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Results in Mathematics

CUBIC FORM GEOMETRY FOR IMMERSIONS IN CENTRO-AFFINE AND GRAPH HYPERSURFACES

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ABSTRACT. We study hypersurfaces whose traceless cubic form vanishes.

1. Introduction and main results

For a hypersurface (M^n, ∇) of $(\mathbb{R}^{n+1}, D)$ with second fundamental form h, the behaviour of ∇h plays a prominent role in both Riemannian geometry and affine differential geometry (ADG).

In the Riemannian setting, M^n is called "parallel" or also "(extrinsic locally) symmetric" if $\nabla h \equiv 0$. If M^n is parallel, then M^n is an open part of a hypersphere, or an open part of a hyperplane, or an open part of a product of an affine subspace $\mathbb{R}^{n-k}$ and a sphere S^k in a $(k+1)$-dimensional affine subspace $\mathbb{R}^{k+1}$, orthogonal to $\mathbb{R}^{n-k}$, see [8]. For a classification of parallel submanifolds of $\mathbb{R}^{n+p}$, which can be defined in a similar way, see [1]. For further generalizations, e.g. parallel submanifolds in space-forms, we refer to the survey [2].

In the classical affine differential geometry, the cubic form C of a Blaschke hypersurface (see below) is given by $C := \nabla h$. The theorem of Pick and Berwald states that a Blaschke hypersurface has vanishing cubic form if and only if it is a non-degenerate hyperquadric. In this way, non-degenerate quadrics are characterized in a differential geometric way.
This theorem has been generalized in many directions, e.g. Simon [7] and Nomizu and Pinkall [4] showed that a non-degenerate hypersurface (with an arbitrary affine structure) of $\mathbb{R}^{n+1}$ with $h \mid C$ (i.e. when there is a one-form ρ such that

$$C(X,Y,Z) = \rho(X)h(Y,Z) + \rho(Y)h(Z,X) + \rho(Z)h(X,Y)$$

for all X, Y and $Z \in \mathfrak{X}(M)$) is a hyperquadric.

Recently, Lusala [3] applied methods from ADG to investigate a cubic form C for non-degenerate hypersurfaces of space-forms as follows. Given a hypersurface (M^n, ∇) of a

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space-form $(\widetilde{M}^{n+1}(c), \widetilde{\nabla})$ with non-degenerate second fundamental form h, put $C(X,Y,Z) = (\nabla h)(X,Y,Z)$ for X,Y,Z in $\mathfrak{X}(M)$.
Since h is non-degenerate, we can define the Tchebychev vector field T by

$$\begin{aligned} h(T,u) &= \frac{1}{n} tr_h C(u,\cdot,\cdot) \\ &= \frac{1}{n}\sum_{i=1}^{n} \epsilon_i C(u,e_i,e_i) \text{ for all } u \ , \end{aligned}$$

where e_i is an orthonormal frame w.r.t. h; i.e. $h(e_i,e_j) = \epsilon_i \delta_{ij}$ with $\epsilon_i = \pm 1$. The symmetric traceless part $\widetilde{C}$ of C is given by

$$\widetilde{C}(X,Y,Z) = C(X,Y,Z) - \frac{n}{n+2}\left(h(X,Y)h(Z,T) + h(Z,X)h(Y,T) + h(Y,Z)h(X,T)\right)$$

for X,Y,Z in $\mathfrak{X}(M^n)$.
One can now try to classify hypersurfaces for which $\widetilde{C}$ vanishes identically, hereby generalizing the notion of parallel submanifolds. In [3], this has been done for M^2 in $S^3(1)$.

In this paper, we prove several generalizations of the main theorem of [3]. Our first results deal with (pseudo-)Riemannian manifolds.

Theorem 1. *Let M^n be a Riemannian hypersurface of $S^{n+1}(1)$ with non-degenerate second fundamental form. Let $\widetilde{C}$ be the symmetric traceless part of its cubic form, then $\widetilde{C}$ vanishes identically if and only if M^n is the intersection of the sphere with a non-degenerate quadratic cone which is centered at the origin.*

In fact, the same proof holds if the surrounding space is pseudo-Riemannian.

