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Results in Mathematics

Decomposition of twisted and warped product nets

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Abstract

By definition an orthogonal net on a pseudoriemannian manifold is a family of complementary foliations which intersect perpendicularly. There are derived generalizations of de Rham's decomposition theorem by characterizing those pseudoriemannian manifolds equipped with an orthogonal net, which locally resp. globally allow a representation as a twisted resp. warped product. The results are applied for studying hypersurfaces with harmonic curvature.

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Introduction

The decomposition theorem of de Rham [1] is considered as an excellent tool in riemannian geometry. To give a formulation adapted to the generalizations presented in this article we introduce the notion of orthogonal nets:

Definition 1. A family $\mathcal{E} = (E_i)_{i=0,\dots,k}$ of non-degenerate, totally integrable subbundles E_i of the tangent bundle TM of a pseudoriemannian manifold M is called an orthogonal net on M, if TM is the orthogonal sum $\bigoplus_{i=0}^{k} E_i$.

Herewith, de Rham's theorem says:

Theorem. *If M is a simply connected and complete* riemannian *manifold and $\mathcal{E} = (E_i)_{i=0,\dots,k}$ an orthogonal net on M such that each subbundle E_i is parallel, then M is isometric to a pseudoriemannian product and $\mathcal{E}$ corresponds to its product net.*

Unfortunately, such "parallel" orthogonal nets do not arise very often. Their existence is even impossible on many riemannian manifolds, for instance if their sectional curvature is strictly positive or strictly negative. In these situations one can ask for properties of an orthogonal net, which guarantee the local or global decomposability of the manifold into a twisted or warped product; the latter products differ from the ordinary ones in so far as at every point the metric of the factors are conformally changed (see Definition 2).

As Nölker's work shows (see [2, Theorem 7]), for instance the euclidean spheres and the hyperbolic spaces are very rich in local representations by warped products. The simplest ones are described by polar coordinates.

In this article we achieve the following: We firstly list the infinitesimal properties of the canonical product net of twisted and of warped products (see Proposition 2), motivated thereby we introduce the notions of TP-nets and WP-nets (see Definition 3) and prove that exactly these nets admit local representations by twisted resp. warped products (see Corollary 1). Since in the theorem of de Rham mainly the global decomposability is of interest, we furthermore prove corresponding global results:

Theorem 1 (Decomposition of TP-netted manifolds). *If $\mathcal{E} = (E_i)_{i=0,\dots,k}$ is a TP-net on a simply connected pseudoriemannian manifold M and for every $i \geq 1$ the leaves of the foliation $L_i^{\mathcal{E}}$ tangential to E_i are geodesically complete and their mean curvature vector field H_i (see section 1) is the gradient of a C^∞-function on M, then M 'is' a twisted product and $\mathcal{E}$ its product net.*

Theorem 2 (Decomposition of WP-netted manifolds). *If $\mathcal{E} = (E_i)_{i=0,\dots,k}$ is a WP-net on a simply connected pseudoriemannian manifold M and at least one of the following conditions is fulfilled:*

- *the leaves* of the foliation tangential to $E_i^\perp$ are geodesically complete for every $i \geq 1$,*
- *there exist a set $I \subset \{0, \dots, k\}$ of k indices such that the leaves of the foliation $L_i^{\mathcal{E}}$ tangential to E_i are geodesically complete for every $i \in I$,*

then M 'is' a warped product and $\mathcal{E}$ its product net.

We emphasize that in the theorems 1 and 2 it was not necessary to assume the "entire geodesical completeness" of M. Herewith, they become applicable to a wider range of TP- and WP-nets, for instance the WP-nets of Nölker mentioned above are of this kind.

That such generalizations of the theorem of de Rham are of general interest can be seen by the fact that many authors worked on it: For *riemannian* manifolds Theorem 1 was proved by Gauchman [3] and Theorem 2 by Hiepko [4] and Nölker [2]. The importance also of the *pseudoriemannian* situation – for instance with regard to general relativity – made Wu prove the pseudoriemannian version of de Rham's theorem [5], [6] and was our motivation to consider this general setting for twisted and warped products. It is not possible in this situation to carry over the riemannian proofs. To show the local results we use Proposition 1 of [7], which there was used to handle the case of nets with two factors. For the proof of the global results we use a decomposition theorem derived in [8].

We show now how to recover de Rham's theorem from our results: If $\mathcal{E} = (E_i)_{i=0,\dots,k}$ is an orthogonal net with parallel subbundles E_i on a pseudoriemannian manifold M, then $\mathcal{E}$ is a WP-net (see Definition 3). According to Corollary 1 the netted manifold $(M, \mathcal{E})$ is locally a warped product, which in fact is an ordinary product of pseudoriemannian manifolds because of Proposition 2 (d). Thus, we have recovered the local version of de Rham's theorem. Applying Theorem 2 we obtain Wu's global pseudoriemannian version of de Rham. Then, of course the assumption "M complete" is to be replaced by "M geodesically complete", that means, the geodesic flow of M is complete. As consequence then every leaf $L_i^{\mathcal{E}}(p)$ is geodesically complete, because it is a totally geodesic submanifold of M, and so the second assumption of Theorem 2 is really satisfied. But Theorem 2 proves *a stronger version of de Rham's theorem*, in which the geodesical completeness only of the leaves $L_i^{\mathcal{E}}(p)$ for $i \geq 1$ is demanded.

*Automatically the subbundles $E_i^\perp$ are integrable, because they are autoparallel.

In the last section of this article we give an example, in which TP-nets and WP-nets occur in a natural way, namely as principal curvature nets of a hypersurface immersion $f : M \to N$ from a pseudoriemannian manifold M with harmonic curvature into a pseudoriemannian space N of constant curvature. Sometimes this leads to a simple decomposition of f.

1 Preliminaries and notations

Firstly let us recall some definitions and results. Let $\mathcal{E} = (E_i)_{i=0,\dots,k}$ be an orthogonal net on a pseudoriemannian manifold M. The foliation of integral manifolds of E_i shall be denoted by $L_i^{\mathcal{E}}$ and its leaf through a point $p \in M$ by $L_i^{\mathcal{E}}(p)$. We denote the orthogonal projections $TM \to E_i$ resp. $TM \to E_i^{\perp}$ by $v \mapsto v^i$ resp. $v \mapsto v^{\perp_i}$. Thus, every vector field $X \in \mathfrak{X}(M)$ can be written as $X = \sum_{i=0}^{k} X^i$.

The net $\mathcal{E}$ is said to be *locally decomposable* iff for every point $p \in M$ there exist a neighbourhood U of p in M and a C^∞-diffeomorphism f from a product manifold $\prod_{i=0}^{k} M_i$ onto U such that for every $q \in \prod_{i=0}^{k} M_i$ and every $i = 0, \dots, k$ the "slice" $(q_1, \dots, q_{i-1}) \times M_i \times (q_{i+1}, \dots, q_k)$ is mapped by f into an integral manifold of E_i. In this situation f is called a *local product representation* of $\mathcal{E}$. If especially we can choose $U = M$, the net $\mathcal{E}$ is said to be *(globally) decomposable* and f is called a *(global) product representation* of $\mathcal{E}$. In [8, Theorem 1] we saw:

Lemma 1.[†] *An orthogonal net $\mathcal{E} = (E_i)_{i=0,\dots,k}$ is locally decomposable if and only if $E_i^{\perp}$ is integrable for every $i = 0, \dots, k$.*

