Result.Math. 35 (1999) 355-379
0378-6218/99/040355-25 $ 1.50+0.20/0

Results in Mathematics

Superstability of the Cauchy, Jensen and Isometry Equations

Jacek Tabor

Abstract

In the first part of our paper we generalize the results obtained by Józef Tabor in [12] concerning the superstability of the Cauchy and Jensen functional equation almost everywhere.

In the second part we prove a general theorem on the superstablity of the Isometry equation in inner product spaces. As a corollary we determine when the Isometry Equation is superstable in the integral norm (this is a partial answer to [14]).

1 Introduction

In 1940 S. M. Ulam posed (cf. [15]) the problem of stability of the Cauchy equation (or in other words the question of the stability of the additive functions). After a slight reformulation we can state this problem in the following way:

Problem U *Assume that G is a group and let $\| \|_{\text{sup}}$ denote the supremum norm in the space of bounded functions defined on G ($G \times G$ respectively) and taking its values in the Banach space X. Let*

$$\mathcal{C}f(x,y) := f(x+y) - f(x) - f(y) \quad \text{for } x, y \in G.$$

Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to X$ satisfies

$$\|\mathcal{C}f\|_{\text{sup}} \leq \delta$$

then an additive function $a : G \to X$ exists with

$$\|f - a\|_{\text{sup}} \leq \varepsilon?$$

This problem was positively solved by Hyers in [6]. It was probably the begining of investigation of the so called "Hyers-Ulam" stability of functional equations. The most

Mathematics Subject Classification (1991): Primary 39B72, Secondary 46B99.
Keywords: stability, superstability, Cauchy functional equation, isometry equation.

important results and the large list of references concerning this theory are included in the survey papers [1], [3].

Józef Tabor noticed (cf. [13]) there is no reason to restrict ourselves to consider the stability of functional equations only in the supremum norm. Other norms on function spaces seem to be equally important and interesting. It occured that in the most commonly used norms the Cauchy Equation is stable (see [12], [9], [10], [11]).

In the following section we are going to generalize main results obtained by Józef Tabor in [12]. We quote Theorem 3.1 from [12].

Theorem T *Let* (G, Σ, λ) *be a completely measurable group such that* $\lambda(G) = \infty$. *Let* $f : G \to \mathbf{R}^n$ *be such that*

$$\mathcal{C}f \in \mathcal{L}_p(G \times G; \mathbf{R}^n).$$

Then there exists a unique additive function $a : G \to \mathbf{R}^n$ *such that*

$$f(x) = a(x) \quad \text{for } \lambda\text{-a.a. } x \in G$$

(the exact formulation of this theorem is more general, however we quote it in the above simplified form as not to introduce more notations and definitions).

We prove in Theorem 3.1 that instead of the space $\mathcal{L}_p(G; \mathbf{R}^n)$ one can take any translation invariant vector space of functions from G into $\mathbf{R}^n$ which does not contain nonzero constants and obtain similair result. We also deal in Theorem 4.1 with a similair result concerning the Jensen functional equation.

The stability of the isometry equation was intensively investigated by mathematicians (for some results and references we refer the reader to [8]). Therefore in Section 3 we partially determine when the Isometry Equation is superstable (this is one of possible answers to [14]). Theorem 3.1 plays an essential role in the proof of these results. Under some weak technical conditions, we show that if the Isometry difference of the function f,

$$\mathcal{I}f(x, y) := |\|f(x) - f(y)\| - \|x - y\||$$

belongs to the space $\mathcal{L}_p(G \times G, \mathbf{R}^n)$ and $\frac{1}{1+\|x\|} \notin \mathcal{L}_p(G, \mathbf{R}^n)$ then f equals almost everywhere to some uniquely determined isometry. As a corollary we obtain that if a given function f satisfies the isometry equation almost everywhere then there exists a unique isometry which is equal to the function f almost everywhere.

Our results are also closely connected to results of J. Chmielinski and J. Rätz (cf. [2]). J. Chmielinski and J. Rätz prove, under some technical conditions, that if a given function satisfies the orthogonality equation almost everywhere, then there exists a unique function which satisfies the orthogonality equation which equals almost everywhere to the given function. We show in Corollary 6.2 similair result, however, in a slightly different setting.

2 Definitions

Let G, V be groups. For $f : G \to V$. we define the *Cauchy difference* of f by

$$\mathcal{C}f(x, y) := f(x + y) - f(y) - f(x) \quad \text{for } x, y \in G.$$

If V is uniquely two divisible, we additionally define the *Jensen difference* of f by

$$\mathcal{J}f(x,y) := f(x) - \frac{f(x+y)+f(x-y)}{2} \quad \text{for } x,y \in G.$$

If both groups G and V are uniquely 2-divisible then we define the *middle Jensen difference* of f by the formula

$$\mathcal{J}_{\mathcal{M}}f(x,y) := f(x+y) - \frac{f(x)+f(y)}{2} \quad \text{for } x,y \in G.$$

We say that a given function f is additive (Jensen) if $\mathcal{C}f = 0$ ($\mathcal{J}f = 0$). It is well known that if G and V are abelian and V uniquely two divisible then every Jensen function is the sum of an additive function and a constant.

Remark 2.1 One can easily notice that if G and V are uniquely 2-divisible abelian groups then a given function f on G is Jensen iff $\mathcal{J}_{\mathcal{M}}f = 0$. This is why both the Jensen difference and the middle Jensen difference are called just the Jensen difference. However, as in this article we deal with the stability of both the Jensen and the middle Jensen difference we had to specify them separetely.

For $a \in G$ we define the translation of by a by the formula

$$T_a(x) := a + x \quad \text{for } x \in G.$$

We say that $\mathcal{L}$ is translation invariant if $f \circ T_a \in \mathcal{L}$ for every $a \in G$, $f \in \mathcal{L}$.

Now we will quote the definition of a p.l.i. ideal and conjugate ideals. For more information about this subject we refer the reader to [7].

Definition 2.1 *Let $\mathcal{I}$ be a family of subsets of G. If*

- $G \notin \mathcal{I}$,
- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
- $A \in \mathcal{I},\ B \subset A \Rightarrow B \in \mathcal{I}$,
- $a \in G,\ A \in \mathcal{I} \Rightarrow a - A \in \mathcal{I}$.

then we say that $\mathcal{I}$ is a proper linearly invariant ideal (p.l.i. ideal for short).

The elements of a p.l.i. ideal are usually understood as some kind of "small sets". The most common example of a p.l.i. ideal is probably the family of all sets with Lebegue measure zero in $\mathbf{R}^n$.

According to the tradition we say that a given condition is satisfied for $\mathcal{I}$-almost all $x \in G$ (in abbreviated form: for $\mathcal{I}$-a.a. $x \in G$) if there exists $A \in \mathcal{I}$ such that this condition holds for $x \in G \setminus A$.

For $x \in G$, $A \subset G \times G$ we define

$$A[x] := \{y \in G \mid (x,y) \in A\}.$$

Definition 2.2 *Let $\mathcal{I}$ be a p.l.i. ideal in G. We define p.l.i. ideal $\Omega(\mathcal{I})$ in $G \times G$ by the formula*

$$\Omega(\mathcal{I}) := \{A \subset G \times G \mid A[x] \in \mathcal{I} \quad \text{for } \mathcal{I}\text{-a.a. } x \in G\}.$$

We make an analogon of this definition for function spaces.

Definition 2.3 *Let $\mathcal{I}$ be a p.l.i. ideal in G and let $\mathcal{L}$ be a group of functions from G into V. We define the group of function $\Omega_{\mathcal{I}}(\mathcal{L})$ from $G \times G$ into V by the formula*

$$\Omega_{\mathcal{I}}(\mathcal{L}) := \{F : G \times G \to V \mid F(x, \cdot) \in \mathcal{L} \quad \text{for } \mathcal{I}\text{-a.a. } x \in G\}.$$

Remark 2.2 We would like to mention that Definition 2.3 can be understood as a generalization of Definition 2.2. Suppose that we are given a p.l.i. ideal $\mathcal{I}$ in G. We put

$$\mathcal{L}_{\mathcal{I}} := \{f : G \to \mathbf{R} \mid f(x) = 0 \quad \text{for } \mathcal{I}\text{-a.a. } x \in G\}.$$

Then one can easily check that

$$\Omega_{\mathcal{I}}(\mathcal{L}_{\mathcal{I}}) = \mathcal{L}_{\Omega(\mathcal{I})}.$$

Example 2.1 Let M, N be normed spaces, let $\mathcal{I} = \emptyset$ be a trivial ideal in M and let

$$\mathcal{L} := \{f : M \to N \mid \lim_{\|y\| \to \infty} f(y) = 0\}.$$

Then

$$\Omega_{\mathcal{I}}(\mathcal{L}) = \{F : M \times M \mid \lim_{\|y\| \to \infty} F(x, y) = 0 \quad \text{for } x \in M\}.$$

For the convenience of the reader we quote the definition of a complete measurable group (see [5]).

Definition 2.4 *We say that (G, Σ, λ) is a* complete measurable group *if*

(a) (G, Σ, λ) is a σ-finite measure space, λ is not identically zero and is complete,

(b) the σ-algebra Σ and the measure λ are invariant with respect to left translations,

(c) $\lambda \times \lambda$ is the completion of the product measure,

(d) the transformation

$$S : G \times G \ni (x, y) \to (x, x + y)$$

is measurability preserving, i.e. S and S^{-1} are measurable.

