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Results in Mathematics

The D_ω – classical orthogonal polynomials

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Abstract. This is an expository paper; it aims to give an essentially self–contained overview of discrete classical polynomials from their characterizations by Hahn's property and a Rodrigues' formula which allows us to construct it. The integral representations of corresponding forms are given.

Key words. Orthogonal polynomials, semi-classical forms, difference operator

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Introduction

The present paper has to be taken as a synthetic work in the domain dealing with discrete orthogonal polynomials satisfying Hahn's property (see the statment b) of Prop. 2.1). We can distinguish three periods in the history of discrete orthogonal polynomials.

During the first period were discovered the now well–known orthogonal polynomials: Charlier [7], Meixner [24],later continuous Hahn [3], period during which Hahn has pointed out one characterization of Jacobi polynomials through the orthogonality of the sequence of derivatives [12], see also [33]. In fact, orthogonality of Charlier polynomials was proved by Meixner who built all the orthogonal sequences possessing a Sheffer–type generating function; similarly, the orthogonality of continuous Hahn polynomials was proved in [2,3] by considering generalized hypergeometric series. In these works, the Hahn's property, by means of the difference operator, is not apparent. See also [1] for historic elements.

The second period is dominated by P. Lesky who studies systematically the discrete (positive definite)orthogonal polynomials through a variational principle due to Gröbner [10], which allows him to build a Rodrigues' formula [16] or through a second–order difference equation fulfilled by these polynomials [17,18]. This last point of view was already pointed out by O.E. Lancaster [15] and continued by the russian school [25]. Likewise, a finite difference Rodrigues' formula was discuted in [32].

At the present time, there are papers whose ambition is to provide a global exposition where the different characterizations are showed equivalent [9,11,14,28], following the framework given in [21]. But, none of precedent authors [9,11,14,28] builds the discrete orthogonal polynomials fulfilling the statements pointed out. Whereas here, we construct the D_ω– classical orthogonal polynomials from the (functional) so–called Rodrigues' formula, by giving the elements of their second–order recurrence relation. We have not considered finite orthogonal sequences; we are only interested by regular forms.

The first section contains material of a preliminary and introductory character. The second section deals with $D_\omega-$ classical orthogonal polynomials, namely these which fulfil Hahn's property : if $\{P_n\}_{n\geq 0}$ is orthogonal then $\{D_\omega P_{n+1}\}_{n\geq 0}$ is also orthogonal where $(D_\omega f)(x) := \frac{f(x+\omega)-f(x)}{\omega}$, for any polynomial f . We give a Rodrigues' formula involving the form itself. This allows us to determine the $D_\omega-$ classical polynomials (Prop.2.4.). In the third section, we exhaustively describe the cases which arise; thus, we meet again Charlier, Meixner, Meixner–Pollaczek, continuous Hahn polynomials and other particular cases [8]. When $\omega = 0$, we rediscover the $(D-)$ classical polynomials: Hermite, Laguerre, Bessel and Jacobi [8,23]. The last section is devoted to the study of the consequences provided by the equation fulfilled by the $D_{-\omega}-$ classical forms. We give the moments of certain so–called canonical forms and their integral representations as solutions of this equation.

It must be noted the remarkable fact that the Meixner polynomials become Meixner–Pollaczek polynomials when essentially $\omega \to i\omega$, ω being real. In the same way, the Charlier polynomials are transformed into another sequence in accordance with [19,20]. In the two cases, what really changes is the representation of the considered form: when ω is real, the Charlier and Meixner forms are represented by a discrete measure whose support lies in $\mathbb{R}$; when ω is pure imaginary, the Meixner–Pollaczek and the transformed Charlier form are represented by an absolute continuous weight–function.

§1. Preliminaries and notations

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $< u, f >$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := < u, x^n >$, $n \geq 0$ the moments of u . For any form u , any polynomial h , we let $Du = u'$ and hu , be the forms defined by duality

$$< u', f > := - < u, f' > \quad ; \quad < hu, f > := < u, hf > \quad , \quad f \in \mathcal{P} .$$

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials , $\deg P_n = n$, $n \geq 0$ and let $\{u_n\}_{n\geq 0}$ be its dual sequence , $u_n \in \mathcal{P}'$ defined by $< u_n, P_m > := \delta_{n,m}$, $n, m \geq 0$.
Let us recall some results [22].

Lemma 1.1. *For any $u \in \mathcal{P}'$ and any integer $m \geq 1$, the following statements are equivalent*
i) $< u, P_{m-1} > \neq 0 \quad , \quad < u, P_n > = 0 \quad , \quad n \geq m$,
ii) $\exists\, \lambda_\nu \in \mathbb{C}$, $0 \leq \nu \leq m-1$, $\lambda_{m-1} \neq 0$ *such that*

$$u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu .$$

As a consequence, the dual sequence $\{u_n^{[1]}\}_{n\geq 0}$ of $\{P_n^{[1]}\}_{n\geq 0}$ where $P_n^{[1]}(x) = (n+1)^{-1} P'_{n+1}(x)$, $n \geq 0$ is given by

$$(1.1) \qquad \left(u_n^{[1]}\right)' = -(n+1) u_{n+1} \quad , \quad n \geq 0 .$$

Similarly, the dual sequence $\{\tilde{u}_n\}_{n\geq 0}$ of $\{\widetilde{P}_n\}_{n\geq 0}$ with $\widetilde{P}_n(x) = a^{-n} P_n(ax+b)$, $n \geq 0$, $a \neq 0$ is given by

$$(1.2) \qquad \tilde{u}_n = a^n (h_{a^{-1}} \circ \tau_{-b}) u_n \quad , \quad n \geq 0$$

where

$$< \tau_{-b} u, f > := < u, \tau_b f > = < u, f(x-b) > \quad , \quad u \in \mathcal{P}' , \ f \in \mathcal{P} , \ b \in \mathbb{C}$$
$$< h_a u, f > := < u, h_a f > = < u, f(ax) > \quad , \quad u \in \mathcal{P}' , \ f \in \mathcal{P} , \ a \in \mathbb{C} - \{0\} .$$

The form u is called regular if we can associate with it a sequence $\{P_n\}_{n\geq 0}$ such that

$$< u, P_m P_n >= r_n \delta_{n,m} \quad , \quad n\,,\, m \geq 0 \quad ; \quad r_n \neq 0 \quad , \quad n \geq 0\,.$$

The sequence $\{P_n\}_{n\geq 0}$ is orthogonal with respect to u . Necessarily , $u = \lambda u_0$, $\lambda \neq 0$. In this case, we have $u_n = \left(< u_0, P_n^2 > \right)^{-1} P_n u_0$, $n \geq 0$. When u is regular, let Φ be a polynomial such that $\Phi u = 0$.
Then $\Phi = 0$ [23].

Let us introduce the Hahn's operator

$$(D_\omega f)(x) := \frac{f(x+\omega) - f(x)}{\omega} \quad , \quad f \in \mathcal{P} \quad , \quad \omega \neq 0\,.$$

We have $D_\omega = \frac{1}{\omega}\big(\tau_{-\omega} - I_{\mathcal{P}}\big)$ where $I_{\mathcal{P}}$ is the identity operator in $\mathcal{P}$. The transposed ${}^tD_\omega$ of D_ω is ${}^tD_\omega = \frac{1}{\omega}\big(\tau_\omega - I_{\mathcal{P}'}\big) = -D_{-\omega}$, leaving out a light abuse of notation without consequence. Thus, we have

$$< D_{-\omega} u, f >= - < u, D_\omega f > \quad , \quad u \in \mathcal{P}' \quad , \quad f \in \mathcal{P} \quad , \quad \omega \in \mathbb{C}\,.$$

When $\omega \rightarrow 0$, we meet again the derivative D . In particular, we have

$$\big(D_{-\omega} u\big)_n = \begin{cases} 0 \quad , \quad n = 0 \\ -\displaystyle\sum_{\nu=0}^{n-1} \frac{n!}{\nu!(n-\nu)!}\omega^{n-1-\nu}(u)_\nu \quad , \quad n \geq 1\,. \end{cases}$$

Lemma 1.2. *The following formulas hold*

(1.3) $\big(D_\omega f_1 f_2\big)(x) = f_1(x)\big(D_\omega f_2\big)(x) + \big(\tau_{-\omega} f_2\big)(x)\big(D_\omega f_1\big)(x) \quad , \quad f_1\,,\, f_2 \in \mathcal{P}$

(1.3)′ $\big(D_\omega f_1 f_2\big)(x) = f_1(x)\big(D_\omega f_2\big)(x) + f_2(x)\big(D_\omega f_1\big)(x) + \omega\big(D_\omega f_1\big)(x)\big(D_\omega f_2\big)(x)$

(1.4) $\big(\tau_{-\omega} f_1 f_2\big)(x) = \big(\tau_{-\omega} f_1\big)(x)\big(\tau_{-\omega} f_2\big)(x) \quad , \quad f_1\,,\, f_2 \in \mathcal{P}$

(1.5) $\tau_\omega(gu) = (\tau_\omega g)(\tau_\omega u) \quad , \quad g \in \mathcal{P}\,,\, u \in \mathcal{P}'$

(1.6) $D_{-\omega}(gu) = g D_{-\omega} u + (D_{-\omega} g)(\tau_\omega u) \quad , \quad g \in \mathcal{P}\,,\, u \in \mathcal{P}'$

(1.6)′ $D_{-\omega}(gu) = (\tau_\omega g)(D_{-\omega} u) + (D_{-\omega} g) u$

(1.7) $\tau_b \circ D_\omega = D_\omega \circ \tau_b$ *in* $\mathcal{P}$ *and in* $\mathcal{P}'$, $b \in \mathbb{C}$

(1.8) $h_a \circ D_\omega = a^{-1} D_{\omega a^{-1}} \circ h_a$ *in* $\mathcal{P}$, $a \in \mathbb{C} - \{0\}$

(1.9) $h_a \circ D_\omega = a D_{a\omega} \circ h_a$ *in* $\mathcal{P}'$, $a \in \mathbb{C} - \{0\}$.

The relations (1.3) – (1.4) are evident. Further, we have

$$\begin{aligned} < \tau_\omega(gu), f > &=< u, g(\tau_{-\omega} f) >=< u, \tau_{-\omega}\big((\tau_\omega g) f\big) > \text{ from (1.4)} \\ &=< (\tau_\omega g)(\tau_\omega u), f > \;,\; \text{ hence (1.5)}\,. \end{aligned}$$

$$\begin{aligned} < D_{-\omega}(gu), f > &= - < u, g(D_\omega f) >= - < u, D_\omega(gf) - (\tau_{-\omega} f)(D_\omega g) > \text{ from (1.3)} \\ &=< g(D_{-\omega} u), f > + < \tau_\omega\big((D_\omega g) u\big), f > \,. \end{aligned}$$

But

$$\begin{aligned} \tau_\omega\big((D_\omega g) u\big) &= \big((\tau_\omega \circ D_\omega) g\big)(\tau_\omega u) \text{ from (1.5)} \\ &= (D_{-\omega} g)(\tau_\omega u) \text{ following the definitions. Hence (1.6)}\,. \end{aligned}$$

With $\tau_\omega = I_{\mathcal{P}'} - \omega D_{-\omega}$, we obtain (1.6)′ . It is easy to prove (1.7) – (1.9) on account of definitions.

Now, consider $\{P_n\}_{n\geq 0}$ as above in section 1 and let

(1.10) $$P_n^{[1]}(x;\omega) = \frac{1}{n+1}\big(D_\omega P_{n+1}\big)(x) \quad , \quad n \geq 0\,.$$

Denoting by $\{u_n^{[1]}(\omega)\}_{n\geq 0}$ the dual sequence of $\{P_n^{[1]}(.;\omega)\}_{n\geq 0}$, we have the result

Lemma 1.3.

$$D_{-\omega}\big(u_n^{[1]}(\omega)\big) = -(n+1)u_{n+1} \quad , \quad n \geq 0\,. \tag{1.11}$$

Indeed, from the definition $< u_n^{[1]}(\omega), P_m^{[1]}(.;\omega) >= \delta_{n,m}$, $n,m \geq 0$, we have $- < D_{-\omega}\big(u_n^{[1]}(\omega)\big), P_{m+1} >= (m+1)\delta_{n,m}$, therefore

$$< D_{-\omega}\big(u_n^{[1]}(\omega)\big), P_m >= 0 \quad , \quad m \geq n+2 \ , \ n \geq 0$$
$$< D_{-\omega}\big(u_n^{[1]}(\omega)\big), P_{n+1} >= -(n+1) \quad , \quad n \geq 0\,.$$

By virtue of lemma 1.1

$$D_{-\omega}\big(u_n^{[1]}(\omega)\big) = \sum_{\nu=0}^{n+1} \lambda_{n,\nu} u_\nu\,.$$

But $< D_{-\omega}\big(u_n^{[1]}(\omega)\big), P_\mu >= \lambda_{n,\mu}$, $0 \leq \mu \leq n+1$ and $\lambda_{n,\mu} = 0$, $0 \leq \mu \leq n$, $\lambda_{n,n+1} = -(n+1)$, $n \geq 0$. Hence (1.11) .