If E is a (metrically) non-degenerate subbundle of TM, then we use the orthogonal splitting $TM = E \oplus E^{\perp}$, $v = v^E + v^{E^\perp}$ to define a bilinear bundle map $h^E : TM \times_M E \to E^{\perp}$ and a linear connection ∇^E for E by $\nabla^M{}_X Y = \nabla^E{}_X Y + h^E(X,Y)$ for all $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$; h^E is called the *second fundamental form* of E, and ∇^E is metric, i.e. $X \cdot \langle Y, Z\rangle = \langle \nabla^E{}_X Y, Z\rangle + \langle Y, \nabla^E{}_X Z\rangle$ for all $X \in \mathfrak{X}(M), Y, Z \in \Gamma(E)$. The subbundle E is integrable if and only if $h^E|(E \times_M E)$ is symmetric. If there exists a section $H \in \Gamma(E^{\perp})$ such that $h^E(X,Y) = \langle X, Y\rangle H$ for all $X, Y \in \Gamma(E)$, then E is said to be *totally umbilic* and H is called the *mean curvature normal* of E. If $\nabla^{E^\perp}{}_X(h^E(Y,Z)) = h^E(\nabla^E{}_X Y, Z) + h^E(Y, \nabla^E{}_X Z)$ for all $X, Y, Z \in \Gamma(E)$, then h^E is said to be *parallel along* E. If E is totally umbilic and the mean curvature normal H^E satisfies $\nabla^{E^\perp}{}_X H^E = 0$ for all $X \in \Gamma(E)$, then E is said to be *spherical*; and that is the case if and only if E is totally umbilic and has parallel second fundamental form along E. E is said to be *autoparallel* iff $\nabla_X Y \in \Gamma(E)$ for all $X, Y \in \Gamma(E)$, in other terms, iff E is spherical with $H^E \equiv 0$. It is well known (see [9]) that E is totally umbilic, resp. spherical, resp. autoparallel if and only if E is integrable and the leaves of the induced foliation are totally umbilic, resp. spherical, resp. totally geodesic submanifolds. If E is integrable, then E has parallel second fundamental form along E if and only if the leaves of the induced foliation are submanifolds with parallel second fundamental form.

If $M = \prod_{i=0}^{k} M_i$ is a pseudoriemannian product, then there exists a canonical globally decomposable net $\mathcal{E} := (E_i)_{i=0,\dots,k}$ on M called the *product net of* M. Furthermore, we define $M^i := M_0 \times \cdots \times M_{i-1} \times M_{i+1} \times \cdots \times M_k$ and the canonical projections $\pi_i : M \to M_i, p \to$

[†]In [8] nets are defined on arbitrary C^∞-manifolds by replacing the term "orthogonal sum" of Definiton 1 by "direct sum"; of course the attribute "othogonal" for nets makes no sense in this general setting; but it is meaningful to speak of the (local) decomposibility and Lemma 1 keeps valid.

p_i, $\pi^i : M \to M^i, p \mapsto p^i$ for every $i = 0, \ldots, k$. By abuse of the language we say that a function $\varphi : M \to \mathbb{R}$ is *independent of* M_i if and only if there exists a function $\tilde{\varphi} : M^i \to \mathbb{R}$ such that $\varphi = \tilde{\varphi} \circ \pi^i$; if M is connected, this is equivalent to $\mathrm{d}\varphi(E_i) = 0$.

2 The geometry of twisted and warped products

Definition 2. Let $M := \prod_{i=0}^{k} M_i$ be the product of C^∞-manifolds $M_0, \ldots, M_k$. A pseudoriemannian metric $\langle \cdot, \cdot \rangle$ on M is called a *twisted product metric* if there exist pseudoriemannian metrics $\langle \cdot, \cdot \rangle_0, \ldots, \langle \cdot, \cdot \rangle_k$ on $M_0, \ldots, M_k$, respectively, and a C^∞-function $\rho := (\rho_0, \ldots, \rho_k) : M \to \mathbb{R}_+^{k+1}$ such that

$$\langle X, Y \rangle := \sum_{i=0}^{k} \rho_i^2 \cdot \langle \pi_{i*} X, \pi_{i*} Y \rangle_i \quad \text{for every} \quad X, Y \in \mathfrak{X}(M) . \tag{1}$$

In this situation the pseudoriemannian manifold $(M, \langle \cdot, \cdot \rangle)$ is called a *twisted product*; we will denote it by ${}^{\rho}\prod_{i=0}^{k} M_i$ and call ρ its *twist function.*

If ρ is independent of $M_1, \ldots, M_k$ and $\rho_0 \equiv 1$, then $\langle \cdot, \cdot \rangle$ is called a *warped product metric* and $(M, \langle \cdot, \cdot \rangle)$ a *warped product*; we will denote it by $M_0 \times_\rho \prod_{i=1}^{k} M_i$ and call $\rho = (\rho_1, \ldots, \rho_k) : M_0 \to \mathbb{R}_+^k$ its *warping function.*

Remark 1. A twisted product in our sense is called *umbilic product* in [3]; in the case $k = 1$ it is called *double twisted product* in [7], while the term "twisted product" is more restrictive in Bishop [10] and Chen [11]. If the twist function $\rho = (\rho_0, \ldots, \rho_k)$ has the property that ρ_j is independent of M_j, then it is called a *mixed warped product* in [12].

In order to give some insight in the geometry of a twisted product ${}^{\rho}\prod_{i=0}^{k} M_i$, we calculate formulas for its Levi-Civita connection and its curvature tensor in comparison with those of the ordinary pseudoriemannian product $\prod_{i=0}^{k} M_i$. For $k = 1$ this result can already be found in [7, Proposition 2].

Proposition 1. *Let $M = {}^{\rho}\prod_{i=0}^{k} M_i$ be a twisted product with twist function $\rho = (\rho_0, \ldots, \rho_k)$ and product net $\mathcal{E} = (E_i)_{i=0,\ldots,k}$, let $\widetilde{\nabla}$ denote the Levi-Civita connection of the ordinary pseudoriemannian product metric $\langle \cdot, \cdot \rangle^\sim$ of $\prod_{i=0}^{k} M_i$ with curvature tensor $\tilde{R}$ and define the vector field $U_i := -\mathrm{grad}(\ln \circ \rho_i)$ for every $i = 0, \ldots, k$, where the gradient is calculated with respect to $\langle \cdot, \cdot \rangle$. Then the Levi-Civita connection ∇ of M is described by*

$$\nabla_X Y \;=\; \widetilde{\nabla}_X Y \;+\; \sum_{i=0}^{k} \left(\langle X^i, Y^i \rangle \cdot U_i - \langle X, U_i \rangle \cdot Y^i - \langle Y, U_i \rangle \cdot X^i \right) \tag{2}$$

and its curvature tensor R by

$$\begin{aligned} R(X,Y) \;=\; & \tilde{R}(X,Y) + \sum_{i=0}^{k} \left((\nabla_X U_i - \langle X, U_i \rangle \, U_i) \wedge Y^i \,+\, X^i \wedge (\nabla_Y U_i - \langle Y, U_i \rangle \, U_i) \right) \\ & + \sum_{i,j=0}^{k} \langle U_i, U_j \rangle \, X^i \wedge Y^j \end{aligned} \tag{3}$$

for every $X, Y \in \mathfrak{X}(M)$*, where for* $u, v \in T_pM$ *the endomorphims* $u \wedge v$ *of* T_pM *is defined by* $(u \wedge v)(w) := \langle w, v\rangle \cdot u - \langle w, u\rangle \cdot v$ *for all* $w \in T_pM$*.*

Proof. The proof is straightforward. For the proof of (3) we use that U_i is the gradient of a C^∞-function, hence $\langle \nabla_X U_i, Y\rangle = \langle \nabla_Y U_i, X\rangle$. □

Remark 2. Nölkers formula of the curvatuve tensor of riemannian warped products in Lemma 2 of [2] is the same, even if it looks different at the first glance.

In the following proposition we discuss the influence of some special properties of the twist function to the geometry of the subbundles E_i.

