Result.Math. 37 (2000) 56-83
0378-6218/00/020056-28 $ 1.50+0.20/0

Results in Mathematics

PERTURBATIONS OF THE HALF–LINEAR EULER DIFFERENTIAL EQUATION

Á. Elbert and A. Schneider

Oscillation/nonoscillation properties of the perturbed half-linear Euler differential equation in the critical case are investigated. Strong connections are found between these half-linear differential equations and some linear differential equations, whose coefficient is the perturbation itself. In addition if the solutions of the corresponding linear differential equation satisfy two integral inequalities, then the asymptotic form of the solutions of the half-linear differential equation is established. Examples are given for the latter case.

0. Preliminaries and new results.

In [E] — among others — the asymptotic behaviour of the half-linear second order differential equation

$$(0.1) \qquad (x'^{n*})' + \frac{n\gamma}{t^{n+1}} x^{n*} = 0, \qquad t > 0$$

is investigated where γ is a constant, n is a fixed real number, $n > 0$, and $u^{n*} = |u|^n \operatorname{sgn} u$. Clearly, (0.1) is a natural generalization of the second order Euler differential equation, known in the theory of the linear second order differential equations, to the half-linear differential equations. It turned out that the differential equation

$$(0.2) \qquad (x'^{n*})' + \frac{n\gamma_0}{t^{n+1}} x^{n*} = 0$$

with the value

$$\gamma_0 = \frac{n^n}{(n+1)^{n+1}}$$

plays a decisive role because for $\gamma \le \gamma_0$ the solutions of (0.1) are non-oscillatory while for $\gamma > \gamma_0$ all solutions are oscillatory which is in complete harmony with the situation known in the linear case.

1980 *Mathematics Subject Classification* (1985 *Revision*). 34C10.

Key words and phrases. half-linear differential equations, principal solutions, Riccati equations.

Research supported by the Department of Mathematics of the University Dortmund and partially by Hungarian Foundation for Scientific Research Grant T 026138 .

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Here we are interested in the asymptotic behaviour of the perturbed half-linear differential equation

$$(x'^{n*})' + \frac{\gamma_0}{t^{n+1}}\left[n + 2(n+1)\delta(t)\right]x^{n*} = 0 \tag{0.3}$$

where the function $\delta(t)$ is piecewise continuous on (t_0, ∞) for some $t_0 \geq 0$. It can be assumed that $\delta(t) \not\equiv 0$ on some interval $[t_1, \infty)$ for $t_1 \geq t_0$ because the properties of (0.2) are already known. Clearly, equation (0.3) is a perturbed version of (0.2).

In case $n = 1$ the differential equation (0.3) is the linear second order differential equation

$$x'' + \frac{1}{4t^2}\left(1 + 4\delta(t)\right)x = 0 \tag{0.4}$$

and making use of the substitutions

$$t = e^s, \quad x(t) = \sqrt{t}\, z(s),$$

we obtain the differential equation

$$z'' + \delta(e^s)z = 0. \tag{0.5}$$

Consequently, any property (e.g. the oscillation) of the solutions of (0.4) can be translated into a corresponding property of the solutions of (0.5). Here we recall a result of A. Wintner [W49] and another result of E. Hille [H48].

Theorem A. *The differential equation* (0.5) *(and consequently* (0.4)*, too) is oscillatory if*

$$\int^{\infty} \delta(e^s)\, ds = \int^{\infty} \delta(t)\frac{dt}{t} = \infty. \tag{0.6}$$

Theorem B. *Let* $\delta(t) \geq 0$ *and*

$$\int^{\infty} \delta(e^s)\, ds = \int^{\infty} \delta(t)\frac{dt}{t} < \infty.$$

Then (0.5) *(consequently* (0.4) *as well) is oscillatory if*

$$\liminf_{s\to\infty} s\int_s^{\infty} \delta(e^\sigma)\, d\sigma > \frac{1}{4}, \tag{0.7}$$

or nonoscillatory if

$$\limsup_{s\to\infty} s\int_s^{\infty} \delta(e^\sigma)\, d\sigma < \frac{1}{4}. \tag{0.8}$$

Returning to the half-linear differential equations, ($n \neq 1$), we know ([E]) that a solution of (0.2) is either equal to $ct^{n/(n+1)}$ with some constant $c \neq 0$ or it has the asymptotic $\mathcal{O}(t^{n/(n+1)}(\log t)^{2/(n+1)})$ for $t \to \infty$.

Concerning the oscillation of the solutions of the half-linear *system*

$$\begin{aligned} x_1' &= a(t)x_1 + b(t)x_2^{\frac{1}{n}*} \\ x_2' &= -c(t)x_1^{n*} + d(t)x_2 \end{aligned}$$

there is a relevant result, namely Theorem 7 in [E82], which can be applied to (0.3). First we rewrite (0.3) into a half-linear system. Let $x_1 = x(t)$, $x_2 = x'^{n*}(t)$, then

$$\begin{aligned} x_1' &= x_2^{\frac{1}{n}*}, \\ x_2' &= -\frac{\gamma_0}{t^{n+1}}\big(n + 2(n+1)\delta(t)\big)x_1{}^{n*}, \end{aligned}$$

therefore we have $b(t) = 1$, $c(t) = \frac{\gamma_0}{t^{n+1}}\big(n + 2(n+1)\delta(t)\big)$, $a(t) = d(t) = 0$, and $\lambda^*(t) = \exp(\int_{t_0}^t (na(s) - d(s))\,ds) = 1$. To apply Theorem 7 we need two auxiliary functions, i.e. a pair $(\lambda(t), \mu(t))$. In our case we can choose them as $\lambda(t) = t^n$, $\mu(t) = 1$ which fulfil all the requirements posed in [E82]. We have to calculate still the integrand in formula (36)† of Theorem 7:

$$\lambda(t)c(t) - \frac{1}{(n+1)^{n+1}} \frac{\left|\lambda'(t) - \lambda(t)\frac{(\lambda^*)'(t)}{\lambda^*(t)}\right|^{n+1}}{\big(b(t)\lambda(t)\big)^n} = 2(n+1)\gamma_0\frac{\delta(t)}{t},$$

hence Theorem 7 mentioned above asserts the following statement.

Theorem C. *If the relation* $\int^\infty \delta(t)\frac{dt}{t} = \infty$ *holds, then* (0.3) *is oscillatory.*

Now we can realize the coincidence of this integral condition with (0.6) in Theorem A. Exactly this coincidence stimulated our investigations to clear up a deeper connection of oscillatory behaviour between the half-linear differential equation (0.3) and the linear differential equation (0.5). We shall work here under two restrictions with different severity on $\delta(t)$:

(R1) the limit $\lim_{T\to\infty} \int_{t_0}^T \delta(t)\frac{dt}{t}$ exists (as a finite number);

(R2) as in (R1) and $\int_t^\infty \delta(s)\frac{ds}{s} \geq 0$ for $t \geq t' > t_0$.

Our main results are the following.

---

† A factor $\lambda(t)$ is missing in this formula in [E82] and once again in the proof on the same page, but from the foregoing treatment it is easy to detect them. Here the correct formula is displayed.

Theorem 1. *Let $n > 1$ and suppose that* (R2) *holds and the linear differential equation* (0.5) *is non-oscillatory. Then the half-linear differential equation* (0.3) *is also non-oscillatory.*

Theorem 2. *Let $0 < n < 1$ and suppose that* (R2) *holds and the half-linear differential equation* (0.3) *is non-oscillatory. Then the linear differential equation* (0.5) *is also non-oscillatory.*

We are not able to provide a general "if and only if" type connection for oscillation or non-oscillation between the differential equations (0.3) and (0.5) if $n \neq 1$. This might be impossible and it is an open problem still. But if we make some restriction on the function $\delta(t)$ or on the corresponding linear differential equation (0.5), then we can establish closer connections. An example is the following.

Theorem 3. *Suppose that* (R2) *holds and* (0.5) *is non-oscillatory. Assume that there exists a constant $\theta \in (0,1)$ such that for a solution $z(s)$ of* (0.5) *the functions $\zeta(s) = z'(s)/z(s)$ and $\mu(s)$, defined by*

$$\mu(s) = \int_s^\infty \zeta^3(\sigma)\, d\sigma,$$

satisfy the relations

$$\int_s^\infty \mu(\sigma)\, d\sigma < \infty, \qquad \int_s^\infty \zeta(\sigma)\mu(\sigma)\, d\sigma \le \frac{\theta}{2}\mu(s) \quad \textit{eventually.} \tag{0.9}$$

Then (0.3) *is non-oscillatory and it has a solution $x(t)$ with the asymptotics*

$$\begin{aligned} x(t) &= t^{\frac{n}{n+1}} z^{\frac{2}{n+1}}(\log t)\left[C + o(1)\right], \quad (C = \text{const.} \neq 0), \\ t\,\frac{x'(t)}{x(t)} &= \frac{n}{n+1} + \frac{2}{n+1}\zeta(\log t) + o\big(\zeta(\log t)\big) \quad \text{as } t \to \infty. \end{aligned} \tag{0.10}$$

Remark 1. By a Hartman theorem ([Hart52]), the function $\zeta(s)$ in Theorem 3 is an L^2-function in some neighbourhood of infinity while the condition (R2) ensures that $\zeta(s) > 0$ and $\lim_{s\to\infty} \zeta(s) = 0$, hence the integral in the definition of $\mu(s)$ and the second integral in (0.9) exist.

Remark 2. The summability of $\mu(s)$ in (0.9) can not be guaranteed in general. In the particular case when $\zeta(s)$ is monotonic (which happens e.g. if $\delta(t) \ge 0$), then one can show that $\mu(s) = \mathcal{O}\big(\zeta^2(s)\big)$, consequently the function $\mu(s)$ is summable and we have the stronger result in (0.10):

$$\begin{aligned} x(t) &= t^{\frac{n}{n+1}} z^{\frac{2}{n+1}}(\log t)\left[C + \mathcal{O}\big(\zeta(\log t)\big)\right], \quad (C = \text{const.} \neq 0), \\ t\,\frac{x'(t)}{x(t)} &= \frac{n}{n+1} + \frac{2}{n+1}\zeta(\log t) + \mathcal{O}\big(\zeta^2(\log t)\big) \quad \text{as } t \to \infty. \end{aligned}$$

In case the linear second order differential equation (0.5) is non-oscillatory, there exists a principal (or small) solution $\bar{z}(s)$ (see [HW] or [Hart], p. 355), unique up to a constant multiple, such that

$$\int^{\infty} \frac{ds}{\bar{z}^2(s)} = \infty, \tag{0.11}$$

while any other solution $z(s)$, linearly independent of $\bar{z}(s)$, has the property

$$\int^{\infty} \frac{ds}{z^2(s)} < \infty, \qquad \lim_{s\to\infty} \frac{\bar{z}(s)}{z(s)} = 0. \tag{0.12}$$

In [EK] the notion of the principal solutions has been extended to all half-linear differential equations — more general than (0.3) — essentially by an extremal property that if $\bar{x}(t)$ and $x(t)$ denote the principal and any other, linearly independent solutions of the half-linear differential equation (0.3), then

$$\frac{x'(t)}{x(t)} > \frac{\bar{x}'(t)}{\bar{x}(t)} \qquad \text{for sufficiently large } t. \tag{0.13}$$

Our main result is formulated in the next theorem.