Let Φ monic and ψ be two polynomials , $\deg \Phi = t$, $\deg \psi = p \geq 1$. We suppose that the pair (Φ, ψ) is admissible, i.e. when $p = t-1$, writing $\psi(x) = a_p x^p + ...$, then a_p is not a positive integer.

Definition. A form u is called $D_{-\omega}$–semi-classical when it is regular and satisfies the equation

$$D_{-\omega}(\Phi u) + \psi u = 0 \tag{1.12}$$

where the pair (Φ, ψ) is admissible. The corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ is called D_ω–semi-classical.

Lemma 1.4. *Consider the sequence $\{\widetilde{P}_n\}_{n\geq 0}$ obtained by shifting P_n i.e. $\widetilde{P}_n(x) = a^{-n}P_n(ax+b) = a^{-n}\big(h_a \circ \tau_{-b} P_n\big)(x)$, $n \geq 0$, $a \neq 0$. If u_0 satisfies (1.12) , then $\tilde{u}_0 = (h_{a^{-1}} \circ \tau_{-b})u_0$ fulfils the equation*

$$D_{-\omega a^{-1}}(\tilde{\Phi}\tilde{u}_0) + \tilde{\psi}\tilde{u}_0 = 0\,,$$

where $\tilde{\Phi}(x) = a^{-t}\Phi(ax+b)$, $\tilde{\psi}(x) = a^{1-t}\psi(ax+b)$.

We need the following formulas easy to prove

$$\begin{cases} g(\tau_b u) = \tau_b\big((\tau_{-b} g)u\big) \\ g(h_a u) = h_a\big((h_a g)u\big) \end{cases} \quad g \in \mathcal{P} \quad , \quad u \in \mathcal{P}'\,. \tag{1.13}$$

Now, with $u_0 = (\tau_b \circ h_a)\tilde{u}_0$, we have

$$\psi u_0 = \psi(\tau_b v) = \tau_b\big((\tau_{-b}\psi)v\big) \quad \text{from (1.13) with } v = h_a \tilde{u}_0\,,$$
$$= \tau_b\big((\tau_{-b}\psi)(h_a\tilde{u}_0)\big) = (\tau_b \circ h_a)\big(h_a \circ \tau_{-b}\psi\big)\tilde{u}_0 = (\tau_b \circ h_a)\big(\psi(ax+b)\tilde{u}_0\big) \quad \text{from (1.13)}\,.$$

Further

$$D_{-\omega}(\Phi u_0) = D_{-\omega}\big(\Phi(\tau_b v) = D_{-\omega}\Big(\tau_b\big((\tau_{-b}\Phi)v\big)\Big) \quad \text{from (1.13)}$$
$$= \tau_b D_{-\omega}\big((\tau_{-b}\Phi)(h_a\tilde{u}_0)\big) = \tau_b D_{-\omega}\Big(h_a\big((h_a \circ \tau_{-b}\Phi)\tilde{u}_0\big)\Big) \quad \text{from (1.13)}$$
$$= a^{-1}(\tau_b \circ h_a) D_{-\omega a^{-1}}\big(\Phi(ax+b)\tilde{u}_0\big) \quad \text{from (1.9)}\,.$$

Equation (1.12) becomes

$$\tau_b \circ h_a\Big(D_{-\omega a^{-1}}\big(\Phi(ax+b)\tilde{u}_0\big) + a\psi(ax+b)\tilde{u}_0\Big) = 0\,.$$

Hence the desired result.

Regarding general semi-classical sequences, we have the following statement that we give for the sake of completeness [21,22]:

Proposition 1.5. *For any monic polynomial Φ and any orthogonal sequence $\{P_n\}_{n\geq 0}$, the following statements are equivalent*
a) There exists an integer $s \geq 0$ such that

$$\Phi(x)P_n^{[1]}(x;\omega) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu}P_\nu(x) \quad , \quad n \geq s \tag{1.14}$$

$$\lambda_{n,n-s} \neq 0 \quad , \quad n \geq s+1 \, . \tag{1.15}$$

b) There exists a polynomial ψ , $\deg \psi = p \geq 1$ such that

$$D_{-\omega}(\Phi u_0) + \psi u_0 = 0 \tag{1.16}$$

where the pair (Φ, ψ) is admissible.
c) There exist an integer $s \geq 0$ and a polynomial ψ , $\deg \psi = p \geq 1$ such that

$$\Phi(x)\big(D_{-\omega}P_m\big)(x) - \psi(x)\big(\tau_\omega P_m\big)(x) = \sum_{\nu=m-t}^{m+s_m} \tilde{\lambda}_{m,\nu}P_{\nu+1}(x) \quad , \quad m \geq t \tag{1.17}$$

$$\tilde{\lambda}_{m,m-t} \neq 0 \quad , \quad m \geq t \tag{1.18}$$

where $s = \max(p-1, t-2)$, the pair (Φ, ψ) is admissible and

$$s_m = \begin{cases} p-1 \ , \ m=0 \\ s \ , \ m \geq 1 \, . \end{cases}$$

We may write

$$\tilde{\lambda}_{m,\nu} = -(\nu+1)\frac{< u_0, P_m^2 >}{< u_0, P_{\nu+1}^2 >}\lambda_{\nu,m} \quad , \quad 0 \leq \nu \leq m+s \, . \tag{1.19}$$

Remarks. 1. We have also the following statement [21]: the form u_0 is $D_{-\omega}$ –semi–classical if and only if the sequence $\{P_n^{[1]}(.;\omega)\}_{n\geq 0}$ is quasi-orthogonal of order s with respect to Φu_0 .
2. When $\{P_n\}_{n\geq 0}$ is orthogonal, it fulfils the standard recurrence relation

$$\begin{aligned} &P_0(x) = 1 \quad , \quad P_1(x) = x - \beta_0 \\ &P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x) \quad , \quad n \geq 0 \, . \end{aligned} \tag{1.20}$$

Likewise, when $\{P_n^{[1]}\}_{n\geq 0}$ is orthogonal $(s=0)$, it fulfils the recurrence relation

$$\begin{aligned} &P_0^{[1]}(x) = 1 \quad , \quad P_1^{[1]}(x) = x - \beta_0^{[1]} \\ &P_{n+2}^{[1]}(x) = (x - \beta_{n+1}^{[1]})P_{n+1}^{[1]}(x) - \gamma_{n+1}^{[1]}P_n^{[1]}(x) \quad , \quad n \geq 0 \, . \end{aligned} \tag{1.21}$$

§2. The D_ω – classical orthogonal polynomials

When $s=0$, the sequence $\{P_n\}_{n\geq 0}$ is called D_ω–classical (discrete classical orthogonal polynomials); moreover, we have the more accurate following statements:

Proposition 2.1. *For any orthogonal sequence $\{P_n\}_{n\geq 0}$, the following statements are equivalent*
a) The sequence $\{P_n\}_{n\geq 0}$ is D_ω-classical.
b) The sequence $\{P_n^{[1]}(.;\omega)\}_{n\geq 0}$ is orthogonal.
c) There exist two polynomials Φ monic , $\deg\Phi\leq 2$, ψ , $\deg\psi=1$ and a sequence $\{\lambda_n\}_{n\geq 0}$, $\lambda_n\neq 0$, $n\geq 0$ such that

$$\Phi(x)\big(D_\omega\circ D_{-\omega}P_{n+1}\big)(x)-\psi(x)\big(D_{-\omega}P_{n+1}\big)(x)+\lambda_nP_{n+1}(x)=0 \quad , \quad n\geq 0\,.$$

a) $\Longrightarrow$ *b)* . From (1.16) and (1.6)$'$, we have

$$\begin{aligned}<u_0,\Phi P_mP_n^{[1]}> &= \frac{1}{n+1}<P_m\Phi u_0,D_\omega P_{n+1}>=-\frac{1}{n+1}<D_{-\omega}\big(P_m\Phi u_0\big),P_{n+1}>\\ &=-\frac{1}{n+1}<(\tau_\omega P_m)D_{-\omega}(\Phi u_0)+(D_{-\omega}P_m)\Phi u_0,P_{n+1}>\\ &=\frac{1}{n+1}<(\tau_\omega P_m)\psi u_0-(D_{-\omega}P_m)\Phi u_0,P_{n+1}>\\ &=\frac{1}{n+1}<u_0,\big((\tau_\omega P_m)\psi-(D_{-\omega}P_m)\Phi\big)P_{n+1}> \;.\end{aligned}$$

Consequently

$$\begin{aligned}&<\Phi u_0,P_mP_n^{[1]}>=0 \quad , \quad 0\leq m\leq n-1\;,\;n\geq 1\\ &<\Phi u_0,\big(P_n^{[1]}\big)^2>=\frac{1}{n+1}\big(\psi'(0)-\frac{1}{2}\Phi''(0)n\big)<u_0,P_{n+1}^2>\neq 0 \;,\; n\geq 0\end{aligned}$$

since (Φ,ψ) is admissible.
b) $\Longrightarrow$ *c)* . From (1.11) and according to assumptions

$$D_{-\omega}\big(P_n^{[1]}u_0^{[1]}\big)=-\chi_nP_{n+1}u_0 \quad , \quad n\geq 0 \tag{2.1}$$

with

$$\chi_n=(n+1)\frac{<u_0^{[1]},\big(P_n^{[1]}\big)^2>}{<u_0,P_{n+1}^2>} \quad , \quad n\geq 0\,.$$

For $n=0$ in (2.1)

$$D_{-\omega}u_0^{[1]}=-\gamma_1^{-1}P_1u_0\,. \tag{2.2}$$

In accordance with (1.6)$'$, we have

$$D_{-\omega}\big(P_n^{[1]}u_0^{[1]}\big)=\big(\tau_\omega P_n^{[1]}\big)D_{-\omega}u_0^{[1]}+\big(D_{-\omega}P_n^{[1]}\big)u_0^{[1]}\;,$$

therefore, on account of (2.2)

$$-\chi_0P_1\big(\tau_\omega P_n^{[1]}\big)u_0+\big(D_{-\omega}P_n^{[1]}\big)u_0^{[1]}=-\chi_nP_{n+1}u_0 \quad , \quad n\geq 0\,. \tag{2.3}$$

Making $n=1$, we get

$$u_0^{[1]}=\gamma_1^{-1}k\Phi u_0 \tag{2.4}$$

where $k\Phi(x) = P_1(x)\big(\tau_\omega P_1^{[1]}\big)(x) - 2\gamma_1^{[1]}\gamma_2^{-1}P_2(x)$ (Φ monic) .
So, equations (2.3) – (2.4) and the regularity of u_0 imply

$$\Phi(x)\big(D_{-\omega}P_n^{[1]}\big)(x) - \psi(x)\big(\tau_\omega P_n^{[1]}\big)(x) + \gamma_1 k^{-1}\chi_n P_{n+1}(x) = 0 \quad , \quad n \geq 0$$

with $\psi(x) = k^{-1}P_1(x)$. Comparing the degrees, we obtain

$$\frac{1}{2}\Phi''(0)n - \psi'(0) + \gamma_1 k^{-1}\chi_n = 0 \quad , \quad n \geq 0 \, ,$$

which means that the pair (Φ, ψ) is admissible. Finally, we have the desired second–order difference equation with $\lambda_n = \gamma_1 k^{-1}(n+1)\chi_n$, $n \geq 0$. In fact, we also have proved that $b) \implies a)$.
$c) \implies a)$. From the given equation, we get

$$< u_0, \Phi\big(D_\omega \circ D_{-\omega}P_{n+1}\big) - \psi\big(D_{-\omega}P_{n+1}\big) >= 0 \quad , \quad n \geq 0 \, .$$

Hence

$$< D_\omega\big(D_{-\omega}(\Phi u_0) + \psi u_0\big), P_{n+1} >= 0 \quad , \quad n \geq 0 \, .$$

Since $< D_\omega\big(D_{-\omega}(\Phi u_0) + \psi u_0\big), 1 >= 0$, we get

$$D_\omega\big(D_{-\omega}(\Phi u_0) + \psi u_0\big) = 0 \, .$$

Hence (1.16) where the pair (Φ, ψ) is admissible on account of $\lambda_n \neq 0$, $n \geq 0$.
Remarks. 1. In the case $s = 0$, when the pair (Φ, ψ) is not admissible, then the solution u of (1.12) is not regular. In other words, when the solution u of (1.12) is regular, then the pair (Φ, ψ) is necessarily admissible.
2. Necessarily, we have

$$\begin{aligned} k\Phi(x) &= \big(1 - 2\gamma_1^{[1]}\gamma_2^{-1}\big)x^2 + \big(2\gamma_1^{[1]}\gamma_2^{-1}(\beta_0 + \beta_1) - \beta_0 - \beta_0^{[1]} - \omega\big)x \\ &\quad + \beta_0(\beta_0^{[1]} + \omega) - 2\gamma_1^{[1]}\gamma_2^{-1}(\beta_0\beta_1 - \gamma_1) \, , \\ k\psi(x) &= P_1(x) \, . \end{aligned} \tag{2.4'}$$

COROLLARY 2.2. *If $\{P_n\}_{n\geq 0}$ is D_ω-classical, the sequence $\{P_n^{[m]}\}_{n\geq 0}$ is D_ω-classical for any $m \geq 1$ and we have*

$$D_{-\omega}\big(\Phi_m u_0^{[m]}\big) + \psi_m u_0^{[m]} = 0 \tag{2.5}$$

with $\Phi_m(x) = \big(\tau_{-m\omega}\Phi\big)(x)$, $\psi_m(x) = \psi(x) - \Big(\sum_{\nu=0}^{m-1} D_\omega \circ \tau_{-\nu\omega}\Phi\Big)(x)$.

$$u_0^{[m]} = \zeta_m\Big(\prod_{\nu=0}^{m-1} \tau_{-\nu\omega}\Phi\Big)u_0 \tag{2.6}$$

where ζ_m is defined by the condition $\big(u_0^{[m]}\big)_0 = 1$.

