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Results in Mathematics

Lipschitz Stability of the Cauchy and Jensen Equations

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Abstract

Let G be an amenable metric semigroup with nonempty center, let E be a reflexive Banach space, and let $f : G \to E$ be a given function. By $\mathcal{C}f : G \times G \to E$ we understand the Cauchy difference of the function f, i.e.:

$$\mathcal{C}f(x,y) := f(x+y) - f(x) - f(y) \quad \text{for } x, y \in G.$$

We prove that if the function $\mathcal{C}(f)$ is Lipschitz then there exists an additive function $A : G \to E$ such that $f - A$ is Lipschitz with the same constant. Analogous result for Jensen equation is also proved.

As a corollary we obtain the stability of the Cauchy and Jensen equations in the Lipschitz norms.

1 Introduction

On the 5-th ICFEI (cf. [13]) Józef Tabor posed the following general problem:

Let G be a group, let E be a normed space and let $\|\cdot\|_G$, $\|\cdot\|_{G\times G}$ be the given norms in the space of functions from G, $G \times G$ respectively, into E. Does there exist a $K \in \mathbf{R}_+$ such that for every function $f : G \to E$ satisfying

$$\|\mathcal{C}f\|_{G\times G} \leq \varepsilon$$

there exists an additive function $A : G \to E$ such that

$$\|f - A\|_G \leq K\varepsilon?$$

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For the supremum norms this question was positively solved by the celebrated Hyers's Theorem (cf. [9]). Józef Tabor in [13] gave an affirmative answer to the question of stability in the case when $\|\cdot\|_G$, $\|\cdot\|_{G\times G}$ denote the integral norms in the spaces $\mathcal{L}^p_\mu(G,E)$, $\mathcal{L}^p_\mu(G,E)$, respectively, where $p \in (0,\infty)$ and G is a group with an invariant meausure μ such that $\mu(G)=\infty$ (in fact there occured even some kind of superstability). This result was completed by S. Siudut in [12] who proved the stability in the case $\mu(G)<\infty$.

The paper contains the positive answer to the Józef Tabor's question in the Lipschitz norms. In Theorem 2.1 and 2.2 we prove some general results concerning the stability of the Cauchy and Jensen equations in the class of generalized Lipschitz functions.

These results are strictly connected with the papers of N. G. de Bruijn (cf. [4],[5]), M. Laczkovich (cf. [10]) and many others (for literature see [10]). The definition originally posed by N. G. de Bruijn reads as follows:

Definition 1.1 *Let G be a semigroup, let E be a normed space, and let $\mathcal{F}$ be a given set of functions from G into E. We say that $\mathcal{F}$ has the difference property if for every $f : G \to E$ such that $\Delta_h f \in \mathcal{F}$ for every $h \in G$ there exists an additive function $A : G \to E$ such that $f - A \in \mathcal{F}$ (where $\Delta_h f(x) := f(x+h) - f(x)$).*

In the papers of N. G. de Bruijn cited above one can find that a large class of functions has the difference property. Surprisingly, P. Erdös proved that under the Continuum Hypothesis the class of Lebesgue measurable functions from $\mathbf{R}$ into $\mathbf{R}$ does not have this property (cf. [10]). However, if we assume that $f : \mathbf{R} \to \mathbf{R}$ is a function such that $\mathcal{C}f : \mathbf{R}^2 \to \mathbf{R}$ is Lebesgue measurable then there exists an additive function $A : G \to E$ such that $f - A$ is Lebesgue measurable ([10], Theorem 5). This leads to another definition:

Definition 1.2 *Let G be a semigroup, and let E be a normed space. Let $\mathcal{F}$ be a given set of functions from G into E, and let $\tilde{\mathcal{F}}$ be a given set of functions from $G \times G$ into E. We say that the pair $(\mathcal{F}, \tilde{\mathcal{F}})$ has the double difference property if for every $f : G \to E$ such that $\mathcal{C}f \in \tilde{\mathcal{F}}$ there exists an additive function $A : G \to E$ such that $f - A \in \mathcal{F}$.*

In the case the vector space $\mathcal{F}$ contains constant functions we can define $\tilde{\mathcal{F}}$ to be the set of all functions f going from $G \times G$ into E such that $f(\cdot, h) \in \mathcal{F}$ for every $h \in G$ and obtain the original definition of de Bruijn. The investigation of pairs is reasonable if we assume that $\tilde{\mathcal{F}}$ is some kind of the "product" space of the space $\mathcal{F}$. In this paper in Corollary 2.4 we prove that if $\mathcal{F}$, $\tilde{\mathcal{F}}$ are the spaces of uniformely continuous functions from G, $G \times G$ respectively, into E, then $(\mathcal{F}, \tilde{\mathcal{F}})$ has the double difference property.