Theorem 4. *Assume* (R2) *holds and* (0.5) *is nonoscillatory. Suppose there exists a constant* $\theta \in (0,1)$ *such that for the principal solution* $\bar{z}(s)$ *of* (0.5) *the functions* $\bar{\zeta}(s)$ *and* $\bar{\mu}(s)$, *defined by*

$$\bar{\zeta}(s) = \bar{z}'(s)/\bar{z}(s), \qquad \bar{\mu}(s) = \int_s^{\infty} \bar{\zeta}^3(\sigma)\, d\sigma,$$

satisfy the relations

$$\int_s^{\infty} \bar{\mu}(\sigma)\, d\sigma < \infty, \qquad \int_s^{\infty} \bar{\zeta}(\sigma)\bar{\mu}(\sigma)\, d\sigma \le \frac{\theta}{2}\bar{\mu}(s) \quad \textit{eventually.}$$

Then (0.3) *is non-oscillatory and the principal solution* $\bar{x}(t)$ *of* (0.3) *has the asymptotics*

$$\begin{aligned} \bar{x}(t) &= t^{\frac{n}{n+1}} \bar{z}^{\frac{2}{n+1}}(\log t)\,[C + o(1)], \quad (C = \text{const.} \neq 0), \\ t\,\frac{\bar{x}'(t)}{\bar{x}(t)} &= \frac{n}{n+1} + \frac{2}{n+1}\bar{\zeta}(\log t) + o\big(\bar{\zeta}(\log t)\big) \quad \text{as } t \to \infty. \end{aligned} \tag{0.14}$$

Let $x(t)$ *be any non-principal solution of* (0.3), *then*

$$\int^{\infty} \frac{dt}{x^2(t)\,|x'(t)|^{n-1}} < \infty, \qquad \lim_{t\to\infty} \frac{\bar{x}(t)}{x(t)} = 0, \tag{0.15}$$

while for the principal solution $\bar{x}(t)$

$$\int^{\infty} \frac{dt}{\bar{x}^2(t)\,|\bar{x}'(t)|^{n-1}} = \infty. \tag{0.16}$$

Moreover, if $x_1(t)$ and $x_2(t)$ are two non-principal solutions of (0.3), then there exists the limit

$$\lim_{t\to\infty} \frac{x_1(t)}{x_2(t)} = C_{1,2}, \qquad C_{1,2} \notin \{0, -\infty, +\infty\}.$$

In connection with the integrals in (0.15) and (0.16) we would like to mention some pioneering results of J. D. Mirzov [M]. In this paper he defined (probably first) the notion of the principal solution for half-linear differential equations and he tried to generalize the Hartman's integral criterion in (0.11)-(0.12) which separates the principal/nonprincipal solutions. Actually he proved the relations

$$\int^{\infty} \frac{dt}{\bar{x}^{m_*}(t)} = \infty, \qquad \int^{\infty} \frac{dt}{x^{m^*}(t)} < \infty$$

for the principal solution $\bar{x}(t)$ and a non-principal solution $x(t)$, resp., where

$$m_* = \min_{0\le x\le 1} \left\{ \frac{1 - x^{1+\frac{1}{n}}}{1-x} + (1-x)^{\frac{1}{n}} \right\}, \qquad m^* = \max_{0\le x\le 1} \left\{ \frac{1 - x^{1+\frac{1}{n}}}{1-x} + (1-x)^{\frac{1}{n}} \right\}.$$

Clearly, for the linear differential equations we have $n = 1$, and we get $m_* = m^* = 2$, and thus the integrals (0.11)-(0.12).

We can apply Theorem 4 to the following differential equations

$$\text{(0.17)} \qquad (x'^{n*})' + \frac{\gamma_0}{t^{n+1}} \left(n + \frac{n+1}{2} \sum_{i=1}^{k} \frac{1}{\log_1^2 t \, \log_2^2 t \cdots \log_i^2 t} \right) x^{n*} = 0 \quad (t > t_k),$$

where $k = 1, 2, \ldots$, $\log_1 t = \log t$, $\log_{i+1} t = \log(\log_i t)$, $t_0 = 0$, $t_{i+1} = \exp(t_i)$, $i = 0, 1, \ldots$.

Corollary 1. *Each nontrivial solution of (0.17) has the asymptotic form*

$$\textit{either} \quad t^{\frac{n}{n+1}} (\log_1 t \, \log_2 t \cdots \log_k t)^{\frac{1}{n+1}} (C + O(\frac{1}{\log t})) \quad \textit{or}$$

$$t^{\frac{n}{n+1}} (\log_1 t \, \log_2 t \cdots \log_k t)^{\frac{1}{n+1}} (\log_{k+1} t)^{\frac{2}{n+1}} (C + O(\frac{1}{\log t})) \quad \textit{as } t \to \infty$$

with some $C \in \setminus \{0\}$.

A generalization of Theorem B to half-linear differential equations is the following.

Theorem 5. *Let (R2) be valid. Then (0.3) is oscillatory if (0.7) is satisfied, or it is non-oscillatory if (0.8) is satisfied.*

An immediate application of this theorem is the following.

Corollary 2. *The solutions of the differential equation*

$$(0.18) \qquad (x'^{n*})' + \frac{\gamma_0}{t^{n+1}}\left(n + \frac{\mu}{\log^\alpha t}\right) x^{n*} = 0 \qquad (t > 1)$$

are oscillatory if and only if either $\mu > 0$ *for* $\alpha < 2$ *or* $\mu > (n+1)/2$ *for* $\alpha = 2$.

The proofs of the above theorems and corollaries are based on the Riccati differential equation technique, generalized to half-linear differential equations. We define this half-linear version of the Riccati equation in Section 1. In Section 2 we introduce a new half-linear differential equation whose solutions are amazingly close to the linear ones. Applying already known results we obtain new criterions for oscillation/nonoscillation of (0.3). Finally, Section 3 is fully devoted to the proofs of theorems and corollaries announced above.

1. Generalized Riccati equations.

The main tool in proving the asymptotic behaviour of the solutions of (0.3) consists in using a Riccati technique developed in [E84].

For the sake of consistency of the notations in the introduction and in the later applications, we use in this section the independent variable s and the functions $z(s)$, $q(s)$, $\dots$. We consider the *general* half-linear differential equation of the form

$$(1.1) \qquad z'' + q(s) f(z, z') = 0 \quad \text{for } s \in I \subset$$

where the function $q(s)$ is piecewise continuous, the function $f(z,u)$ is defined on $\Omega = \times_0$, ${}_0 = \setminus \{0\}$, with the basic properties

i $f(z,u)$ is continuous on Ω;
ii $f(0,u) = 0$ for $u \neq 0$, $zf(z,u) > 0$ if $zu \neq 0$;
iii $f(z,u)$ is a homogeneous function of degree one such that $f(\lambda z, \lambda u) = \lambda f(z,u)$ for $\lambda \in {}_0$, $(z,u) \in \Omega$;
iv $f(z,u)$ is sufficiently smooth in order to ensure the continuous dependence and the uniqueness of the solutions subject to any initial condition posed at any point $s_0 \in I$;
v let $F(T)$ be defined by $F(T) = Tf(T,1)$, then $\int_{-\infty}^{\infty} \frac{dT}{1+F(T)} < \infty$.

Then there exist two continuous functions $H(u)$ and $g(u)$ on such that

a) $H(0) = 0$, $H(u) > 0$ at $u \neq 0$,

$$\int_{-\infty}^{u_1} \frac{dv}{H(v)} < \infty, \qquad \int_{u_2}^{\infty} \frac{dv}{H(v)} < \infty \qquad (u_1 < 0 < u_2);$$

b) $g(0) = 0$, $\lim_{u \to \pm\infty} g(u) = \pm\infty$, $g(u)$ is differentiable on ${}_0$, more precisely

$$f(1,u)\, g'(u) = 1 \qquad (u \in {}_0).$$

The connection between H and g is given by

$$(1.2) \qquad \int_{-\infty}^{g(\sigma)} \frac{dv}{H(v)} = \frac{-1}{\sigma} \quad (\sigma < 0) \quad \text{and} \quad \int_{g(\sigma)}^{\infty} \frac{dv}{H(v)} = \frac{1}{\sigma} \quad (\sigma > 0).$$

The inverse function of $g(u)$ is denoted by $g^{-1}(u)$.

Clearly, in the linear case we have $f(z,u) = z$, $g(u) = u$, $H(u) = u^2$, or in the half-linear case (0.3) $f(z,u) = z^{n*}|u|^{1-n}$, $g(u) = \frac{1}{n}u^{n*}$, $H(u) = |nu|^{\frac{n+1}{n}}$.

Now we introduce the *generalized* Riccati differential equation associated with (1.1) by

$$u' + H(u) + q(s) = 0 \qquad \text{on } I. \tag{1.4}$$

Let $z(s)$ be a solution of (1.1) with $z(s) \neq 0$ on some interval $J \subset I$ and define the function $u(s)$ in J by

$$u(s) = g(\frac{z'(s)}{z(s)}). \tag{1.3}$$

Then $u(s)$ is a solution of (1.3) in J.

As it is customary, an interval $J \subset I$ is called an interval of disconjugacy if there exists a solution $z(s)$ of (1.1) such that $z(s) \neq 0$ on J. In this case (1.1) is disconjugate on J.

There is a close connection between the disconjugacy of (1.1) and the existence of a solution of (1.4). If $u(s)$ is a solution of (1.4) on J, then by (1.3)

$$z(s) = \exp\left(\int_{s_0}^{s} g^{-1}(u(\sigma))d\sigma\right), \quad s, s_0 \in J$$

is a solution of (1.1) with the property $z(s) > 0$ on J.

