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Results in Mathematics

AFFINE TRANSLATION SURFACES

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1. INTRODUCTION

In 1993 Nomizu and Vrancken ([NV]) introduced a new transversal plane for affine surfaces in affine 4-space. Such surfaces come equipped with a metric, g, which is invariant under the group of special affine motions. One class of surfaces, for which the induced metric is Lorentzian, is that of translation surfaces. By definition a *translation surface* is one which can be written, locally, as a sum of two curves. This class coincides with those surfaces which are both Lorentzian and harmonic.

In [MV] we began an investigation of translation surfaces, classifying those whose sectional curvatures are constant. We showed that the curvature must be zero and that this is equivalent to one of the defining curves being planar. In this paper we investigate other, natural, geometric conditions on translation surfaces. In particular, we classify those translation surfaces which are: umbilical, affine spheres, have trivial normal connection or null mean curvature vectors. All of these conditions will imply the surface is flat, but yield, of course, other conditions on the defining curves. One benefit of these classifications is a wider pool of examples of Lorentzian affine surfaces in $\mathbb{R}^4$ on which to test hypotheses.

2. BASIC EQUATIONS FROM A SURFACE IN $\mathbb{R}^4$

In what follows $f\colon M^2 \to \mathbb{R}^4$ will be a surface immersed in $\mathbb{R}^4$. We first give the fundamental equations for a surface in $\mathbb{R}^4$ equipped with an arbitrary transversal plane bundle σ, i.e., $(f_*)(TM)\oplus\sigma = T\mathbb{R}^4$. Eventually we will choose σ to have certain properties.

Given any transversal σ, we have the two fundamental equations.

$$D_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$D_X \xi = -S_\xi X + \nabla^{\perp}_X \xi, \tag{2.2}$$

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where $\nabla_X Y$ and $S_\xi X$ are in TM while $h(X,Y)$ and $\nabla^\perp_X \xi$ are in σ. Note that, in these equations, we have suppressed the mention of f_*.

Because the codimension is two, we can choose a local basis $\{\eta_1, \eta_2\}$ of σ and rewrite $h(X,Y)$ and $\nabla^\perp_X \eta_j$ as follows.

$$h(X,Y) = h^1(X,Y)\eta_1 + h^2(X,Y)\eta_2, \tag{2.3}$$

$$\nabla^\perp_X \eta_j = \tau^1_j(X)\eta_1 + \tau^2_j(X)\eta_2. \tag{2.4}$$

Beginning with $R^D(X,Y)Z = 0 = R^D(X,Y)\eta$, where R^D is the curvature tensor of the standard connection in $\mathbb{R}^4$, using (2.1), (2.2), (2.3), (2.4) and calculating the tangential and σ components, we obtain the structure equations of the immersion. These equations are called the Gauss, Codazzi and Ricci equations of the immersion.

Choose a localframe $u = \{X_1, X_2\}$ on M. We define

$$G_u(Y,Z) = \frac{1}{2}([X_1, X_2, D_Y X_1, D_Z X_2] + [X_1, X_2, D_Z X_1, D_Y X_2]), \tag{2.5}$$

which is the same as

(2.6)
$$G_u(Y,Z) = \frac{1}{2}[X_1, X_2, \xi_1, \xi_2] \left(\begin{vmatrix} h^1(X_1,Y) & h^1(X_2,Z) \\ h^2(X_1,Y) & h^2(X_2,Z) \end{vmatrix} + \begin{vmatrix} h^1(X_1,Z) & h^1(X_2,Y) \\ h^2(X_1,Z) & h^2(X_2,Y) \end{vmatrix} \right).$$

We have used $[X,Y,Z,W]$ to denote the determinant of four vectors in $\mathbb{R}^4$. The second expression is more useful for calculations, while the first shows that $G_u(Y,Z)$ is independent of the choice of σ, and basis $\{\xi_1, \xi_2\}$. Note that the non-degeneracy of G_u is independent of the choice of frame. Thus we will say that M is non-degenerate if G_u is non-degenerate for some choice of frame. In this case, if we set

$$det_u G_u = \begin{vmatrix} G_u(X_1,X_1) & G_u(X_1,X_2) \\ G_u(X_1,X_2) & G_u(X_2,X_2) \end{vmatrix},$$

which is non-zero, then we can define an invariant metric g by

$$g(Y,Z) = \frac{G_u(Y,Z)}{\sqrt[3]{det_u G_u}}. \tag{2.7}$$

We note that the expression on the right-hand side of the equation can be shown to be independent of u. We will only consider surfaces that are nondegenerate with respect to G, and note that, for non-degenerate translation surfaces, the metric is Lorentzian.

