Computational Methods and Function Theory
Volume 9 (2009), No. 2, 633–651

Bohr Type Theorems for Monogenic Power Series

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(Communicated by Fred Brackx)

Abstract. The main goal of this paper is to generalize Bohr's phenomenon from complex one-dimensional analysis to the three-dimensional Euclidean space in the framework of quaternionic analysis.

Keywords. Spherical monogenics, homogeneous monogenic polynomials, Bohr's Theorem.

2000 MSC. Primary 30G35; Secondary 32A05.

1. Introduction

In 1914 Bohr discovered that there exists a radius $r \in (0,1)$ such that if a power series of a holomorphic function converges in the unit disk and its sum has a modulus less than 1, then for $|z| < r$ the sum of the absolute values of its terms is again less than 1. The significance of the theorem is that this radius does not depend on the function.

Theorem 1 (Bohr, 1914)**.** *Let f be a bounded analytic function in the open unit disk, with Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ convergent in the unit disk and with modulus less than* 1*. Then*

$$\sum_{n=0}^{\infty} |a_n||z|^n < 1 \qquad \textit{for } 0 \leq |z| < \frac{1}{3}.$$

This inequality is known as Bohr's inequality and the constant 1/3 cannot be improved.

Originally, this theorem was proved with 1/3 replaced by 1/6 but soon improved to the sharp result by Riesz, Schur and Wiener independently. In Bohr's paper [8]

Received October 28, 2007, in revised forms July 9, 2008, February 4, 2009, and April 21, 2009. Published online June 1, 2009.
The second author is sponsored by Foundation for Science and Technology (FCT) via the PhD/grant SFRH/BD/19174/2004.

ISSN 1617-9447/$ 2.50

his own proof was published as well as a proof by Wiener based on function theoretic methods. Later, Sidon gave a different proof [27], which was subsequently rediscovered by Tomić [29].

Recently, multi-dimensional analogues and other generalizations of Bohr's Theorem have been treated by several authors. We mention, among others, the contributions by Dineen and Timoney [14] (1991), Boas and Khavinson [7] (1997), Aizenberg [1, 2] (2000–2005), Aizenberg, Aytuna and Djakov [3] (2000), Paulsen, Popescu and Singh [25] (2002), Beneteau, Dahlner and Khavinson [6] (2004).

In several of these papers, the proof of Bohr's inequality or of Bohr-type inequalities is based on the orthogonality of the powers of the complex variable(s). Using the standard multi-index notation $\underline{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$ with $|\underline{\alpha}| = \alpha_1 + \cdots + \alpha_n$, $z := (z_1, \dots, z_n)$, $z_i \in \mathbb{C}$, $z^{\underline{\alpha}} := {z_1}^{\alpha_1} {z_2}^{\alpha_2} \cdots {z_n}^{\alpha_n}$, it is shown, for example, in [7] that if a power series $\sum_{\underline{\alpha}} c_{\underline{\alpha}} z^{\underline{\alpha}}$ has a modulus less than 1 in the unit polydisk $\{(z_1, \dots, z_n)\colon \max_{1 \le j \le n} |z_j| < 1\}$, then the sum of the moduli of the terms is less than 1 in the polydisk of radius $1/(3\sqrt{n})$.

Since this work is based on series expansions it is natural to ask the following questions: Is it possible to establish a similar result for monogenic functions? How large can the sum of the moduli of the terms of a convergent power series be? Taylor or Fourier expansions — which of these two different approaches should be considered?

While in the complex case the characterization of complex-valued holomorphic functions by their Taylor or Fourier series expansions is in principle the same, these two series expansions are essentially different in the quaternionic case. The reason is that the Taylor expansion with respect to the Fueter variables does not give us in general orthogonal summands (see [10, pp. 46–53] for a special approach). If we have in mind to adapt results and methods from complex analysis to the quaternionic case then it seems natural to work with the Fourier expansion of a monogenic function. It is not so simple as in the complex case to relate Taylor and Fourier coefficients. Therefore, we will call the final result a *Bohr type theorem.*

It should be also remarked that in some papers (see, e.g. [25]) the idea is to work with the Fourier coefficients of the boundary values of a holomorphic function. We prefer here to consider (analogously to the original formulation of Bohr's Theorem) only monogenic functions in the ball. It follows directly from the supposed boundedness of the monogenic functions that they are also square integrable in the ball and therefore we can work with Fourier series there. The existence of integrable boundary values needs additional assumptions.

In [19] a first Bohr type theorem for reduced-quaternion-valued monogenic functions in the unit ball $B_1(0)$ of the Euclidean space $\mathbb{R}^3$ was proved. Under the additional condition $f(0) = 0$ it is shown that for $r < 0.047$ the desired inequality is satisfied.

