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Results in Mathematics

ON COMMUTATIVITY OF RINGS WITH DERIVATIONS [1] [2]

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Abstract

Let R be a ring and $d : R \longrightarrow R$ a derivation of R. In the present paper we investigate commutativity of R satisfying any one of the properties $(i) d([x,y]) = [x,y], (ii) d(x \circ y) = x \circ y, (iii) d(x) \circ d(y) = 0$, or $(iv) d(x) \circ d(y) = x \circ y$, for all x, y in some apropriate subset of R.

1. Introduction

Let R denote an associative ring with center $Z(R)$. For any $x, y \in R$ we write $[x,y] = xy - yx$ and $x \circ y = xy + yx$. For a non-empty subset S of R, we put $C_R(S) = \{x \in R \mid [x,s] = 0, \text{ for all } s \in S\}$. The set of all commutators of elements of S will be written as $[S,S]$. Recall that R is prime if $aRb = (0)$ implies that $a = 0$ or $b = 0$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, holds for all $x, y \in R$. An additive subgroup U of R is said to be a Lie ideal of R if $[u,r] \in U$, for all $u \in U$ and $r \in R$.

2. Preliminary results

Throughout the present paper we shall make use of the following two basic identities without any specific mention :

$$x \circ (yz) = (x \circ y)z - y[x,z] = y(x \circ z) + [x,y]z$$

$$(x \circ y)z = x(y \circ z) - [x,z]y = (x \circ z)y + x[y,z].$$

We begin with the following known results which will be used extensively to prove our theorems.

Lemma 2.1 ([5, Lemma 3]). Let R be a 2-torsion free prime ring. If U is a nonzero Lie ideal of R and $a \in R$ centralizes $[U,U]$, then a centralizes U, that is $C_R([U,U]) = C_R(U)$.

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Lemma 2.2 ([5, Lemma 4]). If $U \not\subseteq Z(R)$ is Lie ideal of a 2-torsion free prime ring R and $a, b \in R$ such that $aUb = (0)$, then $a = 0$ or $b = 0$.

Lemma 2.3 ([5, Lemma 5]). Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R. If d is a nonzero derivation of R such that $d(U) = 0$, then $U \subseteq Z(R)$.

Lemma 2.4 ([1, Theorem 7]). Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R. If d is a nonzero derivation of R such that $[u, d(u)] \in Z(R)$, for all $u \in U$, then $U \subseteq Z(R)$.

Lemma 2.5 ([4, Lemma 3]). Let R be a prime ring and I a nonzero right ideal of R. If d is a nonzero on R, then d is nonzero on I.

Lemma 2.6 ([4, Theorem 4]). Let R be a prime ring and I a nonzero left ideal of R. If R admits a nonzero derivation d such that $[x, d(x)]$ is central for all $x \in I$, then R is commutative.

Lemma 2.7 ([13, Lemma 3]).If a prime ring R contains a nonzero commutative right ideal, then R is commutative.

Now we prove the following.

Lemma 2.8. Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. If R admits a derivation d such that $d^2(x) = 0$, for all $x \in I$, then $d = 0$.

Proof. We have $d^2(x) = 0$, for all $x \in I$. Replacing x by xy, we get

$$d^2(xy) = 0 = d^2(x)y + 2d(x)d(y) + xd^2(y), \quad \text{for all} \quad x, y \in I.$$

But $d^2(x) = 0 = d^2(y)$ by the hypothesis. The upshot is that $2d(x)d(y) = 0$, for all $x, y \in I$. Sience R is 2-torsion free we find that $d(x)d(y) = 0$. Now, for any $r \in R$, replace y by yr, to get $d(x)yd(r) = 0$, for all $x, y \in I$ and hence $d(x)IRd(r) = (0)$, for all $x \in I$, $r \in R$. Thus, primeness of R forces that either $d = 0$ or $d(x)I = (0)$. If $d(x)I = (0)$, for all $x \in I$, then $d(x)RI = (0)$, for all $x \in I$; since $I \neq 0$, and R is prime the above relation yields that $d(x) = 0$, for all $x \in I$ and by Lemma 2.5, we get the required result.

3. Lie Ideals and Derivations of Prime Rings

There has been a great deal of work recently concerning the relationship between the commutativity of a ring R and the existence of certain specified derivations of R (cf, [1],[3],[4],[8],[9],[12] and [13]). In the year 1992 Daif and Bell [8] established that a semiprime ring R must be commutative if it admits a derivation d such that $d([x, y]) = [x, y]$, for all $x, y \in R$. In view of this result, it is natural to question that: what can we say about the commutativity of R satisfying the property $d([x, y]) = [x, y]$, for all x, y in some distinguished subsets of R ? In the following theorem, it is shown that the conclusion of the above theorem remains true in the case when the ring R is prime and the underlying subset of R is a Lie ideal of R.