Now we introduce some definitions and denotations. For semigroups (not necessarily commutative) we will use the additive notation.

Definition 1.3 *Let G be a semigroup, let E be a vector space and let $f : G \to E$. We define the Cauchy difference of f by*

$$\mathcal{C}f : G \times G \ni (x,y) \to f(x+y) - f(x) - f(y) \in E.$$

In the case where G is uniquely 2-divisible abelian semigroup we define the Jensen difference of f by

$$\mathcal{J}f : G \times G \ni (x,y) \to f(\frac{x+y}{2}) - \frac{1}{2}(f(x)+f(y)) \in E.$$

Let E be a normed space. By $\mathrm{B}(a, r)$ we understand the closed ball with the center at a and the radius r. Now we are going to introduce the definition of a left invariant mean on the space of bounded functions from G into E (cf. [2], [3]).

Definition 1.4 *Let E be a vector space and let $S(E)$ be a given family of subsets of E. We say that this family is linearly invariant iff*

(ı) $x \in E, \alpha \in \mathbf{R}, V \in S(E) \Rightarrow x + \alpha V \in S(E)$.

(ıı) $V, W \in S(E) \Rightarrow V + W \in S(E)$.

As the following proposition shows the class of linearly invariant families is quite large.

Proposition 1.1 *We assume that a vector space E and a family of convex subsets $S(E)$ of E satisfies one of the following conditions:*

(j) $S(E)$ is the family of all convex subsets of E,

(jj) E is a normed space and $S(E)$ is the family of all closed balls,

(jjj) E is a vector lattice and $S(E)$ denotes the family of all intervals,

(jv) E is a semireflexive locally convex topological vector space, and $S(E)$ is the family of closed convex bounded subsets of E,

Then $S(E)$ is a linearly invariant family.

Proof. Cases (j) to (jjj) are trivial. We prove (jv). Let $V, W \in S(E)$. It is enough to show that then $V + W \in S(E)$. Due to Theorem 13.7 from [1] we obtain that because E is a semireflexive locally convex topological vector space, every closed convex subset of E is compact in weak topology. Therefore $V + W$ is compact in weak topology, which implies that $V + W$ is closed. □

Let G be a semigroup and let $S(E)$ be a linearly invariant family. By $\mathcal{L}(G, S(E))$ we denote the set of all functions from G into E such that $\operatorname{im} f \subset V$ for a certain $V \in S(E)$. One can easily check that due to properties of $S(E)$ it is a vector space. If $f \to E$, $a \in G$ then by ${}_af$ we understand the function defined by ${}_af(x) := f(ax)$.

Definition 1.5 *Let G be a semigroup, let E be a vector space, and let $S(E)$ be linearly invariant family of subsets of E. We say that $\mathcal{L}(G, S(E))$ admits a left invariant mean (LIM for short) if there exists a linear operator $M : \mathcal{L}(G, S(E)) \to E$ such that*

(ı) $\operatorname{im} f \subset V \in S(E) \Rightarrow M[f] \in V$,

(ıı) $f \in \mathcal{L}(G, S(E)), a \in G \Rightarrow M[{}_af] = M[f]$.

One can easily give analogous definition for right invariant mean and to translate the results obtained for left invariant mean into analogues for right invariant mean.

Clearly, condition (ı) and the fact that every linearly invariant family contains points implies that if f is a constant function then $M[f] = \operatorname{im} f$.

Example 1.1 Let G be a finite group, let E be a vector space, and let $S(E)$ be any linearly invariant family of convex subsets of E. Let $f \in \mathcal{L}(G, S(E))$ be arbitrary. We define

$$M[f] := \frac{1}{\operatorname{card} G} \sum_{g \in G} f(g).$$

One can easily check that M is a LIM on $\mathcal{L}(G, S(E))$.

Let G be a semigroup, let E be a normed space, and let $\mathrm{B}(E)$ denote the family of balls in E. We say that $\mathcal{L}(G, E)$ admits LIM if and only if $\mathcal{L}(G, \mathrm{B}(E))$ admits LIM. Clearly the semigroup G is left amenable (cf. [8]) if and only if the space $\mathcal{L}(G, \mathbf{R})$ admits LIM. The class of left amenable semigroups is quite large, for example it contains the class of commutative semigroups.

Now we will show a few examples of spaces which admit LIM. The following important theorem was proved by Z. Gajda ([6], Theorem 2.3 and Theorem 2.1).

Theorem G

(i) Let G be a left amenable semigroup, let E be a semireflexive locally convex topological vector space, and let $S(E)$ be the family of closed convex bounded subsets of E. Then $\mathcal{L}(G, S(E))$ admits LIM.

