; 1991)

If we accept Vygotsky's theory of mental development, the educational approach is supposed to be more relevant to intellectual development than the developmental one. But it does not mean that the former approach is to be primarily used as it its cognitive requirements may be beyond learner's zone of proximal development. In other words, the educational approach may be quite appropriate for some topics, whereas the developmental approach may be more suitable for others. For example, the former approach may be suitable for conceptual fields as fractions, decimals, proportionality (Haapasalo, 1993) and two-column proof skills (Anderson, 1983). On the other hand, the latter approach may be appropriate for solving one-step verbal problems in addition and subtraction by means of verbal counting (Bergeron & Herscovics, 1990) as well as for introducing the concept of a limit that promotes its dynamic definition such as "f(x) gets close to b as x gets close to a" (Tall, 1992). However, it should be kept in mind that the choice of an approach is at least person, content and context dependent. After all, a general solution of any learning issue is rarely attainable in a constructivist sense.

To summarize: This study has tried to clarify the relation between procedural and conceptual knowledge. Hoping that this ambitious goal has been at least partially achieved, we are directing our further research toward a framework combining the two presented approaches into a coherent whole. We are closing this paper with a very known proverb: "If you give fish to your fellowman, he will have food for today. If you teach him to fish, he will have food for the rest of his life." It should be used as an invitation to not only comparing different pedagogical approaches to the two types of knowledge, but also thinking about the goals of mathematics education more globally.

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[10] "[T]he distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p. 86)

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