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Results in Mathematics

The eigenvalues of the Laplacian on locally finite networks

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Abstract: We consider the continuous Laplacian on an infinite locally finite network with equal edge lengths under natural transition conditions as continuity at the ramification nodes and classical Kirchhoff conditions at all vertices. It is shown that eigenvalues of the Laplacian in a L^∞–setting are closely related to those of the adjacency and transition operator of the network. In this way the point spectrum is determined completely in terms of combinatorial quantities and properties of the underlying graph as in the finite case [2]. Moreover, the occurrence of infinite geometric multiplicity on trees and some periodic graphs is investigated.

Keywords: Locally finite graphs and networks, Laplacian, eigenvalue problems, adjacency and transition operator.

AMS Subject Classification: 34B45, 05C50, 05C10, 35J05, 34L10, 35P10.

1 Introduction

In [2] a general link between the continuous Laplacian on a finite network and its length adjacency matrix has been established under natural transition conditions at the vertices. In the present paper the corresponding intrinsic relation between the eigenvalues of the Laplacian under the same transition conditions and those of a weighted adjacency operator for any uniformly locally finite infinite network with equal edge lengths is established, as well as the geometric multiplicities of these eigenvalues are investigated. The transition

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conditions at the vertices are the continuity condition at ramification nodes (4) and the classical Kirchhoff zero incident flow sum condition at all vertices (5).

The spectrum of the Laplacian on finite networks has been considered by many authors, see e.g. [1], [2], [4], [6], [15], [16], [17], [20] and the references therein, while it seems that only a few authors treat the infinite case [8], [11]. By the established link between the spectra of the continuous Laplacian and the adjacency operator, the spectral treatment of the latter is of big interest also for the Laplacian. Among the numerous authors, we first refer to the survey article by B. Mohar and W. Woess [19] and the monography by W. Woess [21] and the references therein, and secondly for the ℓ^∞–setting to [5], [7], [8], [12].

In the present infinite context, it is essential that we deal with spaces of bounded functions. From a functional analytic and operator theoretical point of view, a setting in possibly weighted Hilbert Sobolev spaces is more convenient, especially when dealing with random walks [21]. In [11] the Laplacian on a locally finite network is considered in a $\mathcal{H}^2_{K,c}$–setting under Kirchhoff vertex conditions defined by arbitrary edge conductivities c that correspond to the edge lengths. These conditions impose a strong restriction to the corresponding Sobolev spaces in the infinite case. Following the approach in [2], a natural spectral relation to the associated transition operator is established. It is well known [2], [3],[18] that nonreal eigenvalues can occur already in elementary cases, if the conductivities in the Kirchhoff condition are inconsistent with the edge coefficients of the Laplacian. As it stands this has not been taken into account in [11]. For unitary conductivities, the point spectrum can become empty or just an unbounded sequence, see Section 5 in [11]. Therefore, from a combinatorial and geometrical point of view, a L^∞–setting is more appropriate for spectral links with transition or adjacency operators. The results presented here reveal many eigenvalues and -functions of interest and seem to justify the treatment in spaces of bounded functions. In the present investigation only graphs with equal edge lengths will be considered. This implies that the spectrum of the Laplacian under continuity and consistent Kirchhoff vertex conditions is real. As already noted above, for variable edge lengths, the spectral properties can change dramatically. One can also think e.g. of networks with shrinking edges but only finitely many essential ramification nodes that need not to be Liouville spaces, while they bear that property for equal edge lengths, see [8]. A systematic approach of this case is the object of a work under preparation [9].

The present paper is organized as follows. After some graph theoretical preliminaries in Section 2, the circuit space of a uniformly locally finite network and its relation to the incidence matrix is investigated. Its dimension interferes crucially in the eigenvalue multiplicities. In Sections 4 - 6 the main link (Theorems 4.1, 5.1, 6.2, 6.3) between the eigenvalues of the Laplacian on the network and those of its adjacency and transition operator is established. It can be briefly summarized in the characteristic equation

$$\mathcal{A}(\Gamma)\varphi = \cos\sqrt{\lambda}\,\mathrm{Diag}_{i\in\mathrm{N}}\,(\gamma_i)\,\varphi,$$

that relates the adjacency operator $\mathcal{A}$ and the vertex degrees γ_i to an eigenvalue λ of the Laplacian and an admissible node distribution of a corresponding eigenfunction. It turns out that, mutatis mutandis, the geometric multiplicities are the same as in the finite case, where those eigenvalues (Section 6) stemming from a single edge under Neumann or homogeneous Dirichlet boundary conditions have to be distinguished carefully from the immanent eigenvalues that take their rise from the whole combinatorial structure of the network (Section 5). For a γ–regular network the spectral link reduces to the result that a non vanishing bounded sequence φ is the node distribution of an eigenfunction of $-\Delta_G^K$ in $\mathcal{C}_K^2(G) \cap L^\infty(G)$ iff φ is an eigenvector of the adjacency matrix $\mathcal{A}$ of Γ. Moreover, a real number $\lambda \geq 0$ is an eigenvalue of that Laplacian iff $\mu = \gamma \cos\sqrt{\lambda}$ is an eigenvalue of $\mathcal{A}$ in $\ell^\infty(V(\Gamma))$ or $\cos\sqrt{\lambda} = -1$, see Section 7. In Section 8, the point spectrum on the γ–regular tree is shown to be $[0, \infty)$. This result is extended to trees with only a finite number of boundary vertices. Nevertheless there are trees that do not have certain real numbers as eigenvalues, e.g. for $\cos\sqrt{\lambda} = 0$. Moreover, the occurrence of infinite geometric multiplicity is characterized in terms of the essential ramification nodes. Finally, the black hole phenomenon, i.e. the occurrence of eigenspaces isomorphic to $\ell^\infty(\mathbb{Z})$, is investigated in Section 9.

2 Preliminaries and transition conditions

For any graph $\Gamma = (V, E, \in)$, the vertex set is denoted by $V = V(\Gamma)$, the edge set by $E = E(\Gamma)$ and the incidence relation by $\in \subset V \times E$. The valency of each vertex v is denoted by $\gamma(v) = \operatorname{card}\{e \in E \,|\, v \in e\}$. Set

$$\gamma_{\min} = \min\{\gamma(v) \,|\, v \in V\}.$$

Unless otherwise stated, all graphs considered in this paper are assumed to be nonempty, simple, connected and *uniformly locally finite*, i.e.

$$\max_{v \in V(\Gamma)} \gamma(v) =: \gamma_{\max} < \infty. \tag{1}$$

The simplicity property means that Γ contains no loops, and at most one edge can join two vertices in Γ. Moreover, the conditions imply that Γ is countable. For a given numbering of the vertices v_i, $i \in \mathbb{N}$, set $\gamma_i = \gamma(v_i)$ and define the *adjacency matrix* or *adjacency operator* by

$$\mathcal{A}(\Gamma) = (e_{ih})_{i,h \in \mathbb{N}} : \mathbb{R}^{V(\Gamma)} \longrightarrow \mathbb{R}^{V(\Gamma)} \tag{2}$$

where

$$e_{ih} = \begin{cases} 1 & \text{if } v_i \text{ and } v_h \text{ are adjacent in } \Gamma \\ 0 & \text{else} \end{cases}$$

Note that $\mathcal{A}(\Gamma)$ is indecomposable iff Γ is connected. By simplicity, any two adjacent vertices v_i and v_h determine uniquely the edge e_s joining them, and we can set

$$s(i,h) = \begin{cases} s & \text{if } e_s \cap V = \{v_i, v_h\}, \\ 1 & \text{otherwise.} \end{cases}$$

The sequences or vectors with constant entries equal to 1 are denoted by $\mathbf{e}$. For a subgraph Θ in Γ let $\bar{\Theta} = (V(\Theta), E(\bar{\Theta}), \in)$ denote the subgraph of Γ spanned by the vertices in Θ with

$$E(\bar{\Theta}) = \{e \,|\, e \in E(\Gamma),\ e \cap V(\Gamma) \subset V(\Theta)\}.$$

The subgraph Θ is called *induced* if $\bar{\Theta} = \Theta$. Two subgraphs are called *vertex independent* if they have no vertex in common, and *essentially disjoint* if they have only a finite number of edges in common. The (combinatorial) distance between two vertices v_1 and v_2 is defined to be the minimal number of edges of all paths joining v_1 and v_2. For further graph theoretical terminology we refer to [22], and for the algebraic graph theory to [10] and [13].

Moreover, without loss of generality, we consider each graph as a connected *topological graph* in $\mathbb{R}^m$, i.e. $V(\Gamma) \subset \mathbb{R}^m$ and the edge set consists in a collection of Jordan curves

$$E(\Gamma) = \{\pi_j : [0,1] \to \mathbb{R}^m \,|\, j \in \mathbb{N}\}$$

all of length 1 with the following properties: Each support $e_j := \pi_j([0,1])$ has its endpoints in the set $V(\Gamma)$, any two vertices in $V(\Gamma)$ can be connected by a path with arcs in $E(\Gamma)$, and any two edges $e_j \neq e_h$ satisfy $e_j \cap e_h \subset V(\Gamma)$ and $\mathrm{card}(e_j \cap e_h) \leq 1$. The arc length parameter of an edge e_j is denoted by t_j. Unless otherwise stated, we identify the graph $\Gamma = (V, E, \in)$ with its associated network

$$G = \bigcup_{j\in\mathbb{N}} \pi_j([0,1]),$$

especially each edge π_j with its support e_j. G is called a $\mathcal{C}^\nu$-network, if all $\pi_j \in \mathcal{C}^\nu([0,1], \mathbb{R}^m)$. Endowed with the induced topology G is a connected and locally compact space in $\mathbb{R}^m$. We shall distinguish the *boundary vertices* $V_b = \{v_i \in V \,|\, \gamma_i = 1\}$ from the *ramification nodes* $V_r = \{v_i \in V \,|\, \gamma_i \geq 2\}$, especially, we define the *essential ramification nodes* by $V_{\mathrm{ess}} = \{v_i \in V \,|\, \gamma_i \geq 3\}$. The orientation of Γ or the network is given by the incidence factors

$$d_{ij} = \begin{cases} 1 & \text{if } \pi_j(1) = v_i, \\ -1 & \text{if } \pi_j(0) = v_i, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

For a function $u : G \to \mathbb{R}$ we set $u_j := u \circ \pi_j : [0,1] \to \mathbb{R}$ and use the abbreviations

$$u_j(v_i) := u_j(\pi_j^{-1}(v_i)), \quad \partial_j u_j(v_i) := \frac{\partial}{\partial t_j} u_j(t_j)\Big|_{\pi_j^{-1}(v_i)} \quad \text{etc.}$$

As the basic geometric transition condition at ramification nodes we impose the *continuity condition*

$$\forall v_i \in V_r \,:\, e_j \cap e_s = \{v_i\} \implies u_j(v_i) = u_s(v_i), \tag{4}$$

that clearly is contained in the condition $u \in \mathcal{C}(G)$. Moreover, at all vertices we impose the classical Kirchhoff condition

$$\forall i \in \mathbb{N} \,:\, \sum_{j\in\mathbb{N}} d_{ij} \partial_j u_j(v_i) = 0, \tag{5}$$

that includes the Neumann boundary condition at boundary vertices. Note that Condition (5) does not depend on the orientation.