Proposition 2. *Let* $M := {}^{\rho}\prod_{i=0}^{k} M_i$ *be a connected twisted product with product net* $\mathcal{E} = (E_i)_{i=0,\dots,k}$*. With the notations above the following assertions hold true:*

(a) $\mathcal{E}$ *is an orthogonal net with respect to* $\langle \cdot, \cdot \rangle$*.*

(b) For every given point $\bar{p} \in M$*, we can uniquely choose the data* $\langle \cdot, \cdot \rangle_i$ *and* ρ *in such a way that we have* $\rho_i | L_i^{\mathcal{E}}(\bar{p}) = 1$ *for every* $i = 0, \dots, k$*. In this situation we will say that* the representation of M as a twisted product is normalized with respect to $\bar{p}$.

(c) For every i *the subbundle* E_i *is totally umbilic with mean curvature normal* $H_i := (U_i)^{\perp_i}$*.*

(d) The subbundle E_i *is autoparallel if and only if the function* ρ_i *is independent of* M_j *for* $j \neq i$*. If the representation of* M *as a twisted product is normalized with respect to some point* $\bar{p}$*, then the subbundle* E_i *is autoparallel if and only if* $\rho_i \equiv 1$*.*

(e) If for a certain $i \in \{0, \dots, k\}$ *the twist function* ρ *is independent of* M_i*, then the subbundle* $E_i^\perp$ *is autoparallel and the subbundle* E_i *is spherical. If the representation of* M *as a twisted product is normalized with respect to some point* $\bar{p}$*, then the converse is also true.*

(f) If for a certain $i \in \{0, \dots, k\}$ *the function* ρ_i *satisfies* $\rho_i | L_i^{\mathcal{E}}(\bar{p}) = 1$ *for some point* $\bar{p}$*, then the mean curvature normal* H_i *is the gradient of a* C^∞*-function (compare (c)) if and only if* ρ_i *is independent of* M_i*.*

Therefore, if $\langle \cdot, \cdot \rangle$ *is a warped product metric, then for every* $i = 1, \dots, k$ *the subbundle* E_i *is spherical with mean curvature normal* $H_i = U_i \in \Gamma(E_0)$*,* $E_i^\perp$ *is autoparallel and* $H_0 = U_0 = 0$*. (For further properties see Proposition 3 and Corollary 2 taking notice of Proposition 4.)*

Proof. (a) is trivial. For (b) we start with some representation (1) of $\langle \cdot, \cdot \rangle$ and modify $\langle \cdot, \cdot \rangle_i$ resp. ρ_i by multiplying resp. dividing it by the function $\varphi_i : M_i \to \mathbb{R}_+, p_i \to \rho_i(\bar{p}_1, \dots, \bar{p}_{i-1}, p_i, \bar{p}_{i+1}, \dots, \bar{p}_k)$. For (c) take care of (2).

For (d). Because of (c) we have the following sequence of equivalences:

$$E_i \text{ autoparallel} \iff H_i \equiv 0 \iff U_i \in \Gamma(E_i) \iff \rho_i \text{ is independent of } M_j \text{ for } j \neq i \, .$$

Furthermore, if the representation of M as a twisted product is normalized with respect to some point $\bar{p}$, then we claim

$$\rho_i \text{ is independent of } M_j \text{ for } j \neq i \iff \rho_i \equiv 1 \, .$$

Obviously "⇐" is true. For "⇒": Since $\rho_i|L_i^{\mathcal{E}}(\bar{p}) \equiv 1$ by the normalization and ρ_i is constant on every leaf induced by $E_i^{\perp}$ (because of the independence of M_j for $j \neq i$), we get $\rho_i \equiv 1$.

For (e). Formula (2) implies

$$h^{E_i^{\perp}}(X,Y) = (\nabla_X Y)^i = \sum_{j\neq i} \langle X^j, Y^j \rangle U_j^i \quad \text{for} \quad X, Y \in \Gamma(E_i^{\perp}) . \tag{4}$$

For "⇒": If ρ is independent of M_i, then $U_j^i = 0$ for every $j = 0, \ldots, k$, hence $E_i^{\perp}$ is autoparallel because of (4). Furthermore, from (c) we know that E_i is totally umbilic with mean curvature normal $H_i = (U_i)^{\perp_i}$. Since $U_i^i \equiv 0$, H_i coincides with U_i and is therefore a gradient. Thus we obtain $\langle \nabla_X H_i, Y \rangle = \langle \nabla_Y H_i, X \rangle$ for all $X, Y \in \mathfrak{X}(M)$. Because $E_i^{\perp}$ is autoparallel, we get $\langle \nabla_Y H_i, X \rangle = 0$ for all $X \in \Gamma(E_i)$ and $Y \in \Gamma(E_i^{\perp})$; thus $\nabla_X H_i$ is a section in E_i, which means, that E_i is spherical.

For "⇐": Since $E_i^{\perp}$ is assumed to be autoparallel, we get from (4) $U_j^i \equiv 0$ for $j \neq i$, hence ρ_j is independent of M_i. It remains to show that ρ_i is independent of M_i, that means $Z := U_i^i \equiv 0$. To verify this fact we fix $Y \in \Gamma(E_i^{\perp})$ and $X \in \Gamma(E_i)$. Because $E_i^{\perp}$ is autoparallel, we have $\nabla_Y H_i \in \Gamma(E_i^{\perp})$, hence $\langle \nabla_Y H_i, X \rangle = 0$; and as E_i is spherical, we also have $\langle \nabla_X H_i, Y \rangle = 0$. Since Z coincides with $U_i - H_i \in \Gamma(E_i)$, we therefore obtain

$$\langle \nabla_Y Z, X \rangle = \langle \nabla_Y U_i, X \rangle = \langle \nabla_X U_i, Y \rangle = \langle \nabla_X Z, Y \rangle = \langle X, Z \rangle \cdot \langle H_i, Y \rangle ,$$

that means $(\nabla_Y Z)^i = \langle H_i, Y \rangle \cdot Z$. Hence, Z is a $\widehat{\nabla}$-parallel section in E_i along every leaf induced by $E_i^{\perp}$ with respect to the linear connection $\widehat{\nabla}$ for E_i defined by $\widehat{\nabla}_X Y := \nabla^{E_i}{}_X Y - \langle H_i, X \rangle Y$. Since on the other hand we have $\rho_i|L_i^{\mathcal{E}}(\bar{p}) \equiv 1$ by assumption, we get $Z|L_i^{\mathcal{E}}(\bar{p}) = U_i^i|L_i^{\mathcal{E}}(\bar{p}) \equiv 0$ and therefore $Z \equiv 0$.

For (f). If ρ_i is independent of M_i, then obviously $U_i \in \Gamma(E_i^{\perp})$, thus $H_i = U_i = \mathrm{grad}(-\ln \circ \rho_i)$. Conversely, let $\lambda : M \to \mathbb{R}$ be a C^∞-function such that $H_i = \mathrm{grad}(\lambda)$. Since $H_i \in \Gamma(E_i^{\perp})$, thus $\mathrm{d}\lambda(X) = 0$ for every $X \in \Gamma(E_i)$, there exists a C^∞- function $\lambda^i : M^i \to \mathbb{R}$ such that $\lambda = \lambda^i \circ \pi^i$. Moreover, for all $X \in \Gamma(E_i^{\perp})$ we have $\mathrm{d}(\lambda + \ln \circ \rho_i)(X) = \langle H_i - U_i, X \rangle = 0$. Therefore, there exists a C^∞-function $\lambda_i : M_i \to \mathbb{R}$ such that $\lambda_i \circ \pi_i = \lambda^i \circ \pi^i + \ln \circ \rho_i$. Because of the normalization we get that λ_i is constant, which implies that ρ_i is independent of M_i.