Theorem 2. *Let M^n be a pseudo-Riemannian hypersurface of $S^{n+1}_m(\pm 1) \subset \mathbb{R}^{n+2}_k$ with non-degenerate second fundamental form. Let $\widetilde{C}$ be the symmetric traceless part of its cubic form, then $\widetilde{C}$ vanishes identically if and only if M^n is the intersection of $S^{n+1}_m(\pm 1)$ with a non-degenerate quadratic cone which is centered at the origin.*

These results turn out to be a consequence of the fact that the conditions $\widetilde{C} \equiv 0$ and $h \mid C$ are equivalent (cf. the proof of Proposition 1 in section 3), and the following more general result in ADG.

Theorem 3. *Let f be a non-degenerate affine immersion of (M^n, ∇) into $(\widetilde{M}^{n+1}, \widetilde{\nabla})$, where $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ is a centro-affine hypersurface of $\mathbb{R}^{n+2}$ w.r.t. a point o. Let h be the second fundamental form of M^n in $\widetilde{M}^{n+1}$ and C its cubic form. Then $h \mid C$ if and only if $f(M^n)$ is the intersection of $\widetilde{M}^{n+1}$ with a non-degenerate quadratic cone which is centered at the point o.*

Non-degenerate affine immersions into a graph hypersurface $\widetilde{M}$ of $\mathbb{R}^{n+2}$ (i.e. when the connection on $\widetilde{M}$ is induced by a constant transversal vector field) for which $h \mid C$ can be characterized as well.

Theorem 4. *Let f be a non-degenerate affine immersion of (M^n, ∇) into a graph hypersurface $\widetilde{M}^{n+1}$ of $\mathbb{R}^{n+2}$. Let h be the second fundamental form of M^n in $\widetilde{M}^{n+1}$ and C its* *cubic form. Then $h \mid C$ if and only if $f(M^n)$ is the intersection of $\widetilde{M}$ with a cylinder on a non-degenerate quadric of which the rulings are parallel to the normal of $\widetilde{M}^{n+1}$.*

Now, we can deal with affine spheres. If the shape operator of a Blaschke hypersurface $\widetilde{M}$ is identically zero (i.e. its Blaschke normal is constant), $\widetilde{M}$ is called an improper affine sphere; if its shape operator is a constant nonzero multiple of the identity, $\widetilde{M}$ is called a proper affine sphere. In the latter case, one can show that its normals meet at a point, called the center (see e.g. [5]). The two previous theorems then lead to the following result.

Corollary 1. *Let f be a non-degenerate affine immersion of (M^n, ∇) into an affine sphere $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ in $\mathbb{R}^{n+2}$. Let h be the second fundamental form of M^n in $\widetilde{M}^{n+1}$ and C its cubic form. If $h \mid C$, then $f(M^n)$ is the intersection of $\widetilde{M}^{n+1}$ with a hyperquadric.*

Since a central hyperquadric with its Blaschke structure is a centro-affine hypersurface w.r.t. the center of the hyperquadric, and a paraboloid with its Blaschke structure is an improper affine sphere, we also have the following consequence.

Corollary 2. *Let $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ be a non-degenerate hyperquadric in $\mathbb{R}^{n+2}$, equipped with its Blaschke normal. Let f be a non-degenerate affine immersion of (M^n, ∇) into $\widetilde{M}^{n+1}$. Then,*

(a) *in case $\widetilde{M}^{n+1}$ is a central quadric, $h \mid C$ if and only if $f(M^n)$ is the intersection of $\widetilde{M}^{n+1}$ with a non-degenerate cone which vertex coincides with the center of $\widetilde{M}^{n+1}$, or*
(b) *in case $\widetilde{M}^{n+1}$ is a paraboloid, $h \mid C$ if and only if $f(M^n)$ is the intersection of $\widetilde{M}^{n+1}$ with a cylinder on a non-degenerate quadric of which the rulings are parallel to the normal of $\widetilde{M}^{n+1}$.*

The last result proves a conjecture of Lusala (personal communication).