Theorem 3.1. Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R. If R admits a derivation d such that $d([u, v]) = [u, v]$, for all $u, v \in U$, then $U \subseteq Z(R)$.

Proof. We are given that d is a derivation of R such that $d([u,v]) = [u,v]$, for all $u,v \in U$. If $d = 0$ then $[u,v] = 0$, for all $u,v \in U$. Replacing v by $[u,r]$ we get $[u,[u,r]] = 0$, for all $u \in U$, $r \in R$. Again replace r by rs, to get $[u,[u,rs]] = 0$, for all $u \in U$, $r,s \in R$, that is

$$[u,[u,r]]s + r[u,[u,s]] + 2[u,r][u,s] = 0, \quad \text{for} \quad \text{all } u \in U, \ r,s \in R.$$

This implies that $2[u,r][u,s] = 0$, for all $u \in U$, $r,s \in R$. Since char$R \neq 2$, we get $[u,r][u,s] = 0$. Replacing s by sr, we get $[u,r]R[u,r] = (0)$, for all $u \in U$, $r \in R$. Thus primeness of R forces that $[u,r] = 0$, for all $u \in U$, $r \in R$, and hence $U \subseteq Z(R)$. Henceforth, we assume that $d \neq 0$. For any $u,v \in U$, we have $d([u,v]) = [u,v]$. This implies that $d([u,[w,v]]) = [u,[w,v]]$, for all $u,v,w \in U$, and hence $[d(u),[w,v]] + [u,d([w,v])] = [u,[w,v]]$ i.e.

$$[d(u),[w,v]] + [u,d([w,v]) - [w,v]] = 0, \quad \text{for} \quad \text{all } u,v,w \in U.$$

Thus, application of our hypothesis yields that for all $u,v,w \in U$, $[d(u),[w,v]] = 0$, and hence by Lemma 2.1, we find that $d(u) \in C_R(U)$, for all $u \in U$. Thus in particutar $[d(u),u] = 0$, for all $u \in U$. Hence, by application of Lemma 2.4, we get the required result.

Using the same techniques with necessary variations we get the following.

Theorem 3.2. Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R. If R admits a derivation d such that $d([u,v]) + [u,v] = 0$, for all $u,v \in U$, then $U \subseteq Z(R)$.

Corollary 3.1. Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R. If R admits a derivation d such that $d(uv) = uv$, for all $u,v \in U$, then $U \subseteq Z(R)$.

Proof. For any $u,v \in U$ $d(uv - vu) = d(uv) - d(vu) = uv - vu$ and hence by Theorem 3.1, we get the required result.

Similarly, in view of the Theorem 3.2, we get the following.

Corollary 3.2. Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R. If R admits a derivation d such that $d(uv) = vu$, for all $u,v \in U$, then $U \subseteq Z(R)$.

Theorem 3.3. Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$, for all $u \in U$. If R admits a derivation d such that $d(u \circ v) = u \circ v$, for all $u,v \in U$, then $U \subseteq Z(R)$.

Proof. If $d = 0$, then we have

$$u \circ v = 0, \quad \text{for} \quad \text{all } u,v \in U. \tag{3.1}$$

Notice that $vw + wv = (v+w)^2 - v^2 - w^2$, for all $v,w \in U$. Since $u^2 \in U$, for all $u \in U$, $vw+wv \in U$. Also $vw - wv \in U$, for all $v,w \in U$. Hence we find that $2vw \in U$, for all $v,w \in U$. Replacing v by $2vw$ in (3.1) and using (3.1), we have $2v[u,w] = 0$, for all $u,v,w \in U$. This implies that $v[u,w] = 0$, for all $u,v,w \in U$. Again replace v by $[u,r]$, to get $[u,r][u,w] = 0$, for all $u,w \in U$, $r \in R$. For any $s \in R$, replacing r by rs, we get $[u,r]R[u,w] = (0)$, for all $u,w \in U$, $r \in R$. Thus, in particular we have $[u,w]R[u,w] = (0)$, for all $u,w \in U$. Thus primeness of R yields that $[u,w] = 0$, for all $u,w \in U$. Note that the arguments given in the begining of the proof of Theorem 3.1 are still valid in the present situation and hence we get the required result.