For the assertions on warped products. If $\langle \cdot, \cdot \rangle$ is a warped product metric, then from part (e) we get that E_i is spherical and $E_i^{\perp}$ is autoparallel for every $i = 1, \ldots, k$. Since the functions ρ_i only depend on M_0 we get $U_i \in \Gamma(E_0)$ from the definition of U_i and therefore $H_i = U_i$ for $i = 1, \ldots, k$. □

3 Twisted and warped product nets

Motivated by the geometric properties of twisted and warped products we introduce the notions of TP-nets and WP-nets:

Definition 3. Let M be a pseudoriemannian manifold and $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ an orthogonal net on M.

(a) $\mathcal{E}$ is called a *TP-net* if and only if it is locally decomposable and all subbundles E_i are totally umbilic. In this situation the mean curvature normal of E_i is denoted by H_i.

(b) $\mathcal{E}$ is called a *WP-net* if and only if for every $i = 1, \ldots, k$ the subbundle E_i is spherical and its orthogonal complement $E_i^\perp$ is autoparallel.

Proposition 3. *For every WP-net $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ the subbundle E_0 is autoparallel and its orthogonal complement $E_0^\perp$ is integrable; therefore, the net $\mathcal{E}$ is locally decomposable and consequently a TP-net.*

Proof. The autoparallelity of E_0 follows from $E_0 = \bigcap_{i=1}^k E_i^\perp$. In order to show the integrability of $E_0^\perp$ it is sufficient to prove for arbitrary $X \in \Gamma(E_i)$ and $Y \in \Gamma(E_j)$ with $i, j \geq 1$ that $[X, Y] = \nabla_X Y - \nabla_Y X \in \Gamma(E_0^\perp)$. In the case $i = j$ we know that $[X, Y] \in \Gamma(E_i)$, since E_i is integrable. If $i \neq j$, then for every $Z \in \Gamma(E_0)$ we have $\nabla_X Z \in \Gamma(E_j^\perp)$ because $E_0 \oplus E_i \subset E_j^\perp$ and the latter bundle is autoparallel. Therefore, $\langle \nabla_X Y, Z \rangle = X \cdot \langle Y, Z \rangle - \langle Y, \nabla_X Z \rangle = 0$, i.e. $\nabla_X Y \in \Gamma(E_0^\perp)$; thus we have $[X, Y] \in \Gamma(E_0^\perp)$ again.

As the other bundles $E_i^\perp$ are integrable by assumption, the local decomposability follows with Lemma 1. Thus it remains to check that every E_i is totally umbilic. For $i \geq 1$ that follows by assumption, and for $i = 0$ it is true because E_0 even is autoparallel. □

That the notations "TP-net" and "WP-net" are well chosen is demonstrated by the following proposition and corollary:

Proposition 4. *On a connected product manifold $M = \prod_{i=0}^k M_i$ the product net $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ is a TP- resp. WP-net with respect to a pseudoriemannian metric $\langle \cdot, \cdot \rangle$ if and only if $(M, \langle \cdot, \cdot \rangle)$ is a twisted resp. warped product.*

The assertion "$\Leftarrow$" follows immediately from Proposition 2. The proof of "$\Rightarrow$" is based on [7, Proposition 1], which we recall in the special situation important for us:

Lemma 2. *If $(M = M_0 \times M_1, \langle \cdot, \cdot \rangle)$ is a connected pseudoriemannian manifold with orthogonal product net $\mathcal{E} = (E_0, E_1)$ and E_1 is totally umbilic, then there exist a pseudoriemannian metric $\langle \cdot, \cdot \rangle_1$ on M_1 and a C^∞-function $\rho : M \to \mathbb{R}_+$ such that for every $p \in M$ and $v, w \in E_1(p)$ we have*

$$\langle v, w \rangle = \rho^2(p) \cdot \langle \pi_{1*} v, \pi_{1*} w \rangle_1 ,$$

where $\pi_1 : M \to M_1$ denotes the canonical projection.

Indeed, this proposition follows from [7, loc.cit.] by putting $F := \mathrm{id}_M$, choosing some point $p_0 \in M_0$ and defining $\langle \cdot, \cdot \rangle_1$ as the pseudoriemannian metric such that $M_1 \to M$, $q \mapsto (p_0, q)$ becomes an isometric immersion.

Proof for "$\Rightarrow$" of Proposition 4. If $\mathcal{E}$ is a TP-net and $i \in \{0, \ldots, k\}$, then the application of Lemma 2 to the manifold $M = M^i \times M_i$ with product net $(E_i^\perp, E_i)$ gives a pseudoriemannian metric $\langle \cdot, \cdot \rangle_i$ on M_i and a C^∞-function $\rho_i : M \to \mathbb{R}_+$; obviously $(M, \langle \cdot, \cdot \rangle)$ coincides with the twisted product ${}^{(\rho_0,\ldots,\rho_k)}\prod_{i=0}^k (M_i, \langle \cdot, \cdot \rangle_i)$. – Because of Proposition 3 this situation also occurs, if $\mathcal{E}$ is a WP-net. Assuming that the TP-representation ${}^{(\rho_0,\ldots,\rho_k)}\prod_{i=0}^k (M_i, \langle \cdot, \cdot \rangle_i)$ of M is normalized with respect to some point we conclude from Proposition 2 (d) and (e) that then M is a warped product. □

Corollary 1. *Let $(M, \langle\cdot,\cdot\rangle)$ be a pseudoriemannian manifold and $\mathcal{E} = (E_i)_{i=0,\dots,k}$ a TP-net resp. WP-net on M. Then for every point $p \in M$ there exists a local product representation $f : \prod_{i=0}^k M_i \to U$ of $\mathcal{E}$ with $p \in U \subset M$, which is an isometry with respect to a twisted resp. warped product metric on $\prod_{i=0}^k M_i$.*

Proof. The local decomposability of $\mathcal{E}$ implies the existence of a local product representation $f : \prod_{i=0}^k M_i \to U$ as above. After equipping $\prod_{i=0}^k M_i$ with the pseudoriemannian metric with respect to which f becomes an isometry, the product net of $\prod_{i=0}^k M_i$ becomes a TP-net resp. WP-net. Then Proposition 4 completes the proof. □

Corollary 2. *For every WP-net $\mathcal{E} = (E_i)_{i=0,\dots,k}$ on a pseudoriemannian manifold M the following holds besides the assertions of Proposition 3 and Corollary 1:*

(a) For every $i = 1, \dots, k$ the mean curvature normal H_i of E_i is a section in the subbundle E_0 and the tensor field defined by $(X,Y) \mapsto \langle \nabla_X H_i, Y\rangle$ is symmetric; of course, $H_0 \equiv 0$.

(b) By the formula

$$\widetilde{\nabla}_X Y \;:=\; \nabla_X Y - \sum_{i=1}^{k} \left(\langle X^i, Y^i\rangle \cdot H_i - \langle X, H_i\rangle \cdot Y^i - \langle Y, H_i\rangle \cdot X^i \right) \tag{5}$$

there is defined a linear connection on M with respect to which the subbundles $E_0, \dots, E_k$ are parallel; therefore its curvature tensor field satisfies $\widetilde{R}(X,Y)Z = \sum_{i=0}^k \widetilde{R}(X^i, Y^i)Z^i$. The curvature tensor R of $(M, \langle\cdot,\cdot\rangle)$ is related to $\widetilde{R}$ by

$$\begin{aligned} R(X,Y) \;=\; & \widetilde{R}(X,Y) + \sum_{i,j=1}^{k} \langle H_i, H_j\rangle\, X^i \wedge Y^j \\ & + \sum_{i=1}^{k} \left((\nabla_X H_i - \langle X, H_i\rangle\, H_i) \wedge Y^i \,+\, X^i \wedge (\nabla_Y H_i - \langle Y, H_i\rangle\, H_i) \right) . \end{aligned} \tag{6}$$

In particular, if V is a 2-dimensional linear subspace of T_pM ($p \in M$) generated by two unit vectors $v \in E_i(p)$ and $w \in E_j(p)$ with $i \neq j$ and $i, j \geq 1$, the sectional curvature $K_M(V)$ is given by

$$K_M(V) \;=\; -\,\langle H_i(p), H_j(p)\rangle\ . \tag{7}$$

(c) The second fundamental form of $E_0^\perp$ satisfies

$$h^{E_0^\perp}(X,Y) \;=\; \sum_{i=1}^{k} \langle X^i, Y^i\rangle\, H_i \quad \text{for} \quad X, Y \in \Gamma(E_0^\perp)\ ; \tag{8}$$

furthermore, it is parallel along $E_0^\perp$.

