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Results in Mathematics

Orthocentric simplices and their centers

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Abstract

A simplex is said to be orthocentric if its altitudes intersect in a common point, called its orthocenter. In this paper it is proved that if any two of the traditional centers of an orthocentric simplex (in any dimension) coincide, then the simplex is regular. Along the way orthocentric simplices in which all facets have the same circumradius are characterized, and the possible barycentric coordinates of the orthocenter are described precisely. In particular these barycentric coordinates are used to parametrize the shapes of orthocentric simplices. The substantial, but widespread, literature on orthocentric simplices is briefly surveyed in order to place the new results in their proper context, and some of the previously known results are given with new proofs from the present perspective.

Keywords: barycentric coordinates, centroid, circumcenter, equiareal simplex, equifacetal simplex, equiradial simplex, Gram matrix, incenter, Monge point, orthocenter, orthocentric simplex, rectangular simplex, regular simplex

0 Introduction

This paper is a study of the geometric consequences of assumed coincidences of *centers* of a *d-dimensional orthocentric simplex* (or, simply, *orthocentric d-simplex*) S in the d-dimensional Euclidean space, $d \geq 3$, i.e., of a d-simplex S whose $d+1$ altitudes have a common point $\mathcal{H}$, called the *orthocenter* of S. The centers under discussion are the *centroid* $\mathcal{G}$, the *circumcenter* $\mathcal{C}$, and the *incenter* $\mathcal{I}$ of S. For triangles, these centers are mentioned in Euclid's *Elements*, and in fact they are the only centers mentioned there. It is interesting that the triangle's *orthocenter* $\mathcal{H}$, defined as the intersection of the three altitudes, is never mentioned in the *Elements*, and that nothing shows Euclid's awareness of the fact that the three altitudes are concurrent, see [24], p. 58. It is also worth mentioning that one of the most elegant proofs of that concurrence is due to C. F. Gauss, and A. Einstein is said to have prized this concurrence for its nontriviality and beauty. However,

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in contrast to the planar situation, the $d+1$ altitudes of a d-simplex are not necessarily concurrent if $d \geq 3$. We may think of this as a first manifestation of the reality that general d-simplices, $d \geq 3$, *do not have all the nice properties that triangles have.*

It is natural that, besides $\mathcal{G}, \mathcal{C}$, and $\mathcal{I}$, we will also consider the orthocenter $\mathcal{H}$ of a d-simplex S regarding its coincidence with the other three centers. It is well-known that for $d = 2$ the coincidence of any two of the four mentioned centers yields a regular (or equilateral) triangle; see [38], page 78, and for triangle centers in general we refer to [5] and [33]. For $d \geq 3$ this is no longer true, i.e., only weaker degrees of regularity are obtained, see [14] and [15] for recent results on this. One of these weaker degrees is the *equifacetality* of a d-simplex S, i.e., the congruence of its $(d-1)$-faces, which does not imply regularity, see [43] and, for a deeper study of equifacetal simplices, [14]. From [15], Theorem 3.2., it follows that equifacetality implies $\mathcal{G} = \mathcal{C} = \mathcal{I}$, and there it is also shown that the opposite implication, although true for $d \in \{2, 3\}$, does not hold for $d = 4$ (and expectedly also not for $d \geq 5$). Also the coincidence of two of these three centers does not imply that all three coincide. Specifically, the existence of a non-equifacetal 4-simplex with $\mathcal{G} = \mathcal{C} = \mathcal{I}$ follows from [28], [15], Theorem 3.2 (iv) and Theorem 4.5, while the existence of 4-simplices with $\mathcal{G} = \mathcal{C} \neq \mathcal{I}$, $\mathcal{C} = \mathcal{I} \neq \mathcal{G}$, and $\mathcal{G} = \mathcal{I} \neq \mathcal{C}$ follows from [28], [15], Theorems 3.4 and 4.3, respectively. This emphasizes once again the feeling that arbitrary d-simplices, $d \geq 3$, are not the most faithful generalizations of triangles. It turns out that orthocentric d-simplices do resemble triangles closely in significant ways. Namely, we will prove that if S is orthocentric, then

$$\begin{aligned} S \text{ is regular} &\iff S \text{ is equifacetal}, \\ &\iff \mathcal{G} = \mathcal{C} = \mathcal{I} = \mathcal{H}, \\ &\iff \text{any two of these four centers coincide.} \end{aligned}$$

Further similar results, related to other degrees of regularity of S, will be added, and also for a class of special orthocentric simplices, called *rectangular simplices* (for which $\mathcal{H}$ is a vertex), various results will be presented.

The class of orthocentric simplices has a long history. The literature on these special polytopes is large, and there is no satisfactory survey showing the current state of knowledge about them. We therefore give below a short survey of the literature on orthocentric simplices. The reader will observe that there is still much room for finding new properties of this interesting class of simplices.

1 Preliminaries

1.1 Definitions

By $\mathbb{E}^d$ we denote the *d-dimensional Euclidean space* with *origin* O. We write capital letters, e.g. A_i, for *points* or their *position vectors*, and small letters, like a_i, for their *barycentric coordinates* in a suitably defined system. Also we write $\|A_i - A_j\|$ for the usual *distance* between the points A_i and A_j.

A (non-degenerate) *d-simplex* $S = [A_1, \ldots, A_{d+1}]$, $d \geq 2$, is defined as the convex hull of $d+1$ affinely independent points (or position vectors) $A_1, \ldots, A_{d+1}$ in the Euclidean space $\mathbb{E}^d$. The points A_i are the *vertices*, the line segments $E_{ij} = A_iA_j$ (joining two different vertices A_i, A_j) the *edges*, and all k-simplices whose vertices are $k+1$ vertices of S the *k-faces* of S. The *facets* of S are its $(d-1)$-faces, and the *i-th facet* F_i is the *facet opposite to the vertex* A_i.

The *centroid* $\mathcal{G}$ of S is the average of its vertices, the *circumcenter* $\mathcal{C}$ is the center of the unique sphere containing all vertices, and the *incenter* $\mathcal{I}$ is the center of the unique sphere that touches all facets of S. The corresponding radii are called the *circumradius* R and the *inradius* r, respectively. For $d = 2$, the altitudes of S have a common point, the *orthocenter* $\mathcal{H}$ of S. For $d \geq 3$ the altitudes of S are not necessarily concurrent, but if they are, S is called *orthocentric*, and the point $\mathcal{H}$ of concurrence is called the *orthocenter* of S. A special class of orthocentric d-simplices, also investigated here, is that of *rectangular d-simplices*; in this case the orthocenter coincides with a vertex of S. The last center to be defined here exists again for any d-simplex. Namely, for each edge E_{ij} of S there is a unique hyperplane H_{ij} perpendicular to E_{ij} and containing the centroid $\mathcal{G}_{ij}$ of the remaining $d-1$ vertices. These $\binom{d+1}{2}$ hyperplanes have a common point, the *Monge point* $\mathcal{M}$ of S. This point is the reflection of $\mathcal{C}$ in $\mathcal{G}$ and coincides, if S is orthocentric, with the orthocenter. For the history of the Monge point we refer the reader to [55], § 21, and [11].

A d-simplex S is said to be *regular* or *equilateral* if all its edges have the same length. Note that this is the highest degree of symmetry that S can have, in the sense that the group of isometries of a regular d-simplex is the full symmetric group of permutations on the set of its vertices, see [4], Proposition 9.7.1. Several weaker degrees of regularity of simplices are discussed in the present paper, too. In particular, a d-simplex S is said to be *equifacetal* if all its facets are congruent (or isometric), and it is called *equiareal* if all its facets have the same $(d-1)$-volume, i.e., $(d-1)$-dimensional Lebesgue measure. Furthermore, a d-simplex satisfying $\mathcal{G} = \mathcal{C} = \mathcal{I}$ may be referred to as $(\mathcal{G}, \mathcal{C}, \mathcal{I})$*–equicentral*. (Note that for $d = 2$ the latter three degrees of regularity are equivalent to equilaterality.)

1.2 Some results on centers of general simplices

Interesting results on equifacetal, equiareal and related simplices are contained in the papers [12], [13], [22], [44], [54], [43], [14], and [15]. For $d \geq 2$ we have

$$S \text{ is regular} \implies S \text{ is equifacetal} \implies S \text{ is } (\mathcal{G}, \mathcal{C}, \mathcal{I})-\text{equicentral}, \tag{1}$$

$$\mathcal{I} = \mathcal{G} \iff S \text{ is equiareal}, \tag{2}$$

see [15], Theorem 3.2. The other coincidences $\mathcal{G} = \mathcal{C}$ and $\mathcal{C} = \mathcal{I}$ turn out to have other geometric interpretations that are worth recording. Thus, calling a d-simplex *equiradial* if all its facets have equal circumradii, and *of well-distributed edge-lengths* if the sum of the squared edge-lengths is the same for all facets, it follows from [15], Theorem 3.2, that for $d \geq 2$

$$\mathcal{G} = \mathcal{C} \iff S \text{ has well-distributed edge-lengths}, \tag{3}$$

$$\mathcal{C} = \mathcal{I} \iff \mathcal{C} \text{ is interior and } S \text{ is equiradial}. \tag{4}$$

1.3 The Gram matrix of a simplex

We continue with the representation of a tool that relates the geometry of a simplex S to the algebraic properties of a certain matrix associated to S (see [31] and [37]). Namely, for a d-simplex $S = [A_1, \ldots, A_{d+1}]$ in $\mathbb{E}^d$ one defines the *Gram matrix* G to be the symmetric, positive semidefinite $(d+1) \times (d+1)$ matrix of rank d whose (i, j)-th entry is the inner product $A_i \cdot A_j$ (we mean the ordinary inner product, say), cf. [31], p. 407. Given G, one can calculate the distances $\|A_i - A_j\|$ for every i, j using the formula

$$\|A_i - A_j\|^2 = (A_i - A_j) \cdot (A_i - A_j) .$$

According to the last part of Proposition 9.7.1 in [4], G determines S up to an isometry of $\mathbb{E}^d$. Also one recovers S from G via the *Cholesky factorization* $G = HH^t$, where the rows of H are the vectors A_i coordinatized with respect to some orthonormal basis of $\mathbb{E}^d$. In fact, if G is a symmetric, semidefinite, real matrix of rank r, say, then there exists a unique symmetric, positive semidefinite, real matrix of rank r with $H^2 = G$, cf. [31], Theorem 7.2.6, p. 405, and the symmetry of H implies $G = HH^t$.

2 Basic properties of orthocentric simplices and a survey of known results

We start this section with a short survey of known results about orthocentric d-simplices. Since the case $d = 3$ is not in our focus here, we mention only some basic references referring to orthocentric (and closely related) tetrahedra, namely [55], § 21, [52], § 30, [1], [2], [9], [53], [3], Chapters IV and IX, and the recent paper [30]. (Even J. L. Lagrange [36] obtained results about orthocentric tetrahedra over 200 years ago.) So the following short survey refers to results on orthocentric d-simplices for all dimensions $d \geq 3$.

A first basic property of orthocentric simplices is the fact that they are closed under passing down to faces. (This sometimes allows one to use induction on the dimension for establishing certain properties in high dimensions.) More precisely, *each k-face of an orthocentric d-simplex is itself orthocentric*, $2 \leq k < d$. Even more, in this passing-down procedure the feet of all altitudes of any $(k+1)$-face F are the orthocenters of the k-faces of F. These observations were often rediscovered, see [39], [16], [35], [50], [34], and [48], § 1.3, Problems 1.28, 1.29 and their solutions.