By $\Sigma \times \Sigma$ we will denote the family of all measurable sets for the measure $(\lambda \times \lambda)$. It is worth mentioning that under the assumptions (a) - (d) the measure λ is invariant under translations and under symmetry with respect to zero, and that the transformations S and S^{-1} preserve the measure $\lambda \times \lambda$ (cf. [5], §59). By $\lambda^+(A)$ we denote the outer measure of the set A.

Definition 2.5 *We say that* V is a metric group *if* V *is a group with a metric* d *invariant with respect to translations. For the convenience instead of* $d(x,0)$ *we will write* $\|x\|_d$.

The following definition (cf. [12]) is a generalization of the space $\mathcal{L}_p(G,\mathbf{R})$ (the space of p-integrable real functions on G).

Definition 2.6 *Let* (G,Σ,λ) *be a complete measurable group. For* $f: G \to \mathbf{R}_+$ *we define*

$$\int_G^+ f(x)\,\lambda(x) = \inf\left\{\int_G w(x)\,\lambda(x) \mid f(x) \le w(x) \quad \text{for } x \in G,\ w \in \mathcal{L}_1(G,\mathbf{R})\right\}$$

(by the infimum of the empty set we understand as usualy $+\infty$*).*
Let $p > 0$*, and let* V *be a metric abelian group. For* $f: G \to V$ *we put*

$$\|f\|_p^+ := \sqrt[p]{\int_G^+ (\|f(x)\|_d)^p \lambda(x)},$$

$$\mathcal{L}_p^+(G;V) := \left\{f : G \to V \mid \|f(x)\|_p^+ < \infty\right\}.$$

For a given completely measurable group (G,Σ,λ) we define a p.l.i. ideal $\mathcal{I}_\lambda$ by the formula

$$\mathcal{I}_\lambda := \{A \in \Sigma \mid \lambda(A) = 0\}$$

The reader can now easily notice that by the Fubini Theorem $\mathcal{I}_{(\lambda\times\lambda)} \subset \Omega(\mathcal{I}_\lambda)$.

We would like to mention that $\mathcal{L}_p^+(G,V)$ for $p \in (0,\infty)$ is a translation invariant group, and that if $\lambda(G) = \infty$ then $\mathcal{L}_p^+(G,V)$ does not contain nonzero constants. Moreover, by the Fubini Theorem we obtain that

$$\mathcal{L}_p^+(G \times G; N) \subset \Omega_{\mathcal{I}_\lambda}(\mathcal{L}_p^+(G;N)).$$

(the exact proofs one can find in [12])

3 Supertability of the Cauchy equation

From now on we assume that $\mathcal{I}$ is a p.l.i. in G.

To prove the main theorem of this section we need first to quote the well-known result of R. Ger (see [4], and also [7]). For the convenience of the reader we quote this theorem.

Theorem G *Let* G, V *be groups and let* $f : G \to V$ *be such that*

$$\mathcal{C}f(x,y) = 0 \quad \text{for } \Omega(\mathcal{I})\text{-a.a. } (x,y) \in G \times G.$$

Then there exists a unique additive function $a : G \to V$ *such that*

$$f(x) = a(x) \quad \text{for} \quad \text{for } \mathcal{I}\text{-a.a. } x \in G.$$

To show the potential of the following Theorem we would like to mention that it is a generalization of both Theorem G (under the assumption that V is abelian) and Theorem T (or more exactly Theorem 3.1 from [12]).

Theorem 3.1 *Let G be a group and let V be an abelian group.*

Let $\mathcal{L}$ be a translation invariant group of functions from G into V such that $\mathcal{L}$ does not contain nonzero constants.

Then for every $f : G \to V$ such that

$$\mathcal{C}f \in \Omega_{\mathcal{I}}(\mathcal{L})$$

there exists a unique additive function $a : G \to V$ such that

$$f(x) = a(x) \quad \text{for } \mathcal{I}\text{-a.a. } x \in G.$$

Proof. Clearly for $x, u, v \in G$

$$\begin{aligned} \mathcal{C}f(u,v) &= f(u+v) - f(v) - f(u) \\ &= -[f(u+v+x) - f(x) - f(u+v)] \\ &\quad +[f(v+x) - f(x) - f(v)] \\ &\quad +[f(u+v+x) - f(v+x) - f(u)] \\ &= -\mathcal{C}f(u+v,x) + \mathcal{C}f(v,x) + \mathcal{C}f(u,v+x) \end{aligned}$$

This means that

$$\mathcal{C}f(u,v) = -\mathcal{C}f(u+v,\cdot)(x) + \mathcal{C}f(v,\cdot)(x) + \mathcal{C}f(u,\cdot)T_v(x). \tag{3.1}$$

As $\mathcal{C}f \in \Omega_{\mathcal{I}}(\mathcal{L})$ there exists $A \in \mathcal{I}$ such that

$$\mathcal{C}f(x,\cdot) \in \mathcal{L} \quad \text{for } x \in G \setminus A.$$

Let us fix arbitrary $u, v \in G \setminus A$ such that $u + v \in G \setminus A$. As $\mathcal{L}$ is translation invariant vector space we obtain that the right hand side of (3.1) as a function of variable x is an element of $\mathcal{L}$. It means that $\mathcal{C}f(u,v) \in \mathcal{L}$ as a function of variable x. However, $\mathcal{L}$ does not contain nonzero constants, so $\mathcal{C}f(u,v) = 0$.

Thus we have obtained that

$$\mathcal{C}f(u,v) = 0 \quad \text{for } (u,v) \in G \times G \setminus D,$$

where $D = (A \times G) \cup (G \times A) \cup \bigcup_{x \in G} (x, A - x)$.

The reader can easily check that $D \in \Omega(\mathcal{L})$, so by Theorem G there exists a unique additive function $a : G \to V$ such that

$$f(x) = a(x) \quad \text{for } \mathcal{I}\text{-a.a. } x \in G.$$

□

The assumption that $\mathcal{L}$ does not contain nonzero constants is essential, as without this assumption $\mathcal{L}$ can be the set of all functions from G into V. The following example shows that without the assumption that $\mathcal{L}$ is translation invariant Theorem 3.1 fails to hold.

Example 3.1 Let $G = V = \mathbf{R}$, let $\mathcal{I} = \emptyset$ and let $\mathcal{L}$ denote the vector space of all linear functions from G into V. Clearly $\mathcal{L}$ does not contain nonzero constant function.

We define $f(x) = x^2$. Then obviously $\mathcal{C}f \in \Omega_{\mathcal{I}}(\mathcal{L})$ although f is not an additive function.

Remark 3.1 To show that, under the assumption that V is abelian, Theorem G is a trivial corollary from Theorem 3.1 one has to just take $\mathcal{L}_I$ defined as in Remark 2.2.

The following Theorem is a generalization of Theorem 3.1 from [12]:

Theorem 3.2 *Let (G, Σ, λ) be a complete measurable group such that $\lambda(G) = \infty$, and let V be a metric abelian group. Let $f : G \to V$ be a function such that for all $\varepsilon > 0$*

$$(\lambda \times \lambda)^+(\{(x, y) \, : \, \|\mathcal{C}f(x, y)\|_d \geq \varepsilon\}) < \infty.$$

Then there exists a unique additive function $a : G \to N$ such that

$$f(x) = a(x) \quad \text{for } \lambda\text{-a.a. } x \in G.$$

Proof. We define

$$\mathcal{L} := \{g : G \to N \mid \forall \varepsilon > 0 \, : \, \lambda^+(\{x \, : \, \|g(x)\| \geq \varepsilon\}) < \infty\}.$$

Then obviously $\mathcal{L}$ is translation invariant. As $\lambda(G) = \infty$, $\mathcal{L}$ does not contain nonzero constants. By the Fubini Theorem $\mathcal{C}f \in \Omega_{\mathcal{I}_\lambda}(\mathcal{L})$. Theorem 3.1 makes the proof complete. □

Corollary 3.1 *Let E, F be normed spaces, let B be a bounded set in E, and let $f : E \to F$ be such that for every $x \in E \setminus B$*

$$\lim_{\|y\| \to \infty} \mathcal{C}f(x, y) = 0.$$

Then there exists a unique additive function $a : E \to F$ such that

$$f(x) = a(x) \quad \text{for } x \in E \setminus \tilde{B}$$

for certain bounded set $\tilde{B} \subset E$.

Proof. We put

$$\mathcal{I} := \{B \subset E \mid B \text{ is bounded }\}.$$

$$\mathcal{L} := \{g : E \to F \, : \, \lim_{\|x\| \to \infty} g(x) = 0\}.$$

Obviously $\mathcal{L}$ is a translation invariant vector space which does not contain nonzero constants. Clearly $\mathcal{C}f \in \Omega_{\mathcal{I}}(\mathcal{L})$. Theorem 3.1(i) implies that there exists a unique additive function $a : E \to F$ and a set $\tilde{B} \in \mathcal{I}$ such that

$$f(x) = a(x) \quad \text{for } x \in E \setminus \tilde{B}.$$

□

By λ_n we denote as usually the Lebesgue measure on $\mathbf{R}^n$, and by $B(a, s)$ the ball with the center at a and the radius s.

Corollary 3.2 *Let $f : \mathbf{R}^n \to \mathbf{R}^k$ be such that $\mathcal{C}f$ is locally integrable in $\mathbf{R}^n \times \mathbf{R}^n$. We assume that for all $a, x \in \mathbf{R}^n$*

$$\lim_{s\to\infty} \frac{1}{s^n} \int_{B(a,s)} \mathcal{C}f(x,y)\,\lambda_n(y) = 0.$$

Then f is an additive function.