2. Preliminaries

Let M^n and $\widetilde{M}^{n+1}$ be manifolds with a torsion-free affine connection ∇ resp. $\widetilde{\nabla}$. Let $f : M^n \to \widetilde{M}^{n+1}$ be an immersion for which there is a transversal vector field ξ s.t.

$$\widetilde{\nabla}_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi \tag{2.1}$$

for all $X, Y \in \mathfrak{X}(M^n)$. Then f is said to be an affine immersion of (M, ∇) into $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ and h is called the affine fundamental form; see [5]. For an affine immersion f of (M^n, ∇) into $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ with transversal vector field ξ, the (affine) shape operator S and the transversal connection form τ are defined by

$$\widetilde{\nabla}_X \xi = -f_*(S(X)) + \tau(X)\xi. \tag{2.2}$$

Consider an immersion $f : M \to (\widetilde{M}, \widetilde{\nabla})$ and a transversal vector field ξ; from (2.1), the pair (f, ξ) induces a connection ∇ on M. Then, with this choice of ξ, the immersion considered is an affine immersion.
Sometimes we will identify M with its image and refrain from denoting the immersion.

M^n is called non-degenerate if h is non-degenerate (and this condition is independent of the choice of ξ).

Remark. A cone in $\mathbb{R}^n$ is a set consisting of half-lines emanating from some point v, the vertex of the cone. A quadratic cone Q with vertex v is called non-degenerate if it does not contain a entire straight line, i.e. there exists an affine coordinate system $\{x^1, \dots, x^n\}$ on $\mathbb{R}^n$ in which Q is given by $\sum_{i=1}^n a_i(x^i - v^i)^2 = 0$ and $x^n > v^n$ with $a_i \in \mathbb{R} \setminus \{0\}$.
Note that this notion of non-degeneracy of a cone does not correspond to the concept of a non-degenerate immersion. Indeed, for the inclusion $\iota : Q \hookrightarrow \mathbb{R}^{n+1}$ the second fundamental form of a cone Q is always degenerate.

The cubic form of (M^n, ∇) in $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ is defined by

$$C(X, Y, Z) := (\nabla_X h)(Y, Z) + \tau(X) h(Y, Z) ,$$

for $X, Y, Z \in \mathfrak{X}(M^n)$.

The cubic form is called divisible by h (see [4]) if there exists a one-form ρ such that, for all $X, Y, Z \in \mathfrak{X}(M^n)$,

$$C(X, Y, Z) = \rho(X) h(Y, Z) + \rho(Y) h(Z, X) + \rho(Z) h(X, Y),$$

or, equivalently,

$$C(X, X, Y) = 3\rho(X) h(X, X),$$

and this property is denoted by $h \mid C$. One can show that this property does not depend on the choice of the transversal vector field.

Now take $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ to be $(\mathbb{R}^{n+1}, D)$, the affine space with its usual flat connection and fix an affine coordinate system $\{x^1, \dots, x^{n+1}\}$ on $\mathbb{R}^{n+1}$. Let M^n be a hypersurface of $\mathbb{R}^{n+1}$. If the position vector is transversal to the tangent space at each point x of M^n, one can take $\xi = -x$ and consider the induced connection on M^n given by (2.1); with this choice of ξ, (M^n, ∇) is called a centro-affine hypersurface.

Let again $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ be $(\mathbb{R}^{n+1}, D)$. A hypersurface M of $\mathbb{R}^{n+1}$ is called a graph hypersurface if the connection ∇ on M is induced by a constant transversal vector field ξ, i.e. $D_X \xi = 0$. Taking $\xi = (0, \dots, 0, 1)$, then M is locally given by

$$\{(x^1, \dots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} \mid (x^1, \dots, x^n) \in U \text{ and } x^{n+1} = F(x^1, \dots, x^n)\} ,$$

where U is a connected open part of $\mathbb{R}^n$ and F is a smooth function on U.

If $(\widetilde{M}^{n+1}, \widetilde{\nabla})$ is equipped with a parallel volume element ω (e.g. $(\mathbb{R}^{n+1}, D)$ with its volume form given by the determinant), and M is non-degenerate, there exists (up to sign) a unique choice of ξ such that

(i) $\tau = 0$, or equivalently $\nabla \theta = 0$, and
(ii) θ coincides with the volume element ω_h of the non-degenerate metric h,

where θ denotes the induced volume form given by

$$\theta(X_1, \dots, X_n) = \omega(f_*(X_1), \dots, f_*(X_n), \xi)$$

for $X_1, \ldots, X_n \in \mathfrak{X}(M)$. This choice of ξ is called the Blaschke normal and (M^n, ∇) is called a Blaschke hypersurface.