For the sake of simplicity, we shall use the following notations for the point spectra and the geometric multiplicities.

Definition 2.1

$$\begin{aligned}
\ell^\infty(\Gamma) &= \ell^\infty(V(\Gamma)) \\
\mathbf{S}(G) &= \sigma_p(-\Delta_G^K, \mathcal{C}_K^2(G)\cap L^\infty(G)) \\
\mathbf{s}(\mathcal{Z},\Gamma) &= \sigma_p(\mathcal{Z}, \ell^\infty(\Gamma)) \\
M(\lambda) &= M(\lambda; G) = m_g(\lambda, -\Delta_G^K, \mathcal{C}_K^2(G)\cap L^\infty(G)) \\
m(\mu,\mathcal{Z}) &= m_g(\mu, \mathcal{Z}, \ell^\infty(\Gamma))
\end{aligned}$$

3 Incidence and circuits

Using the factors (3), the orientation of the graph Γ is given by the *incidence matrix* or *incidence operator*

$$\mathcal{D}(\Gamma) = (d_{ik})_{i,k\in\mathbb{N}} : \mathbb{R}^{E(\Gamma)} \longrightarrow \mathbb{R}^{V(\Gamma)}. \tag{6}$$

By (1), $\mathcal{D}$ is a bounded operator

$$\mathcal{D}(\Gamma) : \ell^\infty(E(\Gamma)) \longrightarrow \ell^\infty(V(\Gamma)). \tag{7}$$

Definition 3.1 $\operatorname{corank}(\Gamma) = \dim\ker\mathcal{D}(\Gamma)$

By definition, a *circuit* ζ is a 2–regular connected graph. A finite ζ is a classical closed path, while in the infinite case ζ is the two–sided unbounded path Γ_1 with $V(\Gamma) = \mathbb{Z}$ and the adjacency relation

$$e_{ik} = 1 \iff |i-k| = 1.$$

In fact, Γ_1 can be considered as a combinatorial sphere of infinite radius. Accordingly, we define the *circuit space* of the graph Γ by

Definition 3.2 $\mathbf{\Pi}(\Gamma) = \big\langle c \in \ker\mathcal{D}(\Gamma) \,|\, \operatorname{supp}(c) \text{ is a circuit in } \Gamma\big\rangle$.

It is well known, see e.g. [10], that in the finite case, $\mathbf{\Pi}(\Gamma)$ is the classical circuit space and coincides with $\ker\mathcal{D}(\Gamma)$. In the infinite case, this is not true in general, and $\mathbf{\Pi}(\Gamma)$ can contain elements that have a support isomorphic to Γ_1.

Example 3.3 Let Γ consist in a finite graph Σ and exactly one one–sided unbounded path $\pi(\cong\mathbb{N})$. Then

$$\operatorname{corank}(\Gamma) = \operatorname{corank}(\Sigma).$$

This follows easily from the fact that $c \in \ker\mathcal{D}(\Gamma)$ must be constant on π, say c_0 and, thereby, using an appropriate numbering of the edges of Σ,

$$\mathcal{D}(\Sigma)c\big|_{E(\Sigma)} = \begin{pmatrix} c_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

As $0 = \mathbf{e}^*\mathcal{D}(\Sigma)c\big|_{E(\Sigma)} = c_0$, the kernels of $\mathcal{D}(\Gamma)$ and $\mathcal{D}(\Sigma)$ coincide. If, in addition, Σ is a circuit, then $\operatorname{corank}(\Gamma) = 1$.

Lemma 3.4

$$\operatorname{corank}(\Gamma) < \infty \iff \dim \mathbf{\Pi}(\Gamma) < \infty.$$

Proof. As $\mathbf{\Pi}(\Gamma) \leq \ker \mathcal{D}(\Gamma)$ the first implication is plain. If $\dim \mathbf{\Pi}(\Gamma) < \infty$, Γ consists in a finite graph Σ and a finite number of one–sided unbounded paths $\pi_1, \ldots, \pi_r$, since Γ cannot contain an infinite number of essentially disjoint copies of $\mathbb{Z}$. But, on each π_ρ, $c \in \ker \mathcal{D}(\Gamma)$ must be constant, say c_ρ. This leads to an equation

$$\mathcal{D}(\Sigma)c\big|_{E(\Sigma)} = v$$

with a finite vector v including the elements $\pm c_\rho$ and permits to conclude that $\operatorname{corank}(\Gamma)$ is bounded from above by $\operatorname{corank}(\Sigma) + r - 1$. ■

Lemma 3.5 *If* $\operatorname{corank}(\Gamma) < \infty$, *then*

$$\mathbf{\Pi}(\Gamma) = \ker \mathcal{D}(\Gamma).$$

Proof. Using the form of Γ and the notations of the preceding proof, we reason by induction on the number r. In the finite case the circuit space $\mathbf{\Pi}(\Gamma)$ coincides with $\ker \mathcal{D}(\Gamma)$, so we can conclude in the cases $r = 0$ and $r = 1$ with Example 3.3. For $r \geq 2$, find a two–sided unbounded path $\zeta \cong \mathbb{Z}$ such that π_{r-1} and π_r are subgraphs of ζ. Let d be the element in $\mathbf{\Pi}(\Gamma)$ that has ζ as its support and takes the constant value c_r there. By induction and 0–extension to π_r, $c - d$ is shown to belong to $\mathbf{\Pi}(\Gamma)$. ■

We note in passing, that for $\operatorname{corank}(\Gamma) < \infty$, $\ker \mathcal{D}(\Gamma)$ is a subspace of $\ell^\infty(E(\Gamma))$.

4 Laplacian and adjacency

The canonical Laplacian Δ on a $\mathcal{C}^2$–network G is defined as the operator

$$\Delta = \Delta_G^K = \left(u \mapsto \left(\partial_j^2 u_j\right)_{j \in \mathbb{N}}\right)$$

with the domain

$$\mathcal{C}_K^2(G) = \{u \in \mathcal{C}(G) \,|\, \forall j \in \mathbb{N} : u_j \in \mathcal{C}^2([0,1]),\, u \text{ satisfies } (5)\}$$

or in a weighted Sobolev space $\mathcal{H}_{K,c}^2(G)$. The determination of the point spectrum in $\mathcal{C}_K^2(G) \cap L^\infty(G)$ of the Laplacian is the main concern of the present investigation. In general, if Γ is infinite Δ has only a few eigenfunctions in $\mathcal{H}_K^2(G)$, e. g. those eigenfunctions stemming from zero extensions outside a finite subgraph admitting eigenfunctions that vanish at all its vertices adjacent to vertices outside it. But in $\mathcal{C}_K^2(G) \cap L^\infty(G)$ the point spectrum is not only always nonempty, but contains intervals of the real line. Moreover, for that space, $\sigma(\Delta, G) = \sigma_p(\Delta, G)$ in many cases, see e. g. Section 8.

First we note that there are no positive eigenvalues: Let $u \in \mathcal{C}_K^2(G)$ be an eigenfunction belonging to λ and $\zeta \in \mathcal{C}_K^\infty(G)$ be a cutting function that has a bounded support and coincides on a finite network S with u such that the derivatives of u and ζ have the same sign at points where ζ does not vanish. Then

$$\begin{aligned}\sum_{j\in\mathbb{N}}\int_0^1 \zeta_j\partial_j^2 u_j dt_j &= -\sum_{j\in\mathbb{N}}\int_0^1 \partial_j\zeta_j\partial_j u_j dt_j + \sum_{j\in\mathbb{N}}[\zeta_j\partial_j u_j]_0^1 \\ &= -\sum_{j\in\mathbb{N}}\int_0^1 \partial_j\zeta_j\partial_j u_j dt_j + \sum_{i\in\mathbb{N}}\zeta(v_i)\underbrace{\sum_{j\in\mathbb{N}} d_{ij}\partial_j u_j(v_i)}_{=0} \leq 0,\end{aligned}$$

that leads to the conclusion $\lambda \leq 0$. In view of the results in [4] - [6], the transition conditions considered here are natural and constitute a model case for other more general transition conditions. They are the natural generalization of the Neumann domain boundary condition and have the similar minimizing property in the finite network case, see [4].