Suppose $m = 1$. The form u_0 satisfies (1.16) . Multiplying both sides by Φ and on account of (1.6)′ , we get

$$D_{-\omega}\big((\tau_{-\omega}\Phi)(\Phi u_0)\big) + \big(\psi - D_\omega\Phi\big)\Phi u_0 = 0 \, .$$

Therefore , (2.5) and (2.6) are valid for $m = 1$. By induction, we easily obtain the general case.

The previous results allow us to characterize the D_ω–classical sequences through the so-called (functional) Rodrigues formula. See [32,16,9,14].

Proposition 2.3. *The orthogonal sequence $\{P_n\}_{n\geq 0}$ is $D_\omega-$classical if and only if there exist a monic polynomial Φ , $\deg\Phi \leq 2$ and a sequence $\{\lambda_n\}_{n\geq 0}$, $\lambda_n \neq 0$, $n \geq 0$ such that*

$$P_n u_0 = \lambda_n D^n_{-\omega}\Big(\Big\{\prod_{\nu=0}^{n-1}\tau_{-\nu\omega}\Phi\Big\}u_0\Big) \quad , \quad n\geq 0 \tag{2.7}$$

with $\prod_{\nu=0}^{-1} = 1$.

Necessity. Consider $< D^n_{-\omega}u_0^{[n]}, P_m > = (-1)^n < u_0^{[n]}, D^n_\omega P_m >$, n , $m \geq 0$. For $0\leq m \leq n-1$, $n\geq 1$,
we have $D^n_\omega P_m = 0$. For $m \geq n$, put $m = n+\mu$, $\mu \geq 0$. Then

$$< u_0^{[n]}, D^n_\omega P_{n+\mu} > = \prod_{\nu=1}^{n}(\mu+\nu) < u_0^{[n]}, P_\mu^{[n]} > = n!\delta_{0,\mu}$$

following the definitions. Consequently

$$D^n_{-\omega}u_0^{[n]} = (-1)^n n! u_n \quad , \quad n\geq 0 \, .$$

But from the assumption $u_n = (< u_0, P_n^2 >)^{-1}P_n u_0$, $n \geq 0$ so that, in accordance with (2.6) , we obtain (2.7) where

$$\lambda_n = (-1)^n\frac{< u_0, P_n^2 >}{n!}\zeta_n \quad , \quad n\geq 0\, . \tag{2.8}$$

Sufficiency. Making $n=1$ in (2.7) , we have

$$P_1 u_0 = \lambda_1 D_{-\omega}(\Phi u_0)\, .$$

Therefore, the form u_0 is $D_{-\omega}-$classical, since it is regular.

The Rodrigues' formula can serve for describing the D_ω-classical sequences which are completely determined by the knowledge of the sequences $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$. It is doubtless the shortest way for obtaining them. Indeed, on account of (2.7) , the recurrence relation (1.20) is equivalent to

$$\begin{aligned}\lambda_{n+2}D^{n+2}_{-\omega}\Big(\Big[\prod_{\nu=0}^{n+1}\tau_{-\nu\omega}\Phi\Big]u_0\Big) =& \lambda_{n+1}(x-\beta_{n+1})D^{n+1}_{-\omega}\Big(\Big[\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\Big]u_0\Big)\\ &-\lambda_n\gamma_{n+1}D^n_{-\omega}\Big(\Big[\prod_{\nu=0}^{n-1}\tau_{-\nu\omega}\Phi\Big]u_0\Big) \quad , \quad n\geq 0\, .\end{aligned} \tag{2.9}$$

Proposition 2.4. *The sequences $\{\lambda_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_{n+1}\}_{n\geq 0}$ respectively fulfil the equations*

$$\lambda_{n+2}\Big\{\lambda_1^{-1}+(n+1)\Phi''(0)\Big\}\Big\{\lambda_1^{-1}+(n+\frac{1}{2})\Phi''(0)\Big\}-\lambda_{n+1}\Big\{\lambda_1^{-1}+\frac{1}{2}n\Phi''(0)\Big\}=0 \quad , \quad n\geq 0 \tag{2.10}$$

(2.11)

$$\Big\{\lambda_1^{-1}+n\Phi''(0)\Big\}\beta_{n+1}=\frac{1}{2}n^2\omega\Phi''(0)+\lambda_1^{-1}(n\omega-\beta_0)-\frac{\lambda_{n+2}}{\lambda_{n+1}}\Big\{\frac{1}{2}n(n+1)(2n+1)\omega\big(\Phi''(0)\big)^2$$
$$+(n+1)(2n+1)\Phi'(0)\Phi''(0)+\lambda_1^{-1}\Big(\omega\big(n^2-\frac{1}{2}\big)-\beta_0(2n+1)\Big)\Phi''(0)$$
$$+2(n+1)\lambda_1^{-1}\Phi'(0)-\lambda_1^{-2}(2\beta_0+\omega)\Big\}\quad,\quad n\geq 0$$

(2.12)

$$\lambda_n\gamma_{n+1}=\lambda_{n+1}\Big\{\Big(\lambda_1^{-1}\beta_0-n\Phi'(0)-\frac{1}{2}n^2\omega\Phi''(0)\Big)\beta_{n+1}-n\Big(\lambda_1^{-1}\beta_0\omega+\Phi(0)\Big)\Big\}$$
$$-\lambda_{n+2}\Big\{\frac{1}{4}n^2(n+1)^2\omega^2\big(\Phi''(0)\big)^2+\frac{1}{2}n(n+1)(2n+1)\omega\Phi'(0)\Phi''(0)+(n+1)\Phi(0)\Phi''(0)$$
$$+n(n+1)\big(\Phi'(0)\big)^2-\frac{1}{2}(2n^2-1)\lambda_1^{-1}\beta_0\omega\Phi''(0)-(2n+1)\lambda_1^{-1}\beta_0\Phi'(0)+\lambda_1^{-1}\Phi(0)+\lambda_1^{-2}\beta_0(\beta_0+\omega)\Big\}$$
$$n\geq 0\,.$$

The proof will be carried out in three steps.

First step. The relation (2.9) implies

$$D_{-\omega}\left(\lambda_{n+2}D_{-\omega}\left(\Big[\prod_{\nu=0}^{n+1}\tau_{-\nu\omega}\Phi\Big]u_0\right)-\lambda_{n+1}\big(x+(n+1)\omega-\beta_{n+1}\big)\Big[\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\Big]u_0\right)$$
$$+\prod_{\nu=0}^{n-1}\tau_{-\nu\omega}\Phi\Big\{(n+1)\lambda_{n+1}\tau_{-n\omega}\Phi+\lambda_n\gamma_{n+1}\Big\}u_0=0\quad,\quad n\geq 0\,.\tag{2.13}$$

For proving (2.13) , we need the so-called Leibniz's rule corresponding to operator D_ω .

Lemma 2.5. *For any $g\in\mathcal{P}$ and $u\in\mathcal{P}'$, we have*

$$D_{-\omega}^n\big((\tau_{-n\omega}g)u\big)=\sum_{\nu=0}^{n}\binom{n}{\nu}D_\omega^\nu g D_{-\omega}^{n-\nu}u\quad;\quad n\geq 0\,.\tag{2.14}$$

From (1.6)′ where $g\to\tau_{-\omega}g$ and taking account of $D_\omega=D_{-\omega}\circ\tau_{-\omega}$, we get

$$D_{-\omega}\big((\tau_{-\omega}g)u\big)=gD_{-\omega}u+\big(D_\omega g\big)u\,.\tag{2.15}$$

Now, suppose (2.14) for $0 \le m \le n$. Then, taking $g \to \tau_{-\omega} g$ and by virtue of (2.15) , we have

$$\begin{aligned}
D_{-\omega}^{n+1}\Big((\tau_{-(n+1)\omega}g)u\Big) &= \sum_{\nu=0}^{n} \binom{n}{\nu} D_{-\omega}\Big((D_{\omega}^{\nu} \circ \tau_{-\omega}g) D_{-\omega}^{n-\nu}u\Big) \\
&= \sum_{\nu=0}^{n} \binom{n}{\nu} D_{-\omega}\Big((\tau_{-\omega} \circ D_{\omega}^{\nu}g) D_{-\omega}^{n-\nu}u\Big) \\
&= \sum_{\nu=0}^{n} \binom{n}{\nu} \Big\{D_{\omega}^{\nu}gD_{-\omega}^{n+1-\nu}u + D_{\omega}^{\nu+1}gD_{-\omega}^{n-\nu}u\Big\} \\
&= \sum_{\nu=0}^{n} \binom{n}{\nu} D_{\omega}^{\nu}gD_{-\omega}^{n+1-\nu}u + \sum_{\nu=1}^{n+1} \binom{n}{\nu-1} D_{\omega}^{\nu}gD_{-\omega}^{n+1-\nu}u \\
&= gD_{-\omega}^{n+1}u + \sum_{\nu=1}^{n}\Big\{\binom{n}{\nu} + \binom{n}{\nu-1}\Big\}D_{\omega}^{\nu}gD_{-\omega}^{n+1-\nu}u + (D_{\omega}^{n+1}g)u \\
&= \sum_{\nu=0}^{n+1} \binom{n+1}{\nu} D_{\omega}^{\nu}gD_{-\omega}^{n+1-\nu}u\,.
\end{aligned}$$

Hence (2.14) for $n \ge 0$.

As a consequence, following (2.14) we may write

$$\begin{aligned}
D_{-\omega}^{n+1}\Big((x+(n+1)\omega-\beta_{n+1})\Big[\prod_{\nu=0}^{n}(\tau_{-\nu\omega}\Phi)(x)\Big]u_0\Big) =&(x-\beta_{n+1})D_{-\omega}^{n+1}\Big(\Big[\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\Big]u_0\Big) \\
&+ (n+1)D_{-\omega}^{n}\Big(\Big[\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\Big]u_0\Big)\,.
\end{aligned}$$

Therefore , (2.9) becomes

$$\begin{aligned}
D_{-\omega}^{n}\Bigg[\lambda_{n+2}D_{-\omega}^{2}\Big(\Big[\prod_{\nu=0}^{n+1}\tau_{-\nu\omega}\Phi\Big]u_0\Big) - \lambda_{n+1}D_{-\omega}\Big((x+(n+1)\omega-\beta_{n+1})\prod_{\nu=0}^{n}(\tau_{-\nu\omega}\Phi)(x)u_0\Big) \\
+ (n+1)\lambda_{n+1}\Big[\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\Big]u_0 + \lambda_n\gamma_{n+1}\Big[\prod_{\nu=0}^{n-1}\tau_{-\nu\omega}\Phi\Big]u_0\Bigg] = 0 \quad , \quad n \ge 0\,.
\end{aligned}$$

Hence (2.13) .