(ii) Let G be a left amenable semigroup, let E be a boundedly complete linear lattice, and let $S(E)$ denote the family of intervals in E. Then $\mathcal{L}(G, S(E))$ admits LIM.

Let E be a vector space and let $S(E)$ be a given family of subsets of E. We say that $S(E)$ has the so called *binary intersection property* if every subfamily of $S(E)$ for which any two members have non empty intersection, has a non void intersection (cf. [3]).

Remark 1.1 We say that a Banach space E has the Hahn-Banach extension property if we can extend linear operators going into E with preservation of the norm. The fact that the family of balls of E has the binary intersection property is equivalent to the fact that E has the Hahn-Banach extension property (cf. [11]).

The following theorem is due to R. Badora (cf. [2], [3]).

Theorem B *Let G be a left amenable semigroup, let E be a locally convex linear topological space and let $S(E)$ be a linearly invariant subfamily of the family of all bounded closed convex subsets of E, having the binary intersection property. Then $\mathcal{L}(G, S(E))$ admits* LIM.

2 Main Results

Now we are going to prove the main theorems of the paper. We say that $\mathsf{d} : G \times G \to S(E)$ is translation invariant if

$$\mathsf{d}(x+a, y+a) = \mathsf{d}(a+x, a+y) = \mathsf{d}(x,y) \quad \text{for } x, y, a \in G.$$

The function $f : G \to E$ is d-Lipschitz if

$$f(x) - f(y) \in \mathsf{d}(x,y) \quad \text{for } x, y \in G.$$

Remark 2.1 Let G be a semigroup with metric d and let E be a normed space. Let $L \in \mathbf{R}_+$, and let

$$\mathsf{d}(x,y) := Ld(x,y)\mathrm{B}(0,1).$$

Then the function $f : G \to E$ is Lipschitz with the constant L if and only if it is d-Lipschitz.

By $\mathrm{Z}(G)$ we denote the centre of G.

Theorem 2.1 *Let G be a semigroup such that $\mathrm{Z}(G) \neq \emptyset$, let E be a vector space with a linearly invariant family of subsets $S(E)$, such that $\mathcal{L}(G, S(E))$ admits* LIM. *Let $\mathsf{d} : G \times G \to S(E)$ be a translation invariant function on $G \times G$, and let $f : G \to E$ be arbitrary.*

We assume that $\mathcal{C}f(\cdot, y) : G \to E$ is d-Lipschitz for every $y \in G$. Then there exists an additive function $A : G \to E$ such that $A - f$ is d-Lipschitz.

Moreover, if $\mathrm{im}\, \mathcal{C}f \subset V$ *for a certain $V \in S(E)$ then*

$$\mathrm{im}(A - f) \subset V.$$

Proof. Let $M : \mathcal{L}(G, S(E)) \to E$ be a LIM and let a be a fixed element of $\mathrm{Z}(G)$. Let $x, y \in G$. Then

$$\begin{aligned} f(a+x+y) - f(a+y) &= (\mathcal{C}f(a+x,y) - \mathcal{C}f(a,y)) + f(a+x) - f(a) \\ &\in \mathsf{d}(a+x,a) + f(a+x) - f(a). \end{aligned}$$

Therefore $f(a+x+y) - f(a+y) \in \mathcal{L}(G, S(E))$ as a function of y. Thus we may define

$$A(x) := M_z[f(a+x+z) - f(a+z)],$$

where the subscript z next to M indicates the fact that M is applied to a function of variable z. For $x, y \in G$ we have

$$\begin{aligned} A(x) + A(y) &= M_z[f(a+x+z) - f(a+z)] + M_z[f(a+y+z) - f(a+z)] \\ &= M_z[f(a+x+(y+z)) - f(a+(y+z))] + M_z[f(a+y+z) - f(a+z)] \\ &= M_z[f(a+(x+y)+z) - f(a+z)] = A(x+y), \end{aligned}$$

so A is additive. By the fact that $a \in \mathrm{Z}(G)$ we obtain that $f(a+x+z) = f(x+a+z)$, which implies that

$$A(x) - f(x) = M_z[f(x+a+z) - f(a+z)] - f(x) = M_z[\mathcal{C}f(x, a+z)]. \tag{1}$$

Clearly by (1)

$$(A-f)(x) - (A-f)(y) = M_z[\mathcal{C}f(x,a+z) - \mathcal{C}f(y,a+z)]. \tag{2}$$

Moreover we have

$$\mathcal{C}f(x,a+z) - \mathcal{C}f(y,a+z) \in \mathsf{d}(x,y),$$

which by properties of the mean implies that

$$M_z[\mathcal{C}f(x,a+z) - \mathcal{C}f(y,a+z)] \in \mathsf{d}(x,y).$$

Therefore by (2) $A - f$ is d-Lipschitz.