The main purpose of this paper is to check if this theorem can be extended to a larger class of monogenic functions with values in the reduced quaternions and such that $|f(x)| < 1$ in $B_1(0)$, without the restriction $f(0) = 0$.

Having in mind the analogy to one-dimensional complex function theory we want to know if the result can be proved for a ball in the Euclidean space and not just only for a polydisk. We remark that it is not the goal here to find a sharp estimate for the most general class of functions.

The proof of the main result needs a series of technical preparations. After some preliminaries in Section 2 these preparations are done in Section 3. The main result is proved in Section 4.

2. Preliminaries

Let $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard orthonormal basis of the Euclidean vector space $\mathbb{R}^4$. We introduce the following product according to the multiplication rules

$$\begin{aligned} \mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i &= -2\delta_{i,j}\mathbf{e}_0, && i,j = 1,2,3, \\ \mathbf{e}_0\mathbf{e}_i = \mathbf{e}_i\mathbf{e}_0 &= \mathbf{e}_i, && i = 0,1,2,3, \end{aligned}$$

where $\delta_{i,j}$ is the Kronecker symbol. This non-commutative product, together with the extra condition $\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_3$, generates the algebra of real quaternions denoted by $\mathbb{H}$. The real vector space $\mathbb{R}^4$ will be identified with $\mathbb{H}$ by identifying $a := (a_0, a_1, a_2, a_3) \in \mathbb{R}^4$ with the element

$$\mathbf{a} = a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \in \mathbb{H}.$$

Note that $\mathbf{e}_0$ is the multiplicative unit element of $\mathbb{H}$ and by identifying $\mathbf{e}_0$ with 1, it will be neglected in the following notation.

The real number

$$\mathbf{Sc}(\mathbf{a}) := a_0$$

is called the *scalar part* of $\mathbf{a}$ and

$$\mathbf{Vec}(\mathbf{a}) := a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$$

is the *vector part* of $\mathbf{a}$. Analogously to the complex case, the *conjugate* of $\mathbf{a}$ is the quaternion

$$\overline{\mathbf{a}} := a_0 - a_1\mathbf{e}_1 - a_2\mathbf{e}_2 - a_3\mathbf{e}_3.$$

The *norm* of $\mathbf{a}$ is given by

$$|\mathbf{a}| = \left({a_0}^2 + {a_1}^2 + {a_2}^2 + {a_3}^2\right)^{1/2}$$

and coincides with its corresponding Euclidean norm, as a vector in $\mathbb{R}^4$.

Considering the subset

$$\mathcal{A} := \mathrm{span}_{\mathbb{R}}\{1, \mathbf{e}_1, \mathbf{e}_2\}$$

of $\mathbb{H}$, the real vector space $\mathbb{R}^3$ will be identified with $\mathcal{A}$ by the identification of each element $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ with the reduced quaternion

$$\mathbf{x} = x_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \in \mathcal{A}.$$

We emphasize that $\mathcal{A}$ is only a real vector space but not a sub-algebra, of $\mathbb{H}$.

Let us consider now an open set $\Omega \subset \mathbb{R}^3$ with a piecewise smooth boundary. A quaternion-valued function is a mapping $\mathbf{f}\colon \Omega \to \mathbb{H}$. From now on we will restrict the latter to functions with values in $\mathcal{A}$, i.e. such that

$$\mathbf{f}(x) = [\mathbf{f}(x)]_0 + [\mathbf{f}(x)]_1\mathbf{e}_1 + [\mathbf{f}(x)]_2\mathbf{e}_2,$$

where the coordinate-functions $[\mathbf{f}]_i$, $i = 0, 1, 2$, are real-valued functions defined in Ω. Properties such as continuity, differentiability or integrability are ascribed coordinate-wise.

The operator

$$D = \partial_{x_0} + \mathbf{e}_1\partial_{x_1} + \mathbf{e}_2\partial_{x_2} \tag{1}$$

is called *generalized Cauchy-Riemann operator*. In the same way, we define the *conjugate generalized Cauchy-Riemann operator* by

$$\overline{D} = \partial_{x_0} - \mathbf{e}_1\partial_{x_1} - \mathbf{e}_2\partial_{x_2}. \tag{2}$$

A continuously real-differentiable $\mathcal{A}$-valued function $\mathbf{f}$ is called *monogenic* if

$$D\mathbf{f} = 0 \qquad \text{in } \Omega.$$

It is not necessary to distinguish between left and right monogenic functions in the case of $\mathcal{A}$-valued functions because $D\mathbf{f} = 0$ implies $\mathbf{f}D = 0$ and vice versa.