Proof. The assertion $H_0 \equiv 0$ follows from the autoparallelity of E_0 stated in Proposition 3. For the other statements we use strongly Corollary 1 by representating M locally as a warped product. Then according to Proposition 2 H_i corresponds with the gradient field U_i which is a section in E_0. Firstly this completes the proof of (a). Secondly, comparing the formulas

(5) and (2) we recognize the linear connection $\widetilde{\nabla}$ with respect to a local WP-representation of $\mathcal{E}$ as the connection of an ordinary pseudoriemannian product. Therefore, the bundles E_i are parallel with respect to $\widetilde{\nabla}$ and formula (6) follows from (3). If now $X \in \Gamma(E_i)$ and $Y \in \Gamma(E_j)$ with $i \neq j$ and $i, j \geq 1$ are given, then we derive $R(X,Y) = \langle H_i, H_j\rangle \cdot X \wedge Y + (\nabla_X H_j) \wedge Y + X \wedge (\nabla_Y H_i)$ from (6) since $H_i, H_j \in \Gamma(E_0)$. This implies the equation for the sectional curvature, since E_i and E_j are totally umbilic subbundles. Thus (b) is proved completely. Formula (8) is immediately obtained by specializing formula (4) to our situation.

It remains to show that $E_0^\perp$ has parallel second fundamental form along $E_0^\perp$. As E_0 is autoparallel, we have $\nabla_X H_i \in \Gamma(E_0)$ for every $X \in \Gamma(E_0)$, hence $\langle \nabla_Y H_i, X\rangle = \langle \nabla_X H_i, Y\rangle = 0$ for every further section $Y \in \Gamma(E_0^\perp)$. Thus, we obtain

$$\nabla^{E_0}{}_Y H_i = 0 \quad \text{for every } Y \in \Gamma(E_0^\perp) \text{ and } i \geq 1\,. \tag{9}$$

Using (b) we get for every $X, Y, Z \in \Gamma(E_0^\perp)$ and $i \geq 1$

$$\begin{aligned}\langle \nabla_X Y^i, Z^i\rangle &= \langle \widetilde{\nabla}_X Y^i + \langle X^i, Y^i\rangle\, H_i, Z^i\rangle = \langle (\widetilde{\nabla}_X Y)^i, Z^i\rangle \\ &= \langle \big(\widetilde{\nabla}_X Y + \sum_{j=1}^{k} \langle X^j, Y^j\rangle\, H_j\big)^i, Z^i\rangle = \langle (\nabla_X Y)^i, Z^i\rangle\,,\end{aligned} \tag{10}$$

and therefore

$$\begin{aligned}\nabla^{E_0}{}_X (h^{E_0^\perp}(Y,Z)) &\overset{(8)}{=} \sum_{i=1}^{k} \nabla^{E_0}{}_X \big(\langle Y^i, Z^i\rangle\, H_i\big) \\ &= \sum_{i=1}^{k} \Big(\big(\langle \nabla_X Y^i, Z^i\rangle + \langle Y^i, \nabla_X Z^i\rangle\big) H_i + \langle Y^i, Z^i\rangle \nabla^{E_0}{}_X H_i\Big) \\ &\overset{(9),(10)}{=} \sum_{i=1}^{k} \big(\langle (\nabla_X Y)^i, Z^i\rangle + \langle Y^i, (\nabla_X Z)^i\rangle\big)\, H_i \\ &\overset{(8)}{=} h^{E_0^\perp}(\nabla_X Y, Z) + h^{E_0^\perp}(Y, \nabla_X Z)\,.\end{aligned}$$

□

4 Global decomposition of TP- and WP-netted manifolds

The Theorems 1 and 2 formulated in the introduction show under which conditions the local result of Corollary 1 also holds globally. Our proof is based on the following decomposition theorem derived in [8, Theorem 3 (a)]:

Theorem 3. *Let M be a simply connected pseudoriemannian manifold, which is equipped with an orthogonal,* locally *decomposable net $\mathcal{E} = (E_i)_{i=0,\dots,k}$ satisfying* one *of the following conditions:*

(a) For every $i \geq 1$ the leaves of the foliation $L_i^{\mathcal{E}}$ are totally geodesic and geodesically complete.

(b) For every $i \geq 1$ the maximal integral manifolds of the subbundle $E_i^\perp$ are totally geodesic and geodesically complete.

Then $\mathcal{E}$ is globally decomposable.

For the proof of Theorem 1 and 2 it suffices to show the global decomposability of the net $\mathcal{E}$, for then we can use the same argumentation as in the proof of Corollary 1.

Proof of Theorem 1. By assumption, for every $i \geq 1$ there exists a C^∞-function $\rho_i : M \to \mathbb{R}_+$ such that $H_i = -\operatorname{grad}(\ln \circ \rho_i)$. Then we define a pseudoriemannian metric $\langle \cdot, \cdot \rangle^\sim$ on M by

$$\langle X, Y \rangle^\sim := \langle X^0, Y^0 \rangle + \sum_{i=1}^{k} \frac{1}{\rho_i^2} \langle X^i, Y^i \rangle, \tag{11}$$

which is non-degenerate, since it is non-degenerate on every E_i. We prove the global decomposability of $\mathcal{E}$ using Theorem 3 (a) by showing that the subbundle E_i is autoparallel and has geodesically complete leaves with respect to the Levi-Civita connection $\widetilde{\nabla}$ of $\langle \cdot, \cdot \rangle^\sim$ for every $i = 1, \ldots, k$. Because of Corollary 1 for every point $p \in M$ there exists a connected neighbourhood U of p in M such that U 'is' a twisted product ${}^{\sigma}\prod_{i=0}^{k} M_i$ with some C^∞-function $\sigma = (\sigma_0, \ldots, \sigma_k) : U \to \mathbb{R}_+^{k+1}$ and product net $\mathcal{E}|_U$. On U we can represent H_i as $H_i|U = -(\operatorname{grad}(\ln \circ \sigma_i))^{\perp i}$ (see Proposition 2 (c)). For $i \geq 1$ we obtain $d(\ln \circ \rho_i)(Z) = d(\ln \circ \sigma_i)(Z)$ for every $Z \in \Gamma(E_i^\perp|_U)$; thus, there exist a C^∞-function $\lambda_i : M_i \to \mathbb{R}$ such that $\ln \circ \sigma_i - \ln \circ \rho_i = \lambda_i \circ \pi_i$. Therefore, $\sigma_i = \exp(\lambda_i) \circ \pi_i \cdot \rho_i$ for $i \geq 1$ and thus on U the metric $\langle \cdot, \cdot \rangle$ satisfies

$$\langle X, Y \rangle = \sigma_0^2 \cdot \langle \pi_{0*} X, \pi_{0*} Y \rangle_0 + \sum_{i=1}^{k} \rho_i^2 \cdot \exp^2(\lambda_i) \circ \pi_i \langle \pi_{i*} X, \pi_{i*} Y \rangle_i \,. \tag{12}$$

Hence, on U the metric $\langle \cdot, \cdot \rangle^\sim$ is the twisted product metric

$$\langle X, Y \rangle^\sim = \sigma_0^2 \cdot \langle \pi_{0*} X, \pi_{0*} Y \rangle_0 + \sum_{i=1}^{k} \exp^2(\lambda_i) \circ \pi_i \cdot \langle \pi_{i*} X, \pi_{i*} Y \rangle_i \,.$$

For $i \geq 1$ the function $\exp(\lambda_i) \circ \pi_i$ is independent of M^i, thus by means of Proposition 2 (d) the subbundle $E_i|_U$ is autoparallel with respect to $\widetilde{\nabla}$. Since autoparallelity is a local property, for $i \geq 1$ the subbundle E_i is autoparallel with respect to $\widetilde{\nabla}$.