Now suppose that $\mathrm{im}\mathcal{C}f \subset V$ for a certain $V \in S(E)$. Then due to (1) and properties of the mean

$$A(x) - f(x) = M_z[\mathcal{C}f(x, a+z)] \in V \quad \text{for } x \in G.$$

□

Remark 2.2 There arises a natural question whether the assumption in Theorem 2.1 that $Z(G)$ is nonempty is essential. Although we think that it is superflous, we were not able to omit it.

Let G be a semigroup and let E be a normed space such that $\mathcal{L}(G, E)$ admits LIM. R. Ger proved in [7] that if $\|\mathcal{C}f(x,y)\| \leq g(x)$ for certain $g : G \to \mathbf{R}_+$ then there a exists an additive $A : G \to E$ such that $\|f(x) - A(x)\| \leq g(x)$. The following corollary is an analogue of the theorem of R. Ger for two variables.

Corollary 2.1 *Let G be a group, let E be a normed space such that $\mathcal{L}(G, E)$ admits* LIM. *Let $f : G \to E$, $g : G \to \mathbf{R}_+$ be such functions that*

$$\|\mathcal{C}f(x,y) - \mathcal{C}f(\tilde{x},y)\| \leq g(x - \tilde{x}) \quad \text{for } x, \tilde{x}, y \in G.$$

Then there exists an additive $A : G \to E$ such that

$$\|(f(x) - A(x)) - (f(\tilde{x}) - A(\tilde{x}))\| \leq g(x - \tilde{x}) \quad \text{for } x, \tilde{x} \in G.$$

Proof. We put $\mathsf{d}(x,y) := g(x-y)\mathrm{B}(0,1)$ and make use of Theorem 2.1 as $Z(G) \neq \emptyset$. □

The following corollary is an analogue of Corollary 2.1 for vector lattices.

Corollary 2.2 *Let G be an amenable group, let E be a boundedly complete vector lattice. Let $f : G \to E$, $g_1, g_2 : G \to E$ be such functions that*

$$g_1(x - \tilde{x}) \leq \mathcal{C}f(x,y) - \mathcal{C}f(\tilde{x},y) \leq g_2(x - \tilde{x}) \quad \text{for } x, \tilde{x}, y \in G.$$

Then there exists an additive $A : G \to E$ such that

$$g_1(x - \tilde{x}) \leq (A(x) - f(x)) - (A(\tilde{x}) - f(\tilde{x})) \leq g_2(x - \tilde{x}) \quad \text{for } x, \tilde{x} \in G.$$

Proof. Let $\mathsf{d}(x,y) := [g_1(x-y), g_2(x-y)]$ (where $[a,b]$ denotes the interval with ends a and b in E). Making use of theorem of Z. Gajda (cited before) and Theorem 2.1 we obtain the assertion. □

Now we prove an analogue of Theorem 2.1 for the Jensen equation.

Theorem 2.2 *Let G be a uniquely 2-divisible abelian semigroup, let E be a vector space with a linearly invariant family of subsets $S(E)$ such that $\mathcal{L}(G, S(E))$ admits* LIM. *Let $\mathsf{d} : G \times G \to S(E)$ be a translation invariant function on $G \times G$, and let $f : G \to E$ be arbitrary.*

We assume that $\mathcal{J}f(\cdot, y) : G \to E$ is d-Lipschitz for every $y \in G$. Then there exists an additive function $A : G \to E$ such that $A - f$ is $2\mathsf{d}$-Lipschitz.

Moreover, if $\mathrm{im}\mathcal{J}f \subset V$ for a certain $V \in S(E)$ then there exists a $b \in E$ such that

$$\mathrm{im}(A + b - f) \subset 2V.$$

Proof. Let $M : \mathcal{L}(G, S(E)) \to E$ be a LIM and let $a \in G$ be fixed. Let $x, y \in G$ be arbitrary. Then

$$\begin{aligned} f(\tfrac{a+x+y}{2}) - f(\tfrac{a+y}{2}) &= (\mathcal{J}f(a+x,y) - \mathcal{J}f(a,y)) + \tfrac{1}{2}(f(a+x) - f(a)) \\ &\in \mathsf{d}(a+x,a) + \tfrac{1}{2}(f(a+x) - f(a)). \end{aligned}$$