Call a local frame field $\{Y_1, Y_2\}$ a normalized null frame if $g(Y_j, Y_j) = 0$ and $g(Y_1, Y_2) = 1$. Following [NV] we can find a basis $\{\eta_1, \eta_2\}$ of any transversal bundle σ with the following properties.

$$\begin{aligned}
&[Y_1, Y_2, \eta_1, \eta_2] = 2, && \\
&h^1(Y_1, Y_1) = 1, && h^2(Y_1, Y_1) = 0, \\
&h^1(Y_2, Y_2) = 0, && h^2(Y_2, Y_2) = 1, \\
&h^1(Y_1, Y_2) = 0, && h^2(Y_1, Y_2) = 0.
\end{aligned}$$

There is also a metric $g^{\perp}$ which can be defined on σ such that $g^{\perp}(\eta_j, \eta_j) = 0$ and $g^{\perp}(\eta_1, \eta_2) = -2$. Finally we can fix a transveral plane bundle σ by requiring that

$$(\nabla g)(Y_j, Y_j, Y_j) = 0 = (\nabla g)(Y_i, Y_j, Y_i),$$

where $i, j = 1, 2$. This condition implies that $\nabla \omega_g = 0$, i.e., (∇, ω_g) form an equiaffine structure.

We recall an important lemma from [[]MV], whose proof we include for completeness.

Lemma 2.1. *If M^2 is a translation surface in $\mathbb{R}^4$ then the induced connection ∇ equals the Levi-Civita connection of the affine metric g.*

Proof of Lemma 2.1. Suppose that the surface is given by $f(s,t) = \alpha(s) + \beta(t)$. We note first that

$$D_{\partial s} f_s = \alpha'' = \frac{c^3}{c}\left[\frac{2c_s}{c^3}\alpha' + \eta_1\right],$$

so that$\nabla_{\partial s}\partial s = \frac{2c_s}{c}\partial s$ and, similarly, $\nabla_{\partial t}\partial t = \frac{2c_t}{c}\partial t$. It is clear, of course, that $\nabla_{\partial s}\partial t = \nabla_{\partial t}\partial s = 0$.

Using the fact that $\{\partial s, \partial t\}$ is a null basis with respect to g and $g(\partial s, \partial t) = \epsilon c^2$, one can see that $\hat{\nabla}_X Y = \nabla_X Y$. □

Because the connection is metric, it is easy to see that there are functions a_1, a_8 so that

$$D_{Y_1} Y_1 = a_1 Y_1 + \eta_1, \quad D_{Y_2} Y_1 = -a_8 Y_1, \tag{2.8}$$

$$D_{Y_1} Y_2 = -a_1 Y_2, \quad D_{Y_2} Y_2 = a_8 Y_2 + \eta_2. \tag{2.9}$$

Moreover, from the Codazzi equations we get $\tau_1^1(Y_1) = 2a_1$, $\tau_1^1(Y_2) = -2a_8$, $\tau_2^1(Y_1) = 0 = \tau_1^2(Y_2)$, while the Gauss equation gives $S_1 Y_2 = -kY_1$, $S_2 Y_1 = -kY_2$, where k is the sectional curvature of g. Thus

$$D_{Y_1}\eta_1 = b_1 Y_1 + b_2 Y_2 + 2a_1\eta_1 + b_3\eta_2, \tag{2.10}$$

$$D_{Y_2}\eta_1 = kY_1 - 2a_8\eta_1, \tag{2.11}$$

$$D_{Y_1}\eta_2 = kY_2 - 2a_1\eta_2, \tag{2.12}$$

$$D_{Y_2}\eta_2 = b_4 Y_1 + b_5 Y_2 + b_6\eta_1 + 2a_8\eta_2, \tag{2.13}$$

where $b_1, \dots, b_6$ are additional functions on the surface.