The eigenvalue problem for $\Delta = \Delta_G^K$ in question reads

$$0 \neq u \in \mathcal{C}_K^2(G) \cap L^\infty(G) \quad \text{and} \quad \partial_j^2 u_j = -\lambda u_j \quad \text{for} \quad j \in N. \tag{8}$$

Following the transformations in [2], we formulate Problem (8) as an equivalent infinite matrix differential boundary eigenvalue problem incorporating the adjacency structure of the network. For that purpose we recall that the Hadamard product of matrices of the same size is defined as

$$(a_{ik}) \star (b_{ik}) = (a_{ik}b_{ik})\,.$$

For a function $u : G \to \mathbb{R}$ denote its value distribution in the nodes by

$$\mathbf{n}(u) = (u(v_i))_{i\in\mathbb{N}}\,, \tag{9}$$

and for $x \in [0,1]$ define the matrix

$$\mathbf{U}(x) = (u_{ih}(x))_{i,h\in\mathbb{N}}$$

by

$$u_{ih}(x) = e_{ih}u_{s(i,h)}\left(\frac{1+d_{is(i,h)}}{2} - xd_{is(i,h)}\right). \tag{10}$$

Thus, by denoting $\mathbf{n}(u) = \varphi$, $\mathbf{U}(0)$ has the form

$$\mathbf{U}(0) = \underbrace{\begin{pmatrix} \varphi_0 & \varphi_0 & \cdots & \varphi_0 & \cdots \\ \varphi_1 & \varphi_1 & \cdots & \varphi_1 & \cdots \\ \vdots & \vdots & & \vdots & \vdots \\ \varphi_i & \varphi_i & \cdots & \varphi_i & \cdots \\ \vdots & \vdots & & \vdots & \vdots \end{pmatrix}}_{\varphi\,\mathbf{e}^*} \star \mathcal{A},$$

and Problem (8) becomes equivalent to the following differential boundary eigenvalue problem (11)–(16) for the matrix $\mathbf{U}$:

$$u_{ih} \in C^2([0,1]) \qquad \text{for all } i,h \in \mathbb{N} \tag{11}$$

$$e_{ih} = 0 \Rightarrow u_{ih} = 0 \qquad \text{for all } i,h \in \mathbb{N} \tag{12}$$

$$\mathbf{U}'' = -\lambda \mathbf{U} \qquad \text{in } [0,1] \tag{13}$$

$$\mathbf{U}(0) = \varphi\, \mathbf{e}^* \star \mathcal{A} \qquad \text{(continuity in } V_r(\Gamma)) \tag{14}$$

$$\mathbf{U}^*(x) = \mathbf{U}(1-x) \qquad \text{for } x \in [0,1] \tag{15}$$

$$\mathbf{U}'(0)\ \mathbf{e} = 0 \qquad \text{(K)} \tag{16}$$

We set

$$\Phi := \mathbf{U}(0) = \varphi\, \mathbf{e}^* \star \mathcal{A}, \qquad \Psi := \mathbf{U}'(0),$$

and recall the following elementary rules for a matrix M:

$$(M \star \mathbf{e}\,\varphi^*)\ \mathbf{e} = M\varphi \qquad\qquad (M \star \varphi\ \mathbf{e}^*)\ \mathbf{e} = \mathrm{Diag}(M\,\mathbf{e})\varphi. \tag{17}$$

Finally, introduce the row–stochastic *transition* matrix

$$\mathcal{Z} = \mathrm{Diag}\left(\mathcal{A}(\Gamma)\ \mathbf{e}\right)^{-1} \mathcal{A}(\Gamma) = \mathrm{Diag}_{i\in\mathbb{N}}\left(\gamma_i^{-1}\right) \mathcal{A}(\Gamma), \tag{18}$$

that plays a key role in the characterization of the spectrum of the Laplacian. Note that the stochastic and indecomposable operator $\mathcal{Z}$ is compact iff Γ is finite, and that $\mathcal{A}$ is bounded iff Γ is uniformly locally finite, owing to a result of Mohar [19]. But the operator $\mathcal{Z}$ is bounded for any locally finite graph with operator norm $||\mathcal{Z}||_\infty = 1$. The general link between the spectra of $\mathcal{Z}$ and the Laplacian is contained in the following

Theorem 4.1 *If λ is an eigenvalue of $-\Delta_G^K$ in $\mathcal{C}_K^2(G) \cap L^\infty(G)$ and $\varphi \in \ell^\infty(V(\Gamma))$ a node distribution of an eigenfunction belonging to λ, then*

$$\mathcal{Z}\varphi = \cos\sqrt{\lambda}\,\varphi. \tag{19}$$

Conversely, if $\cos\sqrt{\lambda}$ is an eigenvalue of $\mathcal{Z}$ admitting the eigenvector $\varphi \in \ell^\infty(V(\Gamma))$, then λ is an eigenvalue of $-\Delta_G^K$ in $\mathcal{C}_K^2(G) \cap L^\infty(G)$ and φ the node distribution of some eigenfunction belonging to λ.

The proof of this theorem will be given by the more precise results in the next two sections. It should be noted that, in general, the eigenvalues of Problem (8) for an infinite network are not given by $[0,\infty)$, but are the elements of a countable union of intervals of the real nonnegative half line and in general, complicated to be determined.
We end this section with some examples. First we consider the one–sided unbounded path Γ_0 with $V(\Gamma_0) = \mathbb{N}$ and the adjacency relation

$$e_{ik} = 1 \iff |i-k| = 1 \quad \text{for} \quad i,k \in \mathbb{N}.$$

For each $\lambda \in [0,\infty)$, and for each φ_0 at the boundary vertex, there is a unique solution u of (8). This shows that each $\lambda \in [0,\infty)$ is a simple eigenvalue of $-\Delta$ in $\mathcal{C}_K^2(\Gamma_0) \cap L^\infty(\Gamma_0)$.

Example 4.2 The left graph κ in Figure 1, a "kite", has -1 as a simple eigenvalue for its adjacency matrix. If we connect s kites to the two–sided unbounded path Γ_1 as indicated in the right graph in Figure 1, we obtain a network for which each $\lambda > 0$ with $\cos\sqrt{\lambda} = -\frac{1}{3}$ is an eigenvalue of $-\Delta$ with $s \leq m_g(\lambda) \leq s+2$. The precise multiplicity depends on the position of the kites, since by a support analysis of the vector φ and according to Theorem 4.1, the components of φ correspond to an eigenvector belonging to $-\frac{1}{3} \in \mathrm{s}(\mathcal{Z},\kappa)$, while to such one belonging to $-\frac{1}{2}$ in $\mathrm{s}(\mathcal{Z},\Gamma_1)$.

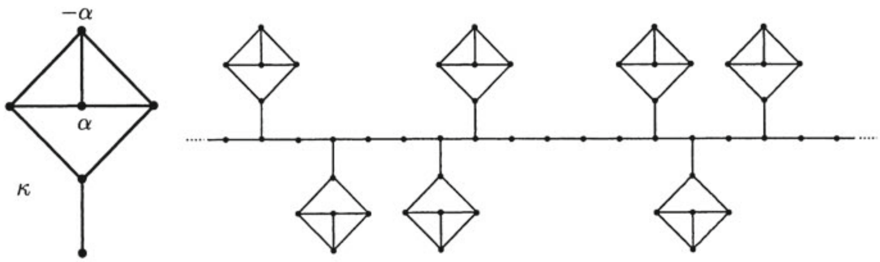

Figure 1: The graph of example 4.2

5 Network immanent eigenvalues

For $\lambda > 0$, a fundamental solution of (13) is given by

$$\mathbf{U}(x) = \cos(x\sqrt{\lambda})\Phi + \frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}}\Psi. \tag{20}$$

In the case $\sin\sqrt{\lambda} \neq 0$, (15) and (16) yield

$$\mathbf{U}(1) = \Phi^* = \Phi\cos\sqrt{\lambda} + \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}\Psi,$$

$$\Psi = \frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}}\left(\mathbf{e}\varphi^* - \cos\sqrt{\lambda}\,\varphi\,\mathbf{e}^*\right) \star \mathcal{A},$$

$$(\mathcal{A} \star \mathbf{e}\varphi^*)\,\mathbf{e} - \cos\sqrt{\lambda}\,(\mathcal{A} \star \varphi\,\mathbf{e}^*)\,\mathbf{e} = 0,$$

and using (17), we are led to the following characteristic equation

$$\mathcal{A}(\Gamma)\varphi = \cos\sqrt{\lambda}\,\mathrm{Diag}_i\,(\gamma_i)\,\varphi. \tag{21}$$

Moreover, we immediately obtain

$$u \in L^\infty(G) \iff \varphi \in \ell^\infty(\Gamma). \tag{22}$$

Thus we can state the following

Theorem 5.1 *Suppose $\lambda > 0$ and $\sin\sqrt{\lambda} \neq 0$. Then λ is an eigenvalue of $-\Delta_G^K$ in $\mathcal{C}_K^2(G) \cap L^\infty(G)$ with node distribution φ of an eigenfunction iff φ is an eigenvector belonging to the eigenvalue $\cos\sqrt{\lambda}$ of the row–stochastic matrix $\mathcal{Z}$ in (18). More precisely,*

$$M(\lambda) = m(\cos\sqrt{\lambda}, \mathcal{Z}).$$

Proof. In fact, the mapping $\mathbf{n}$ defined in (9) establishes an isomorphism between the vector spaces $\ker\left(-\Delta - \lambda I; \mathcal{C}_K^2(G) \cap L^\infty(G)\right)$ and $\ker\left(-\mathcal{Z} - \cos\sqrt{\lambda} I; \ell^\infty(\Gamma)\right)$. As $\sin\sqrt{\lambda} \neq 0$, a vanishing vector φ can only belong to the zero function on the network. Moreover, for the same reason, for a given $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \ell^\infty(V(\Gamma))$ satisfying $\mathcal{Z}(\Gamma)\varphi = \cos\sqrt{\lambda}\varphi$, there exists for each edge e_j a unique $u_j \in \mathcal{C}^2([0,1])$ fulfilling

$$\partial_j^2 u_j = -\lambda u_j \ \text{ in } \ [0,1] \ \text{ and } \ u_j(v_i) = \varphi_i, \ u_j(v_k) = \varphi_k \ \text{ for } \ e_j \cap V(\Gamma) = \{v_i, v_k\}.$$

In this way, a unique continuous function u on G is defined whose matrix U is a solution of (11)–(16). ∎

As an immediate consequence of Theorem 5.1 we obtain the

Corollary 5.2 *Suppose $\lambda > 0$ and $\cos\sqrt{\lambda} = 0$. Then λ is an eigenvalue of $-\Delta_G^K$ in $\mathcal{C}_K^2(G) \cap L^\infty(G)$ iff the node distributions of all corresponding eigenfunctions in $\mathcal{C}_K^2(G) \cap L^\infty(G)$ belong to $\ker\mathcal{A}(\Gamma) \cap \ell^\infty(V(\Gamma))$. Moreover $M(\lambda) = m(0, \Gamma)$.*

6 Single branch eigenvalues

For $\sin\sqrt{\lambda} = 0$, the network displays eigenvalues that also occur on single edges under Neumann or homogeneous Dirichlet boundary conditions, while the eigenvalues of Section 5 cannot belong to those.