Another fundamental property of orthocentric simplices is the perpendicularity of non-intersecting edges. It can also be formulated as follows: *Each edge of an orthocentric d-simplex, $d \geq 3$, is perpendicular to the opposite $(d-2)$-face, and any d-simplex with that property is orthocentric*, cf. [39], [35], [50], [34], [8], and [48], § 1.3, for various approaches. Also in [48], § 1.3, one can find the following property of an orthocentric simplex $S = [A_1, \ldots, A_{d+1}]$ regarding its circumcenter: $\overrightarrow{\mathcal{C}A_1} + \ldots + \overrightarrow{\mathcal{C}A_{d+1}} = (d-1)\overrightarrow{\mathcal{C}\mathcal{H}}$ (cf. Problem 1.29 there). One of the oldest and most elegant discoveries in the geometry of triangle centers is Euler's proof that the centroid $\mathcal{G}$ of a triangle ABC lies on the line segment $\mathcal{C}\mathcal{H}$ and divides it in the ratio $1 : 2$, see [10], p. 17. For an orthocentric d-simplex S we have the analogous situation: *The points $\mathcal{C}, \mathcal{G}$ and $\mathcal{H}$ are on a line (the Euler line of S), and $\mathcal{G}$ divides the segment $\mathcal{C}\mathcal{H}$ in the ratio $(d-1) : 2$.* This result and its analogue for general d-simplices (where the Monge point $\mathcal{M}$ replaces $\mathcal{H}$) were also rediscovered several times, see [45], [39], [16], [29], [35], [46], [8], and [7] for different proofs and extensions. Also in some other situations, theorems on orthocentric simplices have analogues for general simplices if the missing point $\mathcal{H}$ is replaced by $\mathcal{M}$, such as in the case of Feuerbach spheres discussed in the sequel. Once more we mention that if a simplex is orthocentric, then $\mathcal{M}$ coincides with $\mathcal{H}$, see [45] and [46].

It is well known that the $\binom{d+1}{k+1}$ centroids of all k-faces, $k \in \{0, \ldots, d-1\}$, of an orthocentric d-simplex S lie on a sphere, the *Feuerbach k-sphere* of S, see [51] and [35]. Also for general simplices,

H. Mehmke [45] investigated the case $k = d - 1$: the center $\mathcal{F}$ of the respective Feuerbach sphere lies on the Euler line through $\mathcal{G}$ and $\mathcal{C}$, and $\mathcal{G}$ divides the segment $\mathcal{FC}$ in the ratio $1 : d$. The radius of that sphere is $\frac{1}{d}$ times the circumradius of S. If S is orthocentric, then the Feuerbach sphere also contains the feet of the altitudes, and it divides the "upper" parts of the altitudes in the ratio $1 : (d - 1)$, see also [49], [39], [42], [46], [23], [19], and [7] for analogous results. In [16], [17], [29], [35], and [26] the whole sequence of all Feuerbach k-spheres of orthocentric d-simplices is studied. All their corresponding centers $\mathcal{F}_0, \dots, \mathcal{F}_{d-1}$ lie on the Euler line, with $|\mathcal{HF}_k| : |\mathcal{HG}| = (d+1) : 2(k+1)$, and their radii r_k satisfy simple relations depending only on d and k. These papers contain more related results (see also [32]), e.g. observations referring to Feuerbach spheres of so-called orthocentric point systems. Considering the set of $d+2$ points $\mathcal{H}, A_1, \dots, A_{d+1}$ of an orthocentric d-simplex as a whole, one observes that each of them is the orthocenter of the simplex formed by the $d+1$ others. Thus it is natural to speak about *orthocentric systems of $d+2$ points*. Such point sets (and their analogues of larger cardinality) were studied in [49], [16], [19], and [8]. For example, E. Egerváry [16] proved that *a point set $\{P_0, P_1, \dots, P_{d+1}\} \subset \mathbb{E}^d$, $d \geq 2$, is an orthocentric system if and only if the mutual distances $\|P_i - P_j\|\,(i, j = 0, 1, \dots, d+1; i \neq j)$ can be expressed by $d+2$ symmetric parameters λ_i in the form*

$$\|P_i - P_j\|^2 = \lambda_i + \lambda_j\,, \quad \text{with} \ \sum_{i=0}^{d+1} \frac{1}{\lambda_i} = 0\,,\ \lambda_i + \lambda_j > 0\,,\ i \neq j\,, \tag{5}$$

see also [48], § 1.3, Problem 1.28. Based on this, Egerváry showed that $d+1$ points in $\mathbb{E}^d$, whose cartesian coordinates are the elements of an *orthogonal matrix*, form together with the origin an orthocentric point system. If, conversely, the "interior point" of an orthocentric system of $d+2$ points is identified with the origin, then an orthogonal matrix can be found from which the coordinates of the remaining $d+1$ points can be easily described. In [8] it is shown that the $d+2$ Euler lines of an orthocentric system $\{P_0, P_1, \dots, P_{d+1}\} \subset \mathbb{E}^d$, $d \geq 2$, have a common point, called its *orthic point*, and that the $d+2$ centroids as well as the $d+2$ circumcenters of that set form again orthocentric systems, both homothetic to $\{P_0, P_1, \dots, P_{d+1}\}$ with the orthic point as homothety center.

M. Fiedler [19] defines *equilateral d-hyperbolas* as those rational curves of degree d which have all their d asymptotic directions mutually orthogonal. Two such d-hyperbolas are called *independent* if both d-tuples of asymptotic directions satisfy the following: In no k-dimensional linear subspace $(k = 1, \dots, d-2)$, which is determined by k directions from one of these d-tuples, more than k asymptotic directions of the other are contained. He proves that *if there are two independent equilateral d-hyperbolas both containing a system of $d+2$ distinct points in $\mathbb{E}^d$, then this system is orthocentric, and every d-hyperbola containing this system is equilateral.*

Another type of results refers to *characterizations of orthocentric simplices as extreme simplices* regarding certain metrical problems, going back to J. L. Lagrange [36] and W. Borchardt [6], and connected with the symmetric parameters in (5), see [16] and [25]. In the latter paper the following (and further) results are shown: *The maximum [minimum] volume of a d-simplex $S = [A_1, \dots, A_{d+1}]$ containing a point Q and with prescribed distances $\|Q - A_i\| \geq 0$, $i = 1, \dots, d+1$, is attained by an orthocentric d-simplex. The maximum volume of a d-simplex S with given $(d-1)$-volumes of its facets is attained if S is orthocentric.* For getting these results, L. Gerber [25] establishes some purely geometric properties that orthocenters of orthocentric simplices must have, e.g.: The point $\mathcal{H}$ of an orthocentric simplex lies closer to a facet than to the opposite vertex on all except possibly the shortest altitude. Further geometric properties of orthocentric simplices were derived in [19], [20], and [21], § 7. Going back to the parameters λ_i in (5), Fiedler [19] calls an orthocentric simplex *negatively orthocentric* if one of the λ_i's is negative, *positively orthocentric* if all of them are positive, and *singularly orthocentric* if one is zero (the first two cases together are called *non-singularly orthocentric*). He shows that *a d-simplex, $d \geq 2$, with dihedral interior angles ϕ_{ij} is non-singularly orthocentric if and only if there exist real non-zero numbers c_i, $i = 1, \ldots, d+1$, such that $\cos \phi_{ij} = c_i c_j$ for all i, j with $i \neq j$.*

A *reciprocal transformation* with respect to the simplex S is such that for the homogeneous barycentric coordinates of the image $X' = (x'_i)$ of a point $X = (x_i)$ the relation $x'_i = c_i \times x_i$ holds, where the c_i's are fixed non-zero numbers. The *harmonic polar* of $Y = (y_i)$ not contained in any face of S is the hyperplane with equation $\sum \frac{x_i}{y_i} = 0$ in barycentric coordinates. In [19] it is proved that *a d-simplex S is orthocentric if and only if there exists an interior point P of S such that for every selfadjoint point Q (if different from P) of the reciprocal transformation, for which P and the centroid of S correspond, the line through P and Q is perpendicular to the harmonic polar of Q with respect to S. The point P is then the orthocenter of S.* The paper [19] contains also a number of theorems on natural generalizations of positively orthocentric simplices. In the booklet [21] the *polarity* of a point quadric (in a projective space) with equation $\sum a_{ik} x_i x_k = 0$ and a dual quadric $\sum b_{ik} \xi_i \xi_k$ are defined, as usual, by the condition $\sum a_{ik} b_{ik} = 0$, cf. Def. 7.8 there. A point quadric in $\mathbb{E}^d$ is then called *equilateral* (cf. Def. 7.9 in [21]) if it is a polar to the absolute dual quadric. In the case of homogeneous barycentric coordinates the dual absolute quadric has the equation $\sum q_{ik} \xi_i \xi_k = 0$, where the matrix (q_{ik}) is the Gram matrix of the outward normals of the d-simplex S normalized so that the sum of the normals is the zero vector. Fiedler [21] proves that for a *non-singularly orthocentric d-simplex, $d \geq 2$, every equilateral quadric containing all its vertices contains the orthocenter as well, and that, conversely, every quadric containing all vertices and the orthocenter is necessarily equilateral.* And going back to equilateral d-hyperbolas (see above), he proves in [19] the following: *Suppose that in a non-singularly orthocentric d-simplex S, $d \geq 2$, the orthocenter is not contained in any hyperplane orthogonally bisecting an edge. Then there exists exactly one equilateral d-hyperbola containing all vertices, the orthocenter and the centroid of S.*

More general classes of simplices which are still closely related to orthocentric ones were studied by S. R. Mandan (see [40] and [41]) and M. Fiedler (cf. [19] and [20]). The simplices under consideration in the papers of Mandan have two (or more) subsets of their set of altitudes, each subset having a common point. And Fiedler [19] obtains theorems on a family of simplices having the class of positively orthocentric simplices as subfamily. In [20] he investigates the related class of *cyclic simplices*. Also the paper [26] should be mentioned here.

From the literature we also know theorems on special types of orthocentric simplices, in particular on the subfamily of *regular simplices*. So we know that an orthocentric d-simplex $S = [A_1, \ldots, A_{d+1}]$ is regular if and only if $\mathcal{C} = \mathcal{G}$ [46], if and only if $\mathcal{G} = \mathcal{H}$ [23], if and only if $\mathcal{H}$ coincides with the unique point minimizing $\sum_{i=1}^{d+1} \|X - A_i\|$, $X \in \mathbb{E}^d$ (the Fermat-Torricelli point of S, see [47]), and if S is equiareal [25]. Also the paper [7] should be mentioned here. Furthermore, there exist some results on rectangular (or right) simplices as special orthocentric ones, see again [25].

3 The barycentric coordinates of the orthocenter and the Gram matrix of an orthocentric simplex

3.1 Barycentric coordinates and obtuseness

In this section, we show that a non-rectangular orthocentric simplex can essentially be parametrized by the barycentric coordinates of its orthocenter, and we give several useful characterizations of such simplices. We start with a simple but basic theorem.