Proof. Let

$$\mathcal{L} := \{f : \mathbf{R}^n \to E \mid \forall\, a \in \mathbf{R}^n : \lim_{s\to\infty} \frac{1}{s^n} \int_{B(a,s)} f(x)\,\lambda_n(x) = 0\}.$$

We put $\mathcal{I} = \emptyset$. Then one can easily check that $\mathcal{L}$ is translation invariant vector space which does not contain nonzero constants, and by definition $\mathcal{C}f \in \Omega_{\mathcal{I}}(\mathcal{L})$. Theorem 3.1 completes the proof. □

4 Superstability of the Jensen equation

Theorem 4.1 *Let G, V be abelian groups. Let $\mathcal{L}$ be a translation invariant group of functions from G into V. We additionally assume that $\mathcal{L}$ does not contain nonzero constants.*

(i) For every $f : G \to V$ such that

$$\mathcal{J}f \in \Omega_{\mathcal{I}}(\mathcal{L})$$

there exists a unique Jensen function $j : G \to V$ such that

$$f(x) = j(x) \quad \text{for } \mathcal{I}\text{-a.a. } x \in G.$$

(ii) Suppose additionaly that G is uniquely 2-divisible. Then for every $f : G \to V$ such that

$$\mathcal{J}_{\mathcal{M}}f \in \Omega_{\mathcal{I}}(\mathcal{L})$$

there exists a unique Jensen function $j : G \to V$ such that

$$f(x) = j(x) \quad \text{for } \mathcal{I}\text{-a.a. } x \in G.$$

Proof.

(i) As $\mathcal{J}f \in \Omega_{\mathcal{I}}(\mathcal{L})$ there exists $A \in \mathcal{I}$ such that

$$\mathcal{J}f(x,\cdot) \in \mathcal{L} \quad \text{for } x \in G \setminus A.$$

Let $x_0 \in G \setminus A$ be arbitrary and let

$$g(x) := f(x + x_0) - f(x_0) \quad \text{for } x \in G. \tag{4.1}$$

We have for arbitrary $u, v, v \in G$

$$\begin{aligned}
\mathcal{C}g(u,v) &= g(u+v) - g(u) - g(v) \\
&= f(u+v+x_0) - f(u+x_0) - f(v+x_0) + f(x_0) \\
&= \{f(u+v+x_0) - \tfrac{1}{2}(f(-x+u+v+x_0) + f(x+u+v+x_0))\} \\
&\quad -\{f(u+x_0) - \tfrac{1}{2}(f(-x+u-v+x_0) + f(x+u+v+x+x_0))\} \\
&\quad -\{f(v+x_0) - \tfrac{1}{2}(f(-x+u+v+x_0) + f(x-u+v+x_0))\} \\
&\quad +\{f(x_0) - \tfrac{1}{2}(f(-x+u-v+x_0) + f(x-u+v+x_0))\} \\
&= \mathcal{J}f(u+v+x_0, x) - \mathcal{J}f(u+x_0, x+v) \\
&\quad -\mathcal{J}f(v+x_0, x-u) + \mathcal{J}f(x_0, x-u+v).
\end{aligned}$$

Notice that in this equality we use the fact that G is abelian. This means that

$$\begin{aligned}
\mathcal{C}g(u,v) &= \mathcal{J}f(u+v+x_0,\cdot)(x) - \mathcal{J}f(u+x_0,\cdot) \circ T_v(x) \\
&\quad -\mathcal{J}f(v+x_0,\cdot) \circ T_{-u}(x) + \mathcal{J}f(x_0,\cdot) \circ T_{-u+v}(x).
\end{aligned} \tag{4.2}$$

Let $u, v \in G$ be such that $u, v, u+v \in G \setminus (A - x_0)$. Then $u + x_0, v + x_0, u + v + x_0 \in G \setminus A$, and obviously $x_0 \in G \setminus A$. Hence because $\mathcal{L}$ does not contain nonzero constants

$$\mathcal{C}g(u,v) = 0.$$

This yields that

$$g(u+v) = g(u) + g(v)$$

for $u, v \in G \setminus D_{x_0}$, where

$$D_{x_0} = [G \times (A - x_0)] \cup [(A - x_0) \times G] \cup \bigcup_{u \in G} (u, A - x_0 - u).$$

One can easily check that $D_{x_0} \in \Omega(\mathcal{I})$, so by Theorem G there exist a unique additive function $a : G \to V$ such that

$$g(x) = a(x) \quad \text{for} \qquad \text{for } \mathcal{I}\text{-a.a. } x \in G.$$

Let $j(x) := a(x) + f(x_0) - a(x_0)$. Then from (4.1) we get that

$$f(x) = j(x) \quad \text{for } \mathcal{I}\text{-a.a. } x \in G.$$

Clearly j is a unique Jensen function which satisfies the assertion of the theorem.

(ii) The proof of the part (iii) is the exact repetition of the proof of the part (ii), except that instead of equality (4.2) we make use of the following one

$$\begin{aligned}
\mathcal{C}g(u,v) &= 2\mathcal{J}_{\mathcal{M}}f(v+x_0,\cdot) \circ T_{u+x_0}(x) - 2\mathcal{J}_{\mathcal{M}}f(u+v+x_0,\cdot) \circ T_{x_0}(x) \\
&\quad -2\mathcal{J}_{\mathcal{M}}f(x_0,\cdot) \circ T_{u+x_0}(x) + 2\mathcal{J}_{\mathcal{M}}f(u+x_0,\cdot) \circ T_{x_0}(x)
\end{aligned}$$

(for the detailed proof of the above inequality we refer the reader to [12], and therefore we omit it). □

The reader can easily formulate analogons of corollaries from Theorem 3.1 from the previous Section for the Jensen equation and the middle Jensen equation (instead of Theorem 3.1 one should use Theorem 4.1). Thus Theorem 4.1 is a a generalization of Theorem 4.1 from [12].

5 Superstability of Isometries

For an inner product vector space H by $<x,y>$ we denote the inner product of x and y. We will need the following lemma.

Lemma 5.1 *Let H be a inner product space, and let $a, b, c \in H$ be such that*

$$\left| \|a-c\| - \|\frac{a-b}{2}\| \right| \leq \varepsilon, \tag{5.1}$$

$$\left| \|b-c\| - \|\frac{a-b}{2}\| \right| \leq \varepsilon. \tag{5.2}$$

Then

$$\|c - \frac{a+b}{2}\| \leq 2\varepsilon + \sqrt{\|a-b\|\varepsilon}.$$

Proof. If $\|\frac{a-b}{2}\| \leq \varepsilon$, then $\|a-c\| \leq \varepsilon + \frac{\|a-b\|}{2} \leq 2\varepsilon$, $\|b-c\| \leq 2\varepsilon$, so

$$\|c - \frac{a+b}{2}\| \leq \frac{\|a-c\|}{2} + \frac{\|b-c\|}{2} \leq 2\varepsilon.$$

Now suppose that $\|\frac{a-b}{2}\| > \varepsilon$. Then (5.1), (5.2) imply that

$$\begin{array}{rcl} \|a-c\| & \in & [\|\frac{a-b}{2}\| - \varepsilon, \|\frac{a-b}{2}\| + \varepsilon] \subset \mathbf{R}_+, \\ \|b-c\| & \in & [\|\frac{a-b}{2}\| - \varepsilon, \|\frac{a-b}{2}\| + \varepsilon] \subset \mathbf{R}_+. \end{array}$$

Let $K := [(\|\frac{a-b}{2}\| - \varepsilon)^2, (\|\frac{a-b}{2}\| + \varepsilon)^2]$. Thus we obtain that

$$<a-c, a-c> = \|a-c\|^2 \in K, \ <b-c, b-c> \in K.$$

Therefore

$$\frac{<a-c,a-c> + <b-c,b-c>}{2} \in K,$$

and consequently

$$\begin{array}{rcl} <c - \frac{a+b}{2}, c - \frac{a+b}{2}> & = & \frac{<a-c,a-c> + <b-c,b-c>}{2} - <\frac{a-b}{2}, \frac{a-b}{2}> \\ & \in & K - <\frac{a-b}{2}, \frac{a-b}{2}> = K - \|\frac{a-b}{2}\|^2 \\ & = & [\varepsilon^2 - \|a-b\|\varepsilon, \varepsilon^2 + \|a-b\|\varepsilon]. \end{array}$$

So

$$\|c - \frac{a+b}{2}\| \leq \sqrt{\varepsilon^2 + \|a-b\|\varepsilon} \leq 2\varepsilon + \sqrt{\|a-b\|\varepsilon}.$$

□

We say that a metric group G is a *inner product group* if there exists an inner product space H and an additive isometry $i_H : G \to H$. If G is a inner product group with a metric d to shorten the notation instead of $d(x,0)$ we simply write $\|x\|_d$. We additionally define $< a, b>_d := < i_H(a), i_H(b) >$ (one can easily notice that $< a, b>_d$ does not depend on the inner product space H and isometry i_H).