From these equations we can calculate the Gauss, Codazzi and Ricci equations:

$$Y_1 a_8 + Y_2 a_1 + 2a_1 a_8 = -k, \tag{2.14}$$
$$Y_2 b_1 - Y_1 k + 2a_8 b_1 + b_3 b_4 = 0, \tag{2.15}$$
$$Y_2 b_2 + 4a_8 b_2 + b_3 b_5 = 0, \tag{2.16}$$
$$Y_2 b_3 + 5a_8 b_3 + b_2 = 0, \tag{2.17}$$
$$3k = b_3 b_6, \tag{2.18}$$
$$Y_1 b_4 + 4a_1 b_4 + b_1 b_6 = 0, \tag{2.19}$$
$$Y_2 k - Y_1 b_5 - 2a_1 b_5 - b_2 b_6 = 0, \tag{2.20}$$
$$Y_1 b_6 + 5a_1 b_6 + b_4 = 0. \tag{2.21}$$

3. Translation Surfaces which are Affine Umbilical

From (2.10)-(2.13) we see that the shape operators of an affine surface are

$$S_1 = \begin{bmatrix} -b_1 & -k \\ -b_2 & 0 \end{bmatrix}, \quad \text{and} \quad S_2 = \begin{bmatrix} 0 & -b_4 \\ -k & -b_5 \end{bmatrix}. \tag{3.1}$$

Definition 3.1. *A surface is called umbilical iff each shape operator is a multiple of the identity.*

Thus, a translation surface which is affine umbilical has $b_1 = b_2 = b_4 = b_5 = k = 0$. As mentioned above, this is a stronger condition than having zero sectional curvature. To completely classify these surfaces we need the following lemma.

Lemma 3.1. *There is a normalized null frame $\{Z_1, Z_2\}$ so that $\nabla_{Z_i} Z_j = 0$.*

Proof of Lemma 3.1. Define a function $\rho \neq 0$ so that

$$Y_1 \rho = a_1 \rho \qquad \text{and} \qquad Y_2 \rho = -a_8 \rho.$$

Such a ρ exists because the integrability condition for ρ holds, namely

$$Y_1(Y_2 \rho) - Y_2(Y_1 \rho) = [Y_1, Y_2]\rho.$$

Indeed, this is equivalent to

$$-\rho(Y_1 a_8 + Y_2 a_1 + 2a_1 a_8) = 0.$$

We note that the factor inside the parentheses is equivalent to the Gauss equation. Now we set $Z_1 = (1/\rho)Y_1$, $Z_2 = \rho Y_2$, and it is easy to check that $\nabla_{Z_i} Z_j = 0$. □

Now we can show that an umbilical surface is a sum of two planar parabolas plus a single additional function.

Theorem 3.2. *If M^2 is a translation surface which is affine umbilical then M^2 is an affine motion of*

$$f(s,t) = (s, s^2/2 + f(t), t, t^2/2),$$

where $f(t)$ is an arbitrary function.

Proof of Theorem 3.2. From the lemma above, there are null vector fields whose Lie bracket is zero. This implies that we have coordinates $\{s,t\}$ so that $\frac{\partial}{\partial s} = Z_1$ and $\frac{\partial}{\partial t} = Z_2$. We also see that $D_{Z_1}Z_2 = 0$, so that $f(s,t) = \alpha(s) + \beta(t)$, for two curves $\alpha(s)$ and $\beta(t)$. (2.8) and (2.9) give $\eta_1 = \alpha''$ and $\eta_2 = \beta^{\cdot\cdot}$, where $\cdot$ denotes the derivative with respect to t. In addition, from (2.10) and (2.13) we see that

$$\alpha''' = b_3\beta^{\cdot\cdot} \qquad \text{and} \qquad \beta^{\cdot\cdot\cdot} = b_6\alpha''. \tag{3.2}$$