Theorem 3.1: *Let $S = [A_1, \dots, A_{d+1}]$ be a d-simplex.*

(a) *S is orthocentric if and only if for every k the quantity $(A_i - A_k) \cdot (A_j - A_k)$ does not depend on i and j as long as i, j and k are pairwise distinct.*

(b) *If P is a point in the affine hull of S, then S is orthocentric with orthocenter P if and only if the quantity $(A_i - P) \cdot (A_j - P)$ does not depend on i and j as long as $i \neq j$.*

If c_k denotes the quantity in (a) *and c the quantity in* (b), *then we have*

$$c = 0 \iff S \text{ is rectangular at } A_k \text{ for some } k. \tag{6}$$

$$c_k = 0 \iff S \text{ is rectangular at } A_k. \tag{7}$$

Proof: For (a), see [48], Problem 1.28, pages 30, 217. To prove (b), we may assume, without loss of generality, that P is the origin O. If S is orthocentric with orthocenter $\mathcal{H}$, and if i, j, and k are pairwise distinct indices, then A_i is normal to the i-th facet, and therefore $A_i \cdot (A_j - A_k) = 0$, and $A_i \cdot A_j = A_i \cdot A_k$, as desired. Conversely, if $A_i \cdot A_j$ does not depend on i and j as long as $i \neq j$, then $A_i \cdot (A_j - A_k) = 0$, for all pairwise distinct i, j, and k, and therefore A_i is normal to every edge of the i-th facet, and hence to the i-th facet. Therefore O lies on every altitude and has to be the orthocenter. Finally, if $c = 0$, and if $\mathcal{H}$ is the orthocenter of S, then the $d + 1$ vectors $\mathcal{H} - A_1, \dots, \mathcal{H} - A_{d+1}$ in $\mathbb{E}^d$ are normal to each other, and therefore one of them must be the zero vector, i.e., $\mathcal{H} = A_i$ for some i. The other implications follow from the definitions. □

Definition 3.2: For an orthocentric d-simplex $S = [A_1, \dots, A_{d+1}]$ with orthocenter $\mathcal{H}$, we define $\sigma(S)$ of S by

$$\sigma(S) := (\mathcal{H} - A_i) \cdot (\mathcal{H} - A_j), \; i \neq j. \tag{8}$$

By Theorem 3.1(b), $\sigma(S)$ is well-defined. Note that in view of (6), $\sigma(S)$ is zero if and only if S is rectangular. We shall see later that the sign of $\sigma(S)$ is negative if and only if all the angles between any two edges of S are acute. Because of this and for lack of a better term, we propose to call $\sigma(S)$ *the obtuseness of S*. This may conceal the fact that if S is enlarged (by a factor of λ, say), then also $\sigma(S)$ increases (by a factor of λ^2), although the shape of S remains unchanged.

The next technical theorem will be freely used. Among other things, it expresses the edge-lengths of an orthocentric non-rectangular d-simplex S in terms of the obtuseness $\sigma(S)$ of S and the barycentric coordinates of its orthocenter $\mathcal{H}$, showing that these quantities are sufficient for parametrizing such simplices. Clearly, this does not apply to rectangular simplices, since these numbers do not carry any information at all about the simplex other than its being rectangular. This is the main reason why rectangular simplices are temporarily excluded and their study is postponed to a later section.

Theorem 3.3: *Let $S = [A_1, \ldots, A_{d+1}]$ be a non-rectangular orthocentric d-simplex, and let $a_1, \ldots, a_{d+1}$ be the barycentric coordinates of its orthocenter $\mathcal{H}$ with respect to S. Let $c = \sigma(S)$ be the obtuseness of S defined in (8). Then no a_i is equal to 0 or 1 and, for any real numbers $b_1, \ldots, b_{d+1}$,*

$$\left\|\sum b_i(A_i - \mathcal{H})\right\|^2 = c\left[\left(\sum b_i\right)^2 - \sum \frac{b_i^2}{a_i}\right]. \tag{9}$$

In particular,

$$\|A_i - \mathcal{H}\|^2 = \frac{c(a_i - 1)}{a_i} \tag{10}$$

$$\|A_i - A_j\|^2 = -c\left(\frac{1}{a_i} + \frac{1}{a_j}\right). \tag{11}$$

Also, if B_i is the foot of the perpendicular from the vertex A_i to the i-th facet, and if $h_i = \|A_i - B_i\|$ is the corresponding altitude, then

$$B_i - \mathcal{H} = \frac{a_i}{a_i - 1}(A_i - \mathcal{H}) \tag{12}$$

$$h_i^2 = \frac{c}{a_i(a_i - 1)}. \tag{13}$$

Proof: Without loss of generality, we may assume that the orthocenter $\mathcal{H}$ of S lies at the origin O. Taking the scalar product of A_i with $\sum a_i A_i = O$, we see that $a_i\|A_i\|^2 + (1 - a_i)c = 0$. If $a_i = 0$, then $c = 0$. If $a_i = 1$, then $\|A_i\| = 0$. In both cases S would be rectangular. Therefore no a_i is 0 or 1, and

$$\|A_i\|^2 = \frac{c(a_i - 1)}{a_i},$$

as claimed in (10). It follows that

$$\begin{aligned}
\|\textstyle\sum b_i A_i\|^2 &= \textstyle\sum b_i^2\|A_i\|^2 + 2\sum_{i<j} b_i b_j (A_i \cdot A_j) \\
&= c\left[\sum b_i^2\left(\frac{a_i - 1}{a_i}\right) + 2\sum_{i<j} b_i b_j\right] \\
&= c\left[\sum b_i^2 - \sum \frac{b_i^2}{a_i} + 2\sum_{i<j} b_i b_j\right] \\
&= c\left[(\textstyle\sum b_i)^2 - \sum \frac{b_i^2}{a_i}\right],
\end{aligned}$$

as claimed in (9).

Next, let $\mathcal{H}' = B_{d+1}$ be the projection of A_{d+1} on the $(d+1)$-th facet F_{d+1} of S. Since $\mathcal{H}'$ lies on the (well-defined) line joining the vertex A_{d+1} and the origin O, and also in the affine hull of F_{d+1}, it follows that there exist $t, a_1', \ldots, a_d'$ such that

$$\mathcal{H}' = tA_{d+1} \text{ and } \mathcal{H}' = a_1'A_1 + \ldots + a_d'A_d,$$

with $a'_1 + \ldots + a'_d = 1$. Set $a'_{d+1} = -t$. Then we have

$$a'_1 A_1 + \ldots + a'_{d+1} A_{d+1} = O \quad \text{and} \quad a_1 A_1 + \ldots + a_{d+1} A_{d+1} = O\,.$$

From the uniqueness of the dependence relation among $A_1, \ldots, A_{d+1}$ it follows that the $(d+1)$-tuples $(a'_1, \ldots, a'_{d+1})$ and $(a_1, \ldots, a_{d+1})$ are proportional. Since $a'_1 + \cdots + a'_d = 1 \neq 0$, it follows that $a_1 + \cdots + a_d \neq 0$, and that

$$\frac{a'_i}{a'_1 + \cdots + a'_d} = \frac{a_i}{a_1 + \cdots + a_d}\,.$$

Therefore

$$\frac{a'_i}{1} = \frac{a_i}{1 - a_{d+1}}\,,\ \mathcal{H}' = -a'_{d+1} A_{d+1} = \frac{a_{d+1}}{a_{d+1} - 1} A_{d+1}\,. \tag{14}$$

This proves (12) for $i = d+1$, and hence for all i. Finally,

$$h_i^2 = \|A_i - B_i\|^2 = \|A_i - \frac{a_i}{a_i - 1} A_i\|^2 = \frac{c}{(a_i-1)^2} \frac{a_i - 1}{a_i} = \frac{c}{a_i(a_i - 1)}\,,$$

as claimed in (13). □

Theorem 3.4: *Let $S = [A_1, \ldots, A_{d+1}]$ be an orthocentric d-simplex, and let $a_1, \ldots, a_{d+1}$ be the barycentric coordinates of its orthocenter $\mathcal{H}$ with respect to S. Let $\sigma(S)$ be the obtuseness of S defined in* (8). *Then S is non-rectangular if and only if any of the following conditions hold.*

(a) $\sigma(S) \neq 0$.

(b) *None of the faces of S is rectangular.*

(c) *$\mathcal{H}$ is not in the affine hull of any proper face of S.*

(d) *There does not exist any nonempty subset I of $\{1, \ldots, d+1\}$ such that $\Sigma(a_i : i \in I) = 0$.*

(e) *There does not exist any proper subset I of $\{1, \ldots, d+1\}$ such that $\Sigma(a_i : i \in I) = 1$.*

Proof: Without loss of generality, we may assume that the orthocenter $\mathcal{H}$ of S lies at the origin O. Note that property (a) has already been mentioned in (6). To prove (b), suppose that S has a rectangular face F. For simplicity, we may assume that $F = [A_1, \ldots, A_{k+1}], 2 \leq k \leq d$, and that A_1 is the orthocenter of F. Then $(A_1 - A_2) \cdot (A_1 - A_3) = 0$. Since S is orthocentric, it follows from Theorem 3.1 (a) that $(A_1 - A_i) \cdot (A_1 - A_j) = 0$ for all $i \neq 1$ and $j \neq 1$. Therefore the edges of S emanating from A_1 are normal to each other, and hence S is rectangular. To prove (c), suppose that the orthocenter of S is in the affine hull of a proper face F of S. Without loss of generality, we assume that $\mathcal{H} = O$. If A_i is a vertex of F, and A_k is not a vertex of F, then the segment $A_k\mathcal{H}$, being normal to F, is normal to $A_i\mathcal{H}$. Thus $\sigma(S) = 0$, and S would be rectangular. This proves (c). To prove (d), suppose that $a_1 + \ldots + a_k = 0$ for some $k \geq 1$. By (9), we have $\|a_1 A_1 + \ldots + a_k A_k\| = 0$, and therefore $a_1 A_1 + \ldots + a_k A_k = O$ and $\mathcal{H} = a_{k+1} A_{k+1} + \ldots + a_{d+1} A_{d+1}$. Thus $\mathcal{H}$ is in the affine hull of the proper face $[A_{k+1}, \ldots, A_{d+1}]$, contradicting (c). This proves (d) and its equivalent (e). Thus non-rectangular orthocentric simplices satisfy all the conditions above. The converse is trivial. □

3.2 The Gram matrix of an orthocentric simplex and characterizing the barycentric coordinates of its orthocenter

We now characterize those tuples that can occur as the barycentric coordinates of a non-rectanglular orthocentric simplex, and we see how the signs of these coordinates bear on the "acuteness" of its vertex angles. The first theorem describes the Gram matrix (see our Subsection 1.3) of a non-rectangular orthocentric simplex, and the lemma that follows records the value of a determinant that we shall need in several places. This lemma appears as Problem 192 on page 35 (with a hint on page 154, and an answer on page 187) of [18].