Second step. We have the relation

(2.16)

$$\lambda_1^{-1}P_1(x)(\tau_\omega\Omega_n)(x) + \Phi(x)(D_{-\omega}\Omega_n)(x) + \prod_{\nu=0}^{n-1}(\tau_{-\nu\omega}\Phi)(x)\Big\{(n+1)\lambda_{n+1}(\tau_{-n\omega}\Phi)(x) + \lambda_n\gamma_{n+1}\Big\} = 0\,, \quad n \ge 0\,.$$

where

$$\begin{aligned}
\Omega_n(x) = \lambda_{n+2}\Big\{\lambda_1^{-1}P_1(x)\prod_{\nu=0}^{n-1}(\tau_{-(\nu+1)\omega}\Phi)(x) + \Big(D_\omega\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\Big)(x)\Big\} \\
- \lambda_{n+1}\prod_{\nu=0}^{n-1}(\tau_{-(\nu+1)\omega}\Phi)(x)(x+(n+1)\omega-\beta_{n+1}) \quad , \quad n \ge 0\,.
\end{aligned} \tag{2.17}$$

Indeed, we may write

$$\begin{aligned} D_{-\omega}\Big(\Big[\prod_{\nu=0}^{n+1}\tau_{-\nu\omega}\Phi\Big]u_0\Big) &= D_{-\omega}\Big(\Big[\tau_{-\omega}\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\Big]\Phi u_0\Big) \\ &= \Big(\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\Big)D_{-\omega}(\Phi u_0) + D_{\omega}\Big(\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\Big)\Phi u_0 \quad , \quad \text{from} \quad (2.15) \\ &= \lambda_1^{-1}P_1(x)\Big(\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\Big)u_0 + D_{\omega}\Big(\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\Big)\Phi u_0 \quad , \end{aligned}$$

on account of (2.7) where $n=1$. We deduce for (2.13)

$$D_{-\omega}\big(\Omega_n\Phi u_0\big) + \Big(\prod_{\nu=0}^{n-1}\tau_{-\nu\omega}\Phi\Big)\Big((n+1)\lambda_{n+1}\tau_{-n\omega}\Phi + \lambda_n\gamma_{n+1}\Big)u_0 = 0 \quad , \quad n\geq 0\,.$$

Further, in accordance with (1.6)′ and (2.7) , we get

$$D_{-\omega}\big(\Omega_n\Phi u_0\big) = \big(\tau_\omega\Omega_n\big)D_{-\omega}(\Phi u_0) + D_{-\omega}(\Omega_n)\Phi u_0 = \lambda_1^{-1}P_1\big(\tau_\omega\Omega_n\big)u_0 + D_{-\omega}(\Omega_n)\Phi u_0\,.$$

The relation (2.16) follows since u_0 is regular.
Third step. The following condition holds
(2.18)

$$\begin{aligned} &\Big\{\lambda_1^{-1}P_1(x) + \sum_{\nu=1}^{n}\big(D_{-\omega}\circ\tau_{-\nu\omega}\Phi\big)(x)\Big\}\times \\ &\Big\{\lambda_{n+2}\Big[\lambda_1^{-1}\big(\tau_\omega P_1\big)(x) + \sum_{\nu=0}^{n}\big(D_{-\omega}\circ\tau_{-\nu\omega}\Phi\big)(x)\Big] - \lambda_{n+1}\big(x+n\omega-\beta_{n+1}\big)\Big\} \\ &+ \big(\tau_{-n\omega}\Phi\big)(x)\Big\{\lambda_{n+2}\Big[\lambda_1^{-1} + \sum_{\nu=0}^{n}\big(D_{-\omega}\circ D_\omega\circ\tau_{-\nu\omega}\Phi\big)(x)\Big] + n\lambda_{n+1}\Big\} + \lambda_n\gamma_{n+1} = 0 \quad , \quad n\geq 0 \end{aligned}$$

with $\sum_{\nu=1}^{0} = 0$.
In (2.16) , we have to calculate $\tau_\omega\Omega_n$ and $\Phi D_{-\omega}(\Omega_n)$. From (2.17) , we get

$$\begin{aligned} \big(\tau_\omega\Omega_n\big)(x) &= \lambda_{n+2}\Big\{\lambda_1^{-1}\big(\tau_\omega P_1\big)(x)\prod_{\nu=0}^{n-1}\big(\tau_{-\nu\omega}\Phi\big)(x) + \Big(D_{-\omega}\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\Big)(x)\Big\} \\ &\quad - \lambda_{n+1}\prod_{\nu=0}^{n-1}\big(\tau_{-\nu\omega}\Phi\big)(x)\big(x+n\omega-\beta_{n+1}\big)\,. \end{aligned} \tag{2.19}$$

Now, we have

$$\Big(D_{-\omega}\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\Big)(x) = \prod_{\nu=0}^{n-1}\big(\tau_{-\nu\omega}\Phi\big)(x)\sum_{\nu=0}^{n}\big(D_{-\omega}\circ\tau_{-\nu\omega}\Phi\big)(x) \quad , \quad n\geq 0\,. \tag{2.20}$$

We proceed by induction. For $n=1$, we get

$$\Big(D_{-\omega}\Phi\tau_{-\omega}\Phi\Big)(x) = \Phi(x)\Big(\Big(D_{-\omega}\Phi\Big)(x) + \Big(D_{-\omega}\circ\tau_{-\omega}\Phi\Big)(x)\Big)$$

since

$$\Big(D_{-\omega}\big(g\tau_{-\omega}f\big)\Big)(x)=g(x)\big(D_\omega f\big)(x)+f(x)\big(D_{-\omega}g\big)(x) \quad , \quad \text{from (1.3)} . \tag{2.21}$$

We assume (2.20) for $0\leq m\leq n$. Therefore, according to (2.21) and (2.20) , we have

$$\begin{aligned}\Big(D_{-\omega}\prod_{\nu=0}^{n+1}\tau_{-\nu\omega}\Phi\Big)(x)&=\Big(D_{-\omega}\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\big(\tau_{-(n+1)\omega}\Phi\big)\Big)(x)\\&=\prod_{\nu=0}^{n}\big(\tau_{-\nu\omega}\Phi\big)(x)\big(D_\omega\circ\tau_{-n\omega}\Phi\big)(x)+\big(\tau_{-n\omega}\Phi\big)(x)\Big(D_{-\omega}\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\Big)(x)\\&=\prod_{\nu=0}^{n}\big(\tau_{-\nu\omega}\Phi\big)(x)\sum_{\nu=0}^{n+1}\big(D_{-\omega}\circ\tau_{-\nu\omega}\Phi\big)(x) .\end{aligned}$$

On account of (2.20) , the relation (2.19) becomes

$$\begin{aligned}&\big(\tau_\omega\Omega_n\big)(x)=\\&\prod_{\nu=0}^{n-1}\big(\tau_{-\nu\omega}\Phi\big)(x)\Big\{\lambda_{n+2}\Big(\lambda_1^{-1}\big(\tau_\omega P_1\big)(x)+\sum_{\nu=0}^{n}\big(D_{-\omega}\circ\tau_{-\nu\omega}\Phi\big)(x)\Big)-\lambda_{n+1}\big(x+n\omega-\beta_{n+1}\big)\Big\} ,\\&\qquad n\geq 0 .\end{aligned}\tag{2.22}$$

Hence

$$\Phi(x)\Omega_n(x)=\left(\prod_{\nu=0}^{n}\big(\tau_{-\nu\omega}\Phi\big)(x)\right)\Lambda_n(x) \quad \text{with}$$

$$\Lambda_n(x)=\lambda_{n+2}\left\{\lambda_1^{-1}P_1(x)+\sum_{\nu=0}^{n}\big(D_\omega\circ\tau_{-\nu\omega}\Phi\big)(x)\right\}-\lambda_{n+1}\big(x+(n+1)\omega-\beta_{n+1}\big) .$$

According to (1.3) , this yields

$$\big(D_{-\omega}(\Phi\Omega_n)\big)(x)=\Phi(x)\big(D_{-\omega}\Omega_n\big)(x)+\big(\tau_\omega\Omega_n\big)(x)\big(D_{-\omega}\Phi\big)(x)$$

$$\Big(D_{-\omega}\Lambda_n\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\Big)(x)=\prod_{\nu=0}^{n}\big(\tau_{-\nu\omega}\Phi\big)(x)\,\big(D_{-\omega}\Lambda_n\big)(x)+\big(\tau_\omega\Lambda_n\big)(x)\Big(D_{-\omega}\prod_{\nu=0}^{n}\tau_{-\nu\omega}\Phi\Big)(x) .$$

Comparing and in accordance with (2.20) and (2.22) , we deduce

$$\Phi(x)\big(D_{-\omega}\Omega_n\big)(x)=\prod_{\nu=0}^{n-1}\big(\tau_{-\nu\omega}\Phi\big)(x)\Big\{\big(\tau_{-n\omega}\Phi\big)(x)\big(D_{-\omega}\Lambda_n\big)(x)+\big(\tau_\omega\Lambda_n\big)(x)\sum_{\nu=1}^{n}\big(D_{-\omega}\circ\tau_{-\nu\omega}\Phi\big)(x)\Big\} .$$

Taking account of

$$\big(\tau_\omega\Lambda_n\big)(x)=\lambda_{n+2}\Big\{\lambda_1^{-1}\big(\tau_\omega P_1\big)(x)+\sum_{\nu=0}^{n}\big(D_{-\omega}\circ\tau_{-\nu\omega}\Phi\big)(x)\Big\}-\lambda_{n+1}\big(x+n\omega-\beta_{n+1}\big)$$

$$\big(D_{-\omega}\Lambda_n\big)(x)=\lambda_{n+2}\Big\{\lambda_1^{-1}+\sum_{\nu=0}^{n}\big(D_{-\omega}\circ D_\omega\circ\tau_{-\nu\omega}\Phi\big)(x)\Big\}-\lambda_{n+1} ,$$

finally, after simplifying, the relation (2.16) yields (2.18) . Lastly, writing $\Phi(x) = \frac{1}{2}\Phi''(0)x^2 + \Phi'(0)x + \Phi(0)$ and with

$$\sum_{\nu=1}^{n}(D_{-\omega}\circ\tau_{-\nu\omega}\Phi)(x) = \Phi''(0)nx + \Phi'(0)n + \frac{1}{2}\Phi''(0)\omega n^2 \quad , \quad n \geq 0$$
$$\sum_{\nu=0}^{n}(D_{-\omega}\circ\tau_{-\nu\omega}\Phi)(x) = (n+1)\big(\Phi''(0)x + \Phi'(0) + \frac{1}{2}\Phi''(0)\omega(n-1)\big) \quad , \quad n \geq 0$$
$$\sum_{\nu=0}^{n}(D_{-\omega}\circ D_{\omega}\circ\tau_{-\nu\omega}\Phi)(x) = (n+1)\Phi''(0) \quad , \quad n \geq 0\,,$$

an easy computation leads to (2.10) – (2.12) .

§3. The canonical cases

The equation (2.10) yields

$$\frac{\lambda_{n+2}}{\lambda_{n+1}} = \frac{\lambda_1^{-1} + \frac{1}{2}\Phi''(0)n}{\{\lambda_1^{-1} + \Phi''(0)(n+1)\}\{\lambda_1^{-1} + \Phi''(0)(n+\frac{1}{2})\}} \quad , \quad n \geq 0\,.$$

Two cases arise.

A. $\Phi''(0) = 0 \quad , \quad \lambda_n = \lambda_1^n \quad , \quad n \geq 0\,.$

B. $\Phi''(0) = 2 \quad , \quad \lambda_n = \dfrac{\Gamma(\lambda_1^{-1} + n - 1)}{\Gamma(\lambda_1^{-1} + 2n - 1)} \quad , \quad n \geq 0\,.$

Before quoting the different canonical situations, let us proceed to the general transformation

$$\widetilde{P}_n(x) = A^{-n}P_n(Ax + B) \quad , \quad n \geq 0 \tag{3.1}$$

$$\tilde{\beta}_n = \frac{\beta_n - B}{A} \quad , \quad \tilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{A^2} \quad , \quad n \geq 0\,. \tag{3.2}$$

Then the form $\tilde{u}_0 = (h_{A^{-1}}\circ\tau_{-B})u_0$ fulfils

$$D_{-\frac{\omega}{A}}\big(A^{-t}\Phi(Ax+B)\tilde{u}_0\big) + A^{1-t}k^{-1}(Ax + B - \beta_0)\tilde{u}_0 = 0\,. \tag{3.3}$$

where k is given by (2.4)$'$. Any so-called canonical situation will be denoted by $\hat{\beta}_n$, $\hat{\gamma}_{n+1}$, $\hat{u}_0$.