The generalized Cauchy-Riemann operator (1) and its conjugate (2) factorize the Laplace operator in $\mathbb{R}^3$. In fact,

$$\Delta_3\mathbf{f} = D\overline{D}\mathbf{f} = \overline{D}D\mathbf{f},$$

whenever $\mathbf{f} \in C^2$, which implies that any monogenic function is also a harmonic function.

Analogous to the complex one-dimensional case,

$$\frac{1}{2}\overline{D}$$

defines a derivative of monogenic functions. This was shown in [18], where

$$\frac{1}{2}\overline{D}\mathbf{f}$$

was called *hypercomplex derivative* of $\mathbf{f}$. Moreover, an $\mathcal{A}$-valued monogenic function with an identically vanishing hypercomplex derivative is called *hyperholomorphic constant* (see again [18]). It is immediately clear that such function depends only on the variables x_1 and x_2.

Additionally, we introduce the following notation: $B := B_1(0)$ is the unit ball in $\mathbb{R}^3$ centered at the origin, $S = \partial B$ its boundary and $d\sigma$ (resp. dV) the Lebesgue measure on S (resp. B). We will denote by $L_2(S;\mathcal{A};\mathbb{R})$ (resp. $L_2(B;\mathcal{A};\mathbb{R})$) the $\mathbb{R}$-linear Hilbert space of square integrable functions on S (resp. B) with values in $\mathcal{A}$. In the case $L_2(S;\mathbb{R};\mathbb{R})$ (resp. $L_2(B;\mathbb{R};\mathbb{R})$) we abbreviate by $L_2(S)$ (resp. $L_2(B)$). For any $\mathbf{f},\mathbf{g} \in L_2(S;\mathcal{A};\mathbb{R})$ (resp. $L_2(B;\mathcal{A};\mathbb{R})$) the real-valued *inner product* is given by

$$\langle \mathbf{f},\mathbf{g}\rangle_{L_2(S;\mathcal{A};\mathbb{R})} = \int_S \mathbf{Sc}(\overline{\mathbf{f}}\mathbf{g})\, d\sigma, \tag{3}$$

and

$$\langle \mathbf{f},\mathbf{g}\rangle_{L_2(B;\mathcal{A};\mathbb{R})} = \int_B \mathbf{Sc}(\overline{\mathbf{f}}\mathbf{g})\, dV \tag{4}$$

respectively. Each homogeneous harmonic polynomial P_n of order n can be written in spherical coordinates as

$$P_n(x) = r^n P_n(\omega), \qquad \omega \in S. \tag{5}$$

Its restriction, $P_n(\omega)$, to the boundary of the unit ball is called *spherical harmonic of degree n*. From (5), it is clear that a homogeneous polynomial is determined by its restriction to S. Denoting by $\mathcal{H}_n(S)$ the space of real-valued spherical harmonics of degree n on S, it is well-known (see [5, 24]) that

$$\dim \mathcal{H}_n(S) = 2n+1.$$

It is also known (see again [5] and [24]) that if $n \neq m$, the spaces $\mathcal{H}_n(S)$ and $\mathcal{H}_m(S)$ are orthogonal in $L_2(S)$.

Now let $\mathbf{H}_n$ be an $\mathcal{A}$-valued homogeneous monogenic polynomial of degree n. As with the spherical harmonics, the restriction of $\mathbf{H}_n$ to the boundary of the unit ball is called spherical monogenic of degree n. Following the notation of [13], we denote by $\mathcal{M}^+(\Omega;\mathcal{A};n)$ the space of $\mathcal{A}$-valued homogeneous monogenic polynomials of degree n in $\Omega \subset \mathbb{R}^3$. In [21], it is shown that the space $\mathcal{M}^+(\Omega;\mathcal{A};n)$ has dimension $2n+3$. Later, this result was generalized for arbitrary higher dimensions by Delanghe in [13]. We denote further by $\mathcal{M}^+(\Omega;\mathcal{A})$ the space of square integrable $\mathcal{A}$-valued monogenic functions defined in $\Omega \subset \mathbb{R}^3$.