Therefore now on we shall assume that $d \neq 0$. Suppose on contrary that $U \not\subseteq Z(R)$. For any $u, v \in U$, we have $d(u \circ v) = u \circ v$. This can be rewritten as

$$d(u) \circ v + u \circ d(v) = u \circ v, \quad \text{for all } u, v \in U. \tag{3.2}$$

Replacing v by $2vu$ in (3.2) and using the fact that characteristic of R is different from 2, we find that

$$(d(u) \circ v + u \circ d(v) - u \circ v)u + (u \circ v)d(u) = 0, \quad \text{for all } u, v \in U,$$

and hence application of (3.2) gives that $(u \circ v)d(u) = 0$, for all $u, v \in U$. Again replace v by $2wv$, to get $[u, w]vd(u) = 0$, for all $u, v, w \in U$, and hence $[u, w]Ud(u) = 0$, for all $u, w \in U$. Thus, for each $u \in U$ by Lemma 2.2, either $[u, w] = 0$ or $d(u) = 0$. Now let $U_1 = \{u \in U \mid [u, w] = 0, \text{ for all } w \in U\}$, $U_2 = \{u \in U \mid d(u) = 0\}$. Then U_1 and U_2 both are additive subgroups of U and $U_1 \cup U_2 = U$. Thus, either $U_1 = U$ or $U_2 = U$. If $U_1 = U$, then $[u, w] = 0$, for all $u, w \in U$. Hence again using the same arguments as used in the begining of the proof of Theorem 3.1, we get $U \subseteq Z(R)$, a contradiction. On the other hand if $U_2 = U$, then $d(u) = 0$, for all $u \in U$ and again by Lemma 2.3, we get a contradiction. This completes the proof of the theorem.

Using similar arguments as used in the above theorem, we can prove the following:

Theorem 3.4. Let R be a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$, for all $u \in U$. If R admits a derivation d such that $d(u \circ v) + u \circ v = 0$, for all $u, v \in U$, then $U \subseteq Z(R)$.

4. Ideals and derivations of prime rings

If we choose the underlying subset of R as an ideal instead of a Lie ideal of R in the hypothesis of Theorems 3.3 and 3.4, then we can prove the following result even without the characteristic assumption on the ring.

Theorem 4.1. Let R be a prime ring and I a nonzero ideal of R. If R admits a derivation d such that $d(x \circ y) = x \circ y$, holds for all $x, y \in I$, then R is commutative.

Proof. For any $x, y \in I$, we have $d(x \circ y) = x \circ y$. If $d = 0$, then $x \circ y = 0$, for all $x, y \in I$. Replacing y by yz and using the fact that $xy = -yx$, we find that $y[x, z] = 0$ for all $x, y, z \in I$ and hence $IR[x, z] = (0)$, for all $x, z \in I$. Since $I \neq (0)$, and R is prime, we get $[x, z] = 0$, for all $x, z \in I$, and hence by Lemma 2.7, R is commutative. Hence, onward we assume that $d \neq 0$. For any $x, y \in I$, we have $d(x \circ y) = x \circ y$. This can be rewritten as

$$d(x) \circ y + x \circ d(y) = x \circ y, \quad \text{for all } x, y \in I. \tag{4.1}$$

Replacing y by yx in (4.1), we get

$$d(x) \circ (yx) + x \circ (d(y)x + yd(x)) = x \circ (yx), \quad \text{for all } x, y \in I,$$

and hence in view of (3.3) the above relation yields that $(x \circ y)d(x) = 0$, for all $x, y \in I$. Again replace y by zy, to get $z(x \circ y)d(x) + [x, z]yd(x) = 0$, for all $x, y, z \in I$ and hence $[x, z]IRd(x) = (0)$,

for all $x, z \in I$. Thus, primeness of R forces that for each $x \in I$ either $d(x) = 0$ or $[x, z]I = (0)$ for all $z \in I$. The set of $x \in I$ for which these two properties hold are additive subgroups of I whose union is I and therefore $d(x) = 0$,for all $x \in I$ or $[x, z]I = (0)$, for all $x, z \in I$. If $d(x) = 0$, for all $x \in I$, then by Lemma 2.5, $d = 0$ a contradiction. Therefore $[x, z]I = (0)$, for all $x, z \in I$. This implies that $[x, z]RI = (0)$. Since $I \neq (0)$, we find that $[x, z] = 0$, for all $x, z \in I$ and hence again by Lemmma 2.7, R is commutative.
Using similar arguments as above with necessary variations we can prove the following:

Theorem 4.2. Let R be a prime ring and I a nonzero ideal of R. If R admits a derivation d such that $d(x \circ y) + x \circ y = 0$, holds for all $x, y \in I$, then R is commutative.