Moreover, for $i \geq 1$ the function ρ_i is constant on every leaf $L_i^{\mathcal{E}}(p)$ since $\operatorname{grad}(\rho_i) \in \Gamma(E_i^\perp)$. Therefore, the metric $\langle \cdot, \cdot \rangle^\sim|_{L_i^{\mathcal{E}}(p)}$ coincides with the metric $\langle \cdot, \cdot \rangle|_{L_i^{\mathcal{E}}(p)}$ up to a constant factor, which shows that the geodesical completeness of $(L_i^{\mathcal{E}}(p), \langle \cdot, \cdot \rangle|_{L_i^{\mathcal{E}}(p)})$ implies the geodesical completeness of $(L_i^{\mathcal{E}}(p), \langle \cdot, \cdot \rangle^\sim|_{L_i^{\mathcal{E}}(p)})$.

Thus, we can apply Theorem 3 (a) on the manifold $(M, \langle \cdot, \cdot \rangle^\sim)$ and the net $\mathcal{E}$ and get that $(M, \mathcal{E})$ is decomposable. □

Remark 3. The assumption of Theorem 1 that the mean curvature normal H_i is a gradient means that Theorem 1 handles only twisted products $M = {}^{(\rho_0, \ldots, \rho_k)}\prod_{i=0}^{k} M_i$, where each function ρ_i is independent of M_i for every $i \geq 1$, if this representation of M as a twisted product is normalized with respect to some point $\bar{p} \in M$. On the other hand remember that for a twisted product the mean curvature normal H_i is only the *projection of a gradient* (see Proposition 2 (b)) onto $E_i^\perp$; therefore it should be emphasized that Theorem 1 becomes false, if for a TP-net (E_i) we only assume this weaker property of H_i.

Example. Let X_0, $X_1 \in \mathfrak{X}(\mathbb{R}^2)$ be defined by $X_0(x,y) := (2x, x^2-1)$ and $X_1(x,y) := (x^2-1, -2x)$ and $E_i := \mathbb{R}X_i$. Then $\mathcal{E} := (E_0, E_1)$ is a TP-net on the euclidean plane $\mathbb{R}^2$, the leaves of the two foliations are geodesically complete and the C^∞-function $\lambda(x,y) := -1/2 \cdot x^2 + \ln(x^2+1) + 2y$ has the property that the orthogonal projection of $\mathrm{grad}(\lambda)$ onto E_0 is the mean curvature vector field H_1 of E_1. Nevertheless, $\mathcal{E}$ is not decomposable, since the leaf space induced by $L_1^{\mathcal{E}}$ is not a Hausdorff space (see also [7]).

Proof of Theorem 2. According to Proposition 3 the net $\mathcal{E}$ is locally decomposable. – In the case that the first condition of Theorem 2 is fulfilled part (b) of Theorem 3 can be applied, as by the definition of WP-nets for every $i \geq 1$ the subbundle $E_i^\perp$ is autoparallel. If on the other hand the second condition of Theorem 2 is fulfilled, let $i \in I$ be fixed. Then $\omega_i := \langle H_i, \cdot \rangle$ is a closed 1-form because of Corollary 2. Since M is simply connected, ω_i even is exact, and therefore H_i is a gradient field. Thus, (after renumbering the indices if necessary) all assumptions of Theorem 1 are fulfilled, and we get that $\mathcal{E}$ is decomposable. □

5 Hypersurfaces with harmonic curvature

In this section let $f : M \to N$ be an isometric immersion between connected pseudoriemannian manifolds and suppose that N has dimension $\dim N = 1 + \dim M =: 1 + n \geq 3$ and constant curvature $\varkappa$. Furthermore, suppose that f has a globally defined unit normal field η, denote the shape operator of f with respect to η by A and suppose that on M there exists an orthogonal net $\mathcal{E} = (E_i)_{i=1,\dots,k}$ and a set $\{\lambda_1, \dots, \lambda_k\}$ of C^∞-functions such hat $\lambda_i(p)$ is an eigenvalue of A_p and $E_i(p)$ the corresponding eigenspace for every point p of an open, dense subset $G_A \subset M$ and for every $i \in \{1, \dots, k\}$. In this situation $\mathcal{E}$ is called the *principal curvature net* of f, because each λ_i is a principal curvature function of f (see also [13], where $\mathcal{E}$ is called the curvature net of f). Its multiplicity $\mathrm{rank}(E_i)$ will be denoted by m_i. Although at points $p \in M \setminus G_A$ the eigenspaces of A_p may differ from $E_1(p), \dots, E_k(p)$, nevertheless for every section $X \in \Gamma(E_i)$ the equation $AX = \lambda_i X$ keeps valid on the entire space M, because both sides of the equation are continuous.

It should be mentioned that in the case of a riemannian manifold M there exists an open, dense subset $\widetilde{M}$ of M such that on every connected component of $\widetilde{M}$ the principal curvature net of f is defined uniquely (up to the ordering of the E_i), see [14, §1] and [15, Theorem]; therefore the assumption on the existence of the principal curvature net is not so restrictive in this case. But if the metric of M is indefinite, then this assumption becomes crucial. For instance, it implies that the investigation of [14, 15] keeps valid in the pseudoriemannian case. In particular, the subbundles E_i are totally umbilic and in the case $m_i \geq 2$ even spherical. Nevertheless, in general $\mathcal{E}$ is not a TP-net, because it is not locally decomposable. But we can enforce the local decomposability by demanding that M has harmonic curvature; see Corollary 4. Let us recall that a tensor field of a pseudoriemannian manifold is said to be *harmonic* iff it has vanishing divergence, [16, p. 35]. Notice the following sequence of implications: M has constant curvature $\Rightarrow$ M is an Einstein manifold $\Rightarrow$ M has a parallel Ricci tensor field $\Rightarrow$ M has harmonic curvature, i.e., its curvature tensor is harmonic.

In order to analyze the harmonicity of the curvature of M in our situation we introduce the curvature like tensor field R_A defined by

$$R_A(X,Y)Z = \langle AY,Z\rangle \cdot AX - \langle AX,Z\rangle \cdot AY$$

and its associated Ricci tensor field Ric_A, which can simply be described by

$$Ric_A = -(A - tr_A)\circ A \quad \text{with} \quad tr_A := \operatorname{trace} A\ .$$

Therewith we can write the curvature equation of Gauss (for f) by

$$R(X,Y)Z = \varkappa \cdot \big(\langle Y,Z\rangle \cdot X - \langle X,Z\rangle \cdot Y\big) + R_A(X,Y)Z\ .$$

This representation implies the first assertion of the following proposition.

Proposition 5. *M has harmonic curvature if and only if R_A is harmonic, and the latter is true if and only if Ric_A is a Codazzi tensor field, i.e. $(\nabla_X Ric_A)Y = (\nabla_Y Ric_A)X$, that means, iff*

$$\big((\nabla_X A) - \mathrm{d}tr_A(X)\big)AY = \big((\nabla_Y A) - \mathrm{d}tr_A(Y)\big)AX \tag{13}$$

for all $X,\, Y \in \mathfrak{X}(M)$.

Proof. As R_A satisfies the differential Bianchi identity we get $(\operatorname{div} R_A)(X,Y,Z) = \langle(\nabla_X Ric_A)Y - (\nabla_Y Ric_A)X, Z\rangle$ and obtain thereby the relation between the harmonicity of R_A and the Codazzi equation for Ric_A; using that A, too, is a Codazzi tensor field the latter implies (13) by an easy calculation. □

If M has harmonic curvature, then the principal curvature net and the principal curvature functions of f have further strong properties, as we shall see now.