Now we calculate $(\alpha''')^{\cdot} = 0 = \beta^{\cdot\cdot\cdot\prime}$, and find

$$b_{3t}\beta^{\cdot\cdot} + b_3b_6\alpha'' = 0, \tag{3.3}$$

$$b_{6s}\alpha'' + b_3b_6\beta^{\cdot\cdot} = 0. \tag{3.4}$$

The linear independence of α'' and $\beta^{\cdot\cdot}$ implies $b_3b_6 = 0 = b_{3t} = b_{6s}$. Without loss of generality, assume that $b_6(p) \neq 0$, so that $b_3 = 0$ in a neighborhood of $f(p)$. $\alpha''' = 0$ implies that α is a parabola. So far, then, we have

$$\alpha(s) = (s, s^2/2, 0, 0),$$
$$\beta^{\cdot\cdot\cdot} = (0, b_6(t), 0, 0).$$

This gives the desired result. □

4. Translation Surfaces which are affine spheres

Definition 4.1. *A surface M^2 immersed in $\mathbb{R}^4$ is called an affine sphere if the position vector $x \in \sigma$, so that we have $x = f\eta_1 + g\eta_2$.*

Theorem 4.1. *If M^2 is a translation surface and an affine sphere then M^2 is a sum of planar curves, each of which is either an ellipse or a hyperbola.*

Proof of Theorem 4.1.

By taking multiples of Y_1, Y_2, η_1, and η_2 we may assume that $f = \epsilon = \pm 1$. We use the same letters for the new choice of vector fields. From $x = \epsilon\eta_1 + g\eta_2$ we have

$$Y_1 = \epsilon D_{Y_1}\eta_1 + (Y_1g)\eta_2 + gD_{Y_1}\eta_2, \tag{4.1}$$

$$Y_2 = \epsilon D_{Y_2}\eta_1 + (Y_2g)\eta_2 + gD_{Y_2}\eta_2. \tag{4.2}$$

These last two equations yield

$$b_1 = \epsilon, \tag{4.3}$$
$$b_2 = -\epsilon g k, \tag{4.4}$$
$$a_1 = 0, \tag{4.5}$$
$$Y_1 g = -\epsilon b_3, \tag{4.6}$$
$$g b_4 = -\epsilon k, \tag{4.7}$$
$$g b_5 = 1, \tag{4.8}$$
$$g b_6 = 2\epsilon a_8, \tag{4.9}$$
$$Y_2 g = -2 g a_8. \tag{4.10}$$

From (4.8) we see that g is not equal to zero. Substituting these values in the structure equations then gives

$$-Y_1 k + 2\epsilon a_8 - \epsilon b_3 k/g = 0, \tag{4.11}$$
$$Y_2 k + 2k a_8 - \epsilon b_3/g^2 = 0, \tag{4.12}$$
$$Y_2 b_3 + 5 a_8 b_3 - \epsilon g k = 0, \tag{4.13}$$
$$2\epsilon b_3 a_8/g = 3k, \tag{4.14}$$
$$Y_1 k - \epsilon k b_3/g - 2\epsilon a_8 = 0, \tag{4.15}$$
$$Y_2 k + 2k a_8 - \epsilon b_3/g^2 = 0, \tag{4.16}$$
$$Y_1 a_8 = -k. \tag{4.17}$$

We now have two cases to consider: $a_8 = 0$ and $a_8 \neq 0$. We will first dispose of the case where $a_8 = 0$ and then show that, in fact, this is the only case.