We can now state the theorem of Pick and Berwald.

Theorem 5. *Let $f : (M^n, \nabla) \to (\mathbb{R}^{n+1}, D)$ be a Blaschke hypersurface. If its cubic form C vanishes identically, then $f(M^n)$ is a hyperquadric in $\mathbb{R}^{n+1}$.*

This theorem has been generalized in many directions; e.g.

Theorem 6 ([7]; [4]). *Let $f : (M^n, \nabla) \to (\mathbb{R}^{n+1}, D)$ be a non-degenerate immersion. If C is divisible by h, then $f(M^n)$ lies in a hyperquadric.*

To transfer the concept of the cubic form from ADG to Riemannian geometry for hypersurfaces of space-forms, we take a unit Euclidean normal vector field for the transversal vector field ξ from above and see that the cubic form C is given then by ∇h since $\tau = 0$.

To conclude this section, we recall that the Codazzi equation for h states that the difference $C(X,Y,Z) - C(Y,X,Z)$ equals the transversal (or normal) part of $\widetilde{R}(X,Y)Z$. This implies that the cubic form in all cases under consideration (immersions into centro-affine hypersurfaces or into graph hypersurfaces; hypersurfaces of space-forms) is totally symmetric.

3. Proofs

We deal with the proofs of Theorem 3 and 4 first. Then we show how these lead to Theorem 1 and 2.

3.1. **Proof of Theorem 3.** Let $\widetilde{M}^{n+1}$ be a centro-affine hypersurface of $\mathbb{R}^{n+2}$ w.r.t. the point o and let f be an immersion of M^n into $\widetilde{M}^{n+1}$. Since our considerations will be local, we may assume $M^n \subset \widetilde{M}^{n+1}$ and we will use the notation M and $\widetilde{M}$ throughout for the local situation.

Define $F : \widetilde{M} \subset \mathbb{R}^{n+2} \to \mathbb{R}^{n+2} : x \mapsto \lambda(x)x$ with $\lambda > 0$ such that the image of $\widetilde{M}$ under F is contained in a hyperplane of $\mathbb{R}^{n+2}$ not passing through o. We will denote this hyperplane by $\mathbb{R}^{n+1}$ and the image of M under F by $\overline{M} = F(M)$.

We have the following immersions:

- $(M, \nabla) \hookrightarrow (\widetilde{M}, \widetilde{\nabla})$ with transversal vector field ξ, fundamental form h, shape operator S and transversal connection form τ,
- $(\widetilde{M}, \widetilde{\nabla}) \hookrightarrow (\mathbb{R}^{n+2}, D)$ with transversal vector field $-x$ and fundamental form $\widetilde{h}$ and
- $F : (M, \overline{\nabla}) \hookrightarrow (\mathbb{R}^{n+1}, D)$ with transversal vector field $\overline{\xi} = F_*(\xi)$, fundamental form $\overline{h}$, shape operator $\overline{S}$ and transversal connection form $\overline{\tau}$.

For $V \in T_p\widetilde{M}$, one has

$$F_*(V) = \lambda V + V(\lambda)x.$$

For $X, Y \in T_pM$ we have

$$\begin{aligned} D_X F_*(Y) &= D_X(\lambda Y + Y(\lambda)x) \\ &= \lambda D_X Y + X(\lambda)Y + Y(\lambda)X + X(Y(\lambda))x \end{aligned}$$

and, because of

$$D_X Y = \nabla_X Y + h(X,Y)\xi + \widetilde{h}(X,Y)(-x),$$

we find

$$D_X F_*(Y) = \lambda\left(\nabla_X Y + h(X,Y)\xi - \widetilde{h}(X,Y)x\right) + X(\lambda)Y + Y(\lambda)X + X(Y(\lambda))x.$$