A careful scrutiny of the proof of Theorem 4.1 shows that a prime ring R is commutative if it satisfies the property $x \circ y = 0$, for all $x, y \in R$. Thus, it is natural to explore the behaviour of rings satisfying the property $d(x) \circ d(y) = 0$, for all $x, y \in R$. In the present section we shall study behaviour of the ring satisfying any one of the properties $d(x) \circ d(y) = 0$, $d(x) \circ d(y) = x \circ y$, and $d(x) \circ d(y) + x \circ y = 0$.

Theorem 4.3. Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If R admits a nonzero derivation d such that $d(x) \circ d(y) = 0$, for all $x, y \in I$, then R is commutative.

Proof. For any $x, y \in I$, we have $d(x) \circ d(y) = 0$. Replacing y by yz, we get $d(x) \circ (d(y)z + yd(z)) = 0$, for all $x, y, z \in I$ and hence we find that

$$(d(x) \circ d(y))z - d(y)[d(x), z] + [d(x), y]d(z) + y(d(x) \circ d(z)) = 0, \text{ for all } x, y, z \in I.$$

Now, by our hypothesis the above relation yields that

$$[d(x), y]d(z) - d(y)[d(x), z] = 0, \quad \text{for all } x, y, z \in I. \tag{4.2}$$

Replace z by $zd(x)$ in (4.2) and use (4.2), to get $[d(x), y]zd^2(x) = 0$, for all $x, y, z \in I$ and hence $[d(x), y]IRd^2(x) = (0)$. Thus, primeness of R forces that for each $x \in I$ either $d^2(x) = 0$ or $[d(x), y]I = (0)$. The sets of elements of I for which these two conditions hold are additive subgroups of I whose unioun is I, consequently, we must have either $d^2(I) = 0$ or $[d(x), y]I = (0)$, for all $x, y \in I$. If $d^2(I) = 0$, then by Lemma 2.8, we get $d = 0$, a contradiction. Therefore, consider the remaining possibility that $[d(x), y]I = (0)$, for all $x, y \in I$ and hence $[d(x), y]RI = (0)$, for all $x, y \in I$. Since $I \neq (0)$, and R is prime , we get $[d(x), y] = 0$, for all $x, y \in I$. Hence by Lemma 2.6, R is commutative.

Theorem 4.4. Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If R admits a derivation d such that $d(x) \circ d(y) = x \circ y$, for all $x, y \in I$, then R is commutative.

Proof. For any $x, y \in I$, we have $d(x) \circ d(y) = x \circ y$. If $d = 0$, then $x \circ y = 0$, for all $x, y \in I$. Notice that the arguments given in the begining of the proof of Theorem 4.1 are still valid in the present situation and hence we get the required result. Therefore, we assume that $d \neq 0$. For any $x, y \in I$ we have $d(x) \circ d(y) = x \circ y$. Replacing y by yz, we get $d(x) \circ (d(y)z + yd(z)) = x \circ (yz)$, for all $x, y, z \in I$, and hence

$$(d(x) \circ d(y))z - d(y)[d(x), z] + (d(x) \circ y)d(z) - y[d(x), d(z)] = (x \circ y)z - y[x, z].$$

Now, using our hypothesis we find that

$$(d(x) \circ y)d(z) - d(y)[d(x), z] - y[d(x), d(z)] + y[x, z] = 0. \tag{4.3}$$

For any $r \in R$, replace y by ry in (4.3) and use (4.3), to get

$$[d(x), r]yd(z) - d(r)y[d(x), z] = 0, \text{for all } x, y, z \in I, \quad r \in R.$$

Now, substituting $d(x)$ for r in the above relation, we find that $d^2(x)y[d(x), z] = 0$, for all $x, y, z \in I$ that is $d^2(x)RI[d(x), z] = (0)$, for all $x, z \in I$. Thus primeness of R forces that for each $x \in I$ either $d^2(x) = 0$ or $I[d(x), z] = (0)$, for all $z \in I$. Hence by application of Brauer's trick, we have either $d^2(x) = 0$, for all $x \in I$ or $I[d(x), z] = (0)$ for all $x, z \in I$. If $d^2(x) = 0$, for all $x \in I$, then by Lemma 2.8, $d = 0$, a contradiction. On the other hand let $I[d(x), z] = (0)$, for all $x, z \in I$ that is $IR[d(x), z] = (0)$. Since I is a nonzero ideal of R and R is prime, the above relation yields that $[d(x), z] = 0$, for all $x, z \in I$. Hence by Lemma 2.6, R is commutative.

Using the similar arguments we can prove the following:

Theorem 4.5. Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If R admits a derivation d such that $d(x) \circ d(y) + x \circ y = 0$, for all $x, y \in I$, then R is commutative.

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