For $\lambda = 0$, we are lead to harmonic functions on G, among which there are always the constant functions. Any solution $\mathbf{U}$ of (11)–(16) is of the form

$$\mathbf{U}(x) = \Phi + x\,(\Phi^* - \Phi) \quad \text{ and } \ (\Phi^* - \Phi)\,\mathbf{e} = 0,$$

which implies with (17) the following

Theorem 6.1 *A function $0 \neq u \in \mathcal{C}_K^2(G) \cap L^\infty$ is harmonic in the network Γ iff $\varphi \in \ell^\infty(\Gamma)$ and $\mathcal{Z}\,\varphi = \varphi$. Moreover,*

$$M(0) = m(1, \mathcal{Z}) \geq 1. \tag{23}$$

Note that in general, the multiplicity does not amount to 1 and can often be infinite. For more details and conditions under which the network is a Liouville space see [8]. But in the finite case, 0 is a simple eigenvalue by the Perron–Frobenius–Theorem, see [2].

For $\lambda > 0$ and $\sin\sqrt{\lambda} = 0$, we first observe that by (10) and (20)

$$u \in L^\infty(G) \iff \varphi \in \ell^\infty(\Gamma)\,\&\,\Psi \in \ell^\infty(V(\Gamma) \times V(\Gamma)). \tag{24}$$

The multiplicities are determined by combinatorial characteristics as in the finite case, see Section 5 in [2].

Theorem 6.2 *Suppose $\lambda > 0$, $\sin\sqrt{\lambda} = 0$, and $\cos\sqrt{\lambda} = 1$. Then λ is an eigenvalue of $-\Delta_G^K$ in $\mathcal{C}_K^2(G) \cap L^\infty(G)$ with node distribution $\varphi \in \mathbb{R}\mathbf{e}$ of all eigenfunctions belonging to λ and*

$$M(\lambda) = \operatorname{corank}(\Gamma) + 1.$$

Proof. Note first that $\varphi \in \mathbb{R}\mathbf{e} \iff \Phi = \Phi^*$ and that $\mathbf{U}(x) = \cos(x\sqrt{\lambda})\,\mathbf{e}\,\mathbf{e}^* \star \mathcal{A}$ is a solution of (11)–(16). All remaining eigensolutions are of the form $\frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}}\Psi$ with a matrix Ψ belonging to

$$\mathcal{M}^+(\Gamma) := \{\Psi \in \mathcal{M}(\Gamma) \,|\, \Psi^* = -\Psi,\ \Psi\,\mathbf{e} = 0\},$$

where

$$\mathcal{M}(\Gamma) := \left\{\Psi \,|\, \Psi = (\psi_{ih})_{i,h\in\mathbb{N}},\ \forall i,h \in \mathbb{N}:\ e_{ih} = 0 \Rightarrow \psi_{ih} = 0\right\}.$$

The mapping

$$h_1 = \left(x \longmapsto \left(e_{ih}\, d_{is(i,h)}\, x_{s(i,h)}\right)_{i,h\in\mathbb{N}}\right)$$

defines a monomorphism $h_1 : \mathbb{R}^{E(\Gamma)} \to \mathcal{M}(\Gamma)$ that establishes an isomorphism between $\ker\mathcal{D}(\Gamma)$ and $\mathcal{M}^+(\Gamma)$, since for $\Psi \in \mathcal{M}^+(\Gamma)$, $x_{s(i,h)} = \psi_{ih}\, d_{is(i,h)}$ defines $x \in \mathbb{R}^{E(\Gamma)}$ such that $h_1(x) = \Psi$. Moreover, $\Psi\,\mathbf{e} = 0$ implies $x \in \ker\mathcal{D}(\Gamma)$.

For $\operatorname{corank}(\Gamma) < \infty$, $\ker\mathcal{D}(\Gamma)$ belongs to $\ell^\infty(E(\Gamma))$, and $\ker\mathcal{D}(\Gamma) \cong M^+(\Gamma)$ permits to conclude.

For $\operatorname{corank}(\Gamma) = \infty$, Lemma 3.4 implies $\dim\mathbf{\Pi}(\Gamma) = \infty$. But along any circuit or infinite 2–regular subgraph of G, ζ, one easily defines an eigenfunction belonging to λ that vanishes at all vertices of ζ. Its zero extension to G leads to a matrix Ψ with $h_1^{-1}(\Psi) \in \ker\mathcal{D}(\Gamma)$. Thus the geometric multiplicity is also infinite. ∎

Theorem 6.3 *Suppose $\lambda > 0$, $\sin\sqrt{\lambda} = 0$, and $\cos\sqrt{\lambda} = -1$. If λ is an eigenvalue of $-\Delta_G^K$ in $\mathcal{C}_K^2(G) \cap L^\infty(G)$ then the node distribution φ of any eigenfunction belonging to λ is alternating along any finite circuit in G unless it vanishes. The geometric multiplicities are*

$$M(\lambda) = \begin{cases} \operatorname{corank}(\Gamma) + 1 & \text{if } \Gamma \text{ is bipartite,} \\ \operatorname{corank}(\Gamma) - 1 & \text{if } \Gamma \text{ is not bipartite.} \end{cases} \tag{25}$$

Proof. For $\cos\sqrt{\lambda} = -1$, (20) leads to $\mathbf{U}(0) = \Phi = -\Phi^*$. Thus a node distribution φ of an eigenfunction does not vanish iff Γ is bipartite. In that case φ is alternating along any finite circuit, and for such a vector, $\mathbf{U}(x) = \cos(x\sqrt{\lambda})\,\varphi\,\mathbf{e}^* \star \mathcal{A}$ is a solution of (11)–(16). All remaining eigensolutions are of the form $\frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}}\Psi$ with a matrix Ψ belonging to

$$\mathcal{M}^-(\Gamma) := \{\Psi \in \mathcal{M}(\Gamma) \,|\, \Psi^* = \Psi,\ \Psi\,\mathbf{e} = 0\}.$$

First, let us suppose that Γ is bipartite. Then Γ can be orientated such that each node is either a sink or a source, and

$$h_2 = \left(x \longmapsto \left(e_{ih}\ x_{s(i,h)}\right)_{i,h\in\mathbb{N}}\right)$$

defines an isomorphism $\ker \mathcal{D}(\Gamma) \cong \mathcal{M}^-(\Gamma)$. An argument analogue to the one in the proof of Theorem 6.2 permits to conclude.

Next, we suppose that Γ is not bipartite and of finite corank. If $\mathrm{corank}(\Gamma) = 1$, then Γ contains exactly one odd finite circuit ζ_0 and $\mathbf{\Pi}(\Gamma) = \langle \zeta_0 \rangle$. Thus Γ is of the form 3.3 and $M(\lambda) = 0$. Reasoning inductively on $\mathrm{corank}(\Gamma) > 1$, we have to find an edge $e \in E(\Gamma)$ and $\Psi_0 \in \mathcal{M}^-(\Gamma)$ such that omitting e, the graph

$$\Theta = (V(\Gamma), E(\Gamma)\backslash\{e\}) \tag{26}$$

is connected and either is not bipartite or contains two–sided unbounded paths Γ_1 and, furthermore,

$$\mathcal{M}^-(\Gamma) = \mathbb{R}\,\Psi_0 \oplus \mathcal{M}^-(\Theta). \tag{27}$$

If Γ contains an even finite circuit or contains other odd finite circuits, then we can reason as in [2]. Thus, the remaining case is when Γ contains as finite circuit only ζ_0 and a finite number of copies of Γ_1. Then we choose any $e \in E(\zeta_0)$ and define Ψ_0 on a two–sided unbounded path containing e. This leads to the desired decomposition (26) and (27) and $\mathrm{corank}(\Theta) = \mathrm{corank}(\Gamma) - 2$ permits to conclude.

Finally, suppose that Γ is not bipartite and of infinite corank. Again Lemma 3.4 implies $\dim \mathbf{\Pi}(\Gamma) = \infty$, and, as in the proof of Theorem 6.2, by constructing infinitely many linearly vertex independent eigenfunctions with supports on independent circuits, the geometric multiplicity is shown to be infinite. ■

Note that the multiplicities formulae coincide with those in the finite case [2]. There it has also be shown that a finite graph Γ is bipartite iff $-1 \in \sigma(\mathcal{Z})$. For the infinite case, this is no longer true. Take e.g. a 3–regular tree in which one node is replaced by a triangle such that the resulting graph B is also 3–regular. Then B is not bipartite, but $m(-1, \mathcal{Z}) = \infty$ as well as for all $\lambda \in \mathbf{S}(G)$ with $\cos\sqrt{\lambda} = -1$.

Example 6.4 The corank of the graph in Figure 2 amounts to $1 + s$, where s is the number of squares in it. Thus, for any $\lambda > 0$ with $\sin\sqrt{\lambda} = 0$, $M(\lambda) = s + 2$.

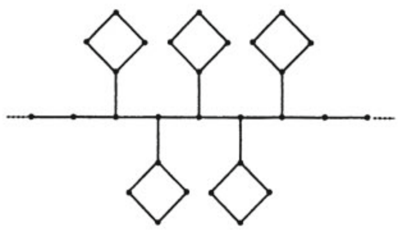

Figure 2: The graph of example 6.4

As a first general application we treat the case of finitely many essential ramification nodes.

Theorem 6.5 *Let G be a uniformly locally finite infinite network that either is bipartite or has corank at least 2. If* card $V_{\mathrm{ess}}(\Gamma) < \infty$, *then* $\mathbf{S}(G) = [0, \infty)$.