More important is however the observation of A. Wintner [W51] which was generalized (see Thm 2.2 in [E84]) to differential equations of the form (1.1) in the following way.

Lemma 1. *Suppose that $u(s)$ is a continuous, piecewise continuously differentiable function on J such that the inequality*

$$u'(s) + H(u(s)) + q(s) \leq 0 \quad \text{for } s \in J \tag{1.5}$$

holds. Then (1.1) is disconjugate on J.

Now we come to a relation between the solutions of the Riccati differential equation (1.4) and an integral equation — called Riccati *integral* equation — which was not stated explicitly in [E84] but can be derived from the results of [E84] as shown in [E87]. This relation reads as follows.

Lemma 2. *Let $H(u)$ be convex and assume that the limit*

$$\lim_{s\to\infty} \int_{s_0}^{s} q(\sigma)\, d\sigma \quad \text{exists (as a finite number).} \tag{R1'}$$

Then $u(s)$ is a solution of (1.4) on an unbounded interval $J = (s_0, \infty)$ if and only if $u(s)$ is a solution of the Riccati integral equation

$$u(s) = \int_{s}^{\infty} q(\sigma)\, d\sigma + \int_{s}^{\infty} H(u(\sigma))\, d\sigma \quad \text{in } J. \tag{1.6}$$

Remark 3. The transition from the Riccati differential equation (1.4) to the Riccati integral equation (1.6) is an immediate consequence of the following Theorem D, generalizing a Hartman theorem [Hart52] and is given by the formulas (2.1)-(2.2) in [E84], p. 234:

Theorem D. *Let $H(u)$ be convex and (1.1) non-oscillatory. Further let*

$$\liminf_{S\to\infty} \frac{\int_{s_0}^{S}\int_{s_0}^{s} q(\sigma)\,d\sigma\,ds}{S} > -\infty.$$

Then there exist the limits

$$\lim_{S\to\infty} \frac{\int_{s_0}^{S}\int_{s_0}^{s} q(\sigma)\,d\sigma\,ds}{S} = C \quad \text{as a finite number and} \tag{1.7}$$

$$\lim_{S\to\infty} \frac{\int_{s_0}^{S} H\left(\frac{C-\int_{s_0}^{s} q(\sigma)\,d\sigma}{2}\right) ds}{S} = 0. \tag{1.8}$$

Obviously, the constant C is a function of s_0.

Now it is clear by Theorem D that the differential equation (1.1) is oscillatory if

$$\lim_{S\to\infty} \int_{s}^{S} q(\sigma)\,d\sigma = \infty$$

which generalizes a result of A. Wintner [W49] to the half-linear differential equation (1.1) (see Corollary 2.1 in [E84]).

If assumption (R1') holds then the integral $\int^{\infty} q(s)\,ds$ converges and (1.7) yields

$$C = \int_{s_0}^{\infty} q(s)\,ds.$$

Integrating (1.4) over $[s_0, s]$, we get equation (2.3) of [E84], p. 234

$$C - \int_{s_0}^{s} q(\sigma)\,d\sigma = u(s) - \int_{s}^{\infty} H(u(\sigma))\,d\sigma,$$

from which the integral equation (1.6) follows.

Making use of this representation of the constant C, also (1.8) can be rewritten in the simpler form

$$\lim_{S\to\infty} \frac{\int_{s_0}^{S} H\left(\frac{1}{2}\int_{s}^{\infty} q(\sigma)\,d\sigma\right) ds}{S} = 0. \tag{1.9}$$

The essence of Lemma 1 on the Riccati differential inequality can be reformulated for the Riccati *integral inequality* as follows.

Lemma 3. *Suppose that $H(u)$ is convex, the function $q(s)$ satisfies (R1') and $\int_{s}^{\infty} q(\sigma)\,d\sigma \geq 0$ for $s \geq s' \geq s_0$ and let $v(s)$ be an absolutely continuous function satisfying the inequality*

$$v(s) \geq \int_{s}^{\infty} q(\sigma)\,d\sigma + \int_{s}^{\infty} H(v(\sigma))\,d\sigma \quad \text{for } s \geq s' \geq s_0.$$

Then the differential equation (1.1) *is non-oscillatory.*

Proof of Lemma 3. Let $u(s)$ be defined by the equality

$$u(s) = \int_s^\infty q(\sigma)\,d\sigma + \int_s^\infty H(v(\sigma))\,d\sigma \quad \text{for } s \geq s' \geq s_0,$$

then $v(s) \geq u(s) \geq 0$ and $H(v(s)) \geq H(u(s))$. Hence

$$u'(s) + q(s) + H(u(s)) \leq 0 \quad \text{for } s \geq s' \geq s_0$$

and the assertion follows from Lemma 1.

Finally we remark that under the restrictions of Lemma 3 we also have either $q(s) \equiv 0$ eventually or

$$0 < \int_s^\infty q(\sigma)\,d\sigma < v(s)$$

hence

$$\int_{s'}^\infty H(\int_s^\infty q(\sigma)\,d\sigma)\,ds < \int_{s'}^\infty H(v(s))\,ds < \infty \tag{1.10}$$

which is a much stronger restriction on the coefficient $q(s)$ than (1.9). We state an even more stringent restriction.

Lemma 4. *Suppose that $H(u)$ is convex, the function $q(s)$ satisfies* (R1') *and* $\int_s^\infty q(\sigma)d\sigma \geq 0$ *for* $s \geq s' \geq s_0$ *and* (1.1) *is non-oscillatory. Then the differential equation*

$$y'' + H(\int_s^\infty q(\sigma)\,d\sigma)\,f(y,y') = 0 \tag{1.11}$$

is also non-oscillatory.

Proof of Lemma 4. The statement is clearly true if $q(s) \equiv 0$. Suppose that this is not the case. Then by Lemma 2 there exists a function $u(s)$ on the unbounded interval J such that $u(s)$ is a solution of the Riccati integral equation (1.6). It is a simple fact that the function $\xi(s)$ defined by

$$\xi(s) = \int_s^\infty H(u(\sigma))\,d\sigma > 0$$

is a solution of the Riccati differential inequality of the differential equation (1.11)

$$\xi' + H(\xi) + H(\int_s^\infty q(\sigma)\,d\sigma) \leq 0$$

because by (1.6)

$$H(\xi) + H(\int_s^\infty q(\sigma)\,d\sigma) \leq H(\xi + \int_s^\infty q(\sigma)\,d\sigma) = H(u(s))$$

and $H(0) = 0$, $H(u)$ is a convex function, $\xi > 0$, and $\int_s^\infty q(\sigma)\,d\sigma \geq 0$. Hence the conditions of Lemma 1 are satisfied thus differential equation (1.11) is non-oscillatory. This completes the proof of Lemma 4.

It is clear that the mere convergence of the integral of the function $H(\int_s^\infty q(\sigma)\,d\sigma)$ in (1.10) is less stringent than the fact that differential equation (1.11) is non-oscillatory.

2. A new half-linear differential equation.

In Section 0 we transformed the linear equation (0.4) into the simpler linear equation (0.5) whose coefficient $\delta(e^s)$ is given by the perturbation $\delta(t)$ of (0.4). The transformation applied for this purpose does not change the oscillation properties of the solutions of these equations. In this section we establish a similar transformation which transforms the half-linear equation (0.3) into a new half-linear differential equation (2.9) whose coefficient is a multiple of $\delta(e^s)$. For the solutions of this new differential equation the notion of the Wronskian can be successfully extended and applied, which confirms a closer connection between the new half-linear differential equation and the linear equations.

Consider the half-linear differential equation (0.3). Let the independent variable s and the function $y(s)$ be introduced in the differential equation (0.3) by

$$s = \log t, \quad y(s) = x(t). \tag{2.1}$$

Then

$$y'(s) = \frac{d}{ds}y(s) = tx'(t), \quad y'^{n*}(s) = t^n x'^{n*}(t)$$

and (0.3) is transformed into

$$(y'^{n*}(s))' - ny'^{n*}(s) + \left(n\gamma_0 + \Gamma\delta(e^s)\right)y^{n*}(s) = 0, \quad \Gamma = 2\left(\tfrac{n}{n+1}\right)^n. \tag{2.2}$$

Let $y(s)$ be a solution of (2.2) such that $y(s) \neq 0$ on J and let the function $\varrho(s)$ be defined by

$$\varrho(s) = \frac{y'^{n*}(s)}{y^{n*}(s)} \quad \text{for } s \in J,$$

then $\varrho(s)$ satisfies the following differential equation

$$\varrho' + n\Phi(\varrho) + \Gamma\delta(e^s) = 0, \tag{2.3}$$

where

$$\Phi(\varrho) = |\varrho|^{\frac{n+1}{n}} - \varrho + \gamma_0. \tag{2.4}$$

Observe that $\Phi'(\varrho) = \frac{n+1}{n}\varrho^{\frac{1}{n}*} - 1$, i.e. $\Phi(\varrho)$ takes on its absolute minimum at $\varrho_0 = (n/(n+1))^n$ and $\Phi(\varrho_0) = 0$ due to the special choice of γ_0. Moreover we find

$$\begin{aligned} \Phi''(\varrho) &= \frac{n+1}{n^2}|\varrho|^{\frac{1-n}{n}} \geq 0, \quad \Phi''(\varrho_0) = \frac{(n+1)^n}{n^{n+1}} = \frac{2}{n\Gamma}, \\ \Phi'''(\varrho) &= \frac{1-n^2}{n^3}\varrho^{(\frac{1}{n}-2)*}. \end{aligned} \tag{2.5}$$

Hence we conclude

$$\Phi(\varrho) - \frac{1}{n\Gamma}(\varrho - \varrho_0)^2 \begin{cases} 0 & \text{if } 0 < n < 1, \\ 0 & \text{if } n > 1 \end{cases} \quad \text{provided } \varrho \;\; \varrho_0 \text{ and } \varrho \geq 0. \tag{2.6}$$

Let $H_n(u)$ be defined by

$$H_n(u) = n\Phi(\varrho_0 + u), \tag{2.7}$$

then the function $u(s) = \varrho(s) - \varrho_0$ is a solution of the differential equation

$$u' + H_n(u) + \Gamma\delta(e^s) = 0. \tag{2.8}$$

By (2.5) the function $H_n(u)$ is convex for $u \in$, $H_n(u) > 0$ if $u \neq 0$ and satifies the relations

$$\int_{-\infty}^{u_1} \frac{du}{H_n(u)} < \infty, \quad \int_{u_2}^{\infty} \frac{du}{H_n(u)} < \infty \quad \text{for any } u_1 < 0 < u_2.$$

In Section 1 we have seen that these properties of $H_n(u)$ define uniquely a function $g_n(u)$ by (1.2), then a function $f_n(z,u)$ in such a way that the differential equation (2.8) is the Riccati differential equation of the *general* half-linear differential equation

$$z'' + \Gamma\delta(e^s) f_n(z, z') = 0. \tag{2.9}$$

This is the new half-linear differential equation announced in the beginning of this section.