3. Homogeneous monogenic polynomials

Based on the Fueter variables $\mathbf{z}_1 = x_1 - \mathbf{e}_1 x_0$ and $\mathbf{z}_2 = x_2 - \mathbf{e}_2 x_0$, several systems of homogeneous monogenic polynomials are constructed and used for different purposes (see, e.g. [9, 12, 15, 16, 23, 28]). Following [23], being $\underline{\gamma} = (\gamma_1,\gamma_2)$ a multi-index such that $\gamma_1+\gamma_2 = n$, the generalized powers (or also Fueter polynomials) of degree n are defined by

$$\begin{aligned}\mathbf{z}_1{}^{\gamma_1} \times \mathbf{z}_2{}^{\gamma_2} &= \underbrace{\mathbf{z}_1 \times \mathbf{z}_1 \times \cdots \times \mathbf{z}_1}_{\gamma_1 \text{times}} \times \underbrace{\mathbf{z}_2 \times \mathbf{z}_2 \times \cdots \times \mathbf{z}_2}_{\gamma_2 \text{times}} \\ &= \frac{1}{n!} \sum_{\pi(i_1,\ldots,i_n)} \mathbf{z}_{i_1} \cdots \mathbf{z}_{i_n},\end{aligned}$$

where the sum is taken over all permutations $\pi(i_1,\ldots,i_n)$ of $(\underbrace{1,\cdots,1}_{\gamma_1},\underbrace{2,\cdots,2}_{\gamma_2})$.

The general form of the Taylor series of a monogenic function $\mathbf{f}\colon \Omega \subset \mathbb{R}^3 \to \mathbb{H}$ in the neighborhood of the origin (see, e.g. [9, 23]) is given by

$$\mathbf{f} = \sum_{n=0}^{\infty} \sum_{|\underline{\gamma}|=n} (\mathbf{z}_1^{\gamma_1} \times \mathbf{z}_2^{\gamma_2})\, \mathbf{c}_{\underline{\gamma}}, \tag{6}$$

where

$$\mathbf{c}_{\underline{\gamma}} = \frac{1}{\gamma_1!\gamma_2!} \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} \mathbf{f}(x)\Big|_{x=0} \in \mathbb{H}$$

are the Taylor coefficients.

In the special case of a monogenic function $\mathbf{f}$ with values in the reduced quaternions it is clear that its associated Taylor coefficients are also $\mathcal{A}$-valued.

In [10, Ch. 3] and [11], a special $\mathbb{R}$-linear complete orthonormal system of $\mathcal{A}$-valued homogeneous monogenic polynomials in the unit ball of $\mathbb{R}^3$ is explicitly constructed. The main idea of this construction is the already mentioned factorization of the Laplace operator by $D\overline{D}$. We take a system of real-valued homogeneous harmonic polynomials and apply the operator $\overline{D}$ to each function of the system to get a system of $\mathcal{A}$-valued homogeneous monogenic polynomials. For a detailed description we introduce spherical coordinates,

$$x_0 = r\cos\theta, \quad x_1 = r\sin\theta\cos\varphi, \quad x_2 = r\sin\theta\sin\varphi,$$

where $0 < r < \infty$, $0 < \theta \leq \pi$, $0 < \varphi \leq 2\pi$. Each point $x = (x_0, x_1, x_2) \in \mathbb{R}^3 \backslash \{0\}$ admits a unique representation $x = r\omega$, where for each $i = 0, 1, 2$ $\omega_i = x_i/r$ and $|\omega| = 1$. We consider the following set of homogeneous harmonic polynomials,

$$\left\{ r^{n+1} U_{n+1}^0, r^{n+1} U_{n+1}^m, r^{n+1} V_{n+1}^m,\, m = 1, \ldots, n+1 \right\}_{n\in\mathbb{N}_0} \tag{7}$$

formed by the extensions in the ball of the spherical harmonics (considered e.g. in [26]),

$$\begin{aligned} U_{n+1}^0(\theta,\varphi) &= P_{n+1}(\cos\theta) \\ U_{n+1}^m(\theta,\varphi) &= P_{n+1}^m(\cos\theta)\cos(m\varphi) \\ V_{n+1}^m(\theta,\varphi) &= P_{n+1}^m(\cos\theta)\sin(m\varphi), \qquad m = 1, \ldots, n+1. \end{aligned} \tag{8}$$

Here, P_{n+1} stands for the Legendre polynomial of degree $n+1$, given by

$$P_{n+1}(t) = \sum_{k=0}^{[(n+1)/2]} a_{n+1,k} t^{n+1-2k},$$
$$P_0(t) = 1,$$

for $t \in (-1,1)$ with

$$a_{n+1,k} = (-1)^k \frac{1}{2^{n+1}} \binom{n+1}{k} \binom{2n+2-2k}{n+1}. \tag{9}$$

The upper bound $[s]$ denotes, as usual, the integer part of $s \in \mathbb{R}$. We stipulate this sum to be zero whenever the upper index is less than the lower one.