If $a_8 = 0$ then by (4.14), $k = 0$, and by (4.12), $b_3 = 0$, so the all the structure equations are of the form $0 = 0$. In addition, (4.3)-(4.10) give $b_1 = \epsilon$, $b_2 = 0$, $a_1 = 0$, $Y_1 g = 0$, $b_4 = 0$, $b_5 = 1/g$, $b_6 = 0$ and $Y_2 g = 0$. Thus we have

$$D_{Y_1} Y_1 = \eta_1, \quad D_{Y_2} Y_1 = 0, \tag{4.18}$$
$$D_{Y_1} Y_2 = 0, \quad D_{Y_2} Y_2 = \eta_2, \tag{4.19}$$

so that we have coordinates $\{s, t\}$ with $Y_1 = \frac{\partial}{\partial s}$ and $Y_2 = \frac{\partial}{\partial t}$. Then (2.10)-(2.13) become

$$\eta_{1s} = \epsilon Y_1, \tag{4.20}$$
$$\eta_{1t} = 0, \tag{4.21}$$
$$\eta_{2s} = 0, \tag{4.22}$$
$$\eta_{2t} = (1/g) Y_2. \tag{4.23}$$

Thus $x(s,t) = \alpha(s) + \beta(t)$, $\eta_1 = \alpha''$, $\eta_2 = \beta^{\cdot\cdot}$, $\alpha''' = \epsilon\alpha'$ and $\beta^{\cdot\cdot\cdot} = 1/g\beta^{\cdot}$. This implies, of course, that α and β are ellipses or hyperbolas.

Now we assume, contrary to fact, that $a_8 \neq 0$. Adding (4.11) and (4.15) one gets

$$\frac{-\epsilon k b_3}{g} = 0, \tag{4.24}$$

so that $kb_3 = 0$. If $k \neq 0$, it follows from (4.14) that $b_3 \neq 0$ and so $kb_3 \neq 0$, a contradiction. Thus $k = 0$ and, from (4.11), $a_8 = 0$. $\square$

5. Translation Surfaces with trivial normal connection

The normal connection for a translation surface is contained in (2.10)-(2.13), namely

$$\nabla^{\perp}_{Y_1}\eta_1 = 2a_1\eta_1 + b_3\eta_2, \tag{5.1}$$

$$\nabla^{\perp}_{Y_2}\eta_1 = -2a_8\eta_1, \tag{5.2}$$

$$\nabla^{\perp}_{Y_1}\eta_2 = -2a_1\eta_2, \tag{5.3}$$

$$\nabla^{\perp}_{Y_2}\eta_2 = b_6\eta_1 + 2a_8\eta_2. \tag{5.4}$$

The normal connection is trivial if the curvature tensor

$$R^{\perp}(X,Y)\eta = 0 = \nabla^{\perp}_X\nabla^{\perp}_Y\eta - \nabla^{\perp}_Y\nabla^{\perp}_X\eta - \nabla^{\perp}_{[X,Y]}\eta \tag{5.5}$$

Theorem 5.1. *If M^2 is a translation surface with trivial normal connection then M^2 is either a sum of two arbitrary planar curves or a sum of a parabola and an arbitrary space curve as follows: $x(s,t) = (s, s^2/2 + f_1(t), f_2(t), f_3(t))$.*

Proof of Theorem 5.1. Applying (5.5) to $X = Y_1$, $Y = Y_2$ and $\eta = \eta_1$ and $\eta = \eta_2$ gives

$$Y_2b_3 = -5a_8b_3, \tag{5.6}$$

$$2Y_1a_8 + 2Y_2a_1 + b_3b_6 + 4a_1a_8 = 0, \tag{5.7}$$

$$Y_1b_6 = -5a_1b_6. \tag{5.8}$$

Combining (5.7), (2.14) and (2.18), we arrive at $k = 0 = b_3b_6$. We may assume that $b_3 = 0$ in a neighborhood of p, with a similar solution if $b_6 = 0$. Since $k = 0$, we can assume that $a_1 = 0 = a_8$, so that our structure equations yield

$$Y_2b_1 = 0, \tag{5.9}$$

$$b_2 = 0, \tag{5.10}$$

$$Y_1b_4 + b_1b_6 = 0, \tag{5.11}$$

$$Y_1b_5 + b_2b_6 = 0, \tag{5.12}$$

$$Y_1b_6 + b_4 = 0, \tag{5.13}$$

while(5.8) reduces to

$$Y_1b_6 = 0.$$

Thus we have $b_2 = 0$, $b_4 = 0$, $b_1b_6 = 0$, $Y_1b_6 = 0$ and $Y_2b_1 = Y_1b_5 = 0$. We can write $x(s,t) = \alpha(s) + \beta(t)$.