Corollary 5.1 *Let G be an inner product group, let H be an inner product space and let $i : G \to H$ be an isometry. Then i is a Jensen function.*

Proof. Let $i : G \to H$ be an isometry and let $a, b \in G$ be arbitrary. Then

$$|\|i(a+b) - i(a)\| - \|(a+b) - b\|_d| = 0,$$

$$|\|i(a) - i(a-b)\| - \|a - (a-b)\|_d| = 0,$$

$$\tfrac{1}{2}|\|i(a+b) - i(a-b)\| - \|(a+b) - (a-b)\|_d| = 0.$$

Inserting the last inequality into the first and the second we obtain that

$$|\|i(a+b) - i(a)\| - \tfrac{1}{2}\|i(a+b) - i(a-b)\|| = 0,$$

$$|\|i(a) - i(a-b)\| - \tfrac{1}{2}\|i(a+b) - i(a-b)\|| = 0.$$

We can now make use of Lemma 5.1 and obtin that

$$\|i(a) - \frac{i(a+b) - i(a-b)}{2}\| = 0.$$

As a, b were arbitrarily fixed this implies that i is jensen. □

We have mentioned in the previous section that under the assumptions (a) - (d) of Definition 2.1 the measure λ is invariant under translations and under symmetry with respect to zero, and that the transformations S and S^{-1} preserve the measure $(\lambda \times \lambda)$. Therefore the measure $(\lambda \times \lambda)$ is also preserved by the transformations

$$I(x, y) = (y, x), \ I_-(x, y) = (x, -y).$$

This means that the space $\mathcal{L}_p^+(G \times G; \mathbf{R})$ is invariant with respect to the operations S, S^{-1}, I, I_-, which leads us to the following definition:

Definition 5.1 *Let $\mathcal{J}$ be a p.l.i. ideal in $G \times G$. We say that $\mathcal{J}$ is* measurable *if $\mathcal{J}$ is invariant with respact to operations S, S^{-1}, I, I_-.*

Let P be a group of functions from $G \times G$ into H. We say that P is measurable *if P is invariant with respect to operations S, S^{-1}, I, I_-.*

Theorem 5.1 *Let $(G, \|\cdot\|_d)$ be an inner product group and let $\mathcal{L}$ be a translation invariant group of functions from G into $\mathbf{R}$. We assume that there does not exist $f \in \mathcal{L}$ with*

$$\frac{1}{1 + \|x\|_d} \le f(x) \quad \text{for } \mathcal{I}\text{-a.a. } x \in G. \tag{5.3}$$

Let Ω be a measurable subgroup of $\Omega_{\mathcal{I}}(\mathcal{L})$. Suppose additionally that Ω is invariant with respect to operation

$$M : G \times G \ni (x, y) \to (x, 2y) \in G \times G.$$

Then for every function $f : G \to H$ such that

$$\mathcal{I}f \in \Omega$$

there exists a unique isometry $i : G \to H$ such that

$$f(x) = i(x) \quad \text{for } \mathcal{I}\text{-a.a. } x \in G.$$

Before we proceed with the proof for the convenience of the reader we will sketch its main idea. Making use of Lemma 5.1 we show that as $\mathcal{I}f \in P$, $\mathcal{J}f \in \Omega_{\mathcal{I}}(\mathcal{K}_{\mathcal{L}})$, for some translation invariant space $\mathcal{K}_{\mathcal{L}}$. The fact that $\frac{1}{1+\|x\|_d} \notin \mathcal{L}$ implies that the space $\mathcal{K}_{\mathcal{L}}$ does not contain nonzero constants. Then we make use of Theorem 4.1 and obtain that there exists a Jensen function i which is equal to f almost everywhere, which enables us to prove that i is an isometry.

Proof. As Ω is measurable and invariant with respect to operation M it is also invariant with respect to operation $\Phi := S^{-1} \circ M \circ I \circ S \circ I$. One can easily check that $\Phi(x,y) = (x+y, x-y)$.

As $\mathcal{I}f \in \Omega$, also $(\mathcal{I}f) \circ (I_{-}S^{-1}), (\mathcal{I}f) \circ S, (\mathcal{I}f) \circ \Phi \in \Omega$. Clearly

$$\begin{aligned} (\mathcal{I}f) \circ (I_{-} \circ S^{-1}) &= |\|f(x) - f(x-y)\| - \|y\|_d|, && (5.4)\\ (\mathcal{I}f) \circ S(x,y) &= |\|f(x+y) - f(x)\| - \|y\|_d|, && (5.5)\\ \tfrac{1}{2}(\mathcal{I}f) \circ \Phi(x,y) &= |\tfrac{1}{2}\|f(x+y) - f(x-y)\| - \|y\|_d|. && (5.6) \end{aligned}$$

Let $\varepsilon := (\mathcal{I}f) \circ (I_{-}S^{-1}) + (\mathcal{I}f) \circ S + \frac{1}{2}(\mathcal{I}f) \circ \Phi$. Applying (5.6) to (5.4) and (5.5) and making use of the triangle inequality we obtain that

$$|\|f(x) - f(x-y)\| - \tfrac{1}{2}\|f(x+y) - f(x-y)\|| \le \varepsilon(x,y),$$
$$|\|f(x+y) - f(x)\| - \tfrac{1}{2}\|f(x+y) - f(x-y)\|| \le \varepsilon(x,y).$$

Then Lemma 5.1 implies that

$$\|f(x) - \frac{f(x+y) + f(x-y)}{2}\| \le 2\varepsilon(x,y) + \sqrt{\varepsilon(x,y)\|f(x+y) - f(x-y)\|}.$$

However, by (5.6), $\|f(x+y) - f(x-y)\| \le 2\varepsilon(x,y) + 2\|y\|_d$, so

$$\begin{aligned} \|\mathcal{J}f(x,y)\| &= \|f(x) - \tfrac{f(x+y)+f(x-y)}{2}\| \\ &\le 2\varepsilon(x,y) + \sqrt{\varepsilon(x,y)(2\varepsilon(x,y) + 2\|y\|_d} && (5.7)\\ &\le 4\varepsilon(x,y) + \sqrt{4\varepsilon(x,y)\|y\|_d}. \end{aligned}$$

(Step 1) We define

$$\mathcal{K}_{\mathcal{L}} := \{f : G \to H \mid \exists \nu \in \mathcal{L} : \|f(x)\| \le \nu(x) + \sqrt{\nu(x)(1 + \|x\|_d)}\}.$$

One can check by easy calculations that as $\mathcal{L}$ is translation invariant group $\mathcal{K}_{\mathcal{L}}$ is also translation invariant group.

We prove that $\mathcal{K}_{\mathcal{L}}$ does not contain nonzero constants. For a contradiction, suppose that there exist a constant function $f \in \mathcal{K}_{\mathcal{L}}$ such that $\|f(x)\| = 1$ for $x \in G$. Then by the definition of $\mathcal{K}_{\mathcal{L}}$ there would exist $\nu \in \mathcal{L}$ such that

$$\|f(x)\| \le \nu(x) + \sqrt{\nu(x)(1 + \|x\|_d)} \quad \text{for } x \in G.$$

Let $\nu_1(x) := \min\{1, \nu(x)\}$. Let $x \in G$ be arbitrary. Then one can easily notice that

$$1 \le \nu_1(x) + \sqrt{\nu_1(x)(1 + \|x\|_d)}. \tag{5.8}$$

As the function ν_1 is bounded from above by 1, we obtain that

$$\nu_1(x) \le \sqrt{\nu_1(x)} \le \sqrt{\nu_1(x)(1 + \|x\|_d)}. \tag{5.9}$$

By (5.8) and (5.9) we get

$$1 \le 2\sqrt{\nu_1(x)(1 + \|x\|_d)},$$

and consequently that

$$\nu_1(x)(1 + \|x\|_d) \ge \tfrac{1}{4}.$$

This means that $\frac{1}{1+\|x\|_d} \le 4\nu_1(x) \le 4\nu(x)$. We have obtained a contradiction with (5.3)) as $4\nu \in \mathcal{L}$.

(Step 2) Now we will show that

$$\mathcal{J}f(x,y) \in \Omega_{\mathcal{I}}(\mathcal{K}_{\mathcal{L}}). \tag{5.10}$$

We know that $\varepsilon \in \Omega_{\mathcal{I}}(\mathcal{L})$, so there exists $A \in \mathcal{I}$ such that

$$g_x(\cdot) := \varepsilon(x, \cdot) \in \mathcal{L} \quad \text{for } x \in G \setminus A.$$

Let $x \in G \setminus A$ be arbitrarily fixed and let $h_x(\cdot) := \mathcal{J}f(x, \cdot)$. We will prove that $h_x \in \mathcal{K}_{\mathcal{L}}$, which will end this this part of the proof. By (5.7)

$$\begin{aligned} \|h_x(y)\| &= \|\mathcal{J}f(x,y)\| \le 4\varepsilon(x,y) + \sqrt{4\varepsilon(x,y)\|y\|_d} \\ &= 4g_x(y) + \sqrt{4g_x(y)\|y\|_d}. \end{aligned}$$

However, $4g_x \in \mathcal{L}$ so by the definition of $\mathcal{K}_{\mathcal{L}}$ we obtain that $h_x \in \mathcal{K}_{\mathcal{L}}$. As x was arbitrary in $G \setminus A$ we obtain that $\mathcal{J}f \in \Omega_{\mathcal{I}}(\mathcal{K}_{\mathcal{L}})$.