This means that $f(\frac{a+x+y}{2}) - f(\frac{a+y}{2}) \in \mathcal{L}(G, S(E))$ as a function of y. Thus the function

$$A(x) := 2M_z[f(\frac{a+x+z}{2}) - f(\frac{a+z}{2})]$$

is well defined. Obviously

$$\begin{aligned} A(x) + A(y) &= 2M_z[f(\tfrac{a+x+z}{2}) - f(\tfrac{a+z}{2})] + 2M_z[f(\tfrac{a+y+z}{2}) - f(\tfrac{a+z}{2})] \\ &= 2M_z[f(\tfrac{a+x+y+z}{2}) - f(\tfrac{a+y+z}{2})] + 2M_z[f(\tfrac{a+y+z}{2}) - f(\tfrac{a+z}{2})] \\ &= 2M_z[f(\tfrac{a+(x+y)+z}{2}) - f(\tfrac{a+z}{2})] = A(x+y) \end{aligned}$$

for $x, y \in G$, so A is additive. We have

$$\begin{aligned} &(A-f)(x) - (A-f)(y) \\ &= 2M_z[(f(\tfrac{a+x+z}{2}) - f(\tfrac{a+z}{2}) - \tfrac{1}{2}f(x)) - (f(\tfrac{a+y+z}{2}) - f(\tfrac{a+z}{2}) - \tfrac{1}{2}f(y)))] \\ &= 2M_z[(f(\tfrac{a+x+z}{2}) - \tfrac{1}{2}f(z) - \tfrac{1}{2}f(x)) - (f(\tfrac{a+x+z}{2}) - \tfrac{1}{2}f(z) - \tfrac{1}{2}f(x)))] \\ &= 2M_z[\mathcal{J}f(x, a+z) - \mathcal{J}f(y, a+z)] \in 2\mathsf{d}(x,y), \end{aligned}$$

which means that $A - f$ is $2\mathsf{d}$-Lipschitz.

Now suppose that $\mathcal{J}f \in V$ for a certain $V \in S(E)$. For $z \in G$ we have the following

$$\begin{aligned} f(\tfrac{a+z}{2}) - \tfrac{1}{2}f(z) &= \mathcal{J}f(z,a) + \tfrac{1}{2}f(a) \subset \mathcal{J}f + \tfrac{1}{2}f(a), \\ f(\tfrac{a+\frac{z}{2}}{2}) - \tfrac{1}{2}f(\tfrac{z}{2}) &= \mathcal{J}f(\tfrac{z}{2}, a) + \tfrac{1}{2}f(a) \subset \mathcal{J}f + \tfrac{1}{2}f(a), \\ f(\tfrac{a+\frac{z}{2}}{2}) - \tfrac{1}{2}f(\tfrac{a+z}{2}) &= \mathcal{J}f(\tfrac{a+z}{2}, \tfrac{a}{2}) + \tfrac{1}{2}f(\tfrac{a}{2}) \subset \mathcal{J}f + \tfrac{1}{2}f(\tfrac{a}{2}), \end{aligned}$$

so

$$\begin{aligned} f(\tfrac{z}{2}) - \tfrac{1}{2}f(z) &= \{f(\tfrac{a+z}{2}) - \tfrac{1}{2}f(z)\} - 2\{f(\tfrac{a+\frac{z}{2}}{2}) - \tfrac{1}{2}f(\tfrac{z}{2})\} + 2\{f(\tfrac{a+\frac{z}{2}}{2}) - \tfrac{1}{2}f(\tfrac{a+z}{2})\} \\ &\in (V - 2V + 2V) + \tfrac{1}{2}f(a) - f(\tfrac{a}{2}) \in S(E). \end{aligned}$$

This implies that we can define

$$b := 2M_z[f(\frac{z}{2}) - \frac{1}{2}f(z)] = 2M_z[f(\frac{a+z}{2}) - \frac{1}{2}f(a+z)].$$

Then for $x \in G$

$$\begin{aligned} A(x) + b - f(x) &= 2M_z[f(\tfrac{a+x+z}{2}) - f(\tfrac{a+z}{2})] + 2M_z[f(\tfrac{a+z}{2}) - \tfrac{1}{2}f(a+z)] - f(x) \\ &= 2M_z[f(\tfrac{a+x+z}{2}) - \tfrac{1}{2}(f(x) + f(a+z))] \\ &= 2M_z[\mathcal{J}f(x, a+z)] \subset 2V. \end{aligned}$$

□

The reader can easily state analogues of Corollaries 2.1 and 2.2 for the Jensen equation. Now we prove corollaries of Theorems 2.1 and 2.2 in the class of uniformely continuous functions. At first we introduce some definitions.