The functions P_{n+1}^m defined as

$$P_{n+1}^m(t) := (1-t^2)^{m/2} \frac{d^m}{dt^m} P_{n+1}(t), \qquad -1 \le t \le 1, \tag{10}$$

where $m = 1, \dots, n+1$, are called the associated Legendre functions, associated to the Legendre polynomial P_{n+1} of degree $n+1$.

We remark that whenever $m = 0$, the corresponding associated Legendre function $P_{n+1}^0(t)$ coincides with the Legendre polynomial $P_{n+1}(t)$.

For the Legendre polynomials and the associated Legendre functions several sharp estimates can be found in the literature. From (e.g. [22, p. 179]) we get that for each $t \in [-1,1]$ and $m = 1, \dots, n$ the inequality

$$|P_{n+1}^m(t)| \le \frac{1}{\sqrt{2}} \sqrt{\frac{(n+m)!}{(n-m)!}} \tag{11}$$

holds. The Legendre polynomials, together with the associated Legendre functions, satisfy several recurrence formulae. We now present the formulae which will be used in the next sections. Both the Legendre polynomials and the associated Legendre functions satisfy the recurrence formula

(12)
$$(1-t^2)(P_{n+1}^m(t))' = (n+m+1)P_n^m(t) - (n+1)tP_{n+1}^m(t), \qquad \text{for } m = 0, \dots, n+1.$$

These functions are mutually orthogonal in $L_2([-1,1])$,

$$\int_{-1}^{1} P_{n+1}^m(t) P_{k+1}^m(t)\, dt = 0, \qquad n \neq k,$$

and their norms are given by

$$\int_{-1}^{1} |P_{n+1}^m(t)|^2\, dt = \frac{2}{2n+3} \frac{(n+1+m)!}{(n+1-m)!}, \qquad m = 0, \dots, n+1. \tag{13}$$

A more detailed study of the Legendre polynomials and their associated Legendre functions can be found, for example, in [4, 26].

As described, we apply for each $n \in \mathbb{N}_0$, the operator $(1/2)\overline{D}$ to the homogeneous harmonic polynomials in (7). We obtain then the following set of homogeneous monogenic polynomials

$$\{\mathbf{X}_n^{0,\dagger},\, \mathbf{X}_n^{m,\dagger},\, \mathbf{Y}_n^{m,\dagger} \colon m = 1,\dots,n+1\}, \tag{14}$$

with the notation

$$\mathbf{X}_n^{0,\dagger} := r^n \mathbf{X}_n^0, \qquad \mathbf{X}_n^{m,\dagger} := r^n \mathbf{X}_n^m, \qquad \mathbf{Y}_n^{m,\dagger} := r^n \mathbf{Y}_n^m.$$

It is interesting to observe that the polynomials $\mathbf{X}_n^{n+1,\dagger}$ and $\mathbf{Y}_n^{n+1,\dagger}$ are hyperholomorphic constants (see [10, p. 73]).

In the last section we will need pointwise estimates for the polynomials in the system (14). The idea is to rewrite these polynomials as Taylor sums with respect to the Fueter polynomials. Because pointwise estimates for the polynomials are available we have only to estimate the Taylor coefficients of the polynomials from (14).

Proposition 2. *Let $\underline{\gamma} = (\gamma_1, \gamma_2)$ be a multi-index with $|\underline{\gamma}| = n$. The Taylor coefficients $\mathbf{a}_{\underline{\gamma}}^0$, $\mathbf{a}_{\underline{\gamma}}^m$ and $\mathbf{b}_{\underline{\gamma}}^m$ associated to the homogeneous monogenic polynomials $\mathbf{X}_n^{0,\dagger}$, $\mathbf{X}_n^{m,\dagger}$ and $\mathbf{Y}_n^{m,\dagger}$ satisfy the inequalities*

$$|\mathbf{a}_{\underline{\gamma}}^0| \leq \frac{1}{\underline{\gamma}!}\frac{(n+1)!}{2}\sqrt{n+1},$$

$$|\mathbf{a}_{\underline{\gamma}}^m| \leq \frac{1}{\underline{\gamma}!}\frac{(n+1)!}{2\sqrt{2}}\sqrt{(n+1)\frac{(n+1+m)!}{(n+1-m)!}},$$

$$|\mathbf{b}_{\underline{\gamma}}^m| \leq \frac{1}{\underline{\gamma}!}\frac{(n+1)!}{2\sqrt{2}}\sqrt{(n+1)\frac{(n+1+m)!}{(n+1-m)!}}$$

for $m = 1, \dots, n+1$.