Now it is clear that Theorem C is a consequence of Theorem D.

Owing to Lemma 2, every solution $u(s)$ of (2.8) for $s \geq s_0$ is also a solution of the Riccati *integral* equation

$$u(s) = \int_s^{\infty} \Gamma\delta(e^{\sigma})\, d\sigma + \int_s^{\infty} H_n(u(\sigma))\, d\sigma. \tag{2.10}$$

if (R1) holds. An immediate consequence of (2.10) is the relation $\lim_{s\to\infty} u(s) = 0$. On the other hand, if the stronger restriction (R2) holds, then $u(s) > 0$.

For we work often under the condition that the function $u(s)$ in (2.8) or (2.10) tends to zero, we need the values of $H_n(u)$ when u is small. By (2.4), (2.5) and (2.7) we have

$$\begin{aligned} H_n(u) &= n^2\gamma_0 \sum_{i=2}^{\infty} \binom{\frac{n+1}{n}}{i} \frac{u^i}{\varrho_0^i} = \frac{1}{\Gamma}u^2 + \frac{1}{2}\int_0^u (u-v)^2 H_n'''(v)\, dv = \\ &= \frac{1}{\Gamma}u^2 + (1-n)\big[Bu^3 + \ldots\big], \quad (|u| < \varrho_0), \end{aligned} \tag{2.11}$$

where

$$B = \frac{(n+1)^{2n}}{6n^{2n+1}}, \qquad H_n'''(0) = 6(1-n)B.$$

Also the following inequality will be useful

$$|H_n(u) - H_n(v) - \frac{1}{\Gamma}(u^2 - v^2)| < \frac{1}{6}b(\varepsilon)|u-v|(u^2+uv+v^2) \quad \text{for } 0 \leq u, v \leq \varepsilon,\ u \neq v \tag{2.12}$$

where $b = b(\varepsilon) = \max\{|H_n'''(s)| : 0 \leq s \leq \varepsilon\}$. Clearly, $b(0) = |H_n'''(0)| = 6|n-1|B$.

Now we need some knowledge on the behaviour of $g_n(v)$ in the neighbourhood of $v = 0$ for $v > 0$. By (1.2) we have for $v_0 > 0$, $u_0 = g_n(v_0) > 0$

$$\int_{u_0}^{\infty} \frac{du}{H_n(u)} = \frac{1}{v_0} \quad \text{or} \quad H_n(u_0) = v_0^2 g_n'(v_0). \tag{2.13}$$

Now we calculate the integral on the left hand side for $u_0 > 0$ when u_0 is small. By (2.11)

$$H_n(u) = \frac{1}{\Gamma}\left[u^2 + C_n u^3 + \cdots\right]$$

where $C_n = \frac{1}{3}\frac{1-n}{n}(\frac{n+1}{n})^n$, and consider the difference

$$h(u) = \frac{1}{H_n(u)} - \frac{\Gamma}{u^2 + C_n u^3 + \frac{1}{2}C_n^2 u^4}.$$

Clearly, the function $h(u)$ is continuous in $[0, \infty)$ and

$$-\infty < \int_0^{\infty} h(u)\, du < \infty,$$

hence

$$\int_{u_0}^{\infty} \frac{du}{H_n(u)} = \int_{u_0}^{\infty} \frac{\Gamma}{u^2 + C_n u^3 + \frac{1}{2}C_n^2 u^4} du + \mathcal{O}(1).$$

The integral on the right hand side is

$$\Gamma\left[-\frac{1}{u} + C_n \log \frac{\sqrt{1 + C_n u + \frac{1}{2}C_n^2 u^2}}{u}\right]_{u_0}^{\infty} = \frac{\Gamma(1 + C_n u_0 \log u_0 + \mathcal{O}(u_0))}{u_0}$$

hence by (2.13)

$$v_0 = g_n^{-1}(u_0) = \frac{u_0}{\Gamma(1 + C_n u_0 \log u_0 + \mathcal{O}(u_0))} = \frac{1}{\Gamma}\left(u_0 - C_n u_0^2 \log u_0 + \mathcal{O}(u_0^2)\right). \tag{2.14}$$

As consequence we obtain the relations

$$\begin{aligned} g_n(v) &= \Gamma v + C_n \Gamma^2 v^2 \log v + \mathcal{O}(v^2), \\ g_n'(0^+) &= \lim_{v \to +0} g_n'(v) = \Gamma. \end{aligned} \tag{2.15}$$

The limit follows from (2.11), (2.13) and (2.14):

$$\lim_{v \to 0} \frac{H_n(u)}{v^2} = \lim_{u \to 0} \frac{H_n(u)}{u^2}\left(\lim_{u \to 0} \frac{u}{g_n^{-1}(u)}\right)^2 = \frac{1}{\Gamma}\Gamma^2 = \Gamma.$$

The following statement shows how close is the differential equation (2.9) to the corresponding linear differential equation (0.5).

Lemma 5. *Let* (2.9) *be nonoscillatory and let* $z_1(s)$ *and* $z_2(s)$ *be two linearly independent solutions (i.e.* $z_1(s)/z_2(s) \neq \text{const}$*) such that* $z_i(s) > 0$ *on* $[s_0, \infty)$ $(i = 1, 2)$ *and let*

$$\zeta_1(s) = z_1'(s)/z_1(s) > \zeta_2(s) = z_2'(s)/z_2(s).$$

Suppose that restriction (R2) *holds. Then there exists the limit*

$$\lim_{s\to\infty} [z_1'(s)z_2(s) - z_1(s)z_2'(s)] = w_0 > 0$$

where w_0 *is a finite number, and the relation*

$$\int^{\infty} \frac{ds}{z_1^2(s)} < \infty$$

is true. In addition if also the relation

$$\int^{\infty} \frac{ds}{z_2^2(s)} < \infty$$

holds, then there exists the limit

$$\lim_{s\to\infty} \frac{z_1(s)}{z_2(s)} = C_{1,2}$$

such that $C_{1,2} \neq 0, \pm\infty$.

In the proof of Lemma 5 the following simple lemma will be of great use.

Lemma 6. *Let* $\varphi(s)$, $\psi(s)$ *be continuous functions on* $[s_0, \infty)$ *such that* $\varphi(s) > 0$, $\psi(s) \geq 0$, $\lim_{s\to\infty} \varphi(s) = 0$ *and*

$$0 < \int_s^{\infty} \varphi(\sigma)\psi(\sigma)\,d\sigma \leq \varphi(s).$$

Let the function $\Psi(u)$ *be positive nonincreasing on* $(0, u_0)$ *such that*

$$\int_0^{u_0} \Psi(u)\,du < \infty.$$

Then

$$\int^{\infty} \varphi(\sigma)\Psi(\varphi(\sigma))\psi(\sigma)\,d\sigma < \infty.$$

Proof of Lemma 6. Without loss of generality, we may assume that $\varphi(s) < u_0$ for $s \geq s_0$. Then by the monotonicity property of $\Psi(u)$ we have

$$\Psi(\varphi(s)) \leq \Psi\Big(\int_s^{\infty} \varphi(\sigma)\psi(\sigma)\,d\sigma\Big), \qquad s \geq s_0,$$

hence

$$\int_{s_0}^{\infty}\varphi(\sigma)\Psi\big(\varphi(\sigma)\big)\psi(\sigma)\,d\sigma \le \int_{s_0}^{\infty}\Psi\Big(\int_{\sigma}^{\infty}\varphi(\tau)\psi(\tau)\,d\tau\Big)\varphi(\sigma)\psi(\sigma)\,d\sigma$$
$$= \int_0^{\int_{s_0}^{\infty}\varphi(\sigma)\psi(\sigma)\,d\sigma}\Psi(u)\,du \le \int_0^{\varphi(s_0)}\Psi(u)\,du \le \int_0^{u_0}\Psi(u)\,du < \infty,$$

which completes the proof of Lemma 6.

Proof of Lemma 5. Taking into consideration the connection between (1.3) and (1.4), we have that the functions $u_i(s) = g_n(\zeta_i(s))$ $(i = 1,2)$ are solutions of (2.8) or (2.10). By the uniqueness of the initial value problem for differential equation (2.8) it follows that we have either $u_1(s) \equiv u_2(s)$ or $u_1(s) \not\equiv u_2(s)$ on $[s_0,\infty)$. We have the latter case now, more precisely $u_1(s) > u_2(s)$. Relation (2.10) implies that

$$u_i(s) > 0, \qquad \lim_{s\to\infty} u_i(s) = 0 \qquad \text{for } i \in \{1,2\},\ s \in [s_0,\infty).$$

Hence by (R2), (2.10) and (2.11), for any $0 < \varepsilon < \varrho_0 < 1$ there exists a value $\bar{s} = \bar{s}(\varepsilon) > s_0$ such that

$$\text{(2.16)} \qquad \frac{1}{2\Gamma}\int_s^{\infty} u_i^2(s)\,ds < u_i(s) < \varepsilon \qquad \text{for } i \in \{1,2\}, \quad s \in [\bar{s},\infty),$$

moreover by (2.14)

$$\text{(2.17)} \qquad \frac{z_i'(s)}{z_i(s)} = g_n^{-1}(u_i(s)) = \frac{1}{\Gamma}\left(u_i(s) - C_n u_i^2(s)\log u_i(s) + \mathcal{O}(u_i^2(s))\right) \qquad \text{for } s \to \infty.$$