Proof. By Theorems 6.1, 6.2 and 6.3, all $\lambda \geq 0$ with $\sin\sqrt{\lambda} = 0$ belong to $\mathbf{S}(G)$. By Theorem 4.1, it suffices to show that for $\sin\sqrt{\lambda} \neq 0$, $\mu = \cos\sqrt{\lambda}$ is an eigenvalue of $\mathcal{Z}(\Gamma)$. By hypothesis, Γ consists in a finite network S and a finite number of one–sided unbounded paths $\pi_1, \ldots, \pi_r$, each isomorphic to Γ_0. Each of the paths π_i is linked with S at a node w_i with $\gamma(w_i, \Gamma) = 2$ and $\gamma(w_i, \pi_i) = 1$. By extending S if necessary, for each w_i we can assume that the neighbor z_i of w_i in S satisfies $\gamma(z_i, S) = 2$. Now, define a nonzero vector $\tilde{\varphi} \in \mathbb{R}^{V(S)}$ as follows: If $\mu \in \sigma(\mathcal{Z}(S))$, then $\tilde{\varphi}$ is an eigenvector in $\mathcal{Z}(S)$ of μ. If $\mu \notin \sigma(\mathcal{Z}(S))$, then set $\tilde{\varphi} = (\mathcal{Z}(S) - \mu I)^{-1} e_b$, where $e_b \in \mathbb{R}^{V(S)}$ is the vector that takes the value 1 at each w_i and vanishes elsewhere. In both cases, $\tilde{\varphi}$ can be extended to an eigenvector $\varphi \in \ell^\infty(\Gamma)$ of $\mathcal{Z}(\Gamma)$ belonging to μ by setting on each π_i with $V(\pi_i) \sim \mathbb{N}$

$$\varphi_{ik} = c_{i1} a_1(2\mu)^k + c_{i2} a_2(2\mu)^k \qquad (k \in \mathbb{N})$$

with

$$c_{i1} + c_{i2} = \varphi(z_i), \quad c_{i1} a_1(2\mu) + c_{i2} a_2(2\mu) = \varphi(w_i),$$

where

$$a_1(x) = \frac{x}{2} + \sqrt{\frac{x^2}{4} - 1}, \qquad a_2(x) = \frac{x}{2} - \sqrt{\frac{x^2}{4} - 1}. \tag{28}$$

This achieves the proof. ∎

7 Regular networks

If Γ is regular of degree γ, then (19) reduces to the characteristic equation

$$\mathcal{A}\varphi = \gamma \cos\sqrt{\lambda}\,\varphi, \tag{29}$$

and the results of Sections 5 and 6 yield immediately the

Corollary 7.1 *Let Γ be a regular graph of degree γ. Then, a non vanishing sequence $\varphi \in \ell^\infty(\Gamma)$ is the node distribution of an eigenfunction of $-\Delta_G^K$ in $C_K^2(G) \cap L^\infty(G)$ iff it is a bounded eigenvector of the adjacency matrix of Γ. Moreover, a nonnegative real number λ is an eigenvalue of $-\Delta_G^K$ in $C_K^2(G) \cap L^\infty(G)$ iff $\mu = \gamma\cos\sqrt{\lambda}$ is an eigenvalue of $\mathcal{A}$ in $\ell^\infty(\Gamma)$ or* $\cos\sqrt{\lambda} = -1$. *The geometric multiplicities fulfill*

$$M(\lambda) = \begin{cases} m(\gamma, \mathcal{A}) & \text{if } \lambda = 0, \\ m(\gamma\cos\sqrt{\lambda}, \mathcal{A}) & \text{if } \sin\sqrt{\lambda} \neq 0, \\ \operatorname{corank}(\Gamma) + 1 & \text{if } \cos\sqrt{\lambda} = 1, \\ \operatorname{corank}(\Gamma) + 1 & \text{if } \cos\sqrt{\lambda} = -1 \text{ and } \Gamma \text{ is bipartite}, \\ \operatorname{corank}(\Gamma) - 1 & \text{if } \cos\sqrt{\lambda} = -1 \text{ and } \Gamma \text{ is not bipartite}. \end{cases} \tag{30}$$

Moreover, as the odd circuits are precisely the regular graphs with $M(\lambda) = 0$ for $\cos\sqrt{\lambda} = -1$, we conclude

Corollary 7.2 *Let Γ be a regular graph of degree γ. If Γ is bipartite or an odd circuit, then $\gamma \cos\sqrt{\mathbf{S}(G)} = \mathbf{s}(\mathcal{A}, \Gamma)$. In all other cases $\gamma \cos\sqrt{\mathbf{S}(G)} = \mathbf{s}(\mathcal{A}, \Gamma) \cup \{-\gamma\}$.*

Example 7.3 Let Γ_1 be the two–sided unbounded path with $V(\Gamma) = \mathbb{Z}$ as introduced in Section 3. Let $\mu \in \mathbf{s}(\mathcal{A}, \Gamma_1)$. Then the node sequence $\mathbf{x}$ of an eigenvector of μ obeys the recurrence $x_{i+1} + x_{i-1} = \mu x_i$. It has no bounded solution for $\mu^2 > 4$. For $\mu = 2$ the only bounded solutions are $x_i = c$ with c some constant number. For $\mu = -2$, the same result holds with $x_i = (-1)^i c$. For $\mu^2 < 4$ there are two independent bounded sequences due to (28) that lead to the solutions $\cos(k\beta)$ and $\sin(k\beta)$, where $\beta = \arctan\sqrt{4/\mu^2 - 1}$. So $\mathbf{s}(\mathcal{A}, \Gamma_1) = [-2, 2]$, with multiplicities 2 for $-2 < \mu < 2$ and 1 for $\mu = \pm 2$, and therefore

$$\mathbf{S}(\Gamma_1) = [0, \infty)$$

with multiplicity 1 for $\lambda = n^2\pi^2$, where $n \in \mathbb{Z}$, and multiplicity 2 otherwise. Note further that this implies $\sigma(-\Delta, \mathcal{C}_K^2(\Gamma_1) \cap L^\infty(\Gamma_1)) = [0, \infty)$.

Example 7.4 The example of Γ_1 can be extended to the infinite m-dimensional grid generated by the unit cube in $\mathbb{R}^m$: Let Γ_m denote the $2m$-regular graph in $\mathbb{R}^m$ with $V(\Gamma_m) = \mathbb{Z}^m$ and the edges defined by the adjacency

$$e_{zw} = 1 \iff \sum_{j=1}^{m} |z_j - w_j| = 1.$$

This includes as a special case for $m = 2$ the regular graph $\mathbf{K}_1$ belonging to the Keplerian plane tiling with squares. Γ_m is a regular graph of valency $2m$ and the spectrum of the adjacency operator in $\ell^\infty(\mathbb{Z}^m)$ is $[-2m, 2m]$, see [5]. Thus

$$\mathbf{S}(\Gamma_m) = [0, \infty).$$

The multiplicities can be finite and infinite, take e.g. $m(0) = 1$, see [8], while $M(\lambda) = \infty$ for $\cos\sqrt{\lambda} = 0$, see Section 9. Again we note that $\sigma(-\Delta, \mathcal{C}_K^2(\Gamma_m) \cap L^\infty(\Gamma_m)) = [0, \infty)$.

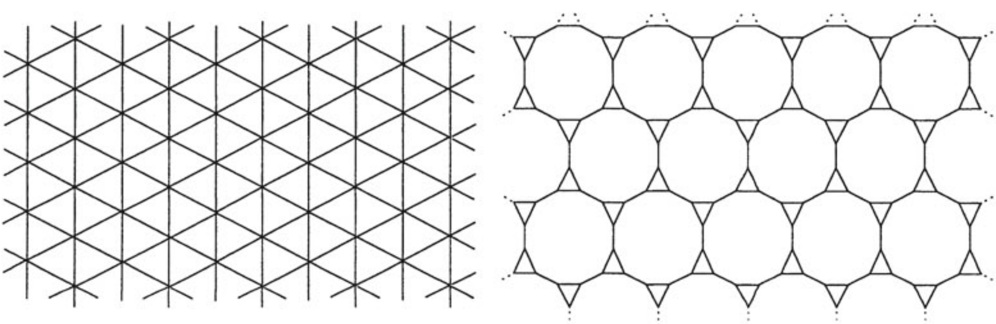

Figure 3: The plane tilings $\mathbf{K}_3$ and $\mathbf{K}_5$

Example 7.5 Consider the graph $\mathbf{K}_3$ formed by Kepler's plane tiling with regular triangles, see Figure 3. $\mathbf{K}_3$ is a regular graph of valency 6 and the spectrum of the adjacency operator in $\ell^\infty(V(\mathbf{K}_3))$ is $[-3,6]$, see [5]. Thus

$$\mathbf{S}(\mathbf{K}_3) = \{\lambda \in [0,\infty) |\ \cos\sqrt{\lambda} \in [-1/2, 1] \cup \{-1\}\}.$$

Again, the multiplicities can be finite and infinite, take e.g. $M(0) = 1$, see [8], while $M(\lambda) = \infty$ for $6\cos\sqrt{\lambda} = -2$, see Section 9. Note that the singletons $(2k+1)^2\pi^2$ can only have eigenfunctions with $\mathbf{n}(u) = 0$. The square roots of the eigenvalues of $-\Delta$ in $\mathcal{C}^2_K(\mathbf{K}_3) \cap L^\infty(\mathbf{K}_3)$ are displayed by bold lines and points in the left part of Figure 4.