Because $\mathcal{K}_{\mathcal{L}}$ is translation invariant and it does not contain nonzero constant functions so we may use Theorem 4.1 and obtain that there exists a unique Jensen function $i : G \to H$ such that

$$f(x) = i(x) \quad \text{for } \mathcal{I}\text{-a.a. } x \in G. \tag{5.11}$$

(Step 3) Now we will prove that there exists $A \in \mathcal{I}$ such that

$$\|i(x) - i(0)\| = \|x\|_d \quad \text{for } x \in G \setminus A. \tag{5.12}$$

By (5.11) we obtain that there exists $B \in \mathcal{I}$ such that $f(x) = i(x)$ for $x \in G \setminus B$. We know that $\mathcal{I}f \in \Omega$, which implies that $\mathcal{I}f \circ (S \circ I) \in \Omega$, so there exists $C \in \mathcal{I}$ such that

$$\mathcal{I}f \circ (S \circ I)(x, \cdot) \in \mathcal{L} \quad \text{for } x \in G \setminus C. \tag{5.13}$$

We put $A := B \cup C$. Let $x \in G \setminus A$ be arbitrarily fixed. Then for $y \in G \setminus ((B - x) \cup B)$

$$\begin{aligned} \mathcal{I}f \circ (S \circ I)(x,y) &= |\|f(x+y) - f(y)\| - \|x + y - y\|_d| \\ &= |\|i(x+y) - i(y)\| - \|x\|_d| \\ &= |\|i(x) - i(0)\| - \|x\|_d|. \end{aligned} \tag{5.14}$$

Suppose for contradiction that $\|i(x) - i(0)\| \neq \|x\|_d$. Let

$$g_x(y) := \frac{\mathcal{I}f \circ (S \circ I)(x,y)}{|\|i(x) - i(0\| - \|x\|_d|}.$$

Then by (5.13) $g_x \in \mathcal{L}$ and by (5.14)

$$g_x(y) = 1 \geq \frac{1}{1 + \|y\|_d} \quad \text{for } \mathcal{I}\text{-a.a. } y \in G.$$

We have obtained contradiction with (5.3)).

(Step 4) Now we will prove that i is an isometry.

Let $x, y \in G$ be arbitrary. Let $a := x - y$. and let A be chosen as to satisfy (5.12).

As $\mathcal{I}$ is a p.l.i ideal $A \cup (a - A) \cup (A - a) \in \mathcal{I}$, so there exists $b \in G \setminus (A \cup (a - A) \cup (A - a))$. Then from (5.12) and the facts that G is an inner product group, H is an inner product space and i is Jensen we obtain

$$\begin{aligned} \|i(x) - i(y)\|^2 &= \|i(x - y) - i(0)\|^2 = \|i(a) - i(0)\|^2 \\ &= \tfrac{1}{2}(\|i(a + b) - i(0)\|^2 + \|i(a - b) - i(0)\|^2) - \|i(b) - i(0)\|^2 \\ &= \tfrac{1}{2}(\|a + b\|_d^2 + \|a - b\|_d^2) - \|b\|_d^2 = \|a\|_d^2 = \|x - y\|_d^2, \end{aligned}$$

which means that i is an isometry. □

6 Applications

In the following Section we will show corollaries of Theorem 5.1 concerning the stability of isometry eqaution and orthogonal equation.

Theorem 6.1 *(i) Let G be an inner product group and let $f : G \to \mathbf{R}$ be such that*

$$\lim_{\|x\|_d + \|y\|_d \to \infty} (\|x\|_d + \|y\|_d) \cdot \mathcal{I}f(x,y) = 0.$$

Then f is an isometry.

(ii) For every inner product group G and every $\varepsilon > 0$ there exists an inner product space H and a function $f : G \to H$ such that

$$\sup_{x,y \in G} (\|x\|_d + \|y\|_d) \cdot \mathcal{I}f(x,y) \leq \varepsilon, \tag{6.1}$$

but f is not an isometry.

Proof.

(i) We define

$$\mathcal{L} := \{g : G \to H \mid \lim_{\|x\|_d \to \infty} \|x\|_d \cdot \|g(x)\| = 0\},$$

$$\Omega := \{\phi : G \times G \to H \mid \lim_{\|x\|_d + \|y\|_d \to \infty} (\|x\|_d + \|y\|_d) \cdot \|\phi(x,y)\| = 0\},$$

and put $\mathcal{I} := \emptyset$. One can easily notice that $\mathcal{L}$ and Ω satisfy the assumptions of Theorem 3.1. Thus we obtain that f is an isometry.

(ii) Without loss of generality we may assume that $\varepsilon < 1$. As G is an inner product group there exists an inner product space H_G and a Jensen isometry $j : G \to H_G$. Let $H := H_G \oplus \mathbf{R}$ be a inner product space with the inner product norm given by the formula $\|(x,r)\| := \sqrt{\|x\|^2 + r^2}$. We define

$$f(x) := \begin{cases} (j(x), \frac{\varepsilon}{4}) & \text{if } \|x\|_d < 1, \\ (j(x), 0) & \text{if } \|x\|_d \geq 1. \end{cases}$$

We show that f satisfies (6.1).

Let $x, y \in G$ be arbitrary. As the equation is symmetric with respect to variables x, y we may assume that $\|x\|_d \leq \|y\|_d$.

If either $\|x\|_d < 1, \|y\|_d < 1$ or $\|x\|_d \geq 1, \|y\|_d \geq 1$ then (6.1) holds trivially.

Now suppose that $\|x\|_d < 1$, $1 \leq \|y\|_d \leq 3$. Then

$$\begin{aligned} (\|x\|_d + \|y\|_d) \cdot \mathcal{I}f(x,y) &= (\|x\|_d + \|y\|_d) \cdot |\|J(x) - j(y)\| - \|x-y\|_d| \\ &\leq 4 \cdot |\|(j(x),0) - (j(y), \tfrac{\varepsilon}{4})\| - \|x-y\|_d| \\ &= 4 \cdot (\sqrt{\|j(x) - j(y)\|^2 + (\tfrac{\varepsilon}{4})^2} - \|j(x) - j(y)\|) \\ &\leq 4 \cdot (\|j(x) - j(y)\| + \tfrac{\varepsilon}{4} - \|j(x) - j(y)\|) = \varepsilon. \end{aligned}$$

The only case left to consider is $\|x\|_d < 1$, $\|y\|_d \geq 3$. Then

$$\|x\|_d + \|y\|_d \leq 1 + \|y\|_d \leq 2(\|y\|_d - 1) \leq 2(\|y\|_d - \|x\|_d) \leq 2\|x-y\|_d.$$

This implies that

$$\begin{aligned} (\|x\|_d + \|y\|_d) \cdot \mathcal{I}f(x,y) &\leq 2\|x-y\|_d \cdot |\|(j(x), \tfrac{\varepsilon}{4}) - (j(y),0)\| - \|x-y\|_d| \\ &= 2\|x-y\|_d \cdot |\sqrt{\|j(x) - j(y)\|^2 + (\tfrac{\varepsilon}{4})^2} - \|x-y\|_d| \\ &= 2\|x-y\|_d \cdot (\sqrt{\|x-y\|_d^2 + (\tfrac{\varepsilon}{4})^2} - \|x-y\|_d) \\ &\leq 2\|x-y\|_d \cdot (\|x-y\|_d + \tfrac{1}{2\|x-y\|_d}(\tfrac{\varepsilon}{4})^2 - \|x-y\|_d) \\ &\leq 2\|x-y\|_d \cdot \tfrac{\varepsilon}{2\|x-y\|_d} \leq \varepsilon. \end{aligned}$$

□

Lemma 6.1 *Let X be a set, let $f : X \to X$, and let $A \subset X$. Then*

$$X \setminus f^{-1}(X \setminus A) = f^{-1}(A).$$

Proof. Obviously $f^{-1}(X \setminus A) = f^{-1}(X) \setminus f^{-1}(A) = X \setminus f^{-1}(A)$, so $X \setminus f^{-1}(X \setminus A) = f^{-1}(A)$. □

Theorem 6.2 *Let G be an inner product group, let H be an inner product space. We assume that $\mathcal{J} \subset \Omega(\mathcal{I})$ is a measurable p.l.i. ideal and that $\mathcal{J}$ is invariant with respect to M^{-1}, that is*

$$M^{-1}(J) \in \mathcal{J} \quad \text{for } J \in \mathcal{J}, \tag{6.2}$$

where M is the operation defined in Theorem 5.1.

Let $f : G \to H$ be such that

$$\|f(x) - f(y)\| = \|x - y\|_d \quad \text{for } \mathcal{J}\text{-a.a. } (x, y) \in G \times G.$$

Then there exists a unique isometry $i : G \to H$ such that

$$f(x) = i(x) \quad \text{for } \mathcal{I}\text{-a.a. } x \in G.$$

Proof. Let

$$\mathcal{L} := \{f : G \to H \mid f(x) = 0 \quad \text{for } \mathcal{I}\text{-a.a. } x \in G\},$$

$$\Omega := \{F : G \times G \to H \mid F(x, y) = 0 \quad \text{for } \mathcal{J}\text{-a.a. } (x, y) \in G.$$

As G is a measurable group, Ω is measurable. To show that the assumptions of Theorem 5.1 are satisfied we have to prove that Ω is invariant with respect to operation M.

So let $F \in \Omega$ be arbitrary. Then by the definition of Ω there exists $A_F \in \mathcal{J}$ such that $F|_{G \setminus A_F} = 0$. This implies that $F \circ M|_{M^{-1}(G \setminus A_F)} = 0$. Hence to prove that $F \circ M \in \Omega$ it is enough to show that $G \setminus M^{-1}(G \setminus A_F) \in J$. However, this holds as by Lemma 6.1 and by (6.2) we get

$$G \setminus M^{-1}(G \setminus A_F) = M^{-1}(A_F) \in \mathcal{J}.$$

□

For $A \subset G$ we put $\frac{1}{2}A := \{x \in G \ : \ 2x \in A\}$. To proceed further we will need the following lemma. By $\Sigma \times \Sigma$ we denote the σ-field of all measurable subsets of $G \times G$ with respect to the measure $(\lambda \times \lambda)$.

Lemma 6.2 *Let (G, Σ, λ) be a complete measurable group such that there exists $r \in \mathbf{R}$ with*

$$\tfrac{1}{2}A \in \Sigma, \ \lambda^+(\tfrac{1}{2}A) \le r\lambda(A) \quad \text{for } A \in \Sigma.$$

Then

$$M^{-1}(A) \in \Sigma \times \Sigma, \ (\lambda \times \lambda)(M^{-1}A) \le r(\lambda \times \lambda)(A) \quad \text{for } A \in \Sigma \times \Sigma.$$

Proof.