A_1. $\Phi(x) = 1$

From (2.11) and (2.12) , we get

$$\beta_n = \beta_0 + \omega n \quad , \quad \gamma_{n+1} = -\lambda_1(n+1) \quad , \quad n \geq 0\,.$$

We may choose $\beta_0 = 0$ and $\lambda_1 = -1/2$ which is equivalent to the choice $A^2 = -2\lambda_1$, $B = \beta_0$ and afterwards $\omega \longrightarrow A\omega$. Then

$$\begin{cases} \beta_n = \omega n \quad , \quad \gamma_{n+1} = \frac{1}{2}(n+1) \quad , \quad \lambda_n = \dfrac{(-1)^n}{2^n} \quad , \quad n \geq 0 \\ D_{-\omega}(u_0) + 2xu_0 = 0\,. \end{cases} \tag{3.4}$$

When $\omega = 0$, we rediscover the Hermite form.

Another choice is $A = \omega$, $B = \beta_0 - \omega a$ where we have put $-\lambda_1/A^2 = a$; then we obtain the following canonical case

$$\text{(3.5)} \qquad \begin{cases} \hat{\beta}_n = a + n \quad , \quad \hat{\gamma}_{n+1} = a(n+1) \quad , \quad n \geq 0 \\ D_{-1}(\hat{u}_0) + a^{-1}(x-a)\hat{u}_0 = 0 \quad , \quad a \neq 0 \, . \end{cases}$$

It is the definition of the Charlier polynomials [8].
Between $\{P_n\}_{n\geq 0}$ associated with u_0 and $\{\widehat{P}_n\}_{n\geq 0}$ associated with $\hat{u}_0$, we have the relation

$$\text{(3.6)} \qquad \widehat{P}_n(x) = (2a)^{n/2} P_n\Big(\frac{x-a}{\sqrt{2a}}\Big) \quad , \quad n \geq 0$$

since $\omega = (2a)^{-1/2}$.
When $\omega \to i\omega$, we choose $A = \omega$, $B = \beta_0 - \omega b$ and we put $\lambda_1 \omega^{-2} = a$, then

$$\text{(3.7)} \qquad \begin{cases} \hat{\beta}_n = b + in \quad , \quad \hat{\gamma}_{n+1} = -a(n+1) \quad , \quad n \geq 0 \\ D_{-i}(\hat{u}_0) - a^{-1}(x-b)\hat{u}_0 = 0 \, . \end{cases}$$

where it is possible to choose b .

$\mathbf{A_2}$. $\mathbf{\Phi(x) = x}$

From (2.11) , (2.12) and (2.4)′ , we have

$$\beta_n = (\omega - 2\lambda_1)n + \beta_0 \quad , \quad \gamma_{n+1} = \lambda_1(n+1)\big((\lambda_1 - \omega)n - \beta_0\big) \quad , \quad n \geq 0 \quad ; \quad k = -\lambda_1 \, .$$

Two cases arise.

$\mathbf{A_{21}}$. $\boldsymbol{\lambda_1 = \omega \neq 0}$. Consequently

$$\beta_n = \beta_0 - \omega n \quad , \quad \gamma_{n+1} = -\beta_0 \omega(n+1) \quad , \quad n \geq 0 \quad , \quad \beta_0 \neq 0 \, .$$

The form u_0 satisfies the following equation

$$D_{-\omega}(xu_0) - \omega^{-1}(x - \beta_0)u_0 = 0 \, .$$

With $A = -\omega$, $B = 0$ and putting $\beta_0 = -a\omega$, we meet again the Charlier polynomials. Besides (3.5) , the form $\hat{u}_0$ also satisfies

$$\text{(3.8)} \qquad D_1(x\hat{u}_0) + (x-a)\hat{u}_0 = 0 \, .$$

$\mathbf{A_{22}}$. $\boldsymbol{\lambda_1 - \omega \neq 0}$. Then [6,24]

$$\beta_n = \beta_0 + (\omega - 2\lambda_1)n \quad , \quad \gamma_{n+1} = \lambda_1(\lambda_1 - \omega)(n+1)\Big(n + \frac{\beta_0}{\omega - \lambda_1}\Big) \quad , \quad n \geq 0 \, .$$

We choose $\lambda_1(\lambda_1 - \omega) = 1$ with $\beta_0/(\omega - \lambda_1) := \alpha + 1$. Therefore

$$\text{(3.9)} \qquad \begin{cases} \beta_n = -\lambda_1^{-1}(\alpha+1) - (\lambda_1 + \lambda_1^{-1})n \quad , \quad \gamma_{n+1} = (n+1)(n+\alpha+1) \quad , \quad n \geq 0 \\ D_{-\omega}(xu_0) - \lambda_1^{-1}\big(x + \lambda_1^{-1}(\alpha+1)\big)u_0 = 0 \end{cases}$$

where $\omega = \lambda_1 - \lambda_1^{-1}$.
When $\lambda_1 = -1$ then $\omega = 0$: we rediscover the Laguerre polynomials.Putting $\lambda_1 := -e^{-\varphi}$, we have $\omega = 2\sinh\varphi$, then

$$(3.9)' \qquad \begin{cases} \beta_n = e^{\varphi}(\alpha+1) + 2n\cosh\varphi \quad , \quad \gamma_{n+1} = (n+1)(n+\alpha+1) \quad , \quad n \geq 0 \\ D_{-\omega}(xu_0) + e^{\varphi}\big(x - (\alpha+1)e^{\varphi}\big)u_0 = 0 \, . \end{cases}$$

We obtain the following canonical case by taking $A = \omega$, $B = (\alpha+1)\omega$ and putting $c := e^{-2\varphi}$, $\varphi \neq 0$

$$(3.10) \qquad \begin{cases} \hat{\beta}_n = \dfrac{c}{1-c}(\alpha+1) + \dfrac{1+c}{1-c}n \quad , \quad \hat{\gamma}_{n+1} = \dfrac{c}{(1-c)^2}(n+1)(n+\alpha+1) \quad , \quad n \geq 0 \\ D_{-1}\big((x+\alpha+1)\hat{u}_0\big) - \Big((1-c^{-1})x + (\alpha+1)\Big)\hat{u}_0 = 0 \, . \end{cases}$$

When $c \in \mathbb{R} - \{0,1\}$, we rediscover the so-called Meixner polynomials of the first kind ($\varphi \in \mathbb{R} - \{0\}$) . In the case $\alpha + 1 > 0$, the form $\hat{u}_0$ is positive definite. It is regular for $c \in \mathbb{C} - \{0,1\}$, $\alpha + 1 \neq -n$, $n \geq 0$.
It is easy to see that $\hat{u}_0 = \hat{u}_0(c)$ fulfils the relation

$$(3.11) \qquad \big(h_{-1} \circ \tau_{\alpha+1}\big)\hat{u}_0(c) = \hat{u}_0(c^{-1}) \, .$$

Whence the Meixner form $\hat{u}_0(c)$ also satisfies

$$(3.12) \qquad D_1(x\hat{u}_0) + \big\{(1-c)x - c(\alpha+1)\big\}\hat{u}_0 = 0 \, .$$

When $\omega \to i\omega$, we put $\omega := 2\sin\phi$, $0 < \phi < \pi$, then $\lambda_1 = e^{i\phi}$ (for $\varphi \to i\varphi$, $\phi = \pi - \varphi$) and

$$\beta_n = -e^{-i\phi}(\alpha+1) - 2n\cos\phi \quad , \quad \gamma_{n+1} = (n+1)(n+\alpha+1) \quad , \quad n \geq 0 \, .$$

We choose $A = \omega$, $B = 2ia\sin\phi$ with $2a := 1 + \alpha$. Consequently

$$(3.13) \qquad \begin{cases} \hat{\beta}_n = -(n+a)\cot\phi \quad , \quad \hat{\gamma}_{n+1} = \dfrac{1}{4}\dfrac{(n+1)(n+2a)}{\sin^2\phi} \quad , \quad n \geq 0 \\ D_{-i}\big((x+ai)\hat{u}_0\big) - 2e^{-i\phi}\big(x\sin\phi + a\cos\phi\big)\hat{u}_0 = 0 \, . \end{cases}$$

We rediscover the so–called Meixner–Pollaczek polynomials. They are not essentially different from the Meixner polynomials of the first kind [27,29].

$\mathbf{B_1}$. $\mathbf{\Phi(x) = x^2}$

From (2.11) , (2.12) and (2.4)′ , we obtain

$$\beta_n = \frac{1}{2}\alpha\omega - \frac{\alpha(\alpha-1)\tau}{(n+\alpha-1)(n+\alpha)} \quad , \quad \gamma_{n+1} = -\frac{(n+1)(n+2\alpha-1)\big(\omega(n+\alpha)^2 - 2\alpha\tau\big)^2}{(2n+2\alpha-1)(2n+2\alpha)^2(2n+2\alpha+1)} \quad , \quad n \geq 0$$

where $\lambda_1^{-1} = 2\alpha$, and τ is an arbitrary parameter.
Two cases arise.

$\mathbf{B_{11}}$. $\boldsymbol{\tau \neq 0}$. Then choosing $\alpha\tau = 1$, which is equivalent to the choice $A = \alpha\tau$, $B = 0$ and afterwards $\omega \to A\omega$, we get

(3.14)

$$\begin{cases} \beta_n = \frac{1}{2}\alpha\omega + \frac{1-\alpha}{(n+\alpha-1)(n+\alpha)} \;, \quad \gamma_{n+1} = -\frac{(n+1)(n+2\alpha-1)\left(\frac{\omega}{2}(n+\alpha)^2-1\right)^2}{(2n+2\alpha-1)(n+\alpha)^2(2n+2\alpha+1)} \;, \quad n \geq 0 \\ D_{-\omega}\big(x^2u_0\big) - 2\big(\alpha x + 1 - \frac{1}{2}\alpha^2\omega\big)u_0 = 0 \,. \end{cases}$$

When $\omega = 0$, we rediscover the Bessel form. The form u_0 is regular if and only if $\alpha \neq -\frac{n}{2}$, $\omega \neq \frac{2}{(n+\alpha)^2}$, $n \geq 0$.

$\mathbf{B_{12}}$. $\omega \neq \mathbf{0}$. We obtain a specific canonical case by taking $A = i\omega$, $B = \frac{1}{2}\alpha\omega$ and putting $\frac{2\alpha\tau}{\omega} := \mu^2$

(3.15)

$$\begin{cases} \hat{\beta}_n = \frac{1}{2}i\frac{(\alpha-1)\mu^2}{(n+\alpha-1)(n+\alpha)} \;, \quad \hat{\gamma}_{n+1} = \frac{1}{4}\frac{(n+1)(n+2\alpha-1)(n+\alpha-\mu)^2(n+\alpha+\mu)^2}{(2n+2\alpha-1)(n+\alpha)^2(2n+2\alpha+1)} \;, \quad n \geq 0 \\ D_i\Big((x-\frac{1}{2}i\alpha)^2\hat{u}_0\Big) + \big(-2\alpha x + i\mu^2\big)\hat{u}_0 = 0 \,. \end{cases}$$

When $\mu = 0$, $\alpha > 0$, the form $\hat{u}_0$ is symmetric and positive definite. In general, it is regular if and only if $\alpha \neq -\frac{n}{2}$, $\alpha - \mu \neq -n$, $\alpha + \mu \neq -n$, $n \in \mathbb{N}$.
The following sequence $\{\widetilde{P}_n\}_{n\geq 0}$ with $\widetilde{P}_n(x) = \widehat{P}_n\Big(x - i(\frac{\alpha}{2} + \mu)\Big)$, $n \geq 0$ is given through

$$(3.16) \qquad \begin{cases} \tilde{\beta}_n = i\left\{\frac{1}{2}\alpha + \mu + \frac{1}{2}\frac{(\alpha-1)\mu^2}{(n+\alpha-1)(n+\alpha)}\right\} \;, \quad n \geq 0 \\ \tilde{\gamma}_{n+1} = \frac{1}{4}\frac{(n+1)(n+2\alpha-1)(n+\alpha+\mu)^2(n+\alpha-\mu)^2}{(2n+2\alpha-1)(n+\alpha)^2(2n+2\alpha+1)} \;, \quad n \geq 0 \\ D_i\left(\big(x - i(\alpha+\mu)\big)^2\tilde{u}_0\right) + \big(-2\alpha x + i(\alpha+\mu)^2\big)\tilde{u}_0 = 0 \,. \end{cases}$$

$\mathbf{B_2}$. $\mathbf{\Phi(x) = (x+1)(x-c)}$, $\mathbf{c \in \mathbb{C} - \{-1\}}$
Following (2.11) , (2.12) and (2.4)$'$, and after some calculations, we obtain
(3.17)

$$\beta_n = \vartheta\frac{\omega}{4} - \frac{1}{2}(1-c) + \frac{\vartheta(\vartheta-2)\tau}{(2n+\vartheta-2)((2n+\vartheta)}$$

$$\gamma_{n+1} = -\frac{(n+1)(n+\vartheta-1)\{\omega n(n+\vartheta)-(1+c)n+\vartheta(\beta_0-c)\}\{\omega n(n+\vartheta)+(1+c)n+\vartheta(\beta_0+1)\}}{(2n+\vartheta-1)(2n+\vartheta)^2(2n+\vartheta+1)}$$

$$n \geq 0$$

where $\vartheta = \lambda_1^{-1}$ and τ is another arbitrary parameter.