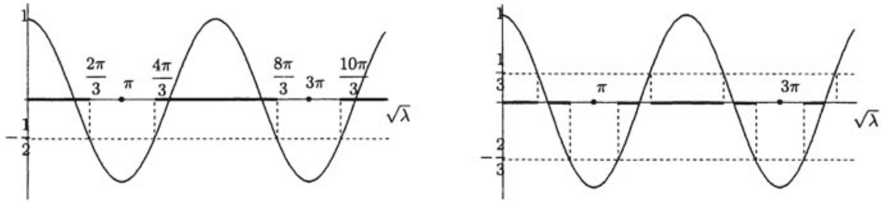

Figure 4: The point spectrum of $-\Delta$ on $\mathbf{K}_3$ and $\mathbf{K}_5$

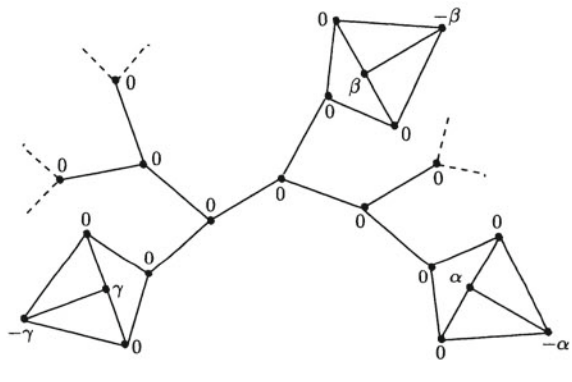

Figure 5: Example 7.7

Example 7.6 Consider the graph $\mathbf{K}_5$ formed by Kepler's plane tiling with regular triangles and dodecagons, see Figure 3. $\mathbf{K}_5$ is a regular graph of valency 3 and the spectrum of the adjacency operator in $\ell^\infty(V(\mathbf{K}_5))$ is $[-2,0] \cup [1,3]$, see [7], [14]. Thus

$$\mathbf{S}(\mathbf{K}_5) = \{\lambda \in [0,\infty) |\ \cos\sqrt{\lambda} \in [-2/3, 0] \cup [1/3, 3] \cup \{-1\}\}.$$

The square roots of the eigenvalues of $-\Delta$ in $\mathcal{C}^2_K(\mathbf{K}_5) \cap L^\infty(\mathbf{K}_5)$ are displayed by bold lines and points in the right part of Figure 4.

Example 7.7 If a 3–regular network contains s subgraphs isomorphic to a kite as in Figure 5, then each $\lambda > 0$ with $\cos\sqrt{\lambda} = -\frac{1}{3}$ is an eigenvalue of multiplicity at least s, since the -1 is a simple eigenvalue of the adjacency matrix of each kite with the indicated support.

8 Trees

Trees display special spectral properties. First let us discuss the regular case and let $\mathbb{T}_\gamma$ denote the γ–regular tree.

Theorem 8.1 *For* $\gamma \geq 2$

$$\sigma\left(\mathcal{A}(\mathbb{T}_\gamma), \ell^\infty\right) = \mathbf{s}\left(\mathbb{T}_\gamma\right) = [-\gamma, \gamma]$$

and

$$\sigma\left(-\Delta, \mathcal{C}_K^2(\mathbb{T}_\gamma) \cap L^\infty(\mathbb{T}_\gamma)\right) = \mathbf{S}\left(\mathbb{T}_\gamma\right) = [0, \infty).$$

Proof. It is well known that $\sigma\left(\mathcal{A}(\mathbb{T}_\gamma), \ell^\infty\right) \subset [-\gamma, \gamma]$. The second assertion is an application of Corollary 7.2 and the first one in the case of $\Gamma_1 = T^2$ is shown in Example 7.3. For $\gamma \geq 3$, we construct spectral embeddings

$$\iota : \mathbf{s}(\Gamma_1) \to \mathbf{s}(\mathbb{T}_\gamma)$$

that associate to each eigenvalue μ of Γ_1 an eigenvalue $\iota(\mu)$ of $\mathbb{T}_\gamma$ such that the images of all ι cover the interval $[-\gamma, \gamma]$. Let $\mathbf{x} = (x_k)_{k \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ be an eigensequence of Γ_1 belonging to the eigenvalue μ:

$$\mu x_k = x_{k-1} + x_{k+1} \qquad (k \in \mathbb{Z})$$

Bearing in mind the tree character, we construct recursively a bounded eigenvector of $\mathbb{T}_\gamma$ with an arbitrary initial edge using the elements of $\mathbf{x}$. In the corresponding figures, black

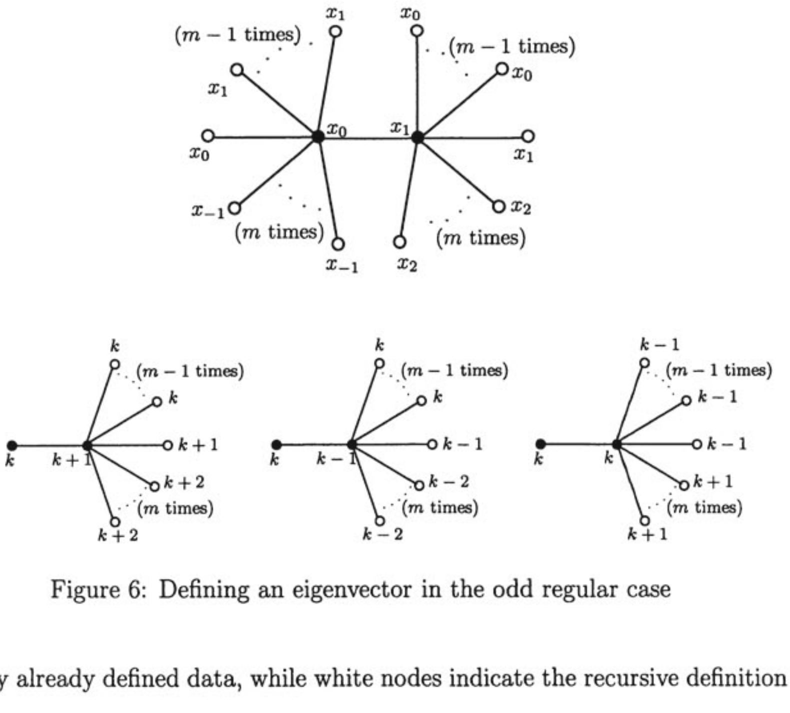

Figure 6: Defining an eigenvector in the odd regular case

nodes display already defined data, while white nodes indicate the recursive definition of the data. Occasionally, for the sake of simplicity, only the index k of x_k is displayed . Since $\gamma \geq 3$, $\mathbb{T}_\gamma$ contains subgraphs isomorphic to Γ_1. First suppose that γ is odd:

$$\gamma = 2m + 1, \qquad m \in \mathbb{N}^*.$$

Choose an arbitrary edge and assign to its vertices the values x_0 and x_1. To their $2m$ remaining neighbors, assign once x_0, $m-1$ times x_1, m times x_{-1} and once x_1, $m-1$ times x_0, m times x_2 respectively as indicated in the upper part of Figure 6. On the remaining edges, assign the values according to the lower part of Figure 6. In this way, a bounded eigenvector of $\mathcal{A}(\mathbb{T}_\gamma)$ is defined belonging to the eigenvalue $\lambda = m\mu + 1$. Thus, we are led to the spectral embedding

$$\iota_1 = (\mu \mapsto m\mu + 1)$$

that yields $[-2m+1, \gamma] \subset \mathrm{s}(\mathbb{T}_\gamma)$. Using the mapping $\tau = (\mu \mapsto -\mu)$ in $\sigma(\Gamma_1)$ and the corresponding eigenvector transformation $\left((x_k)_{k\in\mathbb{Z}} \mapsto \left((-1)^k x_k\right)_{k\in\mathbb{Z}}\right)$, we are led to the spectral embedding $\iota_2 = \iota_1 \circ \tau$ that yields $[-\gamma, 2m-1] \subset \mathrm{s}(\mathbb{T}_\gamma)$. Now suppose that γ is even:

$$\gamma = 2m, \qquad m \in \mathbb{N}^*.$$

Choose an arbitrary edge and assign to its vertices the values x_0 and x_1. To their $2m-1$

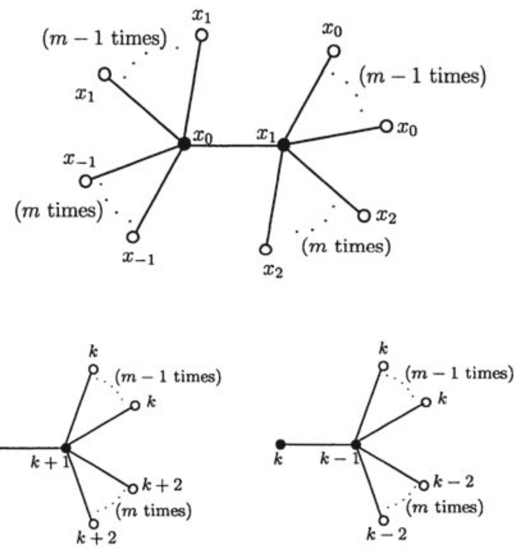

Figure 7: Defining an eigenvector in the even regular case

remaining neighbors, assign $m-1$ times x_1, m times x_{-1} and $m-1$ times x_0, m times x_2 respectively as indicated in the upper part of Figure 7. On the remaining edges, assign the values according to the lower parts of Figure 7. Note that in this construction, the corresponding indices of two adjacent vertices in $\mathbb{T}_\gamma$ can never be the same. In this way, a bounded eigenvector of $\mathcal{A}(\mathbb{T}_\gamma)$ is defined belonging to the eigenvalue $\lambda = m\mu$. Thus, we are led to the spectral embedding

$$\iota = (\mu \mapsto m\mu)$$

that yields $[-\gamma, \gamma] \subset \mathbf{s}(\mathbb{T}_\gamma)$. ∎

But, if there are boundary vertices, then the spectrum may display holes. In the case $\cos\sqrt{\lambda} = 0$, λ needs not to be an eigenvalue.

Example 8.2 Let Z_1 be the infinite tree obtained by the adding in Γ_1 to each vertex one edge such that all vertices of the resulting tree have valency 1 or 3 as depicted in Fig. 8. For $\cos\sqrt{\lambda} = 0$, an eigenfunction would have to have vanishing slopes at all boundary vertices, and thereby would have to vanish at all ramification nodes, which would imply $\sin\sqrt{\lambda} = 0$. Thus, there is no eigenfunction belonging to a value λ with $\cos\sqrt{\lambda} = 0$. In fact, it readily follows that $[0,\infty)\backslash\left(\frac{\pi}{2} + \pi\mathbb{Z}\right) = \mathbf{S}\,(Z_1)$. Of course, $\sigma(-\Delta_G^K, \mathcal{C}_K^2(G) \cap L^\infty(G)) = [0,\infty)$.