Keeping in mind that $F_*(\xi) = \lambda\xi + \xi(\lambda)x$, we also obtain

$$\begin{aligned} D_X F_*(Y) &= F_*(\overline{\nabla}_X Y) + \overline{h}(X,Y)(\lambda\xi + \xi(\lambda)x) \\ &= \lambda\overline{\nabla}_X Y + X(Y(\lambda))x + \overline{h}(X,Y)(\lambda\xi + \xi(\lambda)x) \\ &= \lambda\overline{\nabla}_X Y + \overline{h}(X,Y)\lambda\xi + (X(Y(\lambda)) + \overline{h}(X,Y)\xi(\lambda))x. \end{aligned}$$

By comparing the coefficients of ξ and x and taking the tangential part of both expressions for $D_X F_*(Y)$, we find

$$\begin{aligned} h(X,Y) &= \overline{h}(X,Y) \\ -\lambda\widetilde{h}(X,Y) &= \overline{h}(X,Y)\xi(\lambda) \\ \lambda\overline{\nabla}_X Y &= \lambda\nabla_X Y + X(\lambda)Y + Y(\lambda)X\ , \end{aligned}$$

in particular

$$\overline{\nabla}_X Y = \nabla_X Y + \rho(X)Y + \rho(Y)X,$$

where $\rho = d\log\lambda$.
Calculating $D_X F_*(\xi)$ in two ways gives

$$\begin{aligned} D_X F_*(\xi) &= D_X(\lambda\xi + \xi(\lambda)x) \\ &= X(\lambda)\xi + \xi(\lambda)X + X(\xi(\lambda))x + \lambda(-S(X) + \tau(X)\xi) \\ D_X F_*(\xi) &= -F_*(\overline{S}(X)) + \overline{\tau}(X)F_*(\xi) \\ &= -\lambda\overline{S}(X) - (\overline{S}(X))(\lambda)x + \overline{\tau}(X)(\lambda\xi + \xi(\lambda)x))\ . \end{aligned}$$

Equating the component of ξ in these formulae gives

$$\lambda\overline{\tau}(X) = X(\lambda) + \lambda\tau(X),$$

hence

$$\overline{\tau}(X) = \rho(X) + \tau(X).$$

The cubic form of $\overline{M}$ in $\mathbb{R}^{n+1}$ is given by

$$\begin{aligned}
\overline{C}(X,Y,Z) &= (\overline{\nabla}_X\overline{h})(Y,Z) + \overline{\tau}(X)\overline{h}(Y,Z) \\
&= (\overline{\nabla}_X\overline{h})(Y,Z) + (\rho(X)+\tau(X))h(Y,Z) \\
&= X(h(Y,Z)) + (\rho(X)+\tau(X))h(Y,Z) \\
&\quad -\overline{h}(\nabla_X Y + \rho(X)Y + \rho(Y)X, Z) \\
&\quad -\overline{h}(Y, \nabla_X Z + \rho(X)Z + \rho(Z)X) \\
&= X(h(Y,Z)) + \rho(X)h(Y,Z) + \tau(X)h(Y,Z) \\
&\quad -h(\nabla_X Y, Z) - \rho(X)h(Y,Z) - \rho(Y)h(X,Z) \\
&\quad -h(Y,\nabla_X Z) - \rho(X)h(Y,Z) - \rho(Z)h(Y,X) \\
&= X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y,\nabla_X Z) + \tau(X)h(Y,Z) \\
&\quad -\rho(X)h(Y,Z) - \rho(Y)h(X,Z) - \rho(Z)h(X,Y) \\
&= (\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) \\
&\quad -\rho(X)h(Y,Z) - \rho(Y)h(Z,X) - \rho(Z)h(X,Y) \\
&= C(X,Y,Z) - \rho(X)h(Y,Z) - \rho(Y)h(Z,X) - \rho(Z)h(X,Y).
\end{aligned}$$

Now assume that the cubic form C of M^n in $\widetilde{M}^{n+1}$ is divisible by h. Since $\overline{h}$ and h coincide, we obtain that $\overline{h} \mid \overline{C}$, hence, by Theorem 6, $\overline{M}$ is an open part of a non-degenerate hyperquadric of $\mathbb{R}^{n+1}$.
Since M can be obtained as the result of a central projection of $\overline{M}$; we see that M can be described as the intersection of $\widetilde{M}$ and the cone on $\overline{M}$ whose vertex is the center of $\widetilde{M}$.