If $b_6 = 0$ then $\alpha'' = \eta_1$, $\beta^{\cdot\cdot} = \eta_2$ and $\alpha''' = b_1(s)\alpha'$ and $\beta^{\cdot\cdot\cdot} = b_5(t)\beta^{\cdot}$, so that both curves are planar.

If $b_1 = 0$ and $b_6 \neq 0$ we have, again $b_2 = 0$, $b_4 = 0$, $Y_1b_6 = 0$ and $Y_1b_5 = 0$. In this case $\alpha''' = 0$ and $\beta^{\cdot\cdot\cdot} = b_5(t)\beta^{\cdot} + b_6(t)\alpha''(s)$. This yields the solution given in the statement of the Theorem. $\square$

6. Translation Surfaces with null mean curvature vector

Those translation surfaces which are minimal have been classified in [K] and the next set of surfaces one would like to classify are those surfaces whose mean curvature vectors have constant length, and are non-minimal. We were able to solve this problem for surfaces with null mean curvature vectors, but must leave the case of non-null mean curvature vectors as an open problem.

In general, the mean curvature vector $H = -b_1\eta_2 - b_5\eta_1$, and we first note that if M has null mean curvature vector we have $b_1 b_5 = 0$ so we can assume that $H = -b_5\eta_1$, i.e. $b_1 = 0$.

Theorem 6.1. *If M^2 is a translation surface with null mean curvature vector then M^2 has the form $x(s,t) = (s, s^2/2, 0, 0) + (f_1(t), f_2(t), f_3(t), f_4(t))$, where $f_{\dot 3} f_{\ddot 4} - f_{\dot 4} f_{\ddot 3}$ is a constant, or $x(s,t) = (h_1(s), h_2(s), s, s^2/2) + (f_1(t), f_2(t), 0, 0)$, where $f_{\dot 1} f_{\ddot 2} - f_{\dot 2} f_{\ddot 1}$ is a constant.*

Proof of Theorem 6.1. We will break the proof up into two main parts, $k = 0$ and $k \neq 0$. As mentioned in the introduction, we show that k must be zero.

k=0. As before, choose coordinates s, t so that $\frac{\partial}{\partial s} = Y_1$ and $\frac{\partial}{\partial t} = Y_2$, which implies, by rescaling, that $a_1 = 0 = a_8$. The structure equations (2.14)-(2.21) then yield

$$b_3 b_4 = 0, \tag{6.1}$$
$$Y_2 b_2 + b_3 b_5 = 0, \tag{6.2}$$
$$Y_2 b_3 + b_2 = 0, \tag{6.3}$$
$$b_3 b_6 = 0, \tag{6.4}$$
$$Y_1 b_4 = 0, \tag{6.5}$$
$$Y_1 b_5 + b_2 b_6 = 0, \tag{6.6}$$
$$Y_1 b_6 + b_4 = 0. \tag{6.7}$$

At this point we consider several subcases. First we look at $b_3 = 0$. This means we have $b_1 = 0$, $b_2 = 0$, $Y_1 b_4 = 0$, $Y_1 b_5 = 0$ and $Y_1 b_6 + b_4 = 0$. Thus $x(s,t) = \alpha(s) + \beta(t)$, with $\alpha''' = 0$ and $\beta^{\cdots} = b_4(t)\alpha'(s) + b_5(t)\beta^{\cdot}(t) + b_6\alpha'''(s)$. Writing

$$\alpha(s) = (s, s^2/2, 0, 0),$$
$$\beta(t) = (f_1(t), f_2(t), f_3(t), f_4(t))$$

we see that $[\alpha', \alpha'', \beta^{\cdot}, \beta^{\cdot\cdot}] = f_{\dot 3} f_{\ddot 4} - f_{\dot 4} f_{\ddot 3}$ is a constant. Now