Suppose that we are given a semigroup G with a metric d, invariant under translations. By the product metric we understand any invariant metric $\tilde{d}$ on $G \times G$ such that

$$\tilde{d}((x,a),(y,a)) = \tilde{d}((a,x),(a,y)) = d(x,y) \quad \text{for } x,y,a \in G.$$

Example 2.1 Let G be a semigroup with an invariant metric d. Let $\tilde{d}$ denote one of the following metrics:

$$\begin{aligned}
\tilde{d}((x_1,x_2),(y_1,y_2)) &:= d(x_1,y_1) + d(x_2,y_2), \\
\tilde{d}((x_1,x_2),(y_1,y_2)) &:= \max\{d(x_1,y_1), d(x_2,y_2)\}, \\
\tilde{d}((x_1,x_2),(y_1,y_2)) &:= \sqrt{d(x_1,y_1)^2 + d(x_2,y_2)^2}.
\end{aligned}$$

Then $\tilde{d}$ is the product metric of the metric d.

Let G be a group with a metric d and let $f : G \to E$. We say that $w : \mathbf{R}_+ \to \mathbf{R}_+$ is the module of continuity of f if for every $\delta \in \mathbf{R}_+$

$$d(x,y) \le \delta \Rightarrow \|f(x) - f(y)\| \le w(\delta).$$

As an application of Theorem 2.1 we obtain:

Corollary 2.3 *Let G be a semigroup with an invariant metric d and such that $\mathrm{Z}(G) \neq \emptyset$, and let E be a normed space such that $\mathcal{L}(G,E)$ admits* LIM. *Let $f : G \to E$ be arbitrary. We assume that $w : \mathbf{R}_+ \to \mathbf{R}_+$ is the module of continuity of the function $\mathcal{C}f : G \times G \to E$ with the product metric on $G \times G$. Then there exists an additive function $A : G \to E$ such that w is the module of continuity of the function $f - A$.*

Moreover, if $\mathcal{C}f \in \mathcal{L}(G,E)$ then

$$\|f - A\|_{\sup} \le \|\mathcal{C}f\|_{\sup}.$$

Proof. We put $\mathsf{d}(x,y) := w(d(x,y))\mathrm{B}(0,1)$ and make use of Theorem 2.1. □

As a corollary from Corollary 2.3 we obtain the following result:

Corollary 2.4 *Let G be a semigroup with an invariant metric d and such that $\mathrm{Z}(G) \neq \emptyset$, and let E be a normed space such that $\mathcal{L}(G,E)$ admits* LIM. *By $\mathcal{F}$ we denote the space of uniformely continuous functions from G into E, and by $\tilde{\mathcal{F}}$ the space of uniformly continuous functions from $G \times G$ into E with the product metric on $G \times G$. Then the pair $(\mathcal{F}, \tilde{\mathcal{F}})$ has the double difference property.*

As a trivial corollary from Theorem 2.1 we obtain the following

Corollary 2.5 *Let G be a uniquely 2-divisible abelian semigroup with an invariant metric d and let E be a normed space such that $\mathcal{L}(G,E)$ admits* LIM. *Let $f: G \to E$ be arbitrary. We assume that $w: \mathbf{R}_+ \to \mathbf{R}_+$ is the module of continuity of the function $\mathcal{J}f: G \times G \to E$ with the product metric on $G \times G$. Then there exists an additive function $A: G \to E$ such that $2w$ is the module of continuity of the function $f - A$.*

Moreover, if $\mathcal{J}f \in \mathcal{L}(G,E)$ then there exists $b \in E$ such that

$$\|f - A - b\|_{\sup} \leq 2\|\mathcal{J}f\|_{\sup}.$$

3 Stability of the Cauchy and Jensen Equations in Lipschitz Norms

Now we are going to show some corollaries concerning the stability of the Cauchy and Jensen equations in the Lipschitz norms. At first we introduce the following definition.

Definition 3.1 *Let G be a semigroup with a metric d, and let E be a normed space. We say that $f: G \to E$ is Lipschitz if there exists an $L \in \mathbf{R}$ such that*

$$\|f(x) - f(y)\| \leq L d(x,y) \quad \text{for } x, y \in G.$$

The smallest constant possesing this property we denote by $\operatorname{lip}(f)$. *By* $\mathbf{Lip}(G,E)$ *we mean the space of all bounded Lipschitz functions with the norm*

$$\|f\|_{\mathbf{Lip}} := \|f\|_{\sup} + \operatorname{lip}(f) \quad \text{for } f \in \mathbf{Lip}(G,E).$$

In the case when G is a semigroup with zero, we additionally define $\mathbf{Lip^0}(G,E)$ *as the space of all Lipschitz functions with the norm*

$$\|f\|_{\mathbf{Lip^0}} := \|f(0)\| + \operatorname{lip}(f) \quad \text{for } f \in \mathbf{Lip^0}(G,E).$$