$\mathbf{B_{21}}$. $\omega = \mathbf{0}$. We may choose $c = 1$ and putting $\frac{1}{2}\vartheta(1+\beta_0) = \alpha + 1$, $\frac{1}{2}\vartheta(1-\beta_0) = \beta + 1$, we meet again the Jacobi case.

$\mathbf{B_{22}}$. $\omega \neq \mathbf{0}$. It is possible to put

$$\omega n^2 + (\vartheta\omega + 1 + c)n + \vartheta(\beta_0 + 1) = \big\{\omega(n+\beta+1) + 1 + c\big\}(n+\alpha+1)$$

$$\omega n^2 + (\vartheta\omega - (1+c))n + \vartheta(\beta_0 - c) = \big\{\omega(n+\alpha+1) - (1+c)\big\}(n+\beta+1) \,.$$

Then, we may write
(3.18)

$$\begin{cases} \beta_n = \frac{1}{4}\vartheta\omega - \frac{1}{2}(1-c) + \frac{1}{2}(\alpha^2-\beta^2)\dfrac{1+c-\frac{1}{2}(\alpha-\beta)\omega}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} \\ \gamma_{n+1} = -\omega^2\dfrac{(n+1)(n+\alpha+\beta+1)\big(n+\alpha+1-\frac{1+c}{\omega}\big)\big(n+\beta+1+\frac{1+c}{\omega}\big)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)} \\ \qquad\qquad n \geq 0\,. \end{cases}$$

The choice $A = i\omega$, $B = \frac{1}{4}\vartheta\omega - \frac{1}{2}(1-c)$ with $\alpha + 1 - \frac{1+c}{\omega} := \delta + 1$ leads to the canonical case
(3.19)

$$\begin{cases} \hat{\beta}_n = \frac{1}{2}i(\alpha^2-\beta^2)\dfrac{\delta - \frac{1}{2}(\alpha+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} \\ \hat{\gamma}_{n+1} = \dfrac{(n+1)(n+\alpha+\beta+1)(n+\delta+1)(n+\alpha+\beta+1-\delta)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)} \quad n \geq 0\,, \\ D_i\big(\hat{\Phi}(x)\hat{u}_0\big) + \hat{\psi}(x)\hat{u}_0 = 0\,, \end{cases}$$

with

$$\text{(3.20)} \qquad \begin{aligned} \hat{\Phi}(x) &= \Big\{x + \frac{i}{4}(2\delta - 3\alpha - \beta - 2)\Big\}\Big\{x - \frac{i}{4}(2\delta - \alpha + \beta + 2)\Big\} \\ \hat{\psi}(x) &= -(\alpha+\beta+2)x + \frac{1}{2}i(\alpha-\beta)\Big(\delta - \frac{1}{2}(\alpha+\beta)\Big)\,. \end{aligned}$$

When $\alpha - \beta = 0$ or $\delta = \frac{1}{2}(\alpha+\beta)$, the sequence $\{\widehat{P}_n\}_{n\geq 0}$ is symmetric. We get successively

1. $\alpha - \beta = 0$

$$\text{(3.21)} \qquad \hat{\gamma}_{n+1} = \frac{1}{4}\frac{(n+1)(n+2\alpha+1)(n+\delta+1)(n+2\alpha+1-\delta)}{(2n+2\alpha+1)(2n+2\alpha+3)} \quad , \quad n \geq 0\,.$$

The form $\hat{u}_0$ is positive definite when $-1 < \delta < 2\alpha+1$ or when $\alpha \in \mathbb{R}$ and $\delta + \overline{\delta} = 2\alpha$, $\alpha + 1 > 0$.

A notable particular case is when $\alpha = 0$. Then $\widehat{P}_n(x)$ is related to Pasternak polynomial $F_n^{(\delta)}(x) := {}_3F_2\big(-n, n+1, \frac{1}{2}(1+\delta+x); 1, \delta+1; 1\big)$, [8, *pp.*192 – 193; 26] through $G_n^{(\delta)}(x) = 2^n \widehat{P}_n\big(\frac{x}{2}\big)$ where

$$G_n^{(\delta)}(x) = \Big(\frac{i}{2}\Big)^n n! \frac{\Gamma(1+\delta+n)}{\Gamma(1+\delta)} \frac{\Gamma(1/2)}{\Gamma(n+1/2)} F_n^{(\delta)}(ix)\,.$$

When $\delta = 0$, we have Touchard polynomials [30,34,4]. These polynomials are particular cases too of continuous Hahn polynomials. See below.

2. $\delta = \dfrac{1}{2}(\alpha+\beta)$

$$\text{(3.22)} \qquad \hat{\gamma}_{n+1} = \frac{1}{4}\frac{(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+3)} \quad , \quad n \geq 0\,.$$

The form $\hat{u}_0$ is positive definite when $\alpha + 1 > 0$, $\beta + 1 > 0$.

Finally, when α and β are real and $\delta + \overline{\delta} = \alpha + \beta$, there exist a , b such that

$$a + \overline{a} = \alpha + 1 \quad , \quad b + \overline{b} = \beta + 1 \quad , \quad a + \overline{b} = \delta + 1 \quad , \quad \overline{a} + b = \alpha + \beta + 1 - \delta\,.$$

This yields $\Re(a+b)=\frac{1}{2}(\alpha+\beta+2)$, $\Re(a-b)=\frac{1}{2}(\alpha-\beta)$, $i\Im(a-b)=\delta-\frac{1}{2}(\alpha+\beta)$. Therefore, we respectively get for (3.19) and (3.20)

(3.23)

$$\begin{cases} \hat{\beta}_n = 2\dfrac{\Re(a-b)\big(1-\Re(a+b)\big)\Im(a-b)}{(2n+a+\bar{a}+b+\bar{b}-2)(2n+a+\bar{a}+b+\bar{b})} \\ \hat{\gamma}_{n+1} = \dfrac{(n+1)(n+a+\bar{a}+b+\bar{b}-1)(n+a+\bar{b})(n+\bar{a}+b)(n+a+\bar{a})(n+b+\bar{b})}{(2n+a+\bar{a}+b+\bar{b}-1)(2n+a+\bar{a}+b+\bar{b})^2(2n+a+\bar{a}+b+\bar{b}+1)} \end{cases} \quad , \quad n \geq 0\,,$$

(3.24)
$$\begin{aligned} \widehat{\Phi}(x) &= \left\{x-\frac{1}{2}\Im(a-b)-i\Re a\right\}\left\{x+\frac{1}{2}\Im(a-b)-i\Re b\right\} \\ \hat{\psi}(x) &= -2\Re(a+b)x-\Re(a-b)\Im(a-b)\,. \end{aligned}$$

The form $\hat{u}_0$ is positive definite when $\Re a>0$, $\Re b>0$. We meet again the continuous Hahn form $\hat{\hat{u}}_0$, when we make the following tranformation $\hat{\hat{u}}_0=\big(h_{-1}\circ\tau_{-B}\big)\hat{u}_0$ where $B=-\frac{1}{2}\Im(a+b)$. Then

(3.25)
$$\begin{cases} \hat{\hat{\beta}}_n = -\dfrac{1}{2}\Im(a+b)-\hat{\beta}_n \quad , \quad \hat{\hat{\gamma}}_{n+1}=\hat{\gamma}_{n+1} \quad , \quad n\geq 0 \\ D_{-i}\Big((x+i\bar{a})(x+i\bar{b})\hat{\hat{u}}_0\Big)-2\Big(\Re(a+b)x+\Im(ab)\Big)\hat{\hat{u}}_0=0\,. \end{cases}$$

§4. Integral representations and moments

4.1 Consider the following sequence

$$\phi_0(x)=1 \quad , \quad \phi_n(x)=\prod_{\nu=0}^{n-1}(x-\nu\omega) \quad , \quad n\geq 1\,.$$

It satisfies

(4.1)
$$\big(D_\omega\phi_{n+1}\big)(x)=(n+1)\phi_n(x) \quad , \quad n\geq 0$$

(4.2)
$$\phi_{n+1}(x)=x\phi_n(x)-n\omega\phi_n(x) \quad , \quad n\geq 0\,.$$

The relation (4.1) shows that it is a D_ω–Appell sequence. Its dual sequence $\{w_n(\omega)\}_{n\geq 0}$ fulfils

(4.3)
$$\begin{cases} D_{-\omega}\big(w_n(\omega)\big)=-(n+1)w_n(\omega) \quad , \quad n\geq 0 \\ w_0(\omega)=\delta\,. \end{cases}$$

Therefore

(4.4)
$$w_n(\omega)=\frac{(-1)^n}{n!}D^n_{-\omega}\delta \quad , \quad n\geq 0\,.$$

Note that for any $u\in\mathcal{P}'$, we have

(4.5)
$$u=\sum_{n\geq 0}<u,\phi_n>\frac{(-1)^n}{n!}D^n_{-\omega}\delta\,.$$

But, it is easy to see that

$$D^n_{-\omega}\delta = \frac{1}{\omega^n}\left(\sum_{\nu=0}^{n}\binom{n}{\nu}(-1)^\nu\delta_{\nu\omega}\right) \quad , \quad n \geq 0 . \tag{4.6}$$

We infer

$$< D^n_{-\omega}\delta, x^m >= \omega^{m-n}\sum_{\nu=0}^{n}\binom{n}{\nu}(-1)^\nu \nu^m \quad , \quad n\ ,\ m \geq 0 , \tag{4.7}$$

which implies the identity

$$\sum_{\nu=0}^{n}\binom{n}{\nu}(-1)^\nu \nu^m = 0 \quad , \quad n \geq m+1 . \tag{4.8}$$

On account of (4.5) and (4.7) , we get

$$< u, x^m >= \sum_{n=0}^{m} < u, \phi_n > \omega^{m-n}\sum_{\nu=0}^{n}\frac{(-1)^{n-\nu}}{\nu!(n-\nu)!}\nu^m \quad , \quad m \geq 0 . \tag{4.9}$$

Remark. On the other hand, by induction we obtain

$$x^n = \sum_{\nu=0}^{n}\alpha^n_\nu \omega^\nu \phi_{n-\nu}(x) \quad , \quad n \geq 0 \tag{4.10}$$

where

$$\begin{array}{l} \alpha^{n+1}_\nu = \alpha^n_\nu + (n+1-\nu)\alpha^n_{\nu-1} \quad , \quad 0 \leq \nu \leq n \quad , \quad n \geq 0 \\ \alpha^n_0 = 1 \quad , \quad \alpha^n_n = \delta_{0,n} \quad , \quad \alpha^n_{-1} = 0 \quad , \quad n \geq 0 . \end{array} \tag{4.11}$$

When $\nu \to n-\nu$, we have

$$\alpha^{n+1}_{n-\nu} - (\nu+1)\alpha^n_{n-\nu-1} = \alpha^n_{n-\nu} \quad , \quad 0 \leq \nu \leq n \quad , \quad n \geq 0 .$$

It follows for $n \geq \nu$: $\quad \alpha^n_{n-\nu-1} = \sum_{\mu=0}^{n-1-\nu}(\nu+1)^{n-\nu-1-\mu}\alpha^{\nu+\mu}_\mu \quad$, or if $\quad n \to n+\nu+1$

$$\alpha^{n+\nu+1}_n = \sum_{\mu=0}^{n}(\nu+1)^{n-\mu}\alpha^{\mu+\nu}_\mu \quad , \quad n \geq 0 .$$

Equivalently, the generating function $\quad A_m(z) = \sum_{n\geq 0}\alpha^{n+m}_n z^n \quad$ fulfils the following relation $A_{m+1}(z) = A_m(z)\big(1-(m+1)z\big)^{-1} \quad , |z| < (m+1)^{-1}$. Therefore

$$A_m(z) = \prod_{\mu=0}^{m}\frac{1}{1-\mu z} \quad , \quad |z| < \frac{1}{m} \quad , \quad m \geq 0 . \tag{4.12}$$

Taking account of (4.7) and (4.10) , we get

$$\alpha_{m-n}^{m}=\frac{1}{n!}\sum_{\nu=0}^{n}\binom{n}{\nu}(-1)^{n-\nu}\nu^{m}\quad ,\quad n,m\geq 0\,. \tag{4.13}$$

So that

$$\prod_{\mu=0}^{m}\frac{1}{1-\mu z}=\sum_{n\geq 0}\left\{\sum_{\nu=0}^{m}\frac{(-1)^{m-\nu}}{\nu!(m-\nu)!}\nu^{m+n}\right\}z^{n}\quad ,\quad m\geq 0\,. \tag{4.14}$$

Also see [5] .