Proof. From [17, pp. 87–88] every monogenic function $\mathbf{f}$ with smooth boundary values can be represented by the Cauchy integral formula as

$$\mathbf{f}(x) = \frac{1}{\sigma_{k-1}} \int_\Gamma \frac{\overline{\mathbf{x}-\mathbf{y}}}{|\mathbf{x}-\mathbf{y}|^k} \mathbf{n}(y)\mathbf{f}(y)\, d\Gamma_y, \tag{15}$$

for $x \in \Omega$, where Γ is the smooth boundary of a bounded open subset Ω of $\mathbb{R}^k$ containing the point x and $\mathbf{n}$ is the outward pointing normal unit vector at $y \in \Gamma$. Here σ_{k-1} is the surface area of the unit sphere in $\mathbb{R}^k$. For simplicity we present the proof only for the homogeneous monogenic polynomials $\mathbf{X}_n^{m,\dagger}$, $m = 1, \dots, n+1$. Let us denote by $\mathbf{a}_\gamma^m$ the Taylor coefficients associated to $\mathbf{X}_n^{m,\dagger}$. Applying the Cauchy integral formula (15) to these polynomials in the ball B

and taking partial derivatives with respect to x_1 and x_2, we get

$$\begin{aligned}
\mathbf{a}_\gamma^m &= \frac{1}{\gamma!}\,\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\mathbf{X}_n^{m,\dagger}(x)\big|_{x=0}\\
&= \frac{1}{\gamma!}\frac{1}{4\pi}\int_S \partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\frac{\overline{\mathbf{x}-\mathbf{y}}}{|\mathbf{x}-\mathbf{y}|^3}\bigg|_{x=0}\mathbf{n}(y)\mathbf{X}_n^m(y)\,dS_y.
\end{aligned}$$

Taking the modulus and applying Schwarz inequality we finally obtain

$$|\mathbf{a}_{\underline{\gamma}}^m| \le \frac{1}{2\sqrt{\pi}}\frac{1}{\underline{\gamma}!}(n+1)!\sqrt{\frac{\pi}{2}(n+1)\frac{(n+1+m)!}{(n+1-m)!}}.$$

The previous inequality is based on [10, p. 91] where for $m = 1, \ldots, n+1$ the equalities

$$\|\mathbf{X}_n^m\|_{L_2(S;\mathcal{A};\mathbb{R})} = \|\mathbf{Y}_n^m\|_{L_2(S;\mathcal{A};\mathbb{R})} = \sqrt{\frac{\pi}{2}(n+1)\frac{(n+1+m)!}{(n+1-m)!}}, \tag{16}$$

are obtained and on the paper [20, p. 38] where the estimate

$$\left|\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\frac{\overline{\mathbf{x}-\mathbf{y}}}{|\mathbf{x}-\mathbf{y}|^3}\bigg|_{x=0}\right| \le \frac{(n+1)!}{|\mathbf{y}|^{n+2}}$$

is obtained. The case $m = 0$ is proved analogously and we have to take into account the equality (see again [10, p. 91]),

$$\|\mathbf{X}_n^0\|_{L_2(S;\mathcal{A};\mathbb{R})} = \sqrt{\pi(n+1)}. \tag{17}$$ ■

The estimates of the Taylor coefficients obtained enable us now to establish pointwise estimates of our basis polynomials (14).

Proposition 3. *For $n \in \mathbb{N}_0$ the homogeneous monogenic polynomials satisfy the inequalities*

$$\begin{aligned}
|\mathbf{X}_n^{0,\dagger}(x)| &\le \frac{1}{2}(n+1)(2r)^n\sqrt{(n+1)},\\
|\mathbf{X}_n^{m,\dagger}(x)| &\le \frac{1}{2}(n+1)(2r)^n\sqrt{\frac{(n+1)}{2}\frac{(n+1+m)!}{(n+1-m)!}},\\
|\mathbf{Y}_n^{m,\dagger}(x)| &\le \frac{1}{2}(n+1)(2r)^n\sqrt{\frac{(n+1)}{2}\frac{(n+1+m)!}{(n+1-m)!}}.
\end{aligned}$$

for $m = 1, \ldots, n+1$.