Proposition 6. *If in the situation described at the beginning of this section the pseudo-riemannian manifold M has harmonic curvature, then the following properties hold for all $i,j \in \{1,\dots,k\}$:*

(a) $m_i \geq 2 \implies \lambda_i \cdot \mathrm{d}tr_A|E_i \equiv 0$

(b) *If $i \neq j$, then*

$$\begin{aligned} &\lambda_j \cdot \mathrm{d}tr_A|E_i = (\lambda_j - \lambda_i)\cdot \mathrm{d}\lambda_j|E_i \quad \textit{and} \\ &\forall\, X \in \Gamma(E_i)\ ,\ Y \in \Gamma(E_j) \quad : \quad \nabla_X Y \in \Gamma(E_i \oplus E_j)\ ; \end{aligned} \tag{14}$$

if in particular $\mathrm{d}\lambda_i|E_j = 0$, then even $\nabla_X Y \in \Gamma(E_j)$.

(c) *If $m_i \leq n-2$ and $\big(m_i \geq 2$ or $\lambda_i \equiv 0\big)$, then $\mathrm{d}tr_A|E_i = 0$.*

(d) *If $\mathrm{d}tr_A|E_i = 0$, then $E_i^{\perp}$ is autoparallel and $\mathrm{d}\lambda_j|E_i = 0$ for every $j \in \{1,\dots,k\}$.*

Proof. For $X \in \Gamma(E_i)$ and $Y \in \Gamma(E_j)$ formula (13) implies

$$\lambda_j \cdot \big((\nabla_X A)Y - \mathrm{d}tr_A(X)\cdot Y\big) = \lambda_i \cdot \big((\nabla_Y A)X - \mathrm{d}tr_A(Y)\cdot X\big)\ .$$

Replacing $(\nabla_Y A)X$ by $(\nabla_X A)Y = \mathrm{d}\lambda_j(X)\cdot Y - (A-\lambda_j)\nabla_X Y$ we obtain

$$\lambda_i \cdot \mathrm{d}tr_A(Y)\cdot X - \lambda_j \cdot \mathrm{d}tr_A(X)\cdot Y = (\lambda_i - \lambda_j)\cdot \big(\mathrm{d}\lambda_j(X)\cdot Y - \underbrace{(A-\lambda_j)\nabla_X Y}_{\in\Gamma(E_j^\perp)}\big)\ . \qquad (15)$$

Choosing $i = j$ we deduce (a). For (b) we suppose $i \neq j$. Splitting (15) into its E_j- and $E_j^\perp$-part we obtain (14) and $\lambda_i \cdot \mathrm{d}tr_A(Y)\cdot X = -(\lambda_i - \lambda_j)\cdot(A-\lambda_j)\nabla_X Y$. Replacing the left hand side of this identity by equation (14), in which we change the role of λ_i and λ_j, we get

$$(A-\lambda_j)\nabla_X Y = -\mathrm{d}\lambda_i(Y)\cdot X\ . \qquad (16)$$

This implies $\nabla_X Y \in \Gamma(E_i \oplus E_j)$. The last assertion of (b) follows immediately from (16).

For (c). In the case $\lambda_i \equiv 0$ we apply formula (14) and get $\mathrm{d}tr_A|E_i = \mathrm{d}\lambda_j|E_i$ for every $j \neq i$ over G_A. As both sides of the last equation are continuous it holds on the entire space M. Since we have $tr_A = \sum_{j\neq i} m_j \cdot \lambda_j$ on M, we therefore obtain $\mathrm{d}tr_A|E_i = (n - m_i)\cdot \mathrm{d}tr_A|E_i$, thus $\mathrm{d}tr_A|E_i = 0$ because of $m_i \leq n-2$.

Now we assume $\lambda_i \not\equiv 0$ and $m_i \geq 2$. On the open subset $G := \{p \in M \mid \lambda_i(p) \neq 0\}$ we can apply part (a) of this proposition and obtain $\mathrm{d}_p tr_A|E_i(p) = 0$ for $p \in G$; of course, this equation even holds for $p \in \overline{G}$. But on $M \setminus \overline{G}$ this equation is also true, namely because of the preceding case. Thus $\mathrm{d}tr_A|E_i = 0$ holds in general.

For (d). We fix some i and choose $j \neq i$. Then we get $\mathrm{d}\lambda_j|E_i = 0$ because of (14). Therefore, part (b) of this proposition shows $\nabla_X Z \in \Gamma(E_i)$ for all $X \in \Gamma(E_j)$ and $Z \in \Gamma(E_i)$. In fact, this result keeeps valid, if we choose $X \in \Gamma(E_i^\perp)$. Then by means of the Ricci-Identity we derive the autoparallelity of $E_i^\perp$. – From $\mathrm{d}\lambda_j|E_i = 0$ for $j \neq i$ and $m_i \cdot \lambda_i = tr_A - \sum_{j\neq i} m_j \cdot \lambda_j$ finally we also obtain $\mathrm{d}\lambda_i|E_i = 0$. Thus the proof is complete. □

As a first application we get:

Corollary 3. *If M has harmonic curvature, then the following is true:*

(a) If $m_i \geq 2$ for every $i = 1,\ldots,k$, then $tr_A = const.$

(b) If for some $i = 1,\ldots,k$ the function λ_i is constant $\neq 0$ and $m_i \geq 2$, then $tr_A = const.$

(c) If $tr_A = const.$, then $\nabla A = 0$, i.e., f has parallel second fundamental form.

The assertions (a) and (c) were already proved by M. UMEHARA in [17] and by U. KI, H. NAKAGAWA and M. UMEHARA in [18] with very different methods.

Proof. For (a). The case $k = 1$ is trivial. For $k \geq 2$ we apply Proposition 6 (c), and obtain $\mathrm{d}tr_A|E_i = 0$ for every $i = 1,\ldots,k$, hence $\mathrm{d}tr_A = 0$.

For (b). We get $\mathrm{d}tr_A|E_j = 0$ from Proposition 6 (a) for $j = i$ and from Proposition 6 (b) for $j \neq i$. Consequently we have $\mathrm{d}tr_A = 0$.

For (c). According to Proposition 6 (d) the λ_i are constant and the $E_i^\perp$ are autoparallel. Hence also the bundles $E_i = \bigcap_{j\neq i} E_j^\perp$ are autoparallel. From $TM = E_i \oplus E_i^\perp$ we then get that the E_i are even parallel. Therefore, we obtain $(\nabla_X A)Y = \nabla_X(AY) - A(\nabla_X Y) = \nabla_X(\lambda_i Y) - \lambda_i \cdot \nabla_X Y = 0$ for every $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E_i)$. That proves $\nabla A = 0$. □

Now we combine Proposition 6 with the theory from the foregoing sections.

Corollary 4. *If M has harmonic curvature, then the principal curvature net $\mathcal{E} = (E_i)_{i=1,\dots,k}$ of f is a TP-net.*

Proof. As stated at the beginning of this section the subbundles E_i are totally umbilical. Therefore, according to Lemma 1, we have only to check the involutivity of the subbundles $E_i^\perp$: Firstly, the involutivity of the E_i and Proposition 6 (b) imply the involutivity of every subbundle $E_i \oplus E_j$, $i \neq j$; and therefrom the involutivity of the subbundles $E_i^\perp$ follows. □

Theorem 4. *Suppose that in the situation described at the beginning of this section M has harmonic curvature and that there exists an integer $\ell \in \{1,\dots,k\}$ such that*

$$\mathrm{d}tr_A|E_i = 0 \quad \textit{for} \quad i = 1,\dots,\ell \quad \text{(see Proposition 6 (c))} \quad .$$

Then $\mathcal{F} := (E_i)_{i=0,\dots,\ell}$ with $E_0 := \left(\bigoplus_{i=1,\dots,\ell} E_i\right)^\perp$ is a WP-net and

$$\mathrm{d}\lambda_j\big|\left(\textstyle\bigoplus_{i=1,\dots,\ell} E_i\right) = 0 \quad \textit{for all} \quad j = 1,\dots,k \; .$$

In particular, locally M is a warped product, whose product net corresponds with the WP-net $\mathcal{F}$.