$$\begin{aligned}\beta^{\cdots}(t) &= (f_1^{\cdots}(t), f_2^{\cdots}(t), f_3^{\cdots}(t), f_4^{\cdots}(t)) \\ &= b_4(t)(1, s, 0, 0) + b_5(t)(f_{\dot 1}(t), f_{\dot 2}(t), f_{\dot 3}(t), f_{\dot 4}(t)) + (\tilde b_6(t) - s b_4(t))(0,1,0,0),\end{aligned}$$

where $b_6 = -s b_4(t) + \tilde b_6(t)$. Thus we have

$$\beta^{\cdots}(t) = (b_4(t), \tilde b_6(t), 0, 0) + b_5(t)\beta^{\cdot}(t).$$

This gives one of the cases in the theorem.

The next subcase is $b_3 \neq 0$. This implies that $b_4 = 0$, $b_6 = 0$, $Y_1 b_5 = 0$, $Y_2 b_3 + b_2 = 0$ and $Y_2 b_2 + b_3 b_5 = 0$. Thus we have

$$\alpha'''(s) = b_2 \beta^{\cdot}(t) + b_3 \beta^{\cdot\cdot}(t),$$
$$\beta^{\cdot\cdot\cdot}(t) = b_5 \beta^{\cdot}(t),$$

so that $\beta(t)$ is planar and, as above, $[Y_1, \eta_1, Y_2, \eta_2] = f_1^{\cdot} f_2^{\cdot\cdot} - f_2^{\cdot} f_1^{\cdot\cdot} = c$, a constant. We also have the system which b_2, b_3 and b_5 satisfy:

$$b_{3t} + b_2 = 0, \tag{6.8}$$
$$b_{2t} + b_3 b_5 = 0. \tag{6.9}$$

We have two pairs of solutions to this system: $(f_1^{\cdot\cdot}(t), -f_1^{\cdot}(t))$ and $(f_2^{\cdot\cdot}(t), -f_2^{\cdot}(t))$. Thus we see that the general solution to the system (6.8)-(6.9) is

$$b_2(t,s) = g_1(s) f_1^{\cdot\cdot}(t) + g_2(s) f_2^{\cdot\cdot}(t), \tag{6.10}$$
$$b_3(t,s) = -g_1(s) f_1^{\cdot}(t) - g_2(s) f_2^{\cdot}(t). \tag{6.11}$$

Thus

$$\alpha'''(s) = (-c g_2(s), c g_1(s), 0, 0),$$

and $x(s,t) = (h_1(s), h_2(s), s, s^2/2) + (f_1(t), f_2(t), 0, 0)$.

General case: $k \neq 0$, $b_3 \neq 0$.

In this case we may assume, by the rescaling of our frame, that $b_1 = 0$ and $b_5 = \epsilon = \pm 1$. From the structure equations (2.15), (2.18), (2.21) we get

$$Y_1 k = b_3 b_4, \tag{6.12}$$
$$Y_1 b_3 = 5 a_1 b_3 + \frac{4 b_3^2 b_4}{3k}. \tag{6.13}$$

Next we define three auxiliary functions g_1, g_2, and g_3 as follows:

$$Y_1 a_8 = g_1 - k/2 - a_1 a_8, \tag{6.14}$$
$$Y_2 a_1 = -g_1 - k/2 - a_1 a_8, \tag{6.15}$$
$$Y_1 a_1 = g_2, \tag{6.16}$$
$$Y_2 a_8 = g_3. \tag{6.17}$$

Using (6.12) and (2.20), which is $Y_2 k = 2 a_1 \epsilon + b_2 b_6$, and the integrability conditions for b_3 and k, we can find, by a lengthy calculation, that

(6.18)
$$Y_1 b_2 = \frac{8\epsilon a_1 b_3^2 b_4 + 20 b_2 b_3 b_4 k - 8\epsilon a_1^2 b_3 k - 8\epsilon b_3 g_2 k + 60 a_1 b_2 k^2 + 15 b_3 k^3}{15 k^2},$$