4.2 Putting $< u,\phi_n >:= (u)_n^{\phi}$, $n \geq 0$, from equation (1.16) where $\Phi(x) = \frac{1}{2}\Phi''(0)x^2 + \Phi'(0)x + \Phi(0)$, $\psi(x) = \psi'(0)x + \psi(0)$, we have $< D_{\omega}(\Phi u_0) + \psi u_0, \phi_n >= 0$, $n \geq 0$ namely, on account of (4.1) and (4.2)

$$\begin{aligned}&\Big\{\psi'(0)-\frac{1}{2}(n+1)\Phi''(0)\Big\}(u_0)_{n+2}^{\phi}\\&+\Big\{\psi(0)+(n+1)\big(\omega\psi'(0)-\Phi'(0)\big)-\frac{1}{2}(2n+1)(n+1)\omega\Phi''(0)\Big\}(u_0)_{n+1}^{\phi}\\&-(n+1)\Big\{\Phi(0)+n\omega\Phi'(0)+\frac{1}{2}n^2\omega^2\Phi''(0)\Big\}(u_0)_{n}^{\phi}=0\quad ,\quad n\geq 0\end{aligned} \tag{4.15}$$

$$\psi'(0)(u_0)_1^{\phi}+\psi(0)=0\,. \tag{4.16}$$

The usual moments are obtained from (4.9) .

When $\omega \in \mathbb{R}$, say $\omega = 1$, equalities (4.5) and (4.6) show that u_0 is represented by a discrete measure with supp $\subset [0,+\infty[$:

$$u_0=\sum_{\nu\geq 0}\frac{1}{\nu!}\Big(\sum_{m\geq 0}\frac{(-1)^m}{m!}< u_0,\phi_{m+\nu} >\Big)\delta_{\nu}\ , \tag{4.17}$$

if the following conditions are satisfied

$$s(\nu)=\left|\sum_{m\geq 0}\frac{(-1)^m}{m!}< u_0,\phi_{m+\nu} >\right|< +\infty\quad\text{for any}\ \ \nu\geq 0\,. \tag{4.18}$$

$$\sum_{\nu\geq 0}\frac{s(\nu)\nu^n}{\nu!}< +\infty\quad\text{for any}\quad n\geq 0\,. \tag{4.19}$$

When $\omega \to i\omega$, $\omega \in \mathbb{R}$, we are looking for a weight function U such that

$$< u_0,f >=\int_{-\infty}^{+\infty}U(x)f(x)dx \tag{4.20}$$

where we suppose that U is regular as far as it is necessary. From (4.16) and (4.17) , we get

$$\int_{-\infty}^{+\infty}\Big\{\Big(D_{-i\omega}(\Phi U)\Big)(x)+\psi(x)U(x)\Big\}f(x)dx=0\quad ,\quad f\in\mathcal{P}$$

with the additional condition

$$(4.21)\qquad \int_{-\infty+i\omega}^{+\infty+i\omega} U(x-i\omega)\Phi(x-i\omega)f(x)dx = \int_{-\infty}^{+\infty} U(x-i\omega)\Phi(x-i\omega)f(x)dx \quad , \quad f \in \mathcal{P}\,.$$

Therefore

$$(4.22)\qquad D_{-i\omega}\big(\Phi U\big) + \psi U = \lambda g$$

where $\lambda \in \mathbb{C}$ and g is a representation of the null form.
When $\lambda = 0$, equation (4.22) becomes

$$\Phi(x-i\omega)U(x-i\omega) = \Big(\Phi(x) + i\omega\psi(x)\Big)U(x)\,,$$

so that, if $\omega = -1$, we have

$$(4.23)\qquad U(x+i) = \frac{\Phi(x) - i\psi(x)}{\Phi(x+i)}U(x) \quad , \quad x \in \mathbb{R}$$

and if $\omega = 1$, with $x \to x+i$, we have

$$(4.24)\qquad U(x+i) = \frac{\Phi(x)}{\Phi(x+i) + i\psi(x+i)}U(x) \quad , \quad x \in \mathbb{R}\,.$$

4.3 Now we are able to calculate the moments and to give the integral representations for any canonical form satisfying (4.15) – (4.16) .

$\mathbf{A_1}$. For the Charlier polynomials, from (3.5) we have $\Phi(x) = 1$ and $\psi(x) = a^{-1}(x-a)$. Then the system (4.15) – (4.16) where $\omega = 1$ becomes

$$\begin{aligned} &a^{-1}(\hat{u}_0)^{\phi}_{n+2} - (\hat{u}_0)^{\phi}_{n+1} + (n+1)\Big(a^{-1}(\hat{u}_0)^{\phi}_{n+1} - (\hat{u}_0)^{\phi}_{n}\Big) = 0 \quad , \quad n \geq 0 \\ &a^{-1}(\hat{u}_0)^{\phi}_{1} - 1 = 0\,. \end{aligned}$$

Consequently

$$(4.25)\qquad (\hat{u}_0)^{\phi}_{n} = a^n \quad , \quad n \geq 0\,.$$

From (4.17) , we obtain the well–known result [8]

$$(4.26)\qquad \hat{u}_0 = e^{-a}\sum_{\nu\geq 0}\frac{a^\nu}{\nu!}\delta_\nu\,.$$

In the case (3.7) , we have $\Phi(x) = 1$, $\psi(x) = -a^{-1}(x-b)$ and $\omega = 1$. Then from (4.24) , we get

$$U(x) = \frac{a^{-ix}}{\Gamma\big(1+a-i(x-b)\big)}A(x) \quad \text{with} \quad A(x+i) = A(x)\,.$$

Taking account of

$$(4.27)\qquad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad ,$$

we have

$$U(x)=\frac{a^{-ix}}{\pi}\sin\Big(\pi\big(-a+i(x-b)\big)\Big)\Gamma\big(-a+i(x-b)\big)A(x)\ .$$

Choosing

$$A(x)=\frac{K}{\sin^2\Big(\pi\big(-a+i(x-b)\big)\Big)}\quad ,$$

we infer, again with (4.27)

$$U(x)=\frac{K}{\pi^2}a^{-ix}\Gamma^2\big(-a+i(x-b)\big)\Gamma\big(1+a-i(x-b)\big)\ .$$

Putting $a+ib=-1/2$, we finally obtain

$$U(x)=\frac{K}{\pi^2}a^{-ix}\left|\Gamma(\frac{1}{2}+ix)\right|^2\Gamma(\frac{1}{2}+ix)\quad ,\quad x\in\mathbb{R}\quad ,\quad a\neq 0\ . \tag{4.28}$$

But by virtue of the Mellin–Barnes representation of the second solution $\Psi(\alpha,\gamma;z)$ of the Kummer's equation [31], we have

$$1=\int_{-\infty}^{+\infty}U(x)dx=\frac{2K}{\pi}a^{1/2}\int_0^{+\infty}\frac{e^{-t}}{a+t}dt\quad ,\quad \text{therefore}$$

$$K=\frac{\pi}{2}a^{-1/2}\left(\int_0^{+\infty}\frac{e^{-t}}{a+t}dt\right)^{-1}\quad ,\quad |\arg a|<\pi\ .$$

For the case (3.7) with $b=i(a-\frac{1}{2})$, we finally obtain

$$U(x)=\frac{1}{2}\left(\int_0^{+\infty}\frac{e^{-t}}{a+t}dt\right)^{-1}a^{-(\frac{1}{2}+ix)}\frac{\Gamma(\frac{1}{2}+ix)}{\cosh\pi x}\quad ,\quad x\in\mathbb{R}\quad ,\quad |\arg a|<\pi\ . \tag{4.29}$$

For $a<0$, slight modifications are necessary.

A_2. For the Meixner polynomials of the first kind, from (3.10) we have $\Phi(x)=x+\alpha+1$, $\psi(x)=-\big((1-c^{-1})x+\alpha+1\big)$ and $\omega=1$. Then (4.15) becomes

$$\begin{aligned}(c^{-1}-1)(\hat{u}_0)^{\phi}_{n+2}-(\alpha+1+n+1)(\hat{u}_0)^{\phi}_{n+1}\\ +(n+1)\Big\{(c^{-1}-1)(\hat{u}_0)^{\phi}_{n+1}-(\alpha+1+n)(\hat{u}_0)^{\phi}_{n}\Big\}=0\quad ,\quad n\geq 0\end{aligned}$$

with $(c^{-1}-1)(\hat{u}_0)^{\phi}_1-(\alpha+1)=0$.
Therefore

$$(\hat{u}_0)^{\phi}_n=\left(\frac{c}{1-c}\right)^n\frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)}\quad ,\quad n\geq 0\ ,\ c\in\mathbb{C}-\{0,1\}\ ,\ \alpha+1\in\mathbb{C}-(-\mathbb{N})\ . \tag{4.30}$$

Suppose $c\in\mathbb{R}-\{0,1\}$. First $0<|c|<1$. The condition (4.18) is fulfilled for $-1<c<0$ or $0<c<1/2$ and

$$\sum_{m\geq 0}\frac{(-1)^m}{m!}(\hat{u}_0)^{\phi}_{m+\nu}=c^{\nu}(1-c)^{\alpha+1}\frac{\Gamma(\alpha+1+\nu)}{\Gamma(\alpha+1)}\quad ,\quad \nu\geq 0\ .$$

But the condition (4.19) is satisfied for $|c| < 1$. It is a matter of easy calculation to prove that the form

$$\hat{u}_0(c) = (1-c)^{\alpha+1} \sum_{\nu \geq 0} \frac{\Gamma(\alpha+1+\nu)}{\Gamma(\alpha+1)} \frac{c^\nu}{\nu!} \delta_\nu \tag{4.31}$$

is available for $0 < |c| < 1$, $\alpha + 1 \neq -n$, $n \geq 0$.
When $|c| > 1$, from (4.31) we have

$$\hat{u}_0(c^{-1}) = (1-c^{-1})^{\alpha+1} \sum_{\nu \geq 0} \frac{\Gamma(\alpha+1+\nu)}{\Gamma(\alpha+1)} \frac{c^{-\nu}}{\nu!} \delta_\nu .$$

Therefore by virtue of (3.11)

$$\hat{u}_0(c) = (1-c^{-1})^{\alpha+1} \sum_{\nu \geq 0} \frac{\Gamma(\alpha+1+\nu)}{\Gamma(\alpha+1)} \frac{c^{-\nu}}{\nu!} \delta_{-(\alpha+1+\nu)} \tag{4.32}$$

with the additional condition $\alpha + 1 \in \mathbb{R} - (-\mathbb{N})$.