Theorem 3.5: *If $S = [A_1, \ldots, A_{d+1}]$ is a non-rectangular orthocentric d-simplex, then its Gram matrix is of the form cG, where every entry of G that lies off the diagonal is 1. Conversely, if G is a $(d+1) \times (d+1)$-matrix of rank d such that its non-zero eigenvalues are real and have the same sign, and such that all off-diagonal entries are equal, say, the entries G_{ij} of G are given by*

$$G_{ij} = \begin{cases} c(1+x_i) & \text{if} \quad i = j\,, \\ c & \text{if} \quad i \neq j\,, \end{cases}$$

then $\pm G$ is the Gram matrix of a non-rectangular orthocentric d-simplex S whose orthocenter lies at the origin and for which

$$\|A_i\|^2 = c(1+x_i)\,,\ \|A_i - A_j\|^2 = c(x_i + x_j) \quad \text{if} \quad i \neq j\,. \tag{15}$$

Proof: This is a restatement of the fact that $[A_1, \ldots, A_{d+1}]$ is orthocentric with orthocenter O if and only if $A_i \cdot A_j$ is a constant c independent of i and j for $i \neq j$, and that $c = 0$ if and only if S is rectangular. □

Lemma 3.6: *Let $\mathbb{J} = \mathbb{J}(a_1, \ldots, a_n; b_1, \ldots, b_n)$ be the $n \times n$ matrix whose (ij)-th entry $\mathbb{J}_{ij}$ is defined by*

$$\mathbb{J}_{ij} = \begin{cases} a_i + b_i & \text{if} \quad i = j\,, \\ a_i & \text{if} \quad i \neq j\,. \end{cases}$$

Then

$$\det(\mathbb{J}(a_1, \ldots, a_n; b_1, \ldots, b_n)) = (b_1 b_2 \cdots b_n)\left(1 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}\right),$$

where the right hand side is appropriately interpreted if $b_i = 0$ for some i. In particular

$$\det(\mathbb{J}(a, \ldots, a; b, \ldots, b)) = b^{n-1}(b + na)\,.$$

Proof. We proceed by induction, assuming that the statement is true for $n = k$. Since det is continuous, we may confine ourselves to the domain where no b_i is 0. By dividing the i–th row by b_i for every i, we may also assume that $b_i = 1$ for all i. Then

$$\det\left(\mathbb{J}(a_1, \cdots, a_{k+1},; 1, \cdots, 1)\right)$$

$$= \begin{vmatrix} 1+a_1 & a_1 & a_1 & \cdots & a_1 \\ a_2 & 1+a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k+1} & a_{k+1} & a_{k+1} & \cdots & 1+a_{k+1} \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_1 & a_1 & \cdots & a_1 \\ a_2 & 1+a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k+1} & a_{k+1} & a_{k+1} & \cdots & 1+a_{k+1} \end{vmatrix} + \begin{vmatrix} 1 & a_1 & a_1 & \cdots & a_1 \\ 0 & 1+a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{k+1} & a_{k+1} & \cdots & 1+a_{k+1} \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k+1} & 0 & 0 & \cdots & 1 \end{vmatrix} + \det(\mathbb{J}(a_2, \cdots, a_{k+1}; 1, \cdots, 1))$$

$$= a_1 + (1 + a_2 + \cdots + a_{k+1}) = 1 + a_1 + a_2 + \cdots + a_{k+1}, \text{ as desired.} \qquad \square$$

The next theorem describes the somewhat surprising restrictions that the barycentric coordinates $a_1, \ldots, a_{d+1}$ of the orthocenter of a non-rectangular orthocentric simplex S must obey. We find it more convenient to include in it the relations among the signs of $a_1, \ldots, a_{d+1}$, the sign of the obtuseness $\sigma(S)$, and the acuteness of the angles between the edges. Note in particular that the sign of the obtuseness $\sigma(S)$ of S can be read off the barycentric coordinates of the orthocenter of S.

Definition 3.7: A polyhedral angle with vertex at O and with "arms" $OV_1, \ldots, OV_n$ is called *strongly acute* (respectively, *strongly obtuse, right*) if and only if $\angle V_iOV_j$ is acute (respectively, obtuse, right) for all $i \neq j$.

Theorem 3.8: *The real numbers $a_1, \ldots, a_{d+1}$ with $a_1 + \cdots + a_{d+1} = 1$ occur as the barycentric coordinates of the orthocenter of a non-rectangular orthocentric d-simplex if and only if all of them are positive, or exactly one of them is positive and the others are negative. In the first case $\sigma(S) < 0$, and all the vertex angles of S are strongly acute. In the second case $\sigma(S) > 0$, and one vertex angle of S is strongly obtuse while the others are strongly acute.*

Proof: Let $S = [A_1, \ldots, A_{d+1}]$ be a non-rectangular orthocentric d-simplex, and $a_1, \ldots, a_{d+1}$ be the barycentric coordinates of the orthocenter $\mathcal{H}$ of S. Assume that $\mathcal{H}$ is at the origin O. We use the facts that if $i \neq j$, then

$$A_i \cdot A_j = c\,, \ \|A_i\|^2 = \frac{c(a_i - 1)}{a_i}\,, \ \|A_i - A_j\|^2 = -c\left(\frac{1}{a_i} + \frac{1}{a_j}\right).$$

If $c < 0$, then $a_i(a_i - 1) < 0$ for all i, and therefore $0 < a_i < 1$ for all i. In this case, we have for pairwise distinct i, j, and k

$$(A_i - A_j) \cdot (A_i - A_k) = \frac{c(a_i - 1)}{a_i} - c - c + c = \frac{-c}{a_i} > 0\,.$$

If $c > 0$, and if a_1 and a_2 are positive, then $\|A_1 - A_2\|^2$ would be negative. Therefore at most one of the a_i's is positive. Since the sum is 1, it follows that exactly one a_i is positive. Here again, if i, j, and k are pairwise distinct, and if $t_i = (A_i - A_j) \cdot (A_i - A_k)$, then $t_i < 0$ if $a_i > 0$, and $t_i > 0$ if $a_i < 0$.

Conversely, suppose that $a_1, \ldots, a_{d+1}$ are non-zero real numbers whose sum is 1. Let $x_i = -1/a_i$, and let G be the $(d+1) \times (d+1)$-matrix whose (ij)-th entry G_{ij} is given by

$$G_{ij} = \begin{cases} 1 + x_i & \text{if} \quad i = j\,, \\ 1 & \text{if} \quad i \neq j\,. \end{cases}$$

By Lemma 3.6, the characteristic polynomial of G is given by

$$\begin{aligned} F(\lambda) &= (\lambda - x_1)\cdots(\lambda - x_{d+1})\left(1 + \frac{-1}{\lambda - x_1} + \cdots + \frac{-1}{\lambda - x_{d+1}}\right) \\ &= f(\lambda) - f'(\lambda), \end{aligned}$$

where

$$f(\lambda) = (\lambda - x_1)\cdots(\lambda - x_{d+1}).$$

Let $g(\lambda) = -e^{-\lambda}f(\lambda)$. Then $g'(\lambda) = e^{-\lambda}(f(\lambda) - f'(\lambda)) = e^{-\lambda}F(\lambda)$. The assumption $a_1 + \cdots + a_{d+1} = 1$ is equivalent to saying that $g'(0) = 0$.

If all the a_i's are positive, then all the x_i's are negative, and therefore g has $d+1$ negative zeros. Hence g' has d negative zeros. Thus d of the zeros of g' are negative and the remaining one is 0. Therefore F has the same property, and hence $-G$ is the Gram matrix of some d-simplex S. It is easy to see that such a simplex S is orthocentric with orthocenter O and with barycentric coordinates of O as desired. If exactly one of the a_i's is positive, say

$$a_1 \leq \ldots \leq a_d < 0 < a_{d+1},$$

then

$$x_1 \geq \ldots \geq x_d > 0 > x_{d+1},$$

and g has d positive zeros $x_1, \ldots, x_d$. Therefore g' has $d-1$ positive zeros in $(0, x_1)$. Also, g' has a zero in (x_1, ∞) since $g(x_1) = g(\infty) = 0$. Thus d of the zeros of g' are positive, and the remaining one is 0. Therefore F has the same property, and hence G is the Gram matrix of some d-simplex S. It is easy to see that such an S is orthocentric with orthocenter O and with barycentric coordinates of O as desired. □

Corollary 3.9: *If a d-simplex $S = [A_1, \ldots, A_{d+1}]$ is orthocentric, then at least d of its vertex polyhedral angles are strongly acute, while the remaining one is either right, strongly acute, or strongly obtuse.*

This gives us a seemingly clear-cut idea of which orthocentric simplices ought to be called acute and which ought to be called obtuse. On the other hand, we do not get the criterion that the orthocenter is interior if and only if the circumcenter is.

Theorem 3.10: *Let $S = [A_1, \ldots, A_{d+1}]$ be an orthocentric d-simplex. Then the circumcenter $\mathcal{C}$ of S is interior if and only if S is non-rectangular and if the barycentric coordinates $a_1, \ldots, a_{d+1}$ of the orthocenter $\mathcal{H}$ of S are such that $0 < a_i < 1/(d-1)$ for all i. Consequently, if the circumcenter is interior, then so is the orthocenter, but not conversely.*

Proof: We will see in Section 5 that the circumcenter of a rectangular d-simplex S lies on the hypotenuse facet if $d = 2$, and lies outside S if $d > 2$. So let $S = [A_1, \ldots, A_{d+1}]$ be non-rectangular orthocentric, and let $a_1, \ldots, a_{d+1}$ be the barycentric coordinates of the orthocenter $\mathcal{H}$ of S. Assume that $\mathcal{H} = O$. Then it follows from the Euler line theorem (see, e.g., [16] and [29]) that the circumcenter $\mathcal{C}$ of S is given by

$$\mathcal{C} = \frac{d+1}{2}\,\mathcal{G} = \frac{1}{2}(A_1 + \cdots + A_{d+1}),$$

and therefore

$$\mathcal{C} = \sum \left(\frac{1}{2} + \frac{(1-d)a_i}{2}\right) A_i.$$

Thus the i-th barycentric coordinate of $\mathcal{C}$ is $(1 + (1 - d)a_i)/2$, and $\mathcal{C}$ is interior if and only if $a_i < 1/(d-1)$. It remains to show that the a_i's must be all positive. If not, then exactly one of them is positive (and less than $1/(d-1)$) and the others are negative, contradicting the fact that $a_1 + \ldots + a_{d+1} = 1$. □

4 Coincidence of centers of non-rectangular orthocentric simplices

4.1 Center coincidences except circumcenter = incenter

We start with a simple application of the fact that orthocentric simplices are closed under passing down to faces. Note that the part of Theorem 4.1 connected with the assumption $\mathcal{H} = \mathcal{G}$ is known, see [23]. Note also that a stronger result will be established later, in Theorem 4.3.

Theorem 4.1: *Let S be an orthocentric d-simplex. If the orthocenter of S coincides with the circumcenter or with the centroid of S, then S is regular.*

Proof: Being trivially true for a triangle, the statement immediately follows for all d by induction, using the facts that the circumcenter and the orthocenter of a facet of S are the orthogonal projections of the respective centers of S on that facet, and that the centroid and the orthocenter of a facet of S are the intersections of the respective cevians of S with the facet. □

Remark 4.2: The proof of Theorem 4.1 above does not work if the circumcenter or the centroid is replaced by the incenter, since the incenter of a facet F of a simplex S is neither the projection of the incenter of S on the facet, nor the intersection of the respective cevian through the incenter of S with that facet. However, we shall see in Theorem 4.3 that the statement itself remains valid also in that case. □

The following theorem will be strengthened later to include the case $\mathcal{C} = \mathcal{I}$ and to include rectangular simplices. Note that the part pertaining to $\mathcal{H} = \mathcal{G}$ is treated in [23], and that the part pertaining to $\mathcal{G} = \mathcal{C}$ is treated in [46]. However, we include them in order to give a unified approach.