Consider the function

$$W(s) = [u_1(s) - u_2(s)]\, z_1(s) z_2(s) > 0.$$

Then by (2.8) and (2.17)

$$\begin{aligned} W' &= [u_1' - u_2']\, z_1 z_2 + [u_1 - u_2]\,[z_1' z_2 + z_1 z_2'] \\ &= -\,[H_n(u_1) - H_n(u_2)]\, z_1 z_2 + (u_1 - u_2)\left[g_n^{-1}(u_1) + g_n^{-1}(u_2)\right] z_1 z_2 \\ &= -\left[H_n(u_1) - H_n(u_2) - \frac{1}{\Gamma}(u_1^2 - u_2^2)\right] z_1 z_2 \\ &\quad + \frac{1}{\Gamma}\left(-C_n u_1^2\log u_1 - C_n u_2^2 \log u_2 + \mathcal{O}(u_1^2) + \mathcal{O}(u_2^2)\right) W \end{aligned}$$

or by (2.12)

$$\left|\frac{W'}{W}\right| < \frac{b(\varepsilon)}{6}(u_1^2 + u_1u_2 + u_2^2) + \frac{|C_n|}{\Gamma}\left(u_1^2|\log u_1| + u_2^2|\log u_2| + \mathcal{O}(u_1^2) + \mathcal{O}(u_2^2)\right).$$

We show that the right hand side is an integrable function on $[\bar{s}, \infty)$. Owing to (2.16) this is evident for the functions u_1^2, u_2^2 and consequently for $u_1 u_2$, too. By (2.16) and Lemma 6, we have by choosing $\Psi(u) = |\log u|$ for $0 < u < 1$, $\varphi(s) = u_i(s)$, $\psi(s) = u_i(s)/2\Gamma$ that

$$\int^\infty u_i^2(s)\,|\log u_i(s)|\,ds < \infty \qquad i \in \{1,2\}.$$

Consequently, the function $\log W(s)$ is of bounded variation, and therefore the limit

$$\lim_{s\to\infty} W(s) = W_0 > 0 \tag{2.18}$$

exists. Let the function $w(s)$ denote the Wronskian of the solutions of $z_1(s)$ and $z_2(s)$:

$$w(s) = z_1'(s)z_2(s) - z_1(s)z_2'(s) = [\zeta_1(s) - \zeta_2(s)]z_1(s)z_2(s) > 0.$$

Then making use of the definition of $W(s)$ and (2.17), we have by the Lagrange mean value theorem

$$\frac{W(s)}{w(s)} = \frac{g_n(\zeta_1(s)) - g_n(\zeta_2(s))}{\zeta_1(s) - \zeta_2(s)} = g_n'(\zeta^*), \qquad \zeta_2(s) < \zeta^* < \zeta_1(s),$$

hence by (2.15) and (2.18)

$$\Gamma = \lim_{s\to\infty} \frac{W(s)}{w(s)} = \frac{W_0}{\lim_{s\to\infty} w(s)},$$

i.e. the function $w(s)$ has the limit $w_0 = W_0/\Gamma$ as $s \to \infty$, which proves the first statement of Lemma 5.

Now there are constants $0 < w_1 < w_0 < w_2 < \infty$ such that the relation

$$0 < w_1 < w(s) < w_2 \qquad \text{for } s \geq s_0 \tag{2.19}$$

holds. Making use of the lower bound for $w(s)$, we find

$$\left(\frac{z_2(s)}{z_1(s)}\right)' = -\frac{w(s)}{z_1^2(s)} < -\frac{w_1}{z_1^2(s)} < 0$$

and integrating this inequality over $[\tilde{s}, s]$ for $s > \tilde{s}$ we obtain

$$w_1 \int_{\tilde{s}}^{s} \frac{d\sigma}{z_1^2(\sigma)} < \frac{z_2(s)}{z_1(s)} + w_1 \int_{\tilde{s}}^{s} \frac{d\sigma}{z_1^2(\sigma)} < \frac{z_2(\tilde{s})}{z_1(\tilde{s})}, \qquad s > \tilde{s},$$

which proves that the integral on the left hand side is convergent when $s \to \infty$, as it was stated.

Finally, if we know that also the integral

$$\int^\infty \frac{ds}{z_2^2(s)}$$

is convergent, then by Cauchy - Schwarz inequality we find

$$\int^{\infty} \frac{ds}{z_1(s)z_2(s)} < \infty.$$

Then by (2.19)

$$\frac{w_1}{z_1(s)z_2(s)} < \frac{w(s)}{z_1(s)z_2(s)} = \frac{d}{ds} \log \frac{z_1(s)}{z_2(s)} < \frac{w_2}{z_1(s)z_2(s)},$$

i.e. the function $\log \frac{z_1(s)}{z_2(s)}$ is also of bounded variation, which guaranties the existence of the limit of the quotient $\frac{z_1(s)}{z_2(s)}$, completing the proof of Lemma 5.

Now we are ready to prove our theorems, only we have to recall that the Riccati equation of the linear differential equation $(p(s)x')' + q(s)x = 0$ has the form

$$u' + \frac{1}{p(s)}u^2 + q(s) = 0,$$

where $u(s) = p(s)x'(s)/x(s)$, and the corresponding Riccati's integral equation is

$$u(s) = \int_s^{\infty} q(\sigma)\, d\sigma + \int_s^{\infty} \frac{1}{p(\sigma)} u^2(\sigma)\, d\sigma \tag{2.20}$$

provided $\int^{\infty} q(s)\, ds$ exists.

3. The proofs.

Proof of Theorem 1. Since (0.5) is non-oscillatory, the same is true for the differential equation

$$(\Gamma z')' + \Gamma\delta(e^s)z = 0, \tag{3.1}$$

too, and there is a solution $z(s)$ which has no zero on $[s_0, \infty)$ provided s_0 is sufficiently large and by (2.20) the function

$$u(s) = \frac{\Gamma z'(s)}{z(s)}$$

is a solution of the Riccati integral equation

$$u(s) = \int_s^{\infty} \Gamma\delta(e^{\sigma})\, d\sigma + \int_s^{\infty} \frac{1}{\Gamma} u^2(\sigma)\, d\sigma \tag{3.2}$$

hence $u(s) > 0$. By (2.6), (2.7), we have the inequality

$$H_n(u) = n\Phi(\varrho_0 + u) < \frac{1}{\Gamma}u^2 \quad \text{for } u > 0.$$

Applying this inequality to $u = u(s)$ in (3.2), we obtain

$$u(s) > \int_s^\infty \Gamma\delta(e^\sigma)\,d\sigma + \int_s^\infty H_n(u(\sigma))\,d\sigma,$$

consequently by Lemma 3 the half-linear differential equation (2.9), or equivalently (0.3) is non-oscillatory.

Proof of Theorem 2. Since (0.3) is non-oscillatory, there exists a function $u(s) > 0$ subject to (2.10). Applying again inequality (2.6) to $H_n(u) = n\Phi(\varrho_0 + u)$, we get

$$H_n(u) > \frac{1}{\Gamma}u^2 \quad \text{for } u > 0.$$

Hence we obtain in (2.10)

$$u(s) > \int_s^\infty \Gamma\delta(e^\sigma)\,d\sigma + \frac{1}{\Gamma}\int_s^\infty u^2(\sigma)\,d\sigma,$$

which is the Riccati integral inequality applied to (3.1), hence (3.1) — and equivalently (0.5) — is non-oscillatory.

Proof of Theorem 3. As in the proof of Theorem 1, the function $u_0(s) = \Gamma\zeta(s)$ is a solution of (3.2) for $s \geq s_0$. Hence $u_0(s)$ is a positive function tending to zero as $s \to \infty$. Then for any arbitrary small $r > 0$ there exists $s_1 = s_1(r) \geq s_0$ such that

$$0 < u_0(s) \leq r \qquad \text{provided } s \geq s_1. \tag{3.3}$$

We are going to show that also (2.10) has a solution $u(s)$ such that

$$|u(s) - u_0(s)| < V\mu(s) \quad (s \geq s_2), \qquad V = \frac{b(0)}{3(1-\theta)}\Gamma^3 \tag{3.4}$$

with the same function $b(\varepsilon)$ as in (2.12) and $s_2 \geq s_1$ sufficiently large (will be specified later).

Let the set $\mathcal{W}$ of the admissible functions $w(s)$ be defined by

$$\mathcal{W} = \{w(s): \; w(s) \in C[s_1, \infty), \; 0 \leq w(s) \leq u_0(s) + V\mu(s)\}.$$

We claim that

$$\int_s^\infty H_n(w(\sigma))\,d\sigma < \infty \quad \text{for all } w(s) \in \mathcal{W}. \tag{3.5}$$

First we observe by (3.2) and (R2) that

$$\int_s^\infty u_0^2(\sigma)\,d\sigma \leq \Gamma u_0(s),$$

and hence by (3.3)

$$\text{(3.6)}\qquad \int_s^\infty u_0^i(\sigma)\,d\sigma < r^{i-2}\int_s^\infty u_0^2(\sigma)\,d\sigma \le \Gamma r^{i-2}u_0(s)\quad \text{for } i=2,3,\dots,\ s\ge s_1.$$

These relations can be rewritten for $\zeta(s) = u_0(s)/\Gamma$ or applied to $\mu(s)$:

$$\text{(3.7)}\qquad \int_s^\infty \zeta^i(\sigma)\,d\sigma < \left(\frac{r}{\Gamma}\right)^{i-2}\zeta(s)\quad i=2,3,\dots,\qquad \mu(s)\le\frac{r}{\Gamma}\zeta(s)\le\frac{r^2}{\Gamma^2}.$$

Making use of the series expansion of $H_n(u)$ in (2.11), we obtain by (3.6)

$$\int_s^\infty H_n(u_0(\sigma))\,d\sigma \le \int_s^\infty \tfrac{1}{\Gamma}u_0^2(\sigma)\,d\sigma + n^2\gamma_0\sum_{i=3}^\infty \left|\binom{\frac{n+1}{n}}{i}\right|\int_s^\infty \frac{u_0^i(\sigma)}{\varrho_0^i}\,d\sigma$$
$$\le u_0(s)\left[1+n^2\gamma_0\sum_{i=3}^\infty\left|\binom{\frac{n+1}{n}}{i}\right|\frac{\Gamma r^{i-2}}{\varrho_0^i}\right]$$

and the numeric series on the right hand side is convergent provided $r<\varrho_0$, i.e. (3.5) holds for $w(s)=u_0(s)$.