(Step 1) At first we will prove that

$$(\lambda \times \lambda)^+(M^{-1}(A)) \le r(\lambda \times \lambda)(A) \quad \text{for } A \in \Sigma \times \Sigma. \tag{6.3}$$

Let $A \in \Sigma \times \Sigma$. By the definition of the product measure $(\lambda \times \lambda)$, for every $n \in \mathbf{N}$ there exist sequences $\{A_{nk}\}_{k \in \mathbf{N}}, \{B_{nk}\}_{k \in \mathbf{N}} \subset \Sigma$ such that

$$A \subset \bigcup_{k \in \mathbf{N}} A_{nk} \times B_{nk},$$

$$\sum_{k\in\mathbf{N}} \lambda(A_{nk})\lambda(B_{nk}) \le (\lambda\times\lambda)(A) + \tfrac{1}{n}.$$

Hence for every $n \in \mathbf{N}$

$$(\lambda \times \lambda)^+(M^{-1}A_f) \le (\lambda \times \lambda)^+(M^{-1}(\bigcup_{k\in\mathbf{N}} A_{nk} \times B_{nk}))$$

$$\le \sum_{k\in\mathbf{N}}(\lambda \times \lambda)^+(M^{-1}(A_{nk} \times B_{nk})) = \sum_{k\in\mathbf{N}}(\lambda \times \lambda)^+(A_{nk} \times (\tfrac{1}{2}B_{nk}))$$

$$\le \sum_{k\in\mathbf{N}} \lambda(A_{nk}) \cdot \lambda^+(\tfrac{1}{2}B_{nk}) \le \sum_{k\in\mathbf{N}} \lambda(A_{nk}) \cdot r\lambda(B_{nk}) \le r((\lambda\times\lambda)(A) + \tfrac{1}{n}).$$

As $n \in \mathbf{N}$ is arbitrary we obtain (6.3).

(Step 2) Now we will show that

$$M^{-1}(A) \in \Sigma \times \Sigma \quad \text{for } A \in \Sigma \times \Sigma.$$

Let

$$\mathcal{A} := \{A \in \Sigma \times \Sigma \mid M^{-1}(A) \in \Sigma \times \Sigma\}.$$

Clearly for $A, B \in \Sigma$, $M^{-1}(A \times B) = A \times (\frac{1}{2}B) \in \mathcal{A}$. Let $\{A_n\}_{n\in\mathbf{N}} \subset \mathcal{A}$ be arbitrary. Then

$$M^{-1}(\bigcup_{n\in\mathbf{N}} A_n) = \bigcup_{n\in\mathbf{N}} M^{-1}(A_n),$$

which implies that $\bigcup_{n\in\mathbf{N}} A_n \in \mathcal{A}$. Now suppose that $A \in \mathcal{A}$. By Lemma 6.1 we obtain that $f^{-1}((G \times G) \setminus A) = (G \times G) \setminus f^{-1}(A)$, so $(G \times G) \setminus A \in \mathcal{A}$. By (6.3) we also get that if $(\lambda\times\lambda)(A) = 0$ then $(\lambda\times\lambda)^+(M^{-1}(A)) = 0$, so $A \in \mathcal{A}$.

To recapitulate: we have shown that $\mathcal{A}$ contains products of measurable sets from Σ, that $\mathcal{A}$ contains sets with measure zero, that the countable sum of elements of $\mathcal{A}$ belong to $\mathcal{A}$ and that $\mathcal{A}$ is closed under the operation $A \to G \times G \setminus A$. However, this means that $\mathcal{A} = \Sigma \times \Sigma$. □

As a trivial corollary of Theorem 6.2 and Lemma 6.2 we obtain the following result.

Corollary 6.1 *Let (G, Σ, λ) be a complete measurable inner product group and let H be an inner product space. We additionally assume that there exists $r \in \mathbf{R}$ such that*

$$\tfrac{1}{2}A \in \Sigma,\ \lambda^+(\tfrac{1}{2}A) \le r\lambda(A) \quad \textit{for } A \in \Sigma.$$

Then for every $f : G \to H$ be satisfying

$$\|f(x) - f(y)\| = \|x - y\|_d \quad \textit{for } (\lambda \times \lambda)\textit{-a.a. } (x, y) \in G \times G.$$

there exists a unique isometry $i : G \to H$ such that

$$f(x) = i(x) \quad \textit{for } \lambda\textit{-a.a. } x \in G.$$

The following corollary is similair to the results obtained by J. Chmieliński and J. Rätz in [2]. What is interesting is that although the method of proof in [2] are completely different an analogoue of condition (6.4) was also needed. Therefore it seems natural to ask whether this codition is really essential.

Corollary 6.2 *Let (G, Σ, λ) be a complete measurable inner product group, let H be a inner product space. We additionally assume that*

$$\tfrac{1}{2}A \in \Sigma,\ \lambda^{+}(\tfrac{1}{2}A) \leq r\lambda(A) \quad \textit{for } A \in \Sigma.$$

Let $f : G \to H$ be such that

$$< f(x), f(y) > = < x, y>_d \quad \textit{for } (x,y) \in G \times G \setminus D.$$

Suppose that $(\lambda \times \lambda)(D) = 0$ and that

$$\lambda(\{x \in G \mid (x,x) \in D\}) = 0. \tag{6.4}$$

Then there exists a unique orthogonal function $i : G \to H$ such that

$$f(x) = i(x) \quad \textit{for } \lambda\textit{-a.a. } x \in G.$$

Proof. Let $A := \{x \in G \mid (x,x) \in D\}$, and let $\widetilde{D} := D \cup A \times G \cup G \times A$. Then $(\lambda\times\lambda)(\widetilde{D}) \leq (\lambda\times\lambda)(D) + (\lambda\times\lambda)(A \times G) + (\lambda\times\lambda)(G \times A) = 0$, and therefore $(\lambda\times\lambda)(\widetilde{D}) = 0$. Let $(x,y) \in G \times G \setminus \widetilde{D}$ be arbitrary. Then

$$< f(x), f(x) > = < x, x>_d, \quad < f(y), f(y) > = < y, y>_d,$$

$$< f(x), f(y) > = < x, x>_d,$$

and therefore

$$\begin{aligned} \|f(x) - f(y)\|^2 &= < f(x), f(x) > + < f(y), f(y) > -2 < f(x), f(y) > \\ &= < x, x>_d + < y, y>_d - 2 < x, y>_d = \|x - y\|_d^2. \end{aligned}$$

Thus we have proved that

$$\|f(x) - f(y)\| = \|x - y\|_d \quad \text{for } (x,y) \in G \times G \setminus \widetilde{D}.$$

As $(\lambda\times\lambda)(\widetilde{D}) = 0$ by the previous corollary we obtain that there exists a unique isometry $i : G \to H$ such that $f(x) = i(x)$ for λ -a.a. $x \in G$.

We will show that i is an orthogonal function. Clearly i is Jensen, and therefore $i - i(0)$ is an additive isometry so it is orthogonal. From the assertions of the Corollary we know that

$$< i(x), i(y) > = < x, y>_d \quad \text{for } (\lambda\times\lambda)(x,y) \in G \times G.$$

However, as $i - i(0)$ is orthogonal function, $< i(x) - i(0), i(y) - i(0) > = < x, y>_d$ for $x, y \in G$, which implies that

$$< i(x), i(y) > = < i(x) - i(0), i(y) - i(0) > \quad \text{for } (\lambda\times\lambda)(x,y) \in G \times G,$$

and consequently that

$$< i(x), i(0) > + < i(0), i(x) > = < i(x) + i(y), i(0) > = 0 \quad \text{for } (\lambda\times\lambda)(x,y) \in G \times G.$$

As i is Jensen, $i(x)+i(y)=i(x+y)+i(0)$, so

$$<i(x+y)+i(0), i(0)>=0 \quad \text{for } (\lambda\times\lambda)(x,y)\in G\times G.$$

This implies that also

$$<i(-x-y)+i(0), i(0)>=0 \quad \text{for } (\lambda\times\lambda)(x,y)\in G\times G.$$

Adding up these two inequalities we obtain that

$$<i(x+y)+i(-x-y)+2i(0), i(0)>=<2i(0)+2i(0), i(0)>=0 \quad \text{for } (\lambda\times\lambda)(x,y)\in G\times G$$

so $\|i(0)\|=\sqrt{<i(0),i(0)>}=0$, which implies that i is additive. □

Now we determine when the isometry equation is superstable in the integral norm.

Theorem 6.3 *Let (G,Σ,λ) be a complete measurable group with $\lambda(G)=\infty$, and let $p\in(0,\infty)$.*

We assume that there exists $r\in\mathbf{R}_+$ such that

$$\tfrac{1}{2}A\in\Sigma,\ \lambda^+(\tfrac{1}{2}A)\le r\lambda(A) \quad \textit{for } A\in\Sigma.$$

(i) Suppose that

$$\frac{1}{1+\|x\|_d}\notin\mathcal{L}_p^+(G,\mathbf{R}).$$

Let $f:G\to H$ be such that

$$\mathcal{I}f\in\mathcal{L}_p^+(G\times G;\mathbf{R})$$

Then there exists a unique isometry $i:G\to H$ such that

$$f(x)=i(x) \quad \textit{for } \lambda\textit{-a.a. } x\in G.$$

(ii) Suppose that

$$\frac{1}{1+\|x\|_d}\in\mathcal{L}_p^+(G,\mathbf{R}).$$

We assume additionally that

$$\lambda(A)=0\Rightarrow\lambda(2A)=0 \quad \textit{for } A\in\Sigma. \tag{6.5}$$

Then there exists a Hilbert space H and a function $f:G\to H$ such that

$$\mathcal{I}f\in\mathcal{L}_p^+(G\times G,\mathbf{R}),$$

but there is no isometry $i:G\to H$ with $f(x)=i(x)$ for λ-a.a. $x\in G$.