(6.19)
$$Y_2 b_4 = \frac{8\epsilon a_1 b_3 b_4 + 20 b_2 b_4 k + 20 a_8 b_3 b_4 k + 2\epsilon a_1^2 k + 2\epsilon g_2 k + 15 k^3}{5 b_3 k}.$$

From the integrability condition for b_2 we obtain

$$\begin{aligned}Y_2g_2 =&2a_1^2a_8 + \frac{5a_1^2b_2}{b_3} - \frac{b_3b_4}{2} + 2a_1g_1 + \frac{5b_2g_2}{b_3} - \frac{12\epsilon a_1^2b_3b_4}{5k^2} - \frac{5a_1b_2b_4}{k}\\ &- \frac{2a_1a_8b_3b_4}{k} + \frac{12\epsilon a_1^3}{5k} - \frac{b_3b_4g_1}{k} + \frac{12\epsilon a_1g_2}{5k} + \frac{23a_1k}{2} + \frac{15\epsilon b_2k^2}{4b_3}.\end{aligned}$$

The integrability of b_4 then yields the other derivative

$$Y_1g_2 = -2a_1g_2 - \frac{4a_1b_3^2b_4^2}{3k^2} + \frac{8a_1^2b_3b_4}{3k} + \frac{8b_3b_4g_2}{3k} - 5\epsilon b_3b_4k.$$

We can now calculate the integrability conditions for g_2 and a_1 which yields the following complicated expression.

$$\begin{aligned}&5a_1b_3b_4 - \frac{4\epsilon a_1^2b_3^2b_4^2}{3k^3} + \frac{8\epsilon a_1^3b_3b_4}{3k^2} + \frac{8\epsilon a_1b_3b_4g_2}{3k^2} - \frac{4\epsilon a_1^4}{3k}\\ &- \frac{8\epsilon a_1^2g_2}{3k} - \frac{4\epsilon g_2^2}{3k} + \frac{15a_1^2k}{2} + \frac{25\epsilon b_2b_4k}{4} + \frac{15g_2k}{2} + \frac{75\epsilon k^3}{4} = 0.\end{aligned} \tag{6.20}$$

By assuming that $b_4 \neq 0$ and differentiating (6.20) by Y_1 we arrive at

$$225k^3(4\epsilon a_1b_3b_4 - 4\epsilon a_1^2k - 4\epsilon g_2k + 3k^3) = 0,$$

so that

$$g_2 = \frac{3\epsilon k^3 - 4a_1^2k + 4a_1b_3b_4}{4k}. \tag{6.21}$$

From (6.20) we then get

$$b_2 = \frac{189k^3 + 100a_1b_3b_4}{-50b_4k}. \tag{6.22}$$

Plugging this value for b_2 into (6.18) gives $b_3k = 0$, a contradiction. Thus we may now assume that $b_1 = 0 = b_4$ and $b_5 = \epsilon$. Now the structure equations yield, with an additional function g_1,

$$Y_1b_3 = 5a_1b_3, \tag{6.23}$$

$$Y_1a_8 = g_1 - k/2 - a_1a_8, \tag{6.24}$$

$$Y_2a_1 = -g_1 - k/2 - a_1a_8, \tag{6.25}$$

$$Y_1k = 0, \tag{6.26}$$

$$Y_2b_2 = -4a_8b_2 - \epsilon b_3, \tag{6.27}$$

$$Y_2b_3 = -5a_8b_3 - b_2, \tag{6.28}$$

$$Y_2k = 2\epsilon a_1 + b_2b_6, \tag{6.29}$$

$$Y_1b_6 = -5a_1b_6. \tag{6.30}$$

The integrability of b_3 gives

$$Y_1 b_2 = 5b_3 k + 4a_1 b_2, \tag{6.31}$$

while the integrability of k gives

$$2Y_1 a_1 = -15\epsilon k^2 - 2a_1^2. \tag{6.32}$$

We can then calculate the integrability of b_2 to get $2\epsilon a_1 b_3 + b_2 k = 0$, or

$$b_2 = -\frac{2\epsilon a_1 b_3}{k}. \tag{6.33}$$

Substitution of this value for b_2 in (6.31) again gives $b_3 k = 0$, a contradiction. □

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