Proof. In the following i resp. j is arbitrary in $\{1,\dots,\ell\}$ resp. in $\{1,\dots,k\}$. Then, according to Proposition 6 (d), the subbundles $E_i^\perp$ are autoparallel and $\mathrm{d}\lambda_j|E_i = 0$. Since particularly $\mathrm{d}\lambda_i|E_i = 0$, the subbundles E_i are spherical because of [15, Theorem 1 (d)]. □

Remark 4. Thus far the results of this section keep valid, if we replace the shape operator A by any selfadjoint Codazzi tensor field; instead of assuming the harmonicity of the curvature of M then we have to demand the harmonicity of the associated curvature tensor R_A.

As the ambient space N has constant curvature it is locally isometric to a standard space of constant curvature (described in [19], namely as pseudoeuclidean space, pseudosphere resp. pseudohyperbolic space). Therefore, Theorem 4 in connection with the following proposition shows that locally the hypersurface immersion f can be decomposed.

Proposition 7. *Let $M = M_0 \times_\rho \prod_{i=1}^{\ell} M_i$ be an n-dimensional connected warped product of pseudoriemannian manifolds, N an $(n+1)$-dimensional pseudoriemannian standard space of constant curvature $\varkappa$ and $f : M \to N$ an isometric immersion. Assume that there exist principal curvature functions $\lambda_1,\dots,\lambda_\ell$ of f and an open, dense subset $G \subset M$ such that the product net $\mathcal{F} = (F_i)_{i=0,\dots,\ell}$ of $M_0 \times_\rho \prod_{i=1}^{\ell} M_i$ has the following property: For every $p \in G$ and every $i = 1,\dots,\ell$ the subspace $F_i(p)$ is the eigenspace of the shape operator A_p of f with respect to $\lambda_i(p)$ and $\mathrm{d}\lambda_i|F_i = 0$* (notice that the hypothesis $\mathrm{d}\lambda_i|F_i = 0$ is fulfilled if $rank(F_i) \geq 2$, see [15, Theorem ii)]). *Then, depending on an arbitrarily chosen point $\bar{p} \in M$ there exists an isometry $\Psi : N_0 \times_\sigma \prod_{i=1}^{\ell} N_i \to U$ from a warped product onto an open subset $U \subset N$, local isometries $f_i : M_i \to N_i$ for $i = 1,\dots,\ell$ and a hypersurface immersion $f_0 : M_0 \to N_0$ such that f can be decomposed in the following manner:*

$$f = \Psi \circ (f_0 \times \cdots \times f_\ell) \; . \tag{17}$$

Sketch of the proof. As over G the subbundles F_i with $i \geq 1$ are eigenbundles of the shape operator A of f, the second fundamental form h of f (related to A by $h_p(v,w) = \varepsilon \langle Av, w\rangle \eta_p$ with $\varepsilon := \langle \eta, \eta\rangle \in \{\pm 1\}$) satisfies the essential decomposition hypothesis (D), namely $h|F_i \times_M F_j = 0$ for $i \neq j$.

Now, we fix some point $\bar{p} \in M$ and assume that the representation of M as a warped product is normalized with respect to $\bar{p}$. Then for every $i = 0, \ldots, \ell$ we define the isometric immersion $f_i := f \circ \varphi_i : M_i \to N$, where φ_i denotes the canonical embedding

$$\varphi_i : M_i \xrightarrow{\sim} L_i^{\mathcal{F}}(\bar{p}) \hookrightarrow M \ , \ p_i \mapsto (\bar{p}_1, \ldots, \bar{p}_{i-1}, p_i, \bar{p}_{i+1}, \ldots, \bar{p}_\ell) \ . \tag{18}$$

For $i \geq 1$ the immersion f_i is spherical and its mean curvature normal is $\zeta_i := f_* H_i \circ \varphi_i + \varepsilon (\lambda_i \eta) \circ f_i$ (e.g. see [14, Satz 2 (d)] or [15, Theorem iii) b)]), where H_i denotes the mean curvature normal of F_i. Now we define a *"WP-net initial data set"* $\mathcal{I} = (\bar{q}, V_0, \ldots, V_\ell, z_1 \ldots, z_\ell)$ by $\bar{q} := f(\bar{p})$, $V_i := f_{i*} T_{\bar{p}_i} M_i = f_* F_i(\bar{p})$ and $z_i := \zeta_i(\bar{p}_i)$ for $i \geq 1$ and $V_0 := \left(\bigoplus_{i=1}^{\ell} V_i\right)^{\perp}$. Furthermore, for $i \geq 1$ let N_i denote the unique extrinsic sphere (= geodesically complete, spherical submanifold) of N with $\bar{q} \in N_i$, $T_{\bar{q}} N_i = V_i$ and whose mean curvature normal at $\bar{q}$ is z_i; then from the construction it follows that f_i is a local isometry onto an open part of N_i; in particular N_i is the "spherical hull" of f_i in the sense of [2, Definition 14]. Therefore, if the metric of M and N is positive definite, the assertion of Proposition 7 follows from Nölker's decomposition theorem [2, Theorem 16]. But, in the indefinite case the general version of this theorem does not keep valid, because then the existence of spherical hulls is not guaranteed and the orthogonality of two linear subspaces do not imply that there intersection is the 0-space. Because of this reason the third author of this article has developped a theory which also works in the indefinite situation by introducing the notion of *quadratic hulls*; for their definition more general quadrics (of pseudoeuclidean spaces) than extrinsic spheres are involved; see [20, Chapter 6] (it is intended to publish this method in a forthcoming paper). In our special situation however Nölker's methods would work. We justify this claim by indicating, how one can gather the missing arguments for Proposition 7 from [20].

Combining the curvature equation of Gauss with formula (7) we get $\langle z_i, z_j\rangle = -\varkappa$ for $i \neq j$ (as in [2]). Therefore, $\mathcal{I}$ is regular in the sense of [20, Definition 21] and we get from [20, Theorems 18 and 19]: There exists a connected totally geodesic submanifold N_0 of N with $\bar{q} \in N_0$ and $T_{\bar{q}} N_0 = V_0$ and an isometry Ψ as described in Proposition 7 which can not be enlarged by enlarging N_0 and which is uniquely determined by the following fact: If $\psi_i : N_i \to \prod_{j=0}^{\ell} N_j$ is defined analogously to (18) (replacing $\bar{p}$ by $(\bar{q}, \ldots, \bar{q})$), then $\Psi \circ \psi_i$ is the canonical inclusion $N_i \hookrightarrow N$. If N is a pseudoeuclidean space, then Ψ turns out to be the decomposition map described in [20, Theorem 23] associated to $\bar{q}$, $N_1, \ldots, N_\ell$, and (17) holds by means of [20, Theorem 24]; in order to verify $f_0(M_0) \subset N_0$ one uses [20, Proposition 36]. How the case of an arbitrary N can be deduced from the last situation is described in [20, Section 6.4]. □

Remark 5. It seems to be a good idea to call the isometric immersion

$$\Psi \circ (f_0 \times \mathrm{id}_{N_1} \times \cdots \times \mathrm{id}_{N_\ell}) \ : \ M_0 \times_{\sigma \circ f_0} \prod_{i=1}^{\ell} N_i \to N$$

a *multi-rotation hypersurface* parametrization (see also [2, Example 11 (b)]). Thus, in the situation of Proposition 7 f parametrizises an open part of such a nice hypersurface.

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