Regarding (3.13) , we have $\Phi(x) = x + ai$, $\psi(x) = -2e^{-i\phi}(x \sin\phi + a\cos\phi)$ and $\omega = 1$. On account of (4.24) , we get

$$U(x) = e^{2\phi x} \frac{\Gamma(a-ix)}{\Gamma(1-a-ix)} A(x) \quad \text{with} \quad A(x+i) = A(x) .$$

From (4.27) and choosing

$$A(x) = K \frac{e^{-\pi x}}{\sin\big(\pi(a+ix)\big)} ,$$

we obtain

$$U(x) = \frac{K}{\pi} e^{(2\phi-\pi)x} \Gamma(a-ix)\Gamma(a+ix) \quad , \quad x \in \mathbb{R} \quad , \quad \Re a \neq -n \, , \, n \geq 0 .$$

But taking account of

$$\int_{-\infty}^{+\infty} \Gamma(a-ix)\Gamma(a+ix) t^{ix} dx = 2\pi t^a (1+t)^{-2a} \Gamma(2a) \quad , \quad |\arg t| < \pi \quad , \quad \Re a > 0 ,$$

the following condition

$$1 = \int_{-\infty}^{+\infty} U(x) dx = \frac{K}{\pi} \int_{-\infty}^{+\infty} e^{(2\phi-\pi)x} \Gamma(a-ix)\Gamma(a+ix) dx$$

gives

$$K = \frac{2^{2a-1}}{\Gamma(2a)} \sin^{2a}\phi .$$

Therefore

$$U(x) = \frac{(2\sin\phi)^{2a}}{2\pi\Gamma(2a)} e^{(2\phi-\pi)x} \Gamma(a-ix)\Gamma(a+ix) \quad , \quad x \in \mathbb{R} \quad , \quad \Re a > 0 . \tag{4.33}$$

When $a > 0$, the form $\hat{u}_0$ is positive definite and we have the so-called Meixner–Pollaczek polynomials [24,27].
B. Here $\Phi''(0) = 2$. When $\omega \neq 0$, the equation (4.15) becomes

$$\xi_{n+1} + \omega(n+1)\xi_n = -\left\{\psi(0) + \frac{\Phi(0)}{\omega}\right\}(u_0)^{\phi}_{n+1} \quad , \quad n \geq 0 \tag{4.34}$$

where

$$\xi_n := \left(\psi'(0) - n\right)(u_0)^{\phi}_{n+1} - \omega\left(n^2 + \frac{\Phi'(0)}{\omega}n + \frac{\Phi(0)}{\omega^2}\right)(u_0)^{\phi}_n \quad , \quad n \geq 0\,. \tag{4.35}$$

B₁. For (3.16) , we have $\widetilde{\Phi}(x) = \left(x - i(\alpha+\mu)\right)^2$, $\tilde{\psi}(x) = -2\alpha x + i(\alpha+\mu)^2$ and $\omega = -i$. Therefore

$$\tilde{\psi}(0) + \frac{\widetilde{\Phi}(0)}{\omega} = 0\,.$$

Since $\tilde{\xi}_0 = 0$ on account of (4.16) , from (4.34) we get $\tilde{\xi}_n = 0$, $n \geq 0$. Thus

$$(2\alpha+n)(\tilde{u}_0)^{\phi}_{n+1} - i(n+\alpha+\mu)^2(\tilde{u}_0)^{\phi}_n = 0 \quad , \quad n \geq 0\,,$$

whence

$$(\tilde{u}_0)^{\phi}_n = i^n\left(\frac{\Gamma(\alpha+\mu+n)}{\Gamma(\alpha+\mu)}\right)^2\frac{\Gamma(2\alpha)}{\Gamma(2\alpha+n)} \quad , \quad n \geq 0\,. \tag{4.36}$$

Here, the condition (4.18) is not fulfilled for all sufficiently large values of ν . Consequently, the form $\tilde{u}_0$ does not possess a representation through a discrete measure with *supp* $\subset \mathbb{R}$.
Regarding (3.15) , from (4.23) we have

$$\widehat{U}(x) = \frac{\Gamma\left(\frac{1}{2}\alpha+\mu-ix\right)\Gamma\left(\frac{1}{2}\alpha-\mu-ix\right)}{\Gamma^2\left(1-\frac{1}{2}\alpha-ix\right)}A(x) \quad , \quad x \in \mathbb{R}$$

with $A(x+i) = A(x)$. By virtue of (4.27) and with the choice

$$A(x) = \frac{K}{\sin^2\left(\pi\left(\frac{1}{2}\alpha+ix\right)\right)} \quad ,$$

we finally obtain
(4.37)

$$\widehat{U}(x) = \frac{K}{\pi^2}\Gamma^2\left(\frac{1}{2}\alpha+ix\right)\Gamma\left(\frac{1}{2}\alpha+\mu-ix\right)\Gamma\left(\frac{1}{2}\alpha-\mu-ix\right) \quad , \quad x \in \mathbb{R} \quad , \quad \Re\frac{1}{2}\alpha\,,\ \Re\left(\frac{1}{2}\alpha\pm\mu\right) \notin -\mathbb{N}\,.$$

The usual normalization condition gives

$$K = \frac{\pi}{2}\frac{\Gamma(2\alpha)}{\left(\Gamma(\alpha+\mu)\Gamma(\alpha-\mu)\right)^2} \quad , \quad \Re\alpha > 0 \quad , \quad \Re(\alpha\pm\mu) > 0\,,$$

taking account of the formula [2]

$$\int_{-\infty}^{+\infty}\Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix)dx = 2\pi\frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)} \tag{4.38}$$

$$\Re a\,,\ \Re b\,,\ \Re c\,,\ \Re d > 0\,.$$

Thus

$$(4.39)\quad \widehat{U}(x) = \frac{1}{2\pi}\frac{\Gamma(2\alpha)}{\left(\Gamma(\alpha+\mu)\Gamma(\alpha-\mu)\right)^2}\Gamma^2\left(\frac{1}{2}\alpha+ix\right)\Gamma\left(\frac{1}{2}\alpha+\mu-ix\right)\Gamma\left(\frac{1}{2}\alpha-\mu-ix\right) ,$$
$$x\in\mathbb{R} ,\ \Re\alpha>0 ,\ \Re\left(\frac{1}{2}\alpha\pm\mu\right)>0 .$$

When $\mu = 0$ and $\alpha > 0$, the form is symmetric and positive definite. Then

$$(4.40)\qquad \widehat{U}(x) = \frac{1}{2\pi}\frac{\Gamma(2\alpha)}{\Gamma^4(\alpha)}\left|\Gamma\left(\frac{1}{2}\alpha+ix\right)\right|^4 \quad , \quad x\in\mathbb{R} .$$

$\mathbf{B_2}$. Among the cases quoted, there is not for which the second side of (4.34) is zero. On the other hand, for the form $\tilde{u}_0 = \tau_{-B}\hat{u}_0$ where $\hat{u}_0$ is defined by (3.20) and $B = -(2\delta+\alpha-\beta+2)i/4$, we have $\tilde{\Phi}(x) = \big(x - i(\alpha+1)\big)\big(x - i(\delta+1)\big)$ and $\tilde{\psi}(x) = -(\alpha+\beta+2)x + i(\alpha+1)(\delta+1)$. Therefore $\tilde{\psi}(0) + i\tilde{\Phi}(0) = 0$ .From (4.34) , (4.35) and (4.16) , we easily infer

$$(4.41)\qquad (\tilde{u}_0)_n^{\phi} = i^n\frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)}\frac{\Gamma(\delta+1+n)}{\Gamma(\delta+1)}\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+\beta+2+n)} \quad , \quad n\geq 0 .$$

Thus the condition (4.18) is not satisfied; this means that the $D_\omega-$ classical forms analogous to Jacobi forms are not represented through a discrete measure.

Now, for the case given by (3.19) – (3.20) , taking account of (4.23) , we obtain

$$\widehat{U}(x) = \frac{\Gamma\big(\frac{1}{4}(\alpha+3\beta+2-2\delta)-ix\big)\Gamma\big(\frac{1}{4}(\alpha-\beta+2+2\delta)-ix\big)}{\Gamma\big(\frac{1}{4}(\alpha-\beta+2-2\delta)-ix\big)\Gamma\big(\frac{1}{4}(2\delta-3\alpha-\beta+2)-ix\big)}A(x) \quad , \quad x\in\mathbb{R}$$

with $A(x+i) = A(x)$. By virtue of (4.27) and with the choice

$$A(x) = \frac{K}{\sin\big(\frac{\pi}{4}(2+2\delta-\alpha+\beta)+i\pi x\big)\sin\big(\frac{\pi}{4}(2-2\delta+3\alpha+\beta)+i\pi x\big)} ,$$

we get, putting $a_1 = \frac{1}{4}\big(2(1+\delta)-(\alpha-\beta)\big)$, $a_2 = \frac{1}{4}\big(2(1+\delta)+\alpha-\beta\big)$, $b_1 = \frac{1}{4}\big(2(1-\delta)+3\alpha+\beta\big)$, $b_2 = \frac{1}{4}\big(2(1-\delta)+\alpha+3\beta\big)$:

$$(4.42)\qquad \widehat{U}(x) = \frac{K}{\pi^2}\Gamma(a_1+ix)\Gamma(a_2-ix)\Gamma(b_1+ix)\Gamma(b_2-ix) ,$$

where on account of (4.38)

$$K = \frac{\pi}{2}\frac{\Gamma(\alpha+\beta+2)}{\Gamma(1+\delta)\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+\alpha+\beta-\delta)}$$
$$\Re(1+\delta)>0 \ ,\ \Re(1+\alpha)>0 \ ,\ \Re(1+\beta)>0 \ ,\ \Re(1+\alpha+\beta-\delta)>0 .$$

The following symmetric and positive definite particular cases are interesting:

1. Corresponding to (3.21) when $\alpha=\beta$:

$\mathbf{1_a}$. $\alpha=\beta\in\mathbb{R}$ and $\delta\in\mathbb{R}$, $-1<\delta<2\alpha+1$

(4.43)
$$\widehat{U}(x) = \frac{1}{2\pi}\frac{\Gamma(2\alpha+2)}{\Gamma(1+\delta)\Gamma^2(1+\alpha)\Gamma(1+2\alpha-\delta)}\left|\Gamma\big(\frac{1}{2}(1+\delta)+ix\big)\right|^2\left|\Gamma\big(\frac{1}{2}(1-\delta+2\alpha)+ix\big)\right|^2 ,\ x\in\mathbb{R}.$$

When $\alpha = 0$, we have

$$\widehat{U}(x) = \frac{\sin(\pi\delta)}{\delta} \frac{1}{\cos(\pi\delta) + \cosh(2\pi x)} .$$

But the form associated with $\{G_n^{(\delta)}\}_{n\geq 0}$ is $\tilde{u}_0 = h_2\hat{u}_0$. Therefore

$$\widetilde{U}(x) = \frac{1}{2}\widehat{U}\left(\frac{x}{2}\right) = \frac{\sin(\pi\delta)}{2\delta} \frac{1}{\cos(\pi\delta) + \cosh(\pi x)} ,$$

in accordance with [8, p.193, (8.8)] . Also see [5,30].

1$_\mathbf{b}$. $\alpha = \beta \in \mathbb{R}$, $\delta + \overline{\delta} = 2\alpha$, $\alpha + 1 > 0$

$$\widehat{U}(x) = \frac{1}{2\pi} \frac{\Gamma(2\alpha+2)}{\Gamma^2(1+\alpha)|\Gamma(2\sigma)|^2} |\Gamma(\sigma+ix)|^2 |\Gamma(\overline{\sigma}+ix)|^2 \quad , \quad x \in \mathbb{R} \tag{4.44}$$

with $\sigma = \frac{1}{2}(1+\delta)$.

2. Corresponding to (3.22) when $\delta = \frac{1}{2}(\alpha+\beta)$, $\alpha + 1 > 0$, $\beta + 1 > 0$

(4.45)

$$\widehat{U}(x) = \frac{1}{2} \frac{\Gamma(\alpha+\beta+2)}{\Gamma(1+\alpha)\Gamma^2(1+\frac{1}{2}(\alpha+\beta))\Gamma(1+\beta)} \left|\Gamma(\frac{1}{2}(1+\alpha)+ix)\right|^2 \left|\Gamma(\frac{1}{2}(1+\beta)+ix)\right|^2 \quad , \quad x \in \mathbb{R}$$

which refers to (4.43) .

For the continuous Hahn form $\hat{u}_0$, we have $\hat{\hat{\Phi}}(x) = (x+i\overline{a})(x+i\overline{b})$, $\hat{\hat{\psi}}(x) = -2\left(\Re(a+b)x + \Im(ab)\right)$ and $\omega = i$. From (4.24) , we get

$$\widehat{\widehat{U}}(x) = \frac{\Gamma(\overline{a}-ix)\Gamma(\overline{b}-ix)}{\Gamma(1-a-ix)\Gamma(1-b-ix)} A(x) \quad ,$$

with $A(x+i) = A(x)$. Taking account of (4.27) and of $\sin\big(\pi(a+ix)\big)\sin\big(\pi(b+ix)\big)A(x) = K$, we obtain

$$\widehat{\widehat{U}}(x) = \frac{K}{\pi^2} |\Gamma(a+ix)\Gamma(b+ix)|^2 \quad , \quad \Re a \, , \, \Re b > 0 \quad , \quad x \in \mathbb{R} \tag{4.46}$$

where

$$K = \frac{\pi}{2} \frac{\Gamma(a+b+\overline{a}+\overline{b})}{\Gamma(a+\overline{a})\,|\Gamma(a+b)|^2\,\Gamma(b+\overline{b})} .$$

We meet again the well–known representation of the continuous Hahn form.

Remark. In any case (4.29) , (4.33) , (4.39) , (4.42) and (4.46) , the condition (4.21) is fulfilled by virtue of the standard asymptotic formula

$$|\Gamma(a+ix)| = \sqrt{2\pi}e^{-\pi|x|/2}|x|^{a-\frac{1}{2}}\big(1+r(a,x)\big)$$

where $r(a,x) \to 0$, as $|x| \to +\infty$, uniformly for bounded $|a|$.

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