Theorem 4.3: *Let S be a non-rectangular orthocentric d-simplex. If $\mathcal{H} = \mathcal{G}, \mathcal{H} = \mathcal{C}, \mathcal{H} = \mathcal{I}, \mathcal{G} = \mathcal{I}$, or $\mathcal{G} = \mathcal{C}$, then S is regular.*

Proof: Let $a_1, \ldots a_{d+1}$ be the barycentric coordinates of the orthocenter $\mathcal{H}$ of S with respect to S and assume, without loss of generality, that $\mathcal{H}$ lies at the origin O. We freely use the facts that $A_i \cdot A_j = c$ for all $i \neq j$ and $\|A_i\|^2 = c(a_i - 1)/a_i$ for all i. The cases $\mathcal{H} = \mathcal{G}$ and $\mathcal{H} = \mathcal{C}$ are dealt with in Theorem 4.1. Alternatively, if $\mathcal{H} = \mathcal{G}$, then the a_i's are all equal, and therefore the edge-lengths are equal, by (11). If $\mathcal{H} = \mathcal{C}$, then $\|A_i\|$ does not depend on i, and therefore $\|A_i - A_j\|$ does not depend on i and j, as long as $i \neq j$, and S is regular. If $\mathcal{H} = \mathcal{I}$, then a_i is proportional to the $(d-1)$-volume of the i-th facet which in turn is inversely proportional to the altitude h_i from the i-th vertex. Thus $a_i h_i$ is independent of i. From (13) we have

$$a_i^2 h_i^2 = \frac{a_i^2 c}{a_i(a_i - 1)} = \frac{c a_i}{a_i - 1}\,.$$

Since $x/(x-1)$ is $1-1$, it follows that the a_i's are equal, and S is again regular.

If $\mathcal{G} = \mathcal{I}$, then S is equiareal by ([15], Theorem 3.2 (iii)), and therefore the altitudes are equal. Hence $a_i(a_i - 1) = a_j(a_j - 1)$ for all i and j. Therefore $(a_i - a_j)(a_i + a_j - 1) = 0$. By Theorem 3.4 (e), $a_i + a_j$ cannot be equal to 1. Therefore the a_i's are all equal, and S is regular. (This case is also treated in [47].)

If $\mathcal{G} = \mathcal{C}$, then S has well-distributed edge-lengths (by [15], Theorem 3.2 (i)). Using (11), one can easily see that the sum of the squares of the edge-lengths of the i-th facet is given by

$$c(d-1)\left[\left(\frac{1}{a_1} + \cdots + \frac{1}{a_{d+1}}\right) - \frac{1}{a_i}\right].$$

Therefore, having well-distributed edge-lengths is equivalent to saying that the a_i's are equal, and that S is regular. Alternatively, if $\mathcal{C} = \mathcal{G}$, then the circumcenter is given by

$$\mathcal{C} = \sum_{i=1}^{d+1} \frac{1}{d+1} A_i,$$

and $\|\mathcal{C} - A_i\|$ does not depend on i. Then

$$\begin{aligned}
(d+1)\|\mathcal{C} - A_i\|^2 &= \|(d+1)\mathcal{C} - (d+1)A_i\|^2 \\
&= (d+1)^2\|\mathcal{C}\|^2 + (d+1)^2\|A_i\|^2 - 2(d+1)\sum_{j=1}^{d+1} A_i \cdot A_j \\
&= (d+1)^2\|\mathcal{C}\|^2 + (d+1)^2\|A_i\|^2 - 2(d+1)(\|A_i\|^2 + dc) \\
&= (d+1)^2\|\mathcal{C}\|^2 + (d^2-1)\|A_i\|^2 - 2(d+1)dc.
\end{aligned}$$

Thus $\|A_i\|$ is independent of i, and O is the circumcenter of S. Therefore $\mathcal{H} = \mathcal{C}$, and S is regular. This completes the proof. □

4.2 Equiradial orthocentric simplices and kites

In this subsection we begin a general study of equiradial orthocentric simplices. In view of the fact that

$$\mathcal{C} = \mathcal{I} \text{ if and only if } \mathcal{C} \text{ is interior and the simplex is equiradial },$$

it is natural to study the broader class of equiradial simplices and then to single out those for which $\mathcal{C}$ is interior.

It is convenient to give a name to those d-simplices in which d vertices form a regular $(d-1)$-simplex T, called the base, and are at equal distances from the remaining vertex, called the apex. We propose to call such a d-simplex a *d-dimensional kite*, or simply a *d-kite*, and to denote it by $K_d[s,t]$, where s is the side-length of each edge of T, and where t is the length of each remaining edge. The subscript d may be omitted if no confusion is caused. The quotient t/s carries all the information about the shape of the kite, and will be called the *eccentricity* of the kite. Note that a kite is automatically orthocentric, since the altitude from its apex meets the base in its orthocenter. Note also that all the facets of a kite are themselves kites.

Lemma 4.4: *For $d \leq 3$ an equiradial orthocentic d-simplex is regular. For $d > 3$ there exists a unique siimilarity class of equiradial d-simplices S such that d of its vertices form a regular $(d-1)$-simplex T (of edge-length s, say) and are at the same distance (t, say) from the remaining vertex. Its eccentricity is given by*

$$\epsilon = t/s = \sqrt{(d-2)/d}. \tag{16}$$

Proof: For $d = 2$, the result is trivial, so suppose $d \geq 3$. Suppose there is a non-regular equiradial d-kite K with base T. Then it has to arise in the following way. Inscribe the regular $(d-1)$-simplex $T = [A_1, \ldots, A_d]$ in a $(d-2)$-hypersphere centered at the origin in $\mathbb{E}^{d-1}$, and having radius ρ, the circumradius of T. Each facet of T is the base of exactly two $(d-1)$-kites of circumradius ρ: T itself, and one other, which we now describe. Let A_i^* be the point diametrically opposite to A_i for $1 \leq i \leq d$. Thus $A_i^* = -A_i$. Let $s = \|A_i - A_j\|$ and $t = \|A_i^* - A_j\|$, where $i \neq j$. By taking the inner product of each side of the equation $A_1 + \cdots + A_d = O$ with itself, we obtain $\rho^2 d + d(d-1)(A_i \cdot A_j) = 0$, and therefore $A_i \cdot A_j = -\rho^2/(d-1)$, and

$$s^2 = \|A_i - A_j\|^2 = 2\rho^2 \left(1 + \frac{1}{d-1}\right) = 2\rho^2 \left(\frac{d}{d-1}\right), \tag{17}$$

$$t^2 = \|A_i^* - A_j\|^2 = 2\rho^2 \left(1 - \frac{1}{d-1}\right) = 2\rho^2 \left(\frac{d-2}{d-1}\right). \tag{18}$$

It is clear that

$$t = \rho \iff d = 3 \text{ and that } t > \rho \iff d > 3.$$

But if $t = \rho$, then K degenerates, with the apex lying at the circumcenter of T. On the other hand, if $t > \rho$, then an actual equiradial d-kite of positive height can be formed from T and the kites over the facets of T and the above calculations of s^2 and t^2 yield the indicated formula for the eccentricity. □

In view of the above calculations, if K is a d-dimensional kite whose regular base has edge-length s and circumradius ρ, then $\rho^2/s^2 = (d-1)/(2d)$, and therefore the eccentricity of a d-kite can take any value larger than $\sqrt{(d-1)/(2d)}$.

Lemma 4.5: *A kite $K[s,t]$ in which the circumcenter coincides with the incenter must be regular (i.e., $s = t$).*

Proof: If h is the altitude of a d-kite $K = K[s,t]$ to its regular base T, and if ρ is the circumradius of T, then, by Lemma 4.4,

$$\begin{aligned}
& \text{the circumcenter of } K \text{ is interior} \\
\iff\ & h^2 > \rho^2 \iff t^2 - \rho^2 > \rho^2 \iff t^2 > 2\rho^2 \\
\iff\ & t^2 > \frac{d-1}{d} s^2 \text{ (by (17))} \\
\iff\ & \frac{t^2}{s^2} > \frac{d-1}{d}.
\end{aligned}$$

By (18), the eccentricity of the non-regular equiradial d-kite is given by $\sqrt{(d-2)/d}$, which is less than $\sqrt{(d-1)/d}$. Therefore such kites do not have interior circumcenters, and their circumcenters and incenters cannot coincide. Thus kites in which $\mathcal{C} = \mathcal{I}$ must be regular. □

We record some basic formulas for quantities associated with a kite. These formulas have fairly simple proofs essentially based upon the Pythagorean Theorem, and they are also based on some related calculations for the regular d-simplex which we give first.

Proposition 4.6: *Let $R = R_d = R_{d,s}$, $r = r_{d,s}$, $h = h_{d,s}$, and $V = V_{d,s}$ denote, respectively, the circumradius, the inradius, the altitude, and the volume of a regular d-simplex of edge-length s. Then*

$$R = s\sqrt{\frac{d}{2(d+1)}} \tag{19}$$

$$h = \sqrt{\frac{1}{2}}\sqrt{\frac{d+1}{d}}s \tag{20}$$

$$V = \frac{1}{d!}\sqrt{\frac{d+1}{2^d}}s^d \tag{21}$$

$$r = \sqrt{\frac{1}{2d(d+1)}}\ s. \tag{22}$$

Proof: To verify these formulas, let $S = [A_1, \cdots, A_{d+1}]$ be our regular simplex, and assume that the center of S is the origin O. Then $A_1 + \cdots + A_{d+1} = O$. Taking the scalar product with A_1, we see that $R^2 + d(A_1 \cdot A_2) = 0$, and therefore $A_1 \cdot A_2 = -R^2/d$. Also, $s^2 = (A_1 - A_2)^2 = 2R^2 + 2R^2/d$. Therefore

$$s^2 = \frac{2R^2(d+1)}{d}\ , \quad R^2 = \frac{s^2 d}{2(d+1)}.$$

This proves (19). By Pythagoras' Theorem, we have

$$\begin{aligned} s^2 &= h^2 + R_{d-1}^2 = h^2 + \frac{s^2(d-1)}{2d} \\ h^2 &= \frac{s^2(d+1)}{2d}. \end{aligned}$$

This proves (20).

Using (20), we have

$$\begin{aligned} V_d &= \frac{1}{d}h_d V_{d-1} = \frac{1}{d}\sqrt{\frac{1}{2}}\sqrt{\frac{d+1}{d}}sV_{d-1} = \frac{1}{d(d-1)}\left(\sqrt{\frac{1}{2}}\right)^2\sqrt{\frac{d+1}{d-1}}s^2V_{d-2} \\ &= \frac{1}{d!}\left(\sqrt{\frac{1}{2}}\right)^{d-1}\sqrt{\frac{d+1}{2}}s^{d-1}V_1 = \frac{1}{d!}\left(\sqrt{\frac{1}{2}}\right)^{d}\sqrt{d+1}s^d. \end{aligned}$$

This proves (21).