Theorem 3.1 *Let G be a semigroup with an invariant metric d and let E be a normed space such that $\mathcal{L}(G,E)$ admits* LIM. *We assume that we are given a product metric on $G \times G$.*

(i) We additionally assume that $Z(G) \neq \emptyset$. Let $f: G \to E$ be a function such that $\mathcal{C}f \in \mathbf{Lip}(G \times G, E)$. Then there exists an additive function $A: G \to E$ such that

$$\|f - A\|_{\mathbf{Lip}} \leq \|\mathcal{C}f\|_{\mathbf{Lip}}.$$

(ii) We assume that G is a semigroup with zero. Let $f: G \to E$ be a function such that $\mathcal{C}f \in \mathbf{Lip^0}(G \times G, E)$. Then there exists an additive function $A: G \to E$ such that

$$\|f - A\|_{\mathbf{Lip^0}} \leq \|\mathcal{C}f\|_{\mathbf{Lip^0}}.$$

Proof. (i) Because $\mathcal{C}f$ is Lipschitz, it is uniformly continuous with the module of continuity $w(x) := \operatorname{lip}(\mathcal{C}f)x$. By Corollary 2.3 we obtain that there exists an additive function such that w is the module of continuity of the function $f - A$ and that $\|f - A\|_{\sup} \leq \|\mathcal{C}f\|_{\sup}$. The fact that $w(x) = \mathrm{lip}(\mathcal{C}f)x$ is the module of continuity of $f - A$ yields that $f - A$ is Lipschitz with constant $\mathrm{lip}(f - A) \leq \mathrm{lip}(\mathcal{C}f)$. Then

$$\begin{aligned}\|f - A\|_{\mathbf{Lip}} &= \|f - A\|_{\sup} + \mathrm{lip}(f - A) = \|\mathcal{C}f\|_{\sup} + \mathrm{lip}(f - A) \\ &\leq \|\mathcal{C}f\|_{\sup} + \mathrm{lip}(\mathcal{C}f) = \|\mathcal{C}f\|_{\mathbf{Lip}}.\end{aligned}$$

(ii) As $\mathcal{C}f$ is Lipschitz, it is uniformly continuous, and its module of continuity is $w(x) := \mathrm{lip}(\mathcal{C}f)x$. By Corollary 2.3 we obtain that there exists an additive function such that w is the module of continuity of the function $f - A$. This implies that $\mathrm{lip}(f - A) \leq \mathrm{lip}(\mathcal{C}f)$. Hence

$$\begin{aligned}\|f - A\|_{\mathbf{Lip^0}} &= \|f(0)\| + \mathrm{lip}(f - A) = \|\mathcal{C}f(0,0)\| + \mathrm{lip}(f - A) \\ &\leq \|\mathcal{C}f(0,0)\| + \mathrm{lip}(\mathcal{C}f) = \|\mathcal{C}f\|_{\mathbf{Lip^0}}.\end{aligned}$$

□

Theorem 3.2 *Let G be a uniquely 2-divisible abelian semigroup with an invariant metric d and let E be a normed space such that $\mathcal{L}(G, E)$ admits* LIM. *We assume that we are given a product metric on $G \times G$.*

(i) Let $f : G \to E$ be a function such that $\mathcal{J}f \in \mathbf{Lip}(G \times G, E)$. Then there exists a Jensen function $J : G \to E$ such that

$$\|f - J\|_{\mathbf{Lip}} \leq 2\|\mathcal{J}f\|_{\mathbf{Lip}}.$$

(ii) We additionally asume that G is a semigroup with zero. Let $f : G \to E$ be a function such that $\mathcal{J}f \in \mathbf{Lip^0}(G \times G, E)$. Then there exists a Jensen function $J : G \to E$ such that

$$\|f - J\|_{\mathbf{Lip^0}} \leq 2\|\mathcal{J}f\|_{\mathbf{Lip^0}}.$$

Proof. (i) The proof is analogous to that of Theorem 3.1(i) (instead of Corollary 2.3 we make use of Corollary 2.5).

(ii) One can make the analogous reasoning as in Theorem 3.1(ii) but making use of Corollary 2.5 instead of Corollary 2.3 and stating $J(x) := f(0) + A(x)$ (where A is the additive function constructed in Corollary 2.5). □

The following examples (the second given by J. Chudziak) show that the estimations of $f - A$ and $f - J$ in Theorems 3.1 and 3.2, repectively, are sharp.

Example 3.1 Let $G := \mathbf{R}_+$ with the standard metric, and by the product metric on $G \times G$ we take the sum metric (the first metric defined in Example 2.1). Let

$$f(x) := \arctan(x) \quad \text{for } x \in G.$$

One can check (by using derivative) that $\mathrm{lip}(f) = 1$, $\mathrm{lip}(\mathcal{C}f) = 1$, $\mathrm{lip}(\mathcal{J}f) = \frac{1}{2}$.