Concerning any other function $w(s)\in\mathcal{W}$, it is sufficient to prove the relation

$$\int_s^\infty H_n(u_0(\sigma)+V\mu(\sigma))\,d\sigma<\infty$$

because $H_n(u)$ is a strictly increasing function for $u>0$. By (2.12) we have

$$H_n(u_0+V\mu)-H_n(u_0)-\frac{1}{\Gamma}(2V\mu u_0+V^2\mu^2)<\frac{1}{6}b(\varepsilon)V\mu\left(3u_0^2+3Vu_0\mu+V^2\mu^2\right)$$

provided $u_0(s)+V\mu(s)\le\varepsilon$. By (3.3) and (3.7) we have

$$u_0(s)+V\mu(s)<r+V\frac{r^2}{\Gamma^2}\quad\text{for } s\ge s_1$$

and we can choose

$$\varepsilon=\varepsilon(r)=r+V\frac{r^2}{\Gamma^2}.$$

Hence

$$\int_s^\infty H_n(u_0(\sigma)+V\mu(\sigma))\,d\sigma-\int_s^\infty H_n(u_0(\sigma))\,d\sigma<2\frac{V}{\Gamma}\int_s^\infty u_0(\sigma)\mu(\sigma)\,d\sigma+\frac{V^2}{\Gamma}\int_s^\infty \mu^2(\sigma)\,d\sigma$$
$$+\frac{b(\varepsilon)}{6}V\left(3\int_s^\infty \mu(\sigma)u_0^2(\sigma)\,d\sigma+3V\int_s^\infty u_0(\sigma)\mu^2(\sigma)\,d\sigma+V^2\int_s^\infty\mu^3(\sigma)\,d\sigma\right)$$

and by virtue of (3.6), (3.7), all the integrals on the right hand side are convergent, hence (3.5) holds for all $w(s)\in\mathcal{W}$.

Let the function $\hat{\theta} = \hat{\theta}(r)$ be defined by

$$\hat{\theta} = \hat{\theta}(r) = \theta + \frac{V}{2\Gamma^2}r + b(\varepsilon(r))\left[\frac{1}{2}\Gamma r + \frac{1}{2}\frac{V}{\Gamma}r^2 + \frac{1}{6}\frac{V^2}{\Gamma^3}r^3\right]. \tag{3.8}$$

Now we fix a value of $r > 0$ for which the relations

$$\varepsilon(r) < \varrho_0, \qquad \theta < \hat{\theta}(r) < 1, \qquad \frac{b(\varepsilon(r))}{1-\hat{\theta}(\varepsilon(r))} < 2\frac{b(0)}{1-\theta} \tag{3.9}$$

hold. Then let $s_2 = s_2(r)$ be specified by the inequality

$$0 < u_0(s) - V\mu(s) < u_0(s) + V\mu(s) < \varepsilon(r) \quad \text{for } s \geq s_2 \geq s_1. \tag{3.10}$$

(Here the left inequality is ensured for sufficiently large s_2 by (3.7) because $u_0(s) = \Gamma\zeta(s)$ and $\mu(s) = o(\zeta(s))$.)

We shall use an iteration technique. Given $u_0(s) = \Gamma\zeta(s)$, then by (3.2)

$$u_0(s) = \int_s^\infty \Gamma\delta(e^\sigma)\,d\sigma + \int_s^\infty \frac{1}{\Gamma}u_0^2(\sigma)\,d\sigma. \tag{3.11}$$

Let the sequence $\{u_i(s)\}_{i=0}^\infty$ be defined successively by

$$u_{i+1}(s) = \int_s^\infty \Gamma\delta(e^\sigma)\,d\sigma + \int_s^\infty H_n(u_i(\sigma))\,d\sigma \quad \text{for } i = 0, 1, \ldots. \tag{3.12}$$

By induction we are going to show that there exists a convergent sequence $\{v_i\}_{i=0}^\infty$ such that

$$|u_i(s) - u_0(s)| < v_i\mu(s) \quad \text{for } s \geq s_2,\ i = 1, 2, \ldots \tag{3.13}$$

and

$$v_0 = 0 < v_1 = \frac{b(\varepsilon)}{6}\Gamma^3 < v_2 < \cdots < V = \tfrac{b(0)}{3(1-\theta)}\Gamma^3. \tag{3.14}$$

For $u_0(s) \in \mathcal{W}$, we have

$$u_1(s) = \int_s^\infty \Gamma\delta(e^\sigma)\,d\sigma + \int_s^\infty H_n(u_0(\sigma))\,d\sigma.$$

Clearly, (2.12) permits the estimate

$$|H_n(u) - \tfrac{1}{\Gamma}u^2| < \frac{b(\varepsilon)}{6}u^3 \quad \text{for } 0 < u < \varepsilon \tag{3.15}$$

in case $v = 0$. Hence by (3.11)

$$|u_1(s) - u_0(s)| < \frac{b(\varepsilon)}{6} \int_s^\infty u_0^3(\sigma)\, d\sigma = \frac{b(\varepsilon)}{6}\Gamma^3 \int_s^\infty \zeta^3(\sigma)\, d\sigma = \frac{b(\varepsilon)}{6}\Gamma^3 \mu(s) \quad \text{for } s \ge s_2,$$

hence by (3.10) $u_1(s) \in \mathcal{W}$. Now the formulas (3.13), (3.14) are valid for $i = 1$. Thus the next step in the induction proof is to find a value v_{i+1} such that $v_i < v_{i+1} < V$ and (3.13) holds also for $i+1$, too.

Since by (3.13)

$$0 < u_i(s) < u_0(s) + v_i \mu(s) < \varepsilon \quad \text{for } s \ge s_2,$$

we get $u_i(s) \in \mathcal{W}$, hence by (3.11) and (3.12) we have

$$\begin{aligned} u_{i+1}(s) - u_0(s) &= \int_s^\infty H_n(u_i(\sigma))\, d\sigma - \int_s^\infty \tfrac{1}{\Gamma} u_0^2(\sigma)\, d\sigma = \\ &= \tfrac{1}{\Gamma} \int_s^\infty \left[u_i^2(\sigma) - u_0^2(\sigma)\right] d\sigma + \int_s^\infty \left[H_n(u_i(\sigma)) - \tfrac{1}{\Gamma} u_i^2(\sigma)\right] d\sigma, \end{aligned}$$

consequently by (3.11), (3.15) and (3.13) with $b = b(\varepsilon)$

$$\begin{aligned} |u_{i+1}(s) - u_0(s)| &< \tfrac{1}{\Gamma} \int_s^\infty \left[u_i(\sigma) + u_0(\sigma)\right] v_i \mu(\sigma)\, d\sigma + \frac{b}{6} \int_s^\infty u_i^3(\sigma)\, d\sigma \le \\ &\le \frac{v_i}{\Gamma} \int_s^\infty \left[2u_0(\sigma) + v_i \mu(\sigma)\right] \mu(\sigma)\, d\sigma + \frac{b}{6} \int_s^\infty \left[u_0(\sigma) + v_i \mu(\sigma)\right]^3 d\sigma = \\ &= 2v_i \int_s^\infty \zeta(\sigma)\mu(\sigma)\, d\sigma + \frac{v_i^2}{\Gamma} \int_s^\infty \mu^2(\sigma)\, d\sigma + \frac{b}{6}\Gamma^3 \int_s^\infty \zeta^3(\sigma)\, d\sigma + \\ &+ \frac{b}{2}\Gamma^2 v_i \int_s^\infty \zeta^2(\sigma)\mu(\sigma)\, d\sigma + \frac{b}{2}\Gamma v_i^2 \int_s^\infty \zeta(\sigma)\mu^2(\sigma)\, d\sigma + \frac{b}{6} v_i^3 \int_s^\infty \mu^3(\sigma)\, d\sigma. \end{aligned}$$

Using (0.9) and the estimate of $\mu(s)$ from (3.7), we get the following upper bounds

$$\begin{aligned} &\int_s^\infty \mu^2(\sigma)\, d\sigma < \frac{r}{\Gamma} \int_s^\infty \zeta(\sigma)\mu(\sigma)\, d\sigma \le \frac{r}{\Gamma}\frac{\theta}{2}\mu(s) < \frac{1}{2}\frac{r}{\Gamma}\mu(s), \\ &\int_s^\infty \zeta^2(\sigma)\mu(\sigma)\, d\sigma < \frac{r}{\Gamma} \int_s^\infty \zeta^3(\sigma)\, d\sigma = \frac{r}{\Gamma}\mu(s), \\ &\int_s^\infty \zeta(\sigma)\mu^2(\sigma)\, d\sigma < \int_s^\infty \zeta(\sigma) \left(\frac{r}{\Gamma}\zeta(\sigma)\right)^2 d\sigma = \left(\frac{r}{\Gamma}\right)^2 \mu(s), \\ &\int_s^\infty \mu^3(\sigma)\, d\sigma < \left(\frac{r}{\Gamma}\right)^3 \int_s^\infty \zeta^3(\sigma)\, d\sigma = \left(\frac{r}{\Gamma}\right)^3 \mu(s) \end{aligned}$$

for $s \ge s_1$, consequently

$$\begin{aligned} |u_{i+1}(s) - u_0(s)| &< \mu(s) v_i \left[\theta + \tfrac{V}{\Gamma}\tfrac{r}{2\Gamma} + \tfrac{b}{2}\Gamma^2 \tfrac{r}{\Gamma} + \tfrac{b}{2}\Gamma V \tfrac{r^2}{\Gamma^2} + \tfrac{b}{6} V^2 \tfrac{r^3}{\Gamma^3}\right] + \tfrac{b}{6}\Gamma^3 \mu(s) \\ &= \left(\hat{\theta} v_i + \tfrac{b}{6}\Gamma^3\right) \mu(s) \end{aligned}$$

with $\hat{\theta}$ as in (3.8). Now the recurrence relation for $\{v_i\}$ is established:

$$(3.16) \qquad v_0 = 0, \quad v_{i+1} = \hat{\theta} v_i + \frac{1}{6} b\Gamma^3 \quad \text{for } i = 0, 1, \ldots,$$

in accordance with (3.14). Clearly, the sequence $\{v_i\}$ in (3.16) is increasing and convergent to the limit value $\frac{1}{6} b\Gamma^3/(1-\hat{\theta})$. By (3.9) we get for this limit

$$\frac{1}{6} \frac{b(\varepsilon(r))}{1-\hat{\theta}(\varepsilon(r))} \Gamma^3 < \frac{1}{3} \frac{b(0)}{1-\theta} \Gamma^3 = V,$$

hence (3.14) is justified. This completes the proof of the induction step and we obtain $\{u_i(s)\}_{i=0}^{\infty} \subset \mathcal{W}$.