Finally, to calculate r, we use the fact that $V_d = (d+1)(r\ V_{d-1}/d)$ to obtain

$$r = \frac{d}{d+1}\frac{V_d}{V_{d-1}} = \frac{d}{d+1}\frac{1}{d}\sqrt{\frac{d+1}{d}}\sqrt{\frac{1}{2}}\ s = \sqrt{\frac{1}{2d(d+1)}}\ s.$$

□

Now we are ready to prove the announced

Theorem 4.7: *Let K be a d-kite whose regular $(d-1)$-base S has side-length s and whose remaining vertex P is at distance t from the vertices of S. Let $R = R_{d,s,t}$, $r = r_{d,s,t}$, $h = h_{d,s,t}$, and $V = V_{d,s,t}$ denote, respectively, the circumradius, the inradius, the altitude, and the volume of K. Then*

$$R = t^2\sqrt{\frac{d}{2\left(2t^2d - s^2(d-1)\right)}}, \tag{23}$$

$$h = \sqrt{\frac{2t^2d - s^2(d-1)}{2d}}, \tag{24}$$

$$V = \frac{1}{d!}\sqrt{\frac{2t^2d - s^2(d-1)}{2^d}}\; s^{d-1}, \tag{25}$$

$$r = s\,\frac{\sqrt{2t^2d - s^2(d-1)}}{\sqrt{2}\left(\sqrt{d}s + d\sqrt{2t^2(d-1) - s^2(d-2)}\right)}. \tag{26}$$

Proof: Let u be the circumradius and v the volume of S. Then

$$h^2 = t^2 - u^2 = t^2 - \frac{s^2(d-1)}{2d} = \frac{2t^2d - s^2(d-1)}{2d}.$$

This proves (24). On the other hand, $h = R \pm \sqrt{R^2 - u^2}$. Therefore,

$$\begin{aligned}
t^2 - u^2 &= \left(R \pm \sqrt{R^2 - u^2}\right)^2 = 2R^2 - u^2 \pm 2R\sqrt{R^2 - u^2}. \\
\left(t^2 - 2R^2\right)^2 &= 4R^2(R^2 - u^2) \\
t^4 - 4R^2t^2 &= -4R^2u^2 = -4R^2\frac{s^2(d-1)}{2d} \\
2t^4d &= 4R^2\left(2t^2d - s^2(d-1)\right) \\
R^2 &= \frac{t^4d}{2\left(2t^2d - s^2(d-1)\right)}.
\end{aligned}$$

This proves (23). Also,

$$\begin{aligned}
V &= \frac{1}{d}hv = \frac{1}{d}\sqrt{\frac{2t^2d - s^2(d-1)}{2d}}\frac{1}{(d-1)!}\sqrt{\frac{d}{2^{d-1}}}\; s^{d-1} \\
&= \frac{1}{d!}\sqrt{\frac{2t^2d - s^2(d-1)}{2^d}}\; s^{d-1}.
\end{aligned}$$

This proves (25). It remains to prove (26). We use

$$\begin{aligned}
V &= V_{d,s,t} = \frac{r}{d}v + d\left(\frac{r}{d}V_{d-1,s,t}\right) \\
&= \frac{r}{d}\frac{1}{(d-1)!}\sqrt{\frac{d}{2^{d-1}}}\; s^{d-1} + d\left(\frac{r}{d}\frac{1}{(d-1)!}\sqrt{\frac{2t^2(d-1) - s^2(d-2)}{2^{d-1}}}\; s^{d-2}\right) \\
&= \frac{r}{d!\sqrt{2^{d-1}}}s^{d-2}\left(s\sqrt{d} + d\sqrt{2t^2(d-1) - s^2(d-2)}\right).
\end{aligned}$$

Therefore

$$\frac{1}{d!}\sqrt{\frac{2t^2d - s^2(d-1)}{2^d}}\; s^{d-1} = \frac{r}{d!\sqrt{2^{d-1}}}s^{d-2}\left(s\sqrt{d} + d\sqrt{2t^2(d-1) - s^2(d-2)}\right),$$

and hence

$$\sqrt{\frac{2t^2d - s^2(d-1)}{2}}\, s \;=\; r\left(s\sqrt{d} + d\sqrt{2t^2(d-1) - s^2(d-2)}\right).$$

Therefore

$$r \;=\; s\,\frac{\sqrt{2t^2d - s^2(d-1)}}{\sqrt{2}\left(s\sqrt{d} + d\sqrt{2t^2(d-1) - s^2(d-2)}\right)},$$

as desired. □

Addendum to Theorem 4.7: The formulas derived above lead to the following statements about the eccentricity $\epsilon = \frac{t}{s}$, interior nature of the circumcenter, equiradiality, and the interior nature of the orthocenter for a general kite:

ϵ^2 can take any value in the interval $\left(\frac{d-1}{2d}, \infty\right)$.
The circumcenter of K is interior if and only if $\epsilon^2 > \dfrac{d-1}{d}$.
K is equiradial if and only if $\epsilon^2 = \dfrac{d-2}{d}$ or $\epsilon = 1$ (i.e., $t = s$).
The orthocenter of K is interior if and only if $\epsilon^2 > \dfrac{1}{2}$.

4.3 Tools

The following theorems are needed in the proof of Theorems 4.11 and 4.13. Theorem 4.8 describes how the barycentric coordinates $a_1, \ldots, a_{d+1}$ of the orthocenter of a non-rectangular orthocentric d-simplex S are related to those of the orthocenter of a face F of S, and also how $\sigma(F)$ and $\sigma(S)$ are related. Theorem 4.9 expresses the circumcenter of such an S in terms of its vertices, and the circumradii of S and of its faces in terms of $a_1, \ldots, a_{d+1}$ and $\sigma(S)$. Lemma 4.10 deals with the Gram matrix of a special type of a non-rectangular orthocentric d-simplex S. This will be used in the proof of Theorem 4.11a,b which gives a characterization of non-rectangular orthocentric d-simplices which are equiradial, and in the proof of Theorem 4.13 which proves that orthocentric simplices with $\mathcal{I} = \mathcal{C}$ are regular.

Theorem 4.8: *Let $\mathcal{H}$ be the orthocenter of a non-rectangular orthocentric d-simplex $S = [A_1, \ldots, A_{d+1}]$, and let $\mathcal{H}'$ be the orthocenter of the face $S' = [A_1, \ldots, A_{k+1}]$ of S. Let $a_1, \ldots, a_{d+1}$ be the barycentric coordinates of $\mathcal{H}$ with respect to S, and $a'_1, \ldots, a'_{k+1}$ be the barycentric coordinates of $\mathcal{H}'$ with respect to S'. Let $\sigma(S)$ be the obtuseness of S as defined in (8). Then $a_1 + \cdots + a_{k+1} \neq 0$, and*

$$a'_j = \frac{a_j}{a_1 + \cdots + a_{k+1}}, \;\; \sigma(S') = \frac{\sigma(S)}{a_1 + \cdots + a_{k+1}}. \tag{27}$$

Proof: It is clearly sufficient to prove our theorem for $k = d-1$. Also, we may assume that $\mathcal{H}$ is the origin O. Then it follows from (14) in the proof of Theorem 3.3 that

$$a'_i = \frac{a_i}{a_1 + \cdots + a_d},$$

as desired.

To compute $\sigma(S')$, we set $c = \sigma(S)$ and $s = a_1 + \cdots + a_d (= 1 - a_{d+1})$. Take i and j such that i, j, and $d+1$ are pairwise distinct. Using (14), we see that

$$\begin{aligned} A_i \cdot A_j &= c \\ \mathcal{H}' \cdot A_i &= \frac{-a_{d+1}}{s}(A_{d+1} \cdot A_i) = \frac{-ca_{d+1}}{s} \\ \|\mathcal{H}'\|^2 &= \left(\frac{-a_{d+1}}{s}\right)^2 \|A_{d+1}\|^2 = \left(\frac{-a_{d+1}}{s}\right)^2 \left(\frac{-cs}{a_{d+1}}\right) - \frac{-ca_{d+1}}{s}, \end{aligned}$$

and therefore

$$\begin{aligned} \sigma(S') &= (\mathcal{H}' - A_i) \cdot (\mathcal{H}' - A_j) \\ &= \left(\frac{c}{s}\right)(-a_{d+1} + 2a_{d+1} + 1 - a_{d+1}) = \frac{c}{s}, \end{aligned}$$

as claimed. □

Theorem 4.9: *Let $S = [A_1, \ldots, A_{d+1}]$ be a non-rectangular orthocentric d-simplex, and let $a_1, \ldots, a_{d+1}$ be the barycentric coordinates of its orthocenter $\mathcal{H}$. Let $c = \sigma(S)$ be the obtuseness of S defined in (8). Let $\mathcal{C}$ be the circumcenter and R the circumradius of S. Then*

$$\mathcal{C} + \frac{d-1}{2} \cdot \mathcal{H} = \frac{1}{2}(A_1 + \cdots + A_{d+1}), \tag{28}$$

$$\frac{4R^2}{c} = (d-1)^2 - \sum_{i=1}^{d+1} \frac{1}{a_i}. \tag{29}$$

Consequently, $\mathcal{C}$ is interior if and only if $a_i < 1/(d-1)$. Also, if $F = [A_1, \ldots, A_{k+1}]$ is a face of S, and if $s = s(F) = a_1 + \cdots + a_{k+1}$, then the circumradius R_F of F is given by

$$\frac{4R_F^2}{c} = \frac{(k-1)^2}{s} - \left(\frac{1}{a_1} + \cdots + \frac{1}{a_{k+1}}\right). \tag{30}$$

Proof: For simplicity, we assume $\mathcal{H} = O$ (by replacing A_i and $\mathcal{C}$ by $A_i - \mathcal{H}$ and $\mathcal{C} - \mathcal{H}$, respectively), and we let $P = (A_1 + \cdots + A_{d+1})/2$. We use the facts that $A_i \cdot A_j = c$ for all $i \neq j$, and that $\|A_i\|^2 = c(a_i - 1)/a_i$. Then $2(P - A_{d+1}) = A_1 + \cdots + A_d - A_{d+1}$ and

$$\begin{aligned} 4\|P - A_{d+1}\|^2 &= \|A_1\|^2 + \cdots + \|A_{d+1}\|^2 + (d^2 - 3d)c \\ &= \frac{c(a_1 - 1)}{a_1} + \cdots + \frac{c(a_{d+1} - 1)}{a_{d+1}} + (d^2 - 3d)c \\ &= -c\left(\frac{1}{a_1} + \cdots + \frac{1}{a_{d+1}}\right) + (d+1)c + (d^2 - 3d)c \\ &= -c\left(\frac{1}{a_1} + \cdots + \frac{1}{a_{d+1}}\right) + c(d-1)^2. \end{aligned}$$

Similarly for $4\|P - A_i\|^2$ for every i. Therefore P is equidistant from the vertices of S, showing that P is the circumcenter, and proving (28) and (29).

Letting

$$b_i = \frac{1}{2} + \frac{(1-d)a_i}{2},$$

we see that

$$\mathcal{C} = b_1 A_1 + \cdots + b_{d+1} A_{d+1} \quad \text{and} \quad b_1 + \cdots + b_{d+1} = 1.$$

Therefore $b_1, \ldots, b_{d+1}$ are the barycentric coordinates of $\mathcal{C}$. Thus $\mathcal{C}$ is interior if and only if $b_i > 0$, i.e., if and only if $a_i < 1/(d-1)$, as claimed.

Finally, the statement pertaining to R_F follows from (29) using Theorem 4.7. □

Lemma 4.10: *Let G be the $(d+1) \times (d+1)$–matrix whose entries G_{ij} are given by*

$$G_{ij} = \begin{cases} 1 & \text{if} \quad i \neq j, \\ 1 + x_i & \text{if} \quad i = j. \end{cases} \tag{31}$$

where the x_i's are non–zero numbers.