Proof.

(i) As G is a measurable group $\mathcal{L}_p^+(G\times G;\mathbf{R})$ is measurable. The use Theorem 3.1 we have to prove that $\mathcal{L}_p^+(G\times G;\mathbf{R})$ is invariant with respect to operation M.

So let $f \in \mathcal{L}_p^+(G\times G;\mathbf{R})$ be arbitrary. Then there exist disjoint subsets $\{K_n\}_{n\in\mathbf{N}} \in \Sigma\times\Sigma$ and $\{k_n\}_{n\in\mathbf{N}} \subset \mathbf{R}_+$ such that

$$|f(x)| \le \sum_{i=0}^{\infty} k_i \cdot \chi_{K_i}(x) := \Psi(x) \quad \text{for } x \in G\times G,$$

and

$$\int_G \Psi(x)^p \lambda(x) = \sum_{i=0}^{\infty} (k_i)^p \cdot (\lambda\times\lambda)(K_i).$$

To show that $f\circ M \in \mathcal{L}_p^+(G\times G;\mathbf{R})$ it is clearly enough to show that $\Psi\circ M \in \mathcal{L}_p^+(G\times G;\mathbf{R})$, but

$$\Psi \circ M = \sum_{i=0}^{\infty} k_i \cdot \chi_{M^{-1}(K_i)},$$

so making use of Lemma 6.2 and the fact that $M^{-1}(K_i)$ are disjoint we obtain that

$$\int^+ \Psi(\lambda\times\lambda)\& = \& \int^+ (\sum_{i=0}^{\infty} k_i \cdot \chi_{M^{-1}K_i})^p \lambda(x)$$
$$\& = \& \sum_{i=0}^{\infty} (k_i)^p \cdot (\lambda\times\lambda)(M^{-1}(K_i))$$
$$\& \le \& \sum_{i=0}^{\infty} (k_i)^p \cdot r(\lambda\times\lambda)((K_i)) \le \infty.$$

(ii) Let H_G be a Hilbert space such that there exists a Jensen isometry $j : G \to H_G$. Let $H := H \oplus \mathbf{R}$ be a Hilbert space with the inner product norm given by the formula $\|(x,r)\| := \sqrt{\|x\|_G^2 + r^2}$. The fact that $\frac{1}{1+\|x\|_d} \in \mathcal{L}_+^p(G,\mathbf{R})$ implies that for every $r \in \mathbf{R}$, $\lambda_+(B(0,r)) < \infty$, where $B(0,r)$ denotes the ball with the center at zero and radius r.

Let $k \in \mathbf{N}$ be fixed such that $\lambda + (B(0,k)\setminus\{0\}) > 0$ (if for every $k \in \mathbf{N}$ we would have $\lambda^+(B(0,k)\setminus\{0\})) = 0$ then $\lambda^+(G) \le \lim_{k\to\infty} \lambda^+(B(0,k)\setminus\{0\}) + \lambda^+(\{0\}) = \lambda^+(\{0\}) < \infty$, a contradiction).

We define

$$f(x) := \begin{cases} (j(x),1) & \text{if } \|x\|_d < k, \\ (j(x),0) & \text{if } \|x\|_d \ge k. \end{cases}$$

(Step 1) Suppose, for contradiction, that there exists an isometry $i : G \to H$ such that

$$f(x) = i(x) \quad \text{for } \lambda\text{-a.a. } x \in G. \tag{6.6}$$

Then there exists $A \subset G$, $\lambda(A) = 0$ such that

$$f(x) = i(x) \quad \text{for } x \in G\setminus A.$$

Let $B = \bigcup_{n\in\mathbf{N}} 2^n A$ and let $x \in G\setminus(B\cup\{0\})$ be arbitrarily chosen. Then there exists $l \in \mathbf{N}$ such that $\|lx\|_d = l\|x\|_d > k$ and therefore

$$\begin{aligned} i(x) &= (i(x)-i(0)) + i(0) = \tfrac{i(lx)-i(0)}{l} + i(0) \\ &= \tfrac{f(lx)-i(0)}{l} + i(0) = \tfrac{(j(lx),0)-i(0)}{l} + i(0) \\ &= \tfrac{(j(lx),0)-(j(0),0)}{l} + \tfrac{(j(0),0)-i(0)}{l} + i(0) \\ &= (j(x),0) - (j(0),0) + \tfrac{(j(0),0)-i(0)}{l} + i(0) \\ &= (j(x),0) + (1-\tfrac{1}{l})(i(0)-(j(0),0)). \end{aligned} \tag{6.7}$$

Replacing in the above inequality l by $l+1$ we obtain that

$$(j(x),0)+(1-\frac{1}{l+1})(i(0)-(j(0),0))=i(x)=(j(x),0)+(1-\frac{1}{l})(i(0)-(j(0),0)).$$

so $i(0)=(j(0),0)$. Thus (6.7) means that

$$i(x)=(j(x),0) \quad \text{for } x\in G\setminus(B\cup\{0\}).$$

This implies that

$$i(x)=(j(x),0)\neq(j(x),1)=f(x) \quad \text{for } x\in B(0,k)\setminus(B\cup\{0\}).$$

However by (6.50 $\lambda(B)=0$, so $\lambda_+(B(0,k)\setminus(B\cup\{0\}))=\lambda_+(B(0,k)\setminus\{0\})>0$ by the way we have chosen k. We have obtained a contradiction with (6.6).

(Step 2) Now we will prove that

$$\mathcal{I}f\in\mathcal{L}^p_+(G\times G,\mathbf{R}), \tag{6.8}$$

that is that

$$\int^+_{G\times G}|\|f(x)-f(y)\|-\|x-y\|_d|^p\,(\lambda\times\lambda)(x,y)<\infty.$$

Let

$$A_1:=\{(x,y)\in G\times G\,|\,\|x\|_d\le k,\|y\|_d>k\},$$
$$A_2:=\{(x,y)\in G\times G\,|\,\|x\|_d>k,\|y\|_d\le k\}.$$

Clearly $\mathcal{I}f|_{(G\times G)\setminus(A_1\cup A_2)}=0$, so to prove (6.8) it is enough to prove that $\mathcal{I}f\cdot\chi_{A_1}\in\mathcal{L}^+_p(G\times G;\mathbf{R})$ and that $\mathcal{I}f\cdot\chi_{A_2}\in\mathcal{L}^+_p(G\times G;\mathbf{R})$. However, the function $\mathcal{I}f$ is symetric with respect to variables x and y, so $\int^+_{G\times G}\mathcal{I}f\cdot\chi_{A_1}(\lambda\times\lambda)(x,y)=\int^+_{G\times G}\mathcal{I}f\cdot\chi_{A_2}(\lambda\times\lambda)(x,y)$, which implies that to prove (6.8) it is enough to prove that

$$\mathcal{I}f\cdot\chi_{A_1}\in\mathcal{L}^+_p(G\times G;\mathbf{R}). \tag{6.9}$$

As $\lambda_+(B(0,4k))<\infty$ there exists a measurable set $B_4\subset G$ such that $B(0,4k)\subset B_4$ and $\lambda(B_4)<\infty$. Let $B_1:=\frac{1}{4}B_4$. Then clearly $\lambda(B_1)<\infty$ and $B(0,k)\subset B_1$. Therefore to prove (6.9) we have to prove that

$$\mathcal{I}f\cdot\chi_{B_1\times G}\in\mathcal{L}^+_p(G\times G;\mathbf{R}).$$

As the set $B_1\times G$ is measurable in $G\times G$ it is equivalent to show that

$$\int^+_{B_1\times G}\mathcal{I}f(x,y)^p\,(\lambda\times\lambda)(x,y)<\infty.$$

Let

$$a(p):=\int^+_{B_1\times B_4}\mathcal{I}f(x,y)^p\,(\lambda\times\lambda)(x,y),$$
$$b(p):=\int^+_{B_1\times(G\setminus B_4)}\mathcal{I}f(x,y)^p\,(\lambda\times\lambda)(x,y).$$

Clearly

$$\int_{B_1\times G}^{+} \mathcal{I}f(x,y)^p \ (\lambda\times\lambda)(x,y) \leq a(p)+b(p),$$

However, as $B_1 \times B_4$ has finite measure and $\mathcal{I}f$ is locally bounded, we obtain that $a(p) < \infty$.