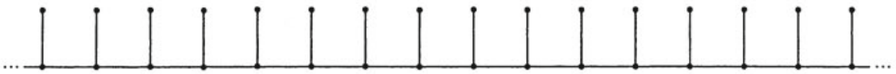

Figure 8: The tree Z_1 having no eigenvalue λ with $\cos\sqrt{\lambda} = 0$

In fact, we can fully characterize the trees admitting these eigenvalues. It is based on a particular property of the corresponding eigenfunctions u. At a boundary vertex v it satisfies the Neumann condition, while at the other vertex of the edge incident with v, u has to vanish, since it must be of the form $u = \varphi\cos(x\sqrt{\lambda})$. The supports of corresponding eigenfunctions bear special properties in trees.

Theorem 8.3 *Let T be a uniformly locally finite tree and $\cos\sqrt{\lambda} = 0$. Then a subtree Σ is the support of an eigenfunction in $\mathcal{C}_K^2(T) \cap L^\infty(T)$ iff Σ fulfills the following conditions:*

(I) $V_b(\Sigma) \subset V_b(T)$

(II) All distances between vertices in $V_b(\Sigma)$ are even.

(III) Adjacencies between vertices in $V(\Sigma)$ and $V(T)\backslash V(\Sigma)$ take only place at vertices in Σ having odd distance to all vertices in $V_b(\Sigma)$.

Thus, $\lambda \in \mathbf{S}(T)$ iff T possesses a subtree Σ satisfying (I)–(III). Moreover, $M(\lambda)$ is bounded from below by the number of vertex independent subgraphs satisfying (I)–(III).

Proof. First let us suppose that Σ is a connected subtree satisfying (I)–(III). We construct an eigenfunction u belonging to λ on T with $\mathrm{supp}(u) = \Sigma$ as follows. Endow Σ with the sink/source–orientation such that its boundary vertices are sources ($x = 0$). On each edge e_j of Σ, set $u_j = a_j\cos(x\sqrt{\lambda})$, with suitable values of a_j in order to fulfill the Kirchhoff condition at all ramification nodes of odd distance to V_b. Note that in V_b, the Neumann condition is fulfilled as well as the Kirchhoff condition at all ramification nodes of even distance to V_b. Moreover, u vanishes at all the vertices of odd distance to V_b. By (III) we can extend u to T with constant value zero .

Next suppose that $u \in \mathcal{C}_K^2(T) \cap L^\infty(T)$ is an eigenfunction belonging to λ and set $\Sigma :=$ $\mathrm{supp}(u)$. Then u cannot vanish identically on an edge in Σ or on some connected part of it. If $v \in \Sigma \cap V(T)$ and $\gamma(v, \Sigma) = 1$, then v must be a boundary vertex in T, since other incident edges in T with v would imply that u vanishes on the incident edge in Σ with v by continuity and by Condition (5) at v.

Since the derivatives at all boundary vertices in Σ vanish, u cannot vanish at those, and along the path in Σ between v_1 and v_2 belonging to $V_b(\Sigma)$, u vanishes at each second node. This shows that the path is of even length. Evidently, edges in T outside Σ can be incident to nodes in Σ only, if u vanishes there. Thus, this can only occur at vertices in Σ having odd distance to all vertices in $V_b(\Sigma)$.

The last two assertions follow immediately with the above construction and the first part. ∎

A similar phenomenon can be observed for $\cos(q\sqrt{\lambda}) = 0$ with $q \in \mathbb{Q}$, and certain classes of trees with respect to the boundary vertices cannot have these eigenvalues. Replace e.g. in Example 8.2 any edge by a path of length 2 and get a tree that does not have eigenvalues with $\cos(2\sqrt{\lambda}) = 0$. A corresponding characterization can be established, but we omit the details.

Nevertheless, in the case card $V_\mathrm{b}(T) = \infty$, it is possible that $\mathbf{S}(T) = [0, \infty)$, see Example 8.8. When the infinite tree has no boundary vertices, then the point spectrum is always $[0, \infty)$. This is part of the

Theorem 8.4 *Let T be a uniformly locally finite infinite tree with* card $V_\mathrm{b}(T) < \infty$. *Then* $\mathbf{S}(T) = [0, \infty)$.

Proof. In the case card $V_\mathrm{ess}(T) < \infty$ we can conclude with Theorem 6.5. Therefore we can assume card $V_\mathrm{ess}(T) = \infty$. Let B denote the smallest connected subtree of T with

$$V_b(T) \subset V(B).$$

Then B is finite, and there are infinitely many essential ramification nodes outside B. Set $u = 0$ on B and on all paths joining $V(B)$ with all nearest by essential ramification nodes outside B. We are led to a finite number of connected and mutually disjoint trees $T_1, \dots, T_r$ satisfying

$$V_b(T_i) = \emptyset \quad \text{for } 1 \le i \le r.$$

Thus, we have to show the assertion in the case $V_b(T) = \emptyset$ under the additional constraint that u vanishes at a ramification node $v_0 \in V_r(T)$.

Orientate T such that v_0 is a source and such that, at all other vertices,

$$\gamma_i^+ := \mathrm{card}\{j \in \mathbb{N} \,|\, d_{ij} = 1\} = 1,$$

$$\gamma_i^- := \mathrm{card}\{j \in \mathbb{N} \,|\, d_{ij} = -1\} = \gamma_i - 1.$$

For an initial step, choose slopes $\psi_1 \dots, \psi_{\gamma(v_0)} \in \mathbb{R}$ satisfying

$$\sum_{j=1}^{\gamma} \psi_j = 0, \quad \sum_{j=1}^{\gamma} |\psi_j| > 0.$$

Then there exists a unique $u \in \mathcal{C}^2(S)$ on the star S centered at v_0 that satisfies (5) at v_0 such that for each $1 \leq j \leq \gamma$,

$$\begin{cases} \partial_j^2 u_j = -\lambda u_j & \text{in } [0,1], \\ u(v_0) = 0, \\ d_{0j}\partial_j u_j(v_0) = \psi_j. \end{cases}$$

Then we extend u to the remaining tree by recurrence following the vertices in steps of the following form:

Suppose at v_{i-1} the value of u and its derivative at v_{i-1} on an incident edge e_1 with v_i are determined. Then the value at v_i and the derivative ψ_{i1} at v_i on e_1 are uniquely determined. Choose slopes $\psi_{i2} \dots, \psi_{i\gamma(v_i)} \in \mathbb{R}$ fulfilling

$$\psi_{i1} = \sum_{j=2}^{\gamma(v_i)} \psi_{ij}, \quad \psi_{ij} = K_{ij}\psi_{i1} \text{ with } 0 \leq K_{ij} \leq 1 \text{ for } 2 \leq j \leq \gamma(v_i). \tag{31}$$

Then there exists a unique $u \in \mathcal{C}^2(S_i)$ on the star S_i centered at v_i that satisfies (5) at v_i such that for each $1 \leq j \leq \gamma(v_i)$,

$$\begin{cases} \partial_j^2 u_j = -\lambda u_j & \text{in } [0,1], \\ u_j(v_i) = u_1(v_i), \\ d_{ij}\partial_j u_j(v_i) = \psi_j. \end{cases}$$

Due to the choice of the coefficients K_{ij} in (8), an easy calculation using (20) yields that the amplitude of the solutions u_j on the outgoing edges does not increase when compared to the one on the ingoing edge. In this way, a bounded eigenfunction belonging to λ is found on T, which achieves the proof.

■

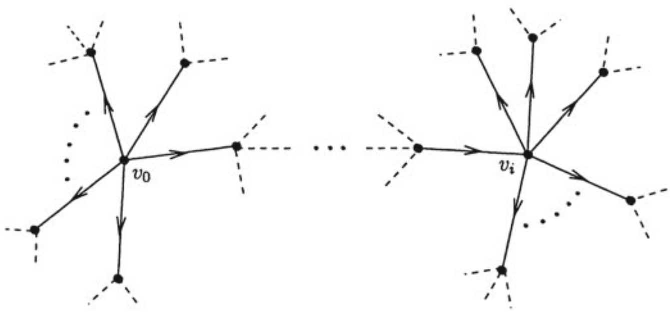

Figure 9: Defining recursively an eigenfunction

As for the multiplicities, one has to distinguish carefully the harmonic case from the strictly positive eigenvalues. In [8] the Liouville property for uniformly locally finite trees has been characterized by means of the reduced tree and the infimal incident length ratio. Here, we cite only the following special case.

Corollary 8.5 *Let T be a uniformly locally finite tree of minimal valency* 3 *and with all edges of equal length. Then*

$$M(0;T) = \infty \iff \text{card } V(T) = \infty. \tag{32}$$

and T is a Liouville network iff it has a finite number of vertices.

For nonzero eigenvalues, the geometric restriction guaranteeing the boundedness of harmonic functions is no longer necessary. In fact, we are led to the following

Theorem 8.6 *Let T be a uniformly locally finite tree with at most a finite number of boundary vertices. Then the following conditions are equivalent:*

(a) $\exists \lambda \in (0,\infty) : M(\lambda;T) = \infty$

(b) $\forall \lambda \in (0,\infty) : M(\lambda;T) = \infty$

(c) $\text{card } V_{\text{ess}}(T) = \infty$

Proof. Note first that $\sigma = [0,\infty)$ by Theorem 8.4. In order to conclude (a) $\Rightarrow$ (c) we observe that card $V_{\text{ess}}(T) < \infty$ implies that T consists in a finite tree F and in a finite number of one–sided unbounded paths Γ_0 that are linked to F at some of its boundary vertices. This implies that all the eigenvalue multiplicities are finite.

It remains to show (c) $\Rightarrow$ (b). In fact we can follow the construction in the proof of Theorem 8.4. The hypothesis on card $V_{\text{ess}}(T)$ guarantees that the above recursive construction leads to infinitely many linearly independent bounded eigensolutions. ∎

This applies immediately to the regular tree $\mathbb{T}^\gamma$ with $\gamma > 2$ that has all eigenvalues of infinite multiplicity. However, there are trees without boundary vertices that are Liouville spaces, but have all other eigenvalues of infinite multiplicity.