Proof. Again, we prove only the case of the polynomials $\mathbf{X}_n^{m,\dagger}$ ($m = 1, \ldots, n+1$), the proof for $\mathbf{X}_n^{0,\dagger}$ and $\mathbf{Y}_n^{m,\dagger}$ is similar. We write these polynomials as a Taylor expansion (6)

$$\mathbf{X}_n^{m,\dagger}(x) = \sum_{|\underline{\gamma}|=n}\left(\mathbf{z}_1{}^{\gamma_1}\times\mathbf{z}_2{}^{\gamma_2}\right)\mathbf{a}_{\underline{\gamma}}^m.$$

Taking now the modulus of $\mathbf{X}_n^{m,\dagger}$ we get

$$|\mathbf{X}_n^{m,\dagger}(x)| \leq \frac{1}{2\sqrt{\pi}}(n+1)!\sqrt{\frac{\pi}{2}(n+1)\frac{(n+1+m)!}{(n+1-m)!}}\frac{(2r)^n}{n!}.$$

The last inequality is based on the previous corollary and the estimate (see [23])

$$|\mathbf{z}_1{}^{\gamma_1} \times \mathbf{z}_2{}^{\gamma_2}| \leq r^n,$$

which holds for every multi-index $\underline{\gamma} = (\gamma_1, \gamma_2)$ such that $|\underline{\gamma}| = n$. ∎

At this point it is important to remark that in the complex case, the proof of Bohr's inequality is also based on the orthogonality of the real parts of the powers of the complex variable(s). In this line of reasoning, we point out an important (analogous) property of the described set (14). For simplicity we will restrict the set to the sphere.

Theorem 4. *For each fixed $n \in \mathbb{N}_0$, the spherical harmonics*

$$\left\{\mathbf{Sc}(\mathbf{X}_n^0), \mathbf{Sc}(\mathbf{X}_n^m), \mathbf{Sc}(\mathbf{Y}_n^m) \colon m = 1, \ldots, n\right\} \tag{18}$$

are orthogonal to each other with respect to the inner product (3).

Proof. According to the results from [10, pp. 63–64] the real parts of the spherical monogenics $\mathbf{X}_n^0$, $\mathbf{X}_n^m$ and $\mathbf{Y}_n^m$ $(m = 1, \ldots, n)$ are given by

$$\begin{aligned} \mathbf{Sc}(\mathbf{X}_n^0) &= A^{0,n}(\theta), \\ \mathbf{Sc}(\mathbf{X}_n^m) &= A^{m,n}(\theta)\cos(m\varphi), \\ \mathbf{Sc}(\mathbf{Y}_n^m) &= A^{m,n}(\theta)\sin(m\varphi) \end{aligned}$$

for $m = 1, \ldots, n$ with

$$A^{l,n}(\theta) = \frac{1}{2}\left(\sin^2\theta\frac{d}{dt}\left[P_{n+1}^l(t)\right]_{t=\cos\theta} + (n+1)\cos\theta P_{n+1}^l(\cos\theta)\right), \tag{19}$$

for $l = 0, \ldots, n$.

The proof uses the orthogonality of $\cos(m_1\varphi)$ and $\sin(m_2\varphi)$, $m_1 = 0, \ldots, n$, $m_2 = 1, \ldots, n$. ∎

In this sense, the previous result reflects one of the most important properties of the chosen system of homogeneous monogenic polynomials and it shows an immediate relationship with the classical complex function theory in the plane, where the real parts of the complex variables z^n are also orthogonal to each other.

For future use in this paper we need also estimates of L_2-norms for the real parts of the spherical monogenics $\mathbf{X}_n^0$, $\mathbf{X}_n^m$ and $\mathbf{Y}_n^m$, $m = 1, \ldots, n$.

Proposition 5. *For $n \in \mathbb{N}_0$, the L_2-norms of the spherical harmonics $\mathbf{Sc}(\mathbf{X}_n^0)$, $\mathbf{Sc}(\mathbf{X}_n^m)$ and $\mathbf{Sc}(\mathbf{Y}_n^m)$ are given by*

$$\|\mathbf{Sc}(\mathbf{X}_n^0)\|_{L_2(S)} = (n+1)\sqrt{\frac{\pi}{2n+1}},$$

$$\|\mathbf{Sc}(\mathbf{X}_n^m)\|_{L_2(S)} = \|\mathbf{Sc}(\mathbf{Y}_n^m)\|_{L_2(S)} = (n+1+m)\sqrt{\frac{\pi}{2}\frac{1}{(2n+1)}\frac{(n+m)!}{(n-m)!}},$$

with $m = 1, \dots, n$.

Proof. According again to the results from [10, pp. 63–64] it follows

$$\mathbf{Sc}(\mathbf{X}_n^l) = A^{l,n}(\theta)\cos(l\varphi),$$

where $A^{l,n}(\theta)$ is given by (19).