Returning to the functions $u_0(s)$ and $u_1(s)$, the inequality (2.6), applied to $H_n(u)$, yields in (3.12) for $i = 0$:

$$u_1(s) \quad u_0(s) \quad \text{if } n \quad 1.$$

On the other hand, the function $H_n(u)$ is strictly increasing for $u > 0$, hence we obtain by (3.12)

$$u_{i+1}(s) \quad u_i(s) \quad \text{if } n \quad 1 \quad \text{for all } i$$

and the sequence $\{u_i(s)\}_{i=0}^{\infty}$ is convergent. Denote the limit function by $u(s)$:

$$\lim_{i \to \infty} u_i(s) = u(s).$$

Then by (3.12), $u(s)$ is a solution of (2.10), and (3.13) implies (3.4). Hence we can write

$$u(s) = \Gamma\zeta(s) + \mathcal{O}(\mu(s)),$$

therefore

$$\varrho(s) = \varrho_0 + u(s) = \left(\frac{n}{n+1}\right)^n + \Gamma\zeta(s) + \mathcal{O}(\mu(s)) > 0$$

is a solution of (2.3). By definition of $\varrho(s)$, this is equivalent to

$$\left(\frac{y'(s)}{y(s)}\right)^n = \varrho(s),$$

and we get

$$(3.17) \qquad \frac{y'(s)}{y(s)} = \frac{n}{n+1} + \frac{2}{n+1}\zeta(s) + \mathcal{O}(\zeta^2(s)) + \mathcal{O}(\mu(s)).$$

Let us observe that by (0.9) and (3.7), the two $\mathcal{O}$ terms are integrable. Using the fact that $\zeta(s) = z'(s)/z(s)$, we obtain by integration in (3.17)

$$\log|y(s)| - \frac{n}{n+1}s - \frac{2}{n+1}\log|z(s)| = C_0 + \mathcal{O}(\zeta(s)) + \mathcal{O}\Big(\int_s^{\infty} \mu(\sigma)\, d\sigma\Big),$$

or

$$y(s) = e^{\frac{n}{n+1}s} z^{\frac{2}{n+1}}(s)\,[C + o(1)]\,, \quad C \neq 0.$$

By (2.1) we have $x(t) = y(\log t)$ hence the first statement of Theorem 3 holds. The second statement of Theorem 3 follows immediately from (3.17) because $tx'(t)/x(t) = y'(s)/y(s)$.

In case when the function $\zeta(s)$ is monotonic (i.e. nonincreasing), then in (3.7) we have the finer estimate $\mu(s) < \zeta^2(s)$, which implies that the function $\mu(s)$ is integrable and there holds the estimate

$$\int_s^\infty \mu(\sigma)\, d\sigma < \zeta(s),$$

and in (3.17) we have the more precise result

$$\frac{y'(s)}{y(s)} = \frac{n}{n+1} + \frac{2}{n+1}\zeta(s) + \mathcal{O}(\zeta^2(s)),$$

and consequently

$$y(s) = e^{\frac{n}{n+1}s} z^{\frac{2}{n+1}}(s)\,\left[C + \mathcal{O}(\zeta(s))\right], \quad C \neq 0.$$

This proves the sharper estimates formulated in Remark 2. Now the proof of Theorem 3 is completed.

Proof of Theorem 4. Applying Theorem 3 to the solution $\bar{z}(s)$ of (0.5), we find by the iteration procedure carried out in the proof of Theorem 3, particularly by (3.4) that (2.8) has a solution $\tilde{u}(s)$ such that

$$\tilde{u}(s) = \Gamma\bar{\zeta}(s) + \mathcal{O}(\bar{\mu}(s)). \tag{3.18}$$

We know that (2.8) is the Riccati equation of the half-linear differential equation (2.9). Let $\tilde{z}(s)$ be a solution of (2.9) for which

$$\tilde{u}(s) = g_n(\frac{\tilde{z}'(s)}{\tilde{z}(s)}), \quad \text{or} \quad \frac{\tilde{z}'(s)}{\tilde{z}(s)} = g_n^{-1}(\tilde{u}(s)).$$

(The solution $\tilde{z}(s)$ is unique up to a constant multiple.) Since by (2.10) $\tilde{u}(s) > 0$, $\lim_{s\to\infty}\tilde{u}(s) = 0$, we find by (2.14) and (3.18)

$$\frac{\tilde{z}'(s)}{\tilde{z}(s)} = g_n^{-1}(\Gamma\bar{\zeta}(s) + \mathcal{O}(\bar{\mu}(s))) = \frac{\bar{z}'(s)}{\bar{z}(s)} - C_n\Gamma\bar{\zeta}^2(s)\log\bar{\zeta}(s) + \mathcal{O}(\bar{\zeta}^2(s)) + \mathcal{O}(\bar{\mu}(s)). \tag{3.19}$$

Clearly, $\bar{\zeta}(s)$ is a solution of the Riccati integral equation

$$\bar{\zeta}(s) = \int_s^\infty \bar{\zeta}^2(\sigma)\, d\sigma + \int_s^\infty \delta(e^\sigma)\, d\sigma,$$

hence by (R2)

$$\int_s^\infty \bar{\zeta}^2(\sigma)\, d\sigma \leq \bar{\zeta}(s).$$

Applying Lemma 6 to $\varphi(s) = \psi(s) = \bar\zeta(s)$ with $\Psi(u) = |\log u|$, we obtain

$$\int^\infty \bar\zeta^2(\sigma)\,|\log\bar\zeta(\sigma)|\,d\sigma < \infty.$$

Also the terms $\mathcal{O}(\bar\zeta^2(s))$ and $\mathcal{O}(\bar\mu(s))$ in (3.19) are summable hence (3.19) implies that the function $\log(\tilde z(s)/\bar z(s))$ is of bounded variation, consequently $\lim_{s\to\infty}\tilde z(s)/\bar z(s)$ exists and is positive. Since in the linear case we have by (0.11) $\int^\infty ds/\bar z^2(s) = \infty$, the same is true for the solution $\tilde z(s)$ of (2.9):

$$\int^\infty \frac{ds}{\tilde z^2(s)} = \infty.$$

By Lemma 5 this integral can be divergent only if the function $\tilde z'(s)/\tilde z(s)$ is minimal, i.e. $\tilde z(s)$ is a constant multiple of the principal solution $\bar z_h(s)$ of the half-linear differential equation (2.9), i.e. $\tilde z(s)/\bar z_h(s)$ =const. Since by (2.1), (2.8), (2.14)

$$(3.20)\qquad \left(t\frac{x'(t)}{x(t)}\right)^{n*} = \varrho(s) = \left(\frac{y'(s)}{y(s)}\right)^{n*} = \varrho_0 + u(s) = (\frac{n}{n+1})^n + g_n(\frac{z'(s)}{z(s)}),$$

the minimal property of $\bar z_h'(s)/\bar z_h(s)$ is valid also to $\bar x'(t)/\bar x(t)$, hence the principal solution $\bar z_h(s)$ belongs to the principal solution $\bar x(t)$ of (0.3).

Consider a non-principal solution $x(t) > 0$ of (0.3) which is connected with a non-principal solution $z(s)$ of (2.9). By Lemma 5 there hold the relations

$$(3.21)\quad \frac{z'(s)}{z(s)} > \frac{\bar z_h'(s)}{\bar z_h(s)},\ \int^\infty \frac{ds}{z^2(s)} < \infty,\ \int^\infty \frac{ds}{\bar z_h^2(s)} = \infty,\ \lim_{s\to\infty}[z'(s)\bar z_h(s) - z(s)\bar z_h'(s)] = \bar w > 0.$$

Particularly we have $z'(s)\bar z_h(s) - z(s)\bar z_h'(s) > \bar w_1 > 0$ for $s \ge s_0$ hence

$$\left(\frac{z(s)}{\bar z_h(s)}\right)' > \frac{\bar w_1}{\bar z_h^2(s)} \quad\text{and}\quad \lim_{s\to\infty}\frac{z(s)}{\bar z_h(s)} \ge \frac{z(s_0)}{\bar z_h(s_0)} + \bar w_1\int_{s_0}^\infty \frac{ds}{\bar z_h^2(s)} = \infty,$$

or

$$(3.22)\qquad \lim_{s\to\infty}\frac{\bar z_h(s)}{z(s)} = 0.$$

By (2.14) and (3.20) we have for $x(t) = y(s)$, $\bar x(t) = \bar y(s)$ $(s = \log t)$:

$$\frac{y'(s)}{y(s)} = \frac{n}{n+1}\left[1 + \frac{1}{n}\left(\frac{n+1}{n}\right)^n \Gamma\frac{z'(s)}{z(s)} + \mathcal{O}(\left(\frac{z'(s)}{z(s)}\right)^2 \log\frac{z'(s)}{z(s)})\right]$$
$$= \frac{n}{n+1} + \frac{2}{n+1}\frac{z'(s)}{z(s)} + \mathcal{O}(\zeta^2(s)\log\zeta(s))$$

hence by Lemma 6

$$(3.23)\qquad \lim_{s\to\infty}\frac{y(s)}{e^{\frac{n}{n+1}s} z^{\frac{2}{n+1}}(s)} = y_0 \ne 0,$$

and similarly for $\bar{y}(s)$:

$$\frac{\bar{y}'(s)}{\bar{y}(s)} = \frac{n}{n+1} + \frac{2}{n+1}\frac{\bar{z}_h'(s)}{\bar{z}_h(s)} + \mathcal{O}(\bar{\zeta}^2(s)\log\bar{\zeta}(s)),$$
$$\lim_{s\to\infty} \frac{\bar{y}(s)}{e^{\frac{n}{n+1}s}\bar{z}_h^{\frac{2}{n+1}}(s)} = \bar{y}_0 \neq 0.$$

Thus we have by (3.22)

$$\lim_{t\to\infty}\frac{\bar{x}(t)}{x(t)} = \lim_{s\to\infty}\frac{\bar{y}(s)}{y(s)} = \lim_{s\to\infty}\frac{\frac{\bar{y}(s)}{e^{\frac{n}{n+1}s}\bar{z}_h^{\frac{2}{n+1}}(s)}}{\frac{y(s)}{e^{\frac{n}{n+1}s}z^{\frac{2}{n+1}}(s)}}\frac{\bar{z}_h^{\frac{2}{n+1}}(s)}{z^{\frac{2}{n+1}}(s)} = \frac{\bar{y}_0}{y_0}\lim_{s\to\infty}\left(\frac{\bar{z}_h(s)}{z(s)}\right)^{\frac{2}{n+1}} = 0,$$

which proves the limit relation in (0.15).