(a) *The characteristic polynomial of G is given by*

$$(\lambda - x_1) \cdots (\lambda - x_{d+1}) \left(1 + \frac{-1}{\lambda - x_1} + \cdots + \frac{-1}{\lambda - x_{d+1}}\right). \tag{32}$$

(b) *Let x and y be distinct non–zero real numbers, and suppose that*

$$x_i = \begin{cases} x & \text{if} \quad i \leq m, \\ y & \text{if} \quad i > m. \end{cases} \tag{33}$$

(i) *If $m = 0$, then cG for some c is the Gram matrix of a d–simplex S if and only if $y = -d-1$. In this case, S is regular.*

(ii) *If $m = 1$, then cG for some c is the Gram matrix of a d–simplex S if and only if*

$$(i)\ xy + dx + my = 0 \quad \text{and} \quad (ii)\ x < 0. \tag{34}$$

In this case, S is a kite having eccentricity $(x+y)/(2yx)$.

(iii) *If $2 \leq m \leq d-1$, then cG for some c is the Gram matrix of a d–simplex S if and only if*

$$(i)\ xy + (d+1-m)x + my = 0 \quad \text{and} \quad (ii)\ x + m < 0. \tag{35}$$

Proof. Statement (a) follows immediately from Lemma 3.6, since the characteristic polynomial of G equals

$$\begin{aligned} & \det(\lambda I - G) \\ = \ & \det(\mathbb{J}(-1, \ldots, -1; \lambda - x_1, \ldots, \lambda - x_{d+1})) \\ = \ & (\lambda - x_1) \cdots (\lambda - x_{d+1}) \left(1 + \frac{-1}{\lambda - x_1} + \cdots + \frac{-1}{\lambda - x_{d+1}}\right). \end{aligned}$$

To prove (b), we use (32) and the fact that $\pm G$ is the Gram matrix of a d–simplex if and only if one of its eigenvalues is 0 and the others all have the same sign.
(i) If $m = 0$, then the characteristic polynomial of G is

$$(\lambda - y)^{d+1}\left(1 - \frac{d+1}{\lambda - y}\right) = (\lambda - y)^d(\lambda - y - (d+1)),$$

and the eigenvalues are y and $y + d + 1$. Also, $y \neq 0$. Therefore, $\pm G$ is the Gram matrix of a d-simplex S that is necessarily regular if and only if $y + d + 1 = 0$.

(ii) If $m = 1$, then the characteristic polynomial of G is

$$\begin{aligned} f(\lambda) &= (\lambda - x)(\lambda - y)^d\left(1 - \frac{1}{\lambda - x} - \frac{d}{\lambda - y}\right) \\ &= (\lambda - y)^{d-1}(\lambda^2 - (x + y + d + 1)\lambda + y + xy + dx), \end{aligned}$$

and one of the eigenvalues is 0 if and only if

$$y + xy + dx = 0. \tag{36}$$

Assuming (36), we see that $x + 1 \neq 0$ (since $dx \neq 0$), and that the remaining eigenvalues of G are

$$y = \frac{-dx}{x+1} \quad \text{and} \quad x + y + d + 1 = x + \frac{-dx}{x+1} + d + 1 = \frac{(x+1)^2 + d}{x+1}.$$

These have the same sign if and only if $x+1$ and $-x(x+1)$ do, which happens if and only if $x < 0$.

One can recover the squares of the edge-length of a simplex from its Gram matrix by the formula $\|A_i - A_j\|^2 = G_{ii} + G_{jj} - 2G_{ij}$. A d-simplex S whose Gram matrix is G is thus a d-kite whose base is a regular $(d-1)$-simplex of side-length $(1 + y) + (1 + y) - 2 = 2y$ and whose remaining edges have edge-lengths $(1 + x) + (1 + y) - 2 = (x + y)$. Its eccentricity is therefore $(x + y)/(2y)$.

(iii) If $2 \leq m \leq d-1$, then, setting $n = d+1-m$, we have $m, n \geq 2$. The characteristic polynomial of G is

$$\begin{aligned} f(\lambda) &= (\lambda - x)^m(\lambda - y)^n\left(1 - \frac{m}{\lambda - x} - \frac{n}{\lambda - y}\right) \\ &= (\lambda - x)^{m-1}(\lambda - y)^{n-1}(\lambda^2 - (x + y + d + 1)\lambda + xy + my + nx)\,, \end{aligned}$$

and one of the eigenvalues is 0 if and only if

$$my + xy + nx = 0\,. \tag{37}$$

Assuming (37), we see that $x + m \neq 0$ (since $nx \neq 0$), and that the remaining eigenvalues of G are

$$x,\, y = \frac{-nx}{x+m}\,, \text{ and } x + y + d + 1 = \frac{(x+m)^2 + mn}{x+m}\,.$$

These have the same sign if and only if x, $-x(x + m)$, and $x + m$ do, which happens if and only if $x + m < 0$. □

4.4 Complete classification of equiradial orthocentric simplices

Here we complete the study of orthocentric simplices in which the circumcenter coincides with the incenter.

Theorem 4.11 a: *Let $S = [A_1, \ldots, A_{d+1}]$ be a non-regular, non-rectangular, equiradial orthocentric d-simplex, and let $a_1, \ldots, a_{d+1}$ be the barycentric coordinates of its orthocenter with respect to S. Then there are two possibilities*

1. S is a kite of eccentricity $\sqrt{(d-2)/d}$, in which case $d \geq 5$, or
2. There exists an m with $2 \leq m \leq d-1$ such that, after relabelling vertices,

$$a_1 = \ldots = a_m\,,\ a_{m+1} = \ldots = a_{d+1} \tag{38}$$

where

$$m(d+1-m) \leq \left(\frac{d^2-3d+4}{2(d-2)}\right)^2, \text{which implies } d \geq 9. \tag{39}$$

Proof: We may assume $d > 3$ since equiradial orthocentric simplices of lower dimensions are known to be regular. Let $S = [A_1, \ldots, A_{d+1}]$ and suppose that the orthocenter is O, and that $a_1, \ldots, a_{d+1}$ are the barycenric coordinates of O with respect to S. Suppose that S is equiradial. Then it follows from (30) in Theorem 4.9 that

$$\frac{(d-2)^2}{1-a_i} + \frac{1}{a_i} = \frac{(d-2)^2}{1-a_j} + \frac{1}{a_j}$$

for every i and j. Therefore

$$\frac{1+(d-3)(d-1)a_i}{a_i(1-a_i)} = \frac{1+(d-3)(d-1)a_j}{a_j(1-a_j)}\,,$$

which simplifies into

$$(a_i - a_j)((d-3)(d-1)a_i a_j + a_i + a_j - 1) = 0\,.$$

If $(d-3)(d-1)a_i a_j + a_i + a_j = 1$ and $(d-3)(d-1)a_i a_k + a_i + a_k = 1$ then, multiplying the first by a_k and the second by a_j and subtracting, we obtain $(a_i - 1)(a_k - a_j) = 0$, and therefore $a_k = a_j$, since no a_i can be 1. Therefore the a_i's can take at most two different values a and b that satisfy

$$(d-3)(d-1)ab + a + b - 1 = 0\,. \tag{40}$$

Since S is not regular, it follows that the a_i's are not all equal, and therefore we may assume that

$$a_1 = \ldots = a_m = a\,,\ a_{m+1} = \ldots = a_{d+1} = b$$

for some m with $1 \leq m \leq d$, and with a and b satisfying (40). Letting $n = d+1-m$, $x = -1/a$ and $y = -1/b$, and using $ma + nb = 1$ and (40), we obtain (37) and

$$xy + x + y = (d-3)(d-1)\,. \tag{41}$$

Also, using $A_i \cdot A_j = c$ for $i \neq j$, and $\|A_i\|^2 = c(a_i - 1)/a_i$, we see that the Gram matrix of S is of the type described in Lemma 4.10 (b), where x and y satisfy (41).

Again we use Lemma 4.10 (b). If $m = 1$, then it follows from (34) and (41) that $x = 3 - d$ and $x < 0$. Substituting $m = 1, n = d$, and $x = 3 - d$ in (37), we see that $d \geq 5$ and that

$$y = \frac{d(d-3)}{d-4}.$$

This corresponds to the kite whose eccentricity ϵ is given by

$$\epsilon^2 = \frac{x+y}{2y} = \frac{(d-2)^2}{d^2},$$

in conformance with Theorem 4.7. Thus we are left with the case

$$2 \leq m \leq d-1,$$

and

$$x + m < 0, \; xy + my + nx = 0, \; x + y + xy = (d-3)(d-1). \tag{42}$$

The last two of these can be rewritten as

$$\begin{aligned} (x+m)(y+n) &= mn, \\ (x+m)(1-n) + (y+n)(1-m) &= d^2 - 3d + 4 - 2mn. \end{aligned}$$

Setting

$$\xi = (x+m)(1-n), \eta = (y+n)(1-m), \tag{43}$$

we see that (42) can be rewritten as

$$\xi > 0, \, \eta > 0, \, \xi + \eta = d^2 - 3d + 4 - 2mn, \xi\eta = mn(mn-d).$$

Now

$$0 \leq (\xi - \eta)^2 = (\xi + \eta)^2 - 4\xi\eta = (d^2 - 3d + 4 - 2mn)^2 - 4mn(mn - d).$$

This simplifies into

$$(d^2 - 3d + 4)^2 - 4mn(d^2 - 3d + 4) + 4mnd \geq 0 \tag{44}$$

or, equivalently,

$$mn \leq \frac{(d^2 - 3d + 4)^2}{4(d-2)^2}, \tag{45}$$

as desired. It remains to prove that $d \geq 9$. Since $mn \geq 2(d-1)$, it follows that

$$2(d-1) \leq \frac{(d^2 - 3d + 4)^2}{4(d-2)^2}, \tag{46}$$

and therefore

$$0 \leq d^4 - 14d^3 + 57d^2 - 88d + 48 = (d-3)\left(d(d-3)(d-8) - 16\right).$$

Since $d > 3$ by assumption, this happens if and only if $d \geq 9$, as claimed. □

The kites have already been completely analyzed in Subsection 4.2. In the other case we have the following converse.

Theorem 4.11 b: *If $2 \leq m \leq d-1$, and if m and d satisfy* (39)*, then there exist exactly two non-similar non-rectangular orthocentric equiradial d-simplices whose orthocenter's barycentric coordinates $a_1, \ldots, a_{d+1}$ satisfy* (38)*. This happens for any given value of m if d is large enough; in particular when $m = 2$ and $d \geq 9$.*

Proof: Suppose that $2 \leq m \leq d-1$ and that (39) holds. Note first that equality cannot take place in (39), since $(d^2 - 3d + 4)/(2(d-2))$ is not an integer. This can be seen by taking the cases $d \equiv 1 \pmod 2$, $d \equiv 0 \pmod 4$, and $d \equiv 2 \pmod 4$. Then it follows immediately that the discriminant of

$$Q(Z) := Z^2 - (d^2 - 3d + 4 - 2mn)Z + mn(mn - d)$$

is strictly positive. Therefore $Q(Z)$ has two distinct real zeros which are necessarily positive since their sum $d^2 - 3d + 4 - 2mn$ and product $mn(mn-d)$ are. Letting X and Y be the zeros of $Q(Z)$, we find x and y by solving the system

$$X = (x+m)(1-n),\ Y = (y+m)(1-n) \tag{47}$$

or the system

$$Y = (x+m)(1-n),\ X = (y+m)(1-n)\,. \tag{48}$$

Since X and Y are distinct, these systems give rise to two distinct pairs (x, y). The Gram matrices corresponding to these values give rise to the desired simplices, as in the proof of Lemma 4.10. This completes the proof. □

Remark 4.12: Consideration of the corresponding Gram matrices shows that these orthocentric equiradial d-simplices may be thought of as generalized kites. They may be described as the join of a regular $(m-1)$-simplex of edge-length a with a regular $(d-m)$-simplex of edge-length b such that all intervening edges have edge-length c, for suitable values of the parameters d, m, a, b, c.