3.2. **Proof of Theorem 4.** Let $\widetilde{M}^{n+1}$ be a graph hypersurface of $\mathbb{R}^{n+2}$ and let f be a non-degenerate affine immersion of M^n into $\widetilde{M}^{n+1}$.
Locally, one can proceed as in the previous proof by considering a projection π of the graph hypersurface $\widetilde{M}$ onto a hyperplane $\mathbb{R}^{n+1}$ that does not contain the direction of the normal of $\widetilde{M}$, where the projection takes place along the normals of $\widetilde{M}$. Calculating the data $(\overline{\nabla}, \overline{h}, \overline{\tau})$ for the projection $\overline{M} = \pi(M)$ in terms of those of M reveals $\overline{\nabla} = \nabla, \overline{h} = h$ and $\overline{\tau} = \tau$, hence $\overline{C} = C$. Since $\overline{h} = h$, $\overline{M}$ is a non-degenerate hyperquadric of $\mathbb{R}^{n+1}$ if $h \mid C$.

3.3. **Proof of Theorem 1 and 2.** To prove Theorem 1, we first note that the Blaschke normal and the Euclidean unit normal coincide for the immersion $S^{n+1}(1) \hookrightarrow (\mathbb{R}^{n+2}, D)$.
To complete the proof of Theorem 1, it now suffices to observe that $\widetilde{C} \equiv 0$ if and only if $h \mid C$; then use Theorem 3.

Proposition 1. *Given a $(0,3)-$tensor C on $\mathbb{R}^n$ and a non-degenerate symmetric $(0,2)-$tensor h on $\mathbb{R}^n$, let T be defined by the condition $nh(T,X) = tr_h C(X,\cdot,\cdot)$ for all $X \in \mathbb{R}^n$. Let $\widetilde{C}$ be the traceless part of C with respect to h, i.e. for all $X, Y, Z \in \mathbb{R}^n$*

$$\widetilde{C}(X,Y,Z) = C(X,Y,Z) - \frac{n}{n+2}\left(h(X,Y)h(Z,T) + h(Z,X)h(Y,T) + h(Y,Z)h(X,T)\right)$$

and call C divisible by h, denoted by $h \mid C$, if there exists a one-form ρ such that

$$C(X,Y,Z) = \rho(X)h(Y,Z) + \rho(Y)h(Z,X) + \rho(Z)h(X,Y) \tag{3.1}$$

for all $X, Y, Z \in \mathbb{R}^n$. Then the conditions $\widetilde{C} \equiv 0$ and $h \mid C$ are equivalent.

Proof. If $\widetilde{C} \equiv 0$, then with $\rho := \frac{n}{n+2} h(T, \cdot)$, it is obvious that (3.1) holds.
On the other hand, assume that C is divisible by h with corresponding one-form ρ. Choose an orthonormal frame $(e_1, \dots, e_n)$ w.r.t. h, i.e. $h(e_i, e_j) = \epsilon_i \delta_{ij}$ with $\epsilon_i = \pm 1$. With $X = \sum_{i=1}^n X^i e_i$, we have

$$\begin{aligned} h(T,X) &= \frac{1}{n} tr_h C(X,\cdot,\cdot) = \frac{1}{n}\sum_{i=1}^n \epsilon_i C(X,e_i,e_i) \\ &= \frac{1}{n}\sum_{i=1}^n \epsilon_i \left(\rho(X)\epsilon_i + 2\rho(e_i)h(X,e_i)\right) \\ &= \frac{1}{n}\sum_{i=1}^n \epsilon_i \left(\rho(X)\epsilon_i + 2\rho(e_i)X^i\epsilon_i\right) \\ &= \frac{1}{n}\sum_{i=1}^n \left(\rho(X) + 2\rho(X^i e_i)\right) = \frac{n+2}{n}\rho(X) . \end{aligned}$$

¿From the definition of the traceless part of C, it is now clear that $\widetilde{C}$ vanishes identically. □

Since Theorem 6 holds true when the signature of the standard metric on $\mathbb{R}^{n+2}$ is changed, the proof of Theorem 2 is similar.

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