On the other hand, the substitution $t = e^s$ and (3.21), (3.23) yield for a non-principal solution $x(t)$:

$$\int^\infty \frac{dt}{x^2(t)\,|x'(t)|^{n-1}} = \int^\infty \frac{t^{n-1}dt}{x^{n+1}(t)\,|t\frac{x'(t)}{x(t)}|^{n-1}} = \int^\infty \frac{e^{ns}ds}{y^{n+1}(s)\left|\frac{y'(s)}{y(s)}\right|^{n-1}}$$
$$= \int^\infty \frac{e^{ns}ds}{\left((y_0+o(1))e^{\frac{n}{n+1}s}z^{\frac{2}{n+1}}(s)\right)^{n+1}\left(\rho_0 + g_n(\frac{z'(s)}{z(s)})\right)^{\frac{n-1}{n}}}$$
$$= \int^\infty \frac{ds}{z^2(s)(y_0+o(1))^{n+1}[(\frac{n}{n+1})^{n-1}+o(1)]} = \mathcal{O}(\int^\infty \frac{ds}{z^2(s)}) < \infty,$$

and similar calculation shows for a principal solution $\bar{x}(t)$ that

$$\int^\infty \frac{dt}{\bar{x}^2(t)\,|\bar{x}'(t)|^{n-1}} = \infty.$$

Finally, if $x_1(t)$ and $x_2(t)$ are two non-principal solutions, then let $z_1(s)$ and $z_2(s)$ be the corresponding solutions of (2.9). By Lemma 5 the function $z_1(s)/z_2(s)$ is convergent when $s \to \infty$, consequently by (3.23) the same is true for the quotient $x_1(t)/x_2(t) = y_1(s)/y_2(s)$, too, which completes the proof of Theorem 4.

Proof of Corollary 1. Consider first the linear differential equation (0.5) for $k \geq 1$:

$$z''(s) + \frac{1}{4s^2}[1 + \sum_{i=1}^{k-1}\frac{1}{\log_1^2 s\log_2^2 s\ldots\log_i^2 s}]z(s) = 0,$$

which has two solutions: a principal solution

$$\bar{z}(s) = (s\log_1 s\ldots\log_{k-1}s)^{1/2}$$

and another — non-principal — solution

$$z(s) = (s\log_1 s \dots \log_{k-1} s)^{1/2} \log_k s.$$

Hence

$$\bar{\zeta}(s) = \frac{1}{2}\left(\frac{1}{s} + \frac{1}{s\log_1 s} + \dots + \frac{1}{s\log_1 s \dots \log_{k-1} s}\right) \qquad s > t_k,$$

$$\bar{\mu}(s) = \int_s^\infty \bar{\zeta}^3(\sigma)\,d\sigma = \frac{1}{8}\int_s^\infty \left(\frac{1}{\sigma} + \frac{1}{\sigma\log_1\sigma} + \dots\right)^3 d\sigma$$
$$= \frac{1}{8}\left(\int_s^\infty \frac{d\sigma}{\sigma^3} + 3\int_s^\infty \frac{d\sigma}{\sigma^3\log\sigma} + \dots\right) = \frac{1}{16}\frac{1}{s^2} + \mathcal{O}(\frac{1}{s^2\log s}),$$

$$\int_s^\infty \bar{\mu}(\sigma)\bar{\zeta}(\sigma)\,d\sigma = \int_s^\infty \left(\frac{1}{16}\frac{1}{\sigma^2} + \mathcal{O}(\frac{1}{\sigma^2\log\sigma})\right)\left(\frac{1}{2\sigma} + \mathcal{O}(\frac{1}{\sigma\log\sigma}\right) d\sigma$$
$$= \frac{1}{64}\frac{1}{s^2} + \mathcal{O}(\frac{1}{s^2\log s}),$$

i.e. the conditions of Theorem 3 and Remark 2 are satisfied for any $\frac{1}{2} < \theta < 1$. Hence Theorem 4 can be applied to (0.17) and the two asymptotics in (0.14) imply Corollary 1.

Proof of Theorem 5.
a) Suppose (0.7) is satisfied. Then there exists a constant $\Theta > 1$ and s_0 such that

$$s\int_s^\infty \delta(e^\sigma)\,d\sigma \ge \frac{\Theta^2}{4} \quad \text{for } s \ge s_0. \tag{3.24}$$

By Theorem B, the linear differential equation (0.5) is oscillatory. Due to Theorem 2, it would be sufficient to provide a proof only for $n > 1$, but we give a general proof, which goes indirect way.

Suppose in contrary that the half-linear differential equation (0.3) is non-oscillatory. Then the Riccati integral equation (2.10) has a solution $u(s) > 0$ for sufficiently large s's, say $s \ge s_1 \ge s_0$, and by (3.24) it follows

$$u(s) = \int_s^\infty \Gamma\delta(e^\sigma)\,d\sigma + \int_s^\infty H_n(u(\sigma))\,d\sigma \ge \frac{\Gamma\Theta^2}{4s} + \int_s^\infty H_n(u(\sigma))\,d\sigma$$

where $\lim_{s\to\infty} u(s) = 0$, and $H_n(u) = \frac{u^2}{\Gamma} + \mathcal{O}(u^3)$. Clearly, there exists a value $\delta > 0$ such that $H_n(u) > u^2/(\Gamma\Theta)$ for $|u| \le \delta$. Consequently, there exists a value $s_2 \ge s_1$ such that $|u(s)| \le \delta$ for $s \ge s_2$, i.e. $u(s)$ satisfies the integral inequality

$$u(s) \ge \frac{\Gamma\Theta^2}{4s} + \int_s^\infty \frac{u^2(\sigma)}{\Gamma\Theta}$$

or equivalently

$$\frac{u(s)}{\Theta} \ge \frac{\Gamma\Theta}{4s} + \int_s^\infty \frac{1}{\Gamma}\left(\frac{u(\sigma)}{\Theta}\right)^2 d\sigma.$$

Hence the function $u(s)/\Theta$ is a solution of the Riccati integral inequality associated to the linear differential equation

$$(3.25) \qquad (\Gamma z')' + \frac{\Gamma\Theta}{4s^2} z = 0.$$

Then the linear version of Lemma 3 yields that (3.25) must be non-oscillatory. But for any $\Theta > 1$, (3.25) is oscillatory, and this contradiction proves the first part of Theorem 5.
b) Suppose (0.8) is satisfied. Then for all $\theta < 1$, sufficiently near to 1, there exists $s_0 = s_0(\theta)$ such that for $s \geq s_0$

$$(3.26) \qquad s \int_s^\infty \delta(e^\sigma)\, d\sigma \leq \frac{\theta}{4}.$$

On the other hand, by Corollary 1 the differential equation

$$(x'^{n*})' + \frac{\gamma_0}{t^{n+1}} \left(n + \frac{n+1}{2} \frac{1}{\log^2 t} \right) x^{n*} = 0$$

is nonoscillatory, i.e. the Riccati integral equation (2.10)

$$v(s) = \frac{\Gamma}{4s} + \int_s^\infty H_n(v(\sigma))\, d\sigma$$

has a solution. By inequality (3.26) we have

$$v(s) \geq \frac{\Gamma}{\theta} \int_s^\infty \delta(e^\sigma)\, d\sigma + \int_s^\infty H_n(v(\sigma))\, d\sigma.$$

Since $H_n(\theta v) < \theta H(v)$ for $0 < \theta < 1$, we get the integral inequality

$$\theta v(s) > \int_s^\infty \Gamma\delta(e^\sigma)\, d\sigma + \int_s^\infty H_n\big(\theta v(\sigma)\big)\, d\sigma,$$

which is a Riccati integral inequality of (2.9), hence (2.9) and also (0.3) is non-oscillatory thus the proof of Theorem 5 is completed.

Acknowledgement. The first named author thanks the warm hospitality to the staff of Fachbereich Mathematik IV of the Dortmund University extended to him during his stay when the paper was written. The authors would like to express their gratitude to Professor Kusano Takashi (Fukuoka University) for calling their attention to the J. D. Mirzov's paper [M] and also for transmitting his own English translation of this paper to them.

References

E82. Á. Elbert, *Oscillation and nonoscillation theorems for some nonlinear ordinary differential equations*, Ordinary and Partial Differential Equations (Dundee, 1982), Lecture Notes in Mathematics, vol. 964, Springer, Berlin – Heidelberg – New York, 1982, pp. 187 – 212.

E. Á. Elbert, *Asymptotic behaviour of autonomous half-linear differential systems on the plane*, Studia Sci. Math. Hungar. **19** (1984), 447 – 464.

E84. Á. Elbert, *Generalized Riccati equation for half-linear second order differential equations*, Differential and Integral Equations: Qualitative Theory. Colloq. Math. Soc. J. Bolyai, vol. 47, 1984, pp. 227–249.

E87. Á. Elbert, Qualitative Theory of Half–linear Differential Equations, Dissertation for Doctoral Degree, 1987. (Hungarian)

EK. Á. Elbert and T. Kusano, *Principal solutions of non-oscillatory half-linear differential equations*, Advances in Mathematical Sciences and Applications **8** (1998), 745 – 759.

Hart52. Ph. Hartman, *On nonoscillatory linear differential equations of second order*, Amer. J. Math. **74** (1952), 389 – 400.

Hart. Ph. Hartman, Ordinary Differential Equations (2nd edit.), Birkhäuser, Boston – Basel – Stuttgart, 1982.

HW. Ph. Hartman and A. Wintner, *On the asignment of asymptotic values for the solutions of linear differential equations of second order*, Amer. J. Math. **77** (1955), 475 – 483.

H48. E. Hille, *Nonoscillation theorems*, Trans. Amer. Math. Soc. **64** (1948), 234 – 252.

M. J. D. Mirzov, *Principal and nonprincipal solutions of a nonlinear system*, Proceedings of I.N. Vekua Institute of Applied Mathematics (Tbilissi State University) **31** (1988), 100 – 117. (Russian)

W49. A. Wintner, *A criterion of oscillatory stability*, Quart. Appl. Math. **7** (1949), 115 – 117.

W51. A. Wintner, *On the nonexistence of conjugate points*, Amer. J. Maths **73** (1951), 368 – 380.

Eingegangen am 5. Oktober 1999

A. Elbert
1124 Nemetvölgyi ut 89
H-1364 Budapest/Ungarn

A. Schneider
Universität Dortmund
Fachbereich Mathematik
D-44221 Dortmund