Now we will show that $b(p) < \infty$ which will end the proof. By the definition of f

$$\begin{aligned}
&\int_{B_1\times(G\setminus B_4)}^{+} \mathcal{I}f(x,y)^p(\lambda\times\lambda)(x,y)\\
&\quad= \int_{B_1\times(G\setminus B_4)}^{+} |\sqrt{\|x-y\|_d^2+1}-\|x-y\|_d|^p(\lambda\times\lambda)(x,y)\\
&\quad\leq \int_{B_1\times(G\setminus B_4)}^{+} (\|x-y\|_d+\frac{1}{2\|x-y\|_d}-\|x-y\|_d)^p(\lambda\times\lambda)(x,y)\\
&\quad= \int_{B_1\times(G\setminus B_4)}^{+} \frac{1}{(2\|x-y\|_d)^p}\,(\lambda\times\lambda)(x,y).
\end{aligned}$$

Obviously if $(x,y) \in B_1 \times (G \setminus G_4)$, then $\|x\|_d \leq k$, $\|y\|_d > 4k$, so

$$2\|x-y\|_d \geq \|y\| + (\|y\| - 2\|x\|) \geq \|y\| + 1.$$

Hence

$$\int_{B_1\times(G\setminus B_4)}^{+} \frac{1}{(2\|x-y\|_d)^p}\,(\lambda\times\lambda)(x,y) \leq \int_{B_1\times(G\setminus B_4)}^{+} \frac{1}{(1+\|y\|_d)^p}\,(\lambda\times\lambda)(x,y).$$

Making use of the Fubini Theorem we now obtain that

$$\int_{B_1\times(G\setminus B_4)}^{+} \frac{1}{(1+\|y\|_d)^p}\,(\lambda\times\lambda)(x,y) \leq \int_{G\setminus B_4}^{+} \left\{\int_{B_1}^{+} \frac{1}{(1+\|y\|_d)^p}\lambda(x)\right\}\lambda(y).$$

$$= \lambda(B_1)\cdot\int_{G\setminus 4B_1}^{+} \frac{1}{1+\|y\|_d}.\lambda(y).$$

However we have chosen B_1 in such a way that $\lambda(B_1) < \infty$ and by the assumptions $\int_G^+ \frac{1}{1+\|y\|_d}\lambda(y) < \infty$, so $\mathcal{I}f \in \mathcal{L}_p^+(G\times G;\mathbf{R})$. □

Corollary 6.3 *Let G be an inner product group. We assume that (G,λ) is either $(\mathbf{R}^n,\lambda_n)$ or $(\mathbf{Z}^n,\delta_n)$, where δ_n denotes the counting measure on $\mathbf{R}^n$. Let $p\in(0,\infty)$ be arbitrary.*

(i) We assume that $p\leq n$. Let H be an inner product space. Let $f: G\to H$ be such that

$$\mathcal{I}f \in \mathcal{L}_p^+(G\times G;\mathbf{R}).$$

Then there exists a unique isometry $i: G\to H$ such that

$$f(x) = i(x) \quad \textit{for } \lambda\textit{-a.a. } x\in G.$$

(ii) If $p>n$ then there exists an inner product space H and a function $f: G\to H$ such that there is no isometry $i: G\to H$ with

$$f(x) = i(x) \quad \textit{for } \lambda\textit{-a.a. } x\in G.$$

Proof. By Theorem 6.3 it is enough to determine whether the following condition holds

$$\frac{1}{1+\|x\|} \in \mathcal{L}^p_+(G,\mathbf{R}) \text{ iff } p > n.$$

(i) Assume that $(G,\lambda) = (\mathbf{R}^n, \lambda_n)$. Making use of the change of variables one can easily notice that

$$\int_{\mathbf{R}^n} \frac{1}{(1+\|x\|)^p} \lambda_n(x) = V(S_{n-1}(0,1)) \int_{\mathbf{R}_+} \frac{1}{(1+r)^p} r^{n-1} dr,$$

where $V(S_{n-1}(0,1))$ denotes the $(n-1)$ dimensional Lebesgue measure of the unit sphere $S_{n-1}(0,1) \subset \mathbf{R}^n$. Clearly

$$\int_{\mathbf{R}_+} \frac{1}{(1+r)^p} r^{n-1} dr < \infty \text{ iff } \int_{(1,\infty)} \frac{1}{(1+r)^p} r^{n-1} dr < \infty.$$

But

$$\int_{(1,\infty)} \frac{1}{(2r)^p} r^{n-1} dr \leq \int_{(1,\infty)} \frac{1}{(1+r)^p} r^{n-1} dr \leq \int_{\mathbf{R}_+} \frac{1}{r^p} r^{n-1} dr,$$

which implies that

$$\int_{(1,\infty)} \frac{1}{(1+r)^p} r^{n-1} dr < \infty \text{ iff } \int_{(1,\infty)} \frac{1}{r^p} r^{n-1} dr.$$

However, $\int_{(1,\infty)} r^{n-p-1} dr < \infty$ iff $n-p-1 < -1$ that is iff $p > n$.

(ii) We assume that $(G,\lambda) = (\mathbf{Z}^n, \delta_n)$. If $a, b \in \mathbf{R}$ then by $[a,b]_{\mathbf{Z}}$ we denote $[a,b] \cap \mathbf{Z}$. For $k \in \mathbf{N}$ let $S_k := [-k,k]^n_{\mathbf{Z}} \setminus [-k+1,k-1]^n_{\mathbf{Z}}$. Clearly $\#K_n = (2k+1)^n - (2k-1)^n$. Let

$$\begin{aligned} m &:= \min\{\|x\| \mid x \in [-1,1]^n \setminus (-1,1)^n\}, \\ M &:= \max\{\|x\| \mid x \in [-1,1]^n \setminus (-1,1)^n\}. \end{aligned}$$

The reader can easily notice that because $S_k/k \subset [-1,1]^n \setminus (-1,1)^n$,

$$\min\{\|x\| \mid x \in S_k\} \geq km,$$

$$\max\{\|x\| \mid x \in S_k\} \leq kM.$$

Then

$$\begin{aligned} \sum_{x\in\mathbf{Z}^n} \frac{1}{(1+\|x\|)^p} &\leq \sum_{k=0}^{\infty} \#S_k \cdot \max_{x\in S_k} \frac{1}{(1+\|x\|)^p} \\ &= \sum_{k=0}^{\infty} \#S_k \cdot \frac{1}{(1+\min\limits_{x\in S_k}\|x\|)^p} \\ &\leq \sum_{k=0}^{\infty} \#S_k \cdot \frac{1}{(1+mk)^p} \\ &= \sum_{k=0}^{\infty} \frac{(2k+1)^n - (2k-1)^n}{(1+km)^p} =: M(p). \end{aligned}$$

Analogously

$$\sum_{x\in\mathbf{Z}^n}\frac{1}{(1+\|x\|)^p}\geq\sum_{k=0}^{\infty}\frac{(2k+1)^n-(2k-1)^n}{(1+kM)^p}=:m(p).$$

One can easily check that $M(p)<\infty$ iff $p>n$ and that $m(p)<\infty$ iff $p>n$. As

$$m(p)\leq\sum_{x\in\mathbf{Z}^n}\frac{1}{(1+\|x\|)^p}\leq M(p)$$

we obtain that $\sum_{x\in\mathbf{Z}^n}\frac{1}{(1+\|x\|)^p}<\infty$ iff $p>n$. □

The previous corollary shows that the isometry equation is stable in integral norm iff $p\leq n$. However there arises a natural question if for $p>n$ this equation is stable in integral norm. This leads to the following problem.

Problem 6.1 Let H be a Hilbert space. Suppose that $p>n$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that if $f:\mathbf{R}^n\to H$ satisfies

$$\|\mathcal{I}f\|_p^+\leq\delta$$

then there exists an isometry $i:\mathbf{R}^n\to H$ with

$$\|f-i\|_p^+\leq\varepsilon?$$

References

[1] J. Aczél, J. Dhombres, *Functional equations in several variables,* Encyclopedia of mathematics and its applications, Cambridge University Press, 1989.

[2] J. Chmieliński, J. Rätz, *Orthogonality equation almost everywhere,* Publicationes Math. Debrecen (submitted).

[3] R. Ger, *A survey of recent results on stability of functional equations,* Proc. of the 4th International Conference on Functional Equations and Inequalities, Pedagogical University in Cracow, 1994, 5-36.

[4] —, *Note on almost additive functions,* Aequationes Math. **24** (1978), 73-76.

[5] R. R. Halmos, *Measure Theory,* Van Nostrand Reinhold Company, New York Cincinnati Toronto London Melbourne, 1950.

[6] D. H. Hyers, *On the stability of the linear functional equation,* Proc. Nat. Acad. Sci. U.S.A **27** (1941), 222–224.

[7] M. Kuczma, *An introduction to the theory of functional equations and inequalities,* Polish Scientific Publishers (PWN) and Silesian University Press, Warszawa-Kraków-Katowice, 1985.

[8] M. Omladič, P. Šemrl, *On non linear perturbations of isometries,* Math. Ann. bf 303, 617–628 (1995).

[9] S. Siudut, *Cauchy difference operator in $\mathcal{L}^p$ spaces,* Functiones et Approximato, Commentari Mathematici, Adam Mickiewicz University in Poznań (to appear) .

[10] Jacek Tabor, *Lipschitz Stability of the Cauchy and Jensen Equations,* Results Math. **32** (1997), no. 1-2, 133–144.

[11] Jacek Tabor, Józef Tabor, *Stability of the Cauchy Type Equations in the Class of Differentiable Functions,* manuscript.

[12] Józef Tabor, *Stability of the Cauchy type equations in $\mathcal{L}^p$ norms,* Results Math. **32** (1997), no. 1-2, 145–158.

[13] Józef Tabor, *Stability of the Cauchy type equations in $\mathcal{L}^p$ norm,* Proceedings of the 5-th International Conference on Functional Equations and Inequalities, Annales de l'Ecole Normale Supérieure A Cracovie (to appear).

[14] Józef Tabor, *Problem 14, The Thirty-fourth International Symposium on Functional Equations, June 10 to 19, 1996, Wisła-Jawornik, Poland,* Aeq. Math. **53**, no. 1-2 (1997), pp. 191.

[15] S. M. Ulam, *A collection of mathematical problems,* Interscience Publ., New York, 1960.

Jacek Tabor
Institute of Mathematics
Jagiellonian University
Reymonta 4 st.
30-059 Kraków
Poland

e-mail: tabor@im.uj.edu.pl

Eingegangen am 10. Mai 1998