Let $J : G \to \mathbf{R}$ be a Jensen function such that $\|f - J\|_{\sup} < \infty$. Then J is constant which means that $\mathrm{lip}(f - J) = \mathrm{lip}(f)$. Moreover, $\|f - J\|_{\sup} \geq \frac{\pi}{2}$. After some tedious but easy calculations one can check that $\|\mathcal{C}f\|_{\sup} = \frac{\pi}{2}$, $\|\mathcal{J}f\|_{\sup} = \frac{\pi}{4}$. This implies that for every Jensen $J : G \to \mathbf{R}$

$$\frac{\|f - J\|_{\mathbf{Lip}}}{\|\mathcal{C}f\|_{\mathbf{Lip}}} \geq 1, \quad \frac{\|f - J\|_{\mathbf{Lip}}}{\|\mathcal{J}f\|_{\mathbf{Lip}}} \geq 2.$$

Example 3.2 Let $G := \mathbf{R}_+$ with the discrete metric, and by the product metric on $G \times G$ we also take discrete metric. Let

$$f(x) := \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x = 0. \end{cases}$$

One can easily check that $\mathrm{lip}(f) = 1$, $\mathrm{lip}(\mathcal{C}f) = 1$, $\mathrm{lip}(\mathcal{J}f) = \frac{1}{2}$.

Let $J : G \to \mathbf{R}$ be a Jensen function such that $\mathrm{lip}(f - J) < \infty$. Then

$$\sup_{x,y \in G} |(f - J)(x) - (f - J)(y)| = \mathrm{lip}(f - J) < \infty.$$

Hence J is a bounded Jensen function, which implies that J is constant, and consequently that $\mathrm{lip}(f - J) = \mathrm{lip}(f)$. Therefore $\|f - J\|_{\mathbf{Lip^0}} = |f(0) - J(0)| + \mathrm{lip}(f - J) \geq \mathrm{lip}(f) = 1$.

This implies that for every Jensen $J : G \to \mathbf{R}$

$$\frac{\|f - J\|_{\mathbf{Lip^0}}}{\|\mathcal{C}f\|_{\mathbf{Lip^0}}} \geq 1, \quad \frac{\|f - J\|_{\mathbf{Lip^0}}}{\|\mathcal{J}f\|_{\mathbf{Lip^0}}} \geq 2.$$

References

[1] A. Alexiewicz, *Analiza funkcjonalna,* Polish Scientific Publishers (PWN), Warszawa 1969.

[2] R. Badora, *On some generalized invariant means and their application to the stability of the Hyers-Ulam type,* Ann. Polon. Math. 58 (1993), 147-159.

[3] R. Badora, *On some generalized invariant means and almost approximately additive functions,* Publ. Math. Debrecen 44 (1994), 123-135.

[4] N. G. de Bruijn, *Functions whose differences belong to a given class,* Nieuw Archief voor Wiskunde, 23 (1951), 194-218.

[5] N. G. de Bruijn, *A difference property for Riemann integrable functions and for some similair classes of functions,* Indagationes Mathematicae, 14 (1962), 145-151.

[6] Z. Gajda, *Invariant means and representations of semigroups in the theory of functional equations,* Prace Naukowe Uniwersytetu Śląskiego, Katowice, 1992.

[7] R. Ger, *The singular case in the stability bahaviour of linear mappings,* Grazer Math. Ber. 316 (1992), 59-70.

[8] F. P. Greenleaf, *Invariant means on topological groups and their applications,* Van Nostrand Mathematical Studies 16, New York - Toronto - London - Melbourne, 1969.

[9] D. H. Hyers, *On the stability of the linear functional equation,* Proc. Nat. Acad. Sci. U.S.A. 27 (1941), 222-224.

[10] M. Laczkovich, *Functions with measurable differences,* Acta Math. Acad. Scient. Hung. 35 (1980), 217-235.

[11] L. Nachbin, *A theorem for Hahn-Banach type for linear transformations,* Trans. Amer. Math. Soc. 72 (1978), 297-300.

[12] S. Siudut, *Cauchy difference operator in $\mathcal{L}^p$ spaces,* manuscript.

[13] Józef Tabor, *Stability of the Cauchy type equations in $\mathcal{L}^p$ norm,* Proceedings of the 5-th International Conference on Functional Equations and Inequalities, Annales de l'Ecole Normale Supérieure a Cracovie (to appear).

[14] Jacek Tabor, Józef Tabor, *Stabilty of the Cauchy Type Equations in the Class of Differentiable Functions,* manuscript.

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