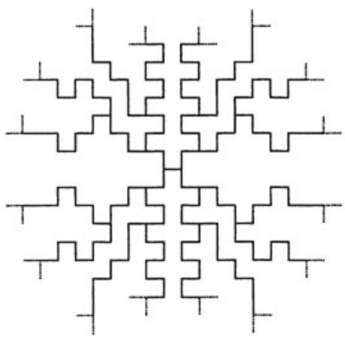

Figure 10: Example 8.7

Example 8.7 The tree in Figure 10 is a Liouville network, but has all other eigenvalues of infinite multiplicity. Let $\beta \geq 2$ denote a parameter and choose an arbitrary edge e_0 in the

3–regular tree T^3. Replace the edge lengths of the four edges being incident to e_0 by β, and, correspondingly, the edge lengths of the 2×2^k edges being incident to one of the edges of the foregoing generation with $2 \times 2^{k-1}$ edges by β^k, as indicated in Fig.10 for $\beta = 2$.

Then the resulting tree T^3_β is a Liouville space, see [8], but for any eigenvalue $\lambda > 0$ we can easily construct an infinite number of independent corresponding eigenfunctions as above.

It should be emphasized that for infinitely many boundary vertices the situation can be much more complicated and the assertion of Theorem 8.6 is no longer true. Let us illustrate this by means of the following

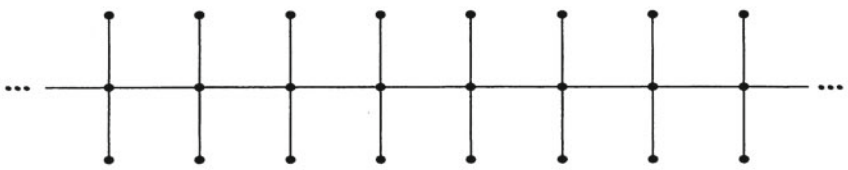

Figure 11: Example 8.8

Example 8.8 Let Γ be the infinite tree as depicted in Figure 11. For $\lambda = 0$, at all boundary vertices, the slopes vanish, so any bounded harmonic function on Γ leads to such a function on Γ_1 and has to be constant, see also [8]. But in the general case, we readily deduce

$$M(\lambda) = \begin{cases} 1 & \text{if } \lambda = 0, \\ 2 & \text{if } \cos\sqrt{\lambda} \neq 0, \\ \infty & \text{if} \cos\sqrt{\lambda} = 0. \end{cases}$$

9 Black holes

In some networks eigenvalues of the Laplacian occur that admit uncountably many independent eigenfunctions. For that purpose we introduce the

Definition 9.1 An eigenvalue of a bounded endomorphism of a Banach space is called a *black hole* if its eigenspace contains a subspace isomorphic to $\ell^\infty(\mathbb{Z})$.

In all the following examples, let $\mathbf{x} = (x_k)_{k\in\mathbb{Z}}$ be an arbitrary element in $\ell^\infty(\mathbb{Z})$.

Example 9.2 Consider the graph $\mathbf{K}_1$ depicted in Figure 12. The indicated value distribution at the nodes gives an eigenfunction of $\mu = 0$ for the adjacency operator. So all the corresponding values of λ are black holes of the Laplacian in $\mathbf{K}_1$.

Example 9.3 Consider now the graph $\mathbf{K}_3$. The value distribution at the nodes displayed in Figure 13 gives an eigenfunction of $\mu = -2$ for the adjacency operator, and the corresponding values of λ are black holes of the Laplacian in $\mathbf{K}_3$.

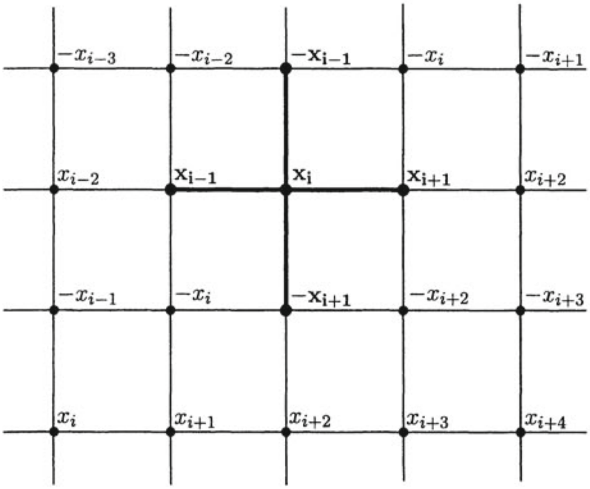

Figure 12: Black hole $\mu = 0$ for $\mathbf{K}_1$

Example 9.4 Consider the 3–regular graph K depicted in Figure 14. The indicated node values distribution gives an eigenfunction of $\mu = -1$ for the adjacency operator, and so all the corresponding values of λ are black holes of the Laplacian in K.

Example 9.5 Consider the 3–regular tree $\mathbb{T}_3$. The value distribution at the nodes displayed in Figure 15 gives an eigenfunction of $\mu = 1$ for the adjacency operator. So all the corresponding values of λ are black holes of the Laplacian in $\mathbb{T}_3$.

Example 9.6 Consider the 4–regular tree $\mathbb{T}_4$. The value distribution at the nodes displayed in Figure 16 gives an eigenfunction of $\mu = 0$ for the adjacency operator. So all the corresponding values of λ are black holes of the Laplacian in $\mathbb{T}_4$.

As it stands, it is not clear under which general condition black holes have to exist. It is not true that a network with infinitely many essential ramification nodes and without boundary vertices must display black holes, as the following example shows.

Example 9.7 The adjacency matrix of the 3–regular infinite band B as depicted in Fig.17 has the point spectrum $\mathbf{s} = [-3, 3]$ and each eigenvalue is of multiplicity at most 4. This can be seen by a symmetry ansatz for a presumed eigenvector $\mathbf{x} = (x_i)_{i \in \mathbb{Z}} \in \ell^\infty$, see Fig.17, that leads to the two cases $x_{2i} = x_{2i+1}$ for all $i \in \mathbb{Z}$ and $x_{2i} = -x_{2i+1}$ for all $i \in \mathbb{Z}$. The first one leads to the recurrence

$$x_{2i+2} = (\lambda - 1)x_{2i} - x_{2i-2}, \qquad i \in \mathbb{Z},$$

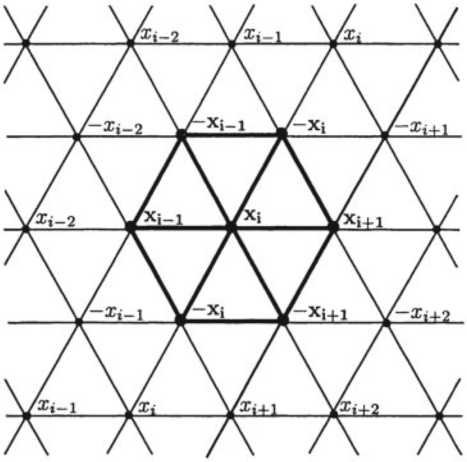

Figure 13: Black hole $\mu = -2$ for $\mathbf{K}_3$

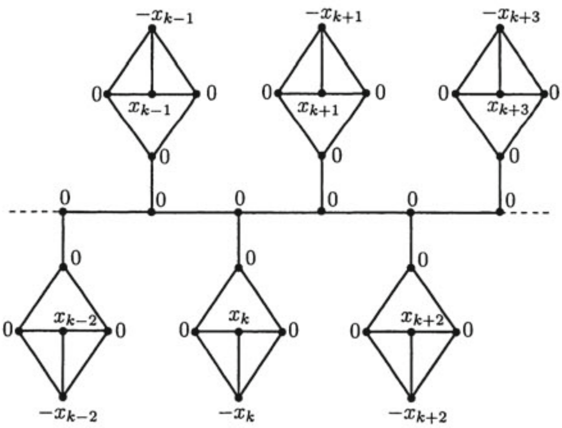

Figure 14: Black hole $\mu = 0$ for K

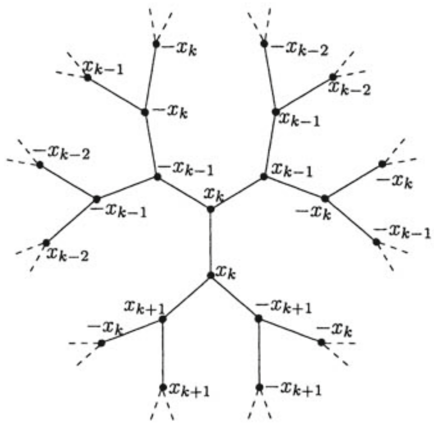

Figure 15: Black hole in $\mathbb{T}_3$ for $\mu = 1$

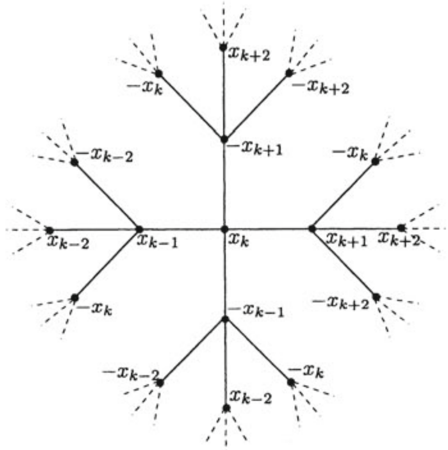

Figure 16: Black hole in $\mathbb{T}_4$ for $\mu = 0$

while the second one yields

$$x_{2i+2} = (\lambda + 1)x_{2i} - x_{2i-2}, \qquad i \in \mathbb{Z}.$$

Using the corresponding multiplicities in Γ_1 (Example 7.3) it follows that those of B cannot exceed 4.

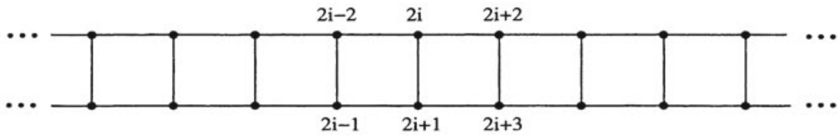

Figure 17: The 3-regular band in 9.7

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