4.5 Orthocentric simplices with circumcenter = incenter are regular

Here we complete the study of orthocentric simplices in which the circumcenter coincides with the incenter.

Theorem 4.13: *If S is an orthocentric d-simplex in which the circumcenter and the incenter coincide, then S is regular.*

Proof: We shall treat the simpler case when S is rectangular in Section 5. So we suppose that S is a non-rectangular orthocentric d-simplex. Let $a_1, \ldots, a_{d+1}$ be the barycentric coordinates of its orthocenter with respect to S. If the incenter and the circumcenter of S coincide, and if S is not regular, then, being necessarily equiradial, S is of one of the two types in Theorem 4.11. The first is a d-kite of eccentricity $\sqrt{(d-2)/d}$, and therefore has an exterior circumcenter by Theorem 4.7. The other type satisfies

$$a_1 = \ldots = a_m = a\,,\ \ a_{m+1} = \ldots = a_{d+1} = b\,,$$

where $x = -1/a$ and $y = -1/b$ are such that

$$x + m < 0\,,\ \ y + n < 0\,,\ \ xy + x + y = (d-3)(d-1)\,. \tag{49}$$

In particular, x and y are negative and

$$\begin{aligned}\text{the circumcenter of } S \text{ is interior} &\iff a, b < \frac{1}{d-1}, \text{by Theorem 3.10,}\\ &\iff x, y < 1 - d\\ &\iff x + 1, y + 1 < 2 - d\\ &\implies (x+1)(y+1) > (d-2)^2\\ &\implies xy + x + y > (d-1)(d-3),\end{aligned}$$

contradicting (49). Therefore here again the circumcenter is not interior, and cannot coincide with the incenter. □

Remark 4.14: We have seen in [15], Theorem 2.2, that there are non-equifacetal tetrahedra whose facets have equal inradii. This prompts the still open question whether there exist non-regular orthocentric tetrahedra (or higher dimensional simplices) with this property.

5 Rectangular simplices

In the previous section, we have coordinatized a non-rectangular orthocentric d-simplex $S = [A_1, \ldots, A_{d+1}]$ by the barycentric coordinates $a_1, \ldots, a_{d+1}$ of its orthocenter, together with the obtuseness $\sigma(S)$ defined in (8). It was noted that $\sigma(S) = 0$ if and only if S is rectangular, and it is easy to see that if S' is similar to S with a similarity factor p, then $\sigma(S') = p^2\sigma(S)$. Thus $|\sigma(S)|$ is not relevant as far as the shape of S is concerned, and S can be scaled so that $\sigma(S) = 0$ or $\sigma(S) = \pm 1$. In view of (10) it is obvious that $\sigma(S) < 0$ if and only if $0 < a_i < 1$ for all i, i.e., if and only if the orthocenter of S is an interior point. Note that, for a triangle, this is equivalent to S being acute-angled.

Rectangular d-simplices are characterized as those orthocentric d-simplices S with $\sigma(S) = 0$. However, the barycentric coordinates of the orthocenter of such a d-simplex S carry no information about S (except for locating the vertex at which the orthocenter occurs), and therefore cannot serve to parametrize S. On the other hand, the lengths $b_1, \ldots, b_d$ of the legs of a rectangular simplex do carry all the essential information about S. Here a *leg* is an edge emanating from the vertex at which S is rectangular. If a d-simplex $S = [A_1, \ldots, A_{d+1}]$ is rectangular, say at A_{d+1}, then one can place it in $\mathbb{E}^d$ in such a way that A_{d+1} lies at the origin and such that the legs $A_{d+1}A_i$ lie on the positive coordinate axes. Then the i-th cartesian coordinate of A_i is b_i and the other coordinates are 0, and

$$\|A_{d+1} - A_i\| = b_i. \tag{50}$$

The volume, the circumradius, the inradius, and other elements of S can be easily computed in terms of the b_i's, as illustrated in Theorem 5.1 below. These formulas will then apply to, but only to, the rectangular faces of S, i.e., to those faces of S that have A_{d+1} as a vertex. To understand the remaining faces, note that they are necessarily faces of the *hypotenuse facet* $[A_1, \cdots, A_d]$, and thus it is sufficient and important to understand this facet. It follows from (57) below that this facet, which is necessarily orthocentric, cannot be rectangular, and therefore it yields to the results of the preceding section. The natural question that arises is whether every non–rectangular orthocentric $(d-1)$-simplex occurs as a facet (necessarily the hypotenuse facet) of a rectangular d-simplex. Theorem 5.3 below provides a satisfactory answer.

Theorem 5.1 *Let $S = [A_1, \cdots, A_{d+1}]$ be a d-simplex that is rectangular at A_{d+1}, and let $b_1, \ldots, b_d$ be the lengths of its legs $A_{d+1}A_1, \ldots, A_{d+1}A_d$, respectively. Let the volume, the circumradius, and the inradius of S be denoted by V, R, and r, respectively. For each i, let V_i be the $(d-1)$-volume of the i-th facet of S, and let h be the altitude of S to the $(d+1)$-th facet $[A_1, \cdots, A_d]$. Then*

$$V = \frac{b_1 \cdots b_d}{d!}, \tag{51}$$

$$V_{d+1} = \frac{b_1 \cdots b_d}{(d-1)!} \sqrt{\frac{1}{b_1^2} + \cdots + \frac{1}{b_d^2}}, \tag{52}$$

$$h = \left(\sqrt{\frac{1}{b_1^2} + \cdots + \frac{1}{b_d^2}} \right)^{-1}, \tag{53}$$

$$r = \left(\frac{1}{b_1} + \cdots + \frac{1}{b_d} + \sqrt{\frac{1}{b_1^2} + \cdots + \frac{1}{b_d^2}} \right)^{-1}, \tag{54}$$

$$R^2 = \frac{b_1^2 + \cdots + b_d^2}{4}. \tag{55}$$

Also, if $A_{d+1} = O$, then the circumcenter $\mathcal{C}$ of S and the orthocenter $\mathcal{B}$ of the facet $[A_1, \cdots, A_d]$ are given by

$$\mathcal{C} = \frac{A_1 + \cdots + A_d}{2}, \tag{56}$$

$$\mathcal{B} = \left(\frac{1}{b_1^2} + \cdots + \frac{1}{b_d^2} \right)^{-1} \left(\frac{1}{b_1^2} A_1 + \cdots + \frac{1}{b_d^2} A_d \right). \tag{57}$$

Proof: The equation (51) is obvious. Using (51) and the d-dimensional Pythagoras' Theorem, we obtain

$$\begin{aligned} V_{d+1}^2 &= \left(\frac{b_1 \cdots b_d}{(d-1)!\, b_d} \right)^2 + \cdots + \left(\frac{b_1 \cdots b_d}{(d-1)!\, b_d} \right)^2 \\ &= \left(\frac{b_1 \cdots b_d}{(d-1)!} \right)^2 \left(\frac{1}{b_1^2} + \cdots + \frac{1}{b_d^2} \right). \end{aligned}$$

This proves (52). Then we use $dV = V_{d+1}h$ to get (53). For (54), we use the preceding formulas and the fact that $dV = r(V_1 + V_2 + \ldots + V_{d+1})$. To prove (56), we use $\|\mathcal{C}\|^2 = \|\mathcal{C} - A_i\|^2$ to conclude that $2\mathcal{C} \cdot A_i = \|A_i\|^2 = b_i^2$, and therefore the i-th coordinate of $\mathcal{C}$ is $b_i/2$. Then we use (56) and Pythagoras' Theorem to obtain (55). Finally, (57) follows from the fact that $\mathcal{B}$ is the projection of A_{d+1} on the facet $[A_1, \ldots, A_d]$ and thus $\mathcal{B} \cdot (A_i - A_j) = 0$ for $1 \le i < j \le d$. □

Theorem 5.2: *If S is rectangular, then its four classical centers $\mathcal{G}, \mathcal{C}, \mathcal{I}$, and $\mathcal{H}$ are pairwise distinct.*

Proof: Let $S = [A_1, \ldots, A_{d+1}]$ be rectangular at A_{d+1}. In view of (55) the circumcenter lies on the hypotenuse facet of a rectangular d-simplex if and only if $d = 2$. Also, the barycentric coordinates of $\mathcal{C}$ with respect to $A_1, \ldots, A_{d+1}$ are $(1/2, \ldots, 1/2, (2-d)/2)$, and therefore $\mathcal{C}$ is never interior (since $(2-d)/2 \le 0$ for all $d \ge 2$). Therefore, of the orthocenter, the circumcenter, the incenter, and the centroid of S, the only two that can possibly coincide are the last two. This cannot happen either. In fact, it is clear that $\mathcal{I} = (r, \ldots, r)$ and $\mathcal{G} = (b_1, \ldots, b_{d+1})/(d+1)$, and therefore the equality $\mathcal{I} = \mathcal{G}$ would imply that $b_i = (d+1)r$ for all i. Using (53), we arrive at the contradiction

$$d + 1 = d + \sqrt{d}. \tag{58}$$

Therefore no two the four classical centers can coincide. □

Theorem 5.3: *Let $T = [A_1, \ldots, A_d]$ be an orthocentric $(d-1)$-simplex. Then the following conditions are equivalent.*

(a) *There exists a d-simplex $S = [A_1, \ldots, A_d, A_{d+1}]$ that is rectangular at A_{d+1}.*

(b) *The orthocenter of T is interior.*

(c) $\sigma(T) < 0$.

Proof: We already know that (b) and (c) are equivalent by Theorem 3.8. Let $t_1, \ldots, t_d$ be the barycentric coordinates of the orthocenter $\mathcal{B}$ of T. If (a) holds, then (57) implies that the t_i's are all positive, and therefore $\mathcal{B}$ is interior. Thus $0 < t_i < 1$ for all i. In view of (10), this implies that $\sigma(T) < 0$. Conversely, if $c = \sigma(T) < 0$, then $0 < t_i < 1$ for all i. Let $b_1, \ldots, b_d$ be the positive numbers defined by

$$b_i^2 t_i = -c, \tag{59}$$

and let $S = [P_1, \ldots, P_{d+1}]$ be the d-simplex that is rectangular at P_{d+1} and whose leg-lengths are $b_1, \ldots, b_d$. Then for $1 \le i < j \le d$ we have

$$\begin{aligned} \|A_i - A_j\|^2 &= -c\left(\frac{1}{t_i} + \frac{1}{t_j}\right) \quad \text{by (11)} \\ &= b_i^2 + b_j^2 \quad \text{by (59)} \\ &= \|P_i - P_j\|^2 \quad \text{by Pythagoras' Theorem.} \end{aligned}$$

Therefore T is congruent to the facet $[P_1, \ldots, P_d]$ of S. This proves (a). □

Finally we mention that the case $d = 3$ of Theorem 5.3 is Problem 1938.3 (pp. 132-134) of [38].

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