First, we prove the equality for $\mathbf{Sc}(\mathbf{X}_n^0)$. By definition of the inner product in $L_2(S)$ we get

$$\begin{aligned}
&\|\mathbf{Sc}(\mathbf{X}_n^0)\|^2_{L_2(S)} \\
&\quad = \int_0^\pi (A^{0,n})^2 \sin\theta d\theta \int_0^{2\pi} d\varphi \\
&\quad = \frac{\pi}{2}\int_0^\pi \Bigg[\sin^4\theta\left(\frac{d}{dt}[P_{n+1}(t)]_{t=\cos\theta}\right)^2 \\
&\qquad\qquad + (n+1)^2\cos^2\theta(P_{n+1}(\cos\theta))^2 \\
&\qquad\qquad + 2\sin^2\theta(n+1)\cos\theta\frac{d}{dt}[P_{n+1}(t)]_{t=\cos\theta}P_{n+1}(\cos\theta)\Bigg]\sin\theta\, d\theta.
\end{aligned}$$

Making the change of variable $t = \cos\theta$ and using the recurrence formula (12), the last expression becomes

$$\begin{aligned}
\|\mathbf{Sc}(\mathbf{X}_n^0)\|^2_{L_2(S)} &= \frac{\pi}{2}\int_{-1}^1 (1-t^2)^2(P'_{n+1}(t))^2\, dt - (n+1)^2\int_{-1}^1 t^2(P_{n+1}(t))^2\, dt \\
&\quad + 2(n+1)^2\int_{-1}^1 tP_n(t)P_{n+1}(t)\, dt \\
&= \frac{\pi}{2}(n+1)^2\int_{-1}^1 (P_n(t))^2\, dt.
\end{aligned}$$

Due to (13) we get

$$\|\mathbf{Sc}(\mathbf{X}_n^0)\|^2_{L_2(S)} = \frac{\pi(n+1)^2}{2n+1}.$$

Now, we prove the equality for the spherical harmonics $\mathbf{Sc}(\mathbf{X}_n^m)$ $(m = 1, \dots, n)$. For $\mathbf{Sc}(\mathbf{Y}_n^m)$ the proof is similar. By definition of the inner product in $L_2(S)$ we have

$$\begin{aligned}
&\|\mathbf{Sc}(\mathbf{X}_n^m)\|_{L_2(S)}^2 \\
&\quad = \int_0^\pi (A^{m,n})^2 \sin\theta\, d\theta \int_0^{2\pi} (\cos(m\varphi))^2\, d\varphi \\
&\quad = \frac{\pi}{4}\int_0^\pi \Bigg[\sin^4\theta \left(\frac{d}{dt}[P_{n+1}^m(t)]_{t=\cos\theta}\right)^2 \\
&\qquad\qquad + (n+1)^2 \cos^2\theta (P_{n+1}^m(\cos\theta))^2 \\
&\qquad\qquad + 2\sin^2\theta (n+1)\cos\theta \frac{d}{dt}[P_{n+1}^m(t)]_{t=\cos\theta} P_{n+1}^m(\cos\theta)\Bigg] \sin\theta\, d\theta.
\end{aligned}$$

Making the change of variable $t = \cos\theta$ and using the recurrence formula (12), the last expression becomes

$$\begin{aligned}
&\|\mathbf{Sc}(\mathbf{X}_n^m)\|_{L_2(S)}^2 \\
&\quad = \frac{\pi}{4}\int_{-1}^1 (1-t^2)^2 \left((P_{n+1}^m(t))'\right)^2 dt - (n+1)^2 \int_{-1}^1 t^2 (P_{n+1}^m(t))^2\, dt \\
&\quad\quad + 2(n+1)(n+1+m)\int_{-1}^1 t P_n^m(t) P_{n+1}^m(t)\, dt \\
&\quad = \frac{\pi}{2}(n+1+m)^2 \int_{-1}^1 (P_n^m(t))^2\, dt.
\end{aligned}$$

Due to (13) we finally get

$$\|\mathbf{Sc}(\mathbf{X}_n^m)\|_{L_2(S)}^2 = \frac{\pi}{2}\frac{(n+1+m)^2}{2n+1}\frac{(n+m)!}{(n-m)!}, \qquad m = 1,\dots,n.$$

∎

4. Bohr's Theorem

In this section, we generalize Bohr's inequality for $\mathcal{A}$-valued monogenic functions defined in the unit ball of the Euclidean space $\mathbb{R}^3$.

From now on we represent by $\mathbf{X}_n^{0,\dagger,*}, \mathbf{X}_n^{m,\dagger,*}, \mathbf{Y}_n^{m,\dagger,*}$, $m = 1,\dots,n+1$, the normalized functions $\mathbf{X}_n^{0,\dagger}, \mathbf{X}_n^{m,\dagger}, \mathbf{Y}_n^{m,\dagger}$ in $L_2(B;\mathcal{A};\mathbb{R})$. The following theorem is from [10, 11].

Theorem 6. *For each n, the set of $2n+3$ homogeneous monogenic polynomials*

$$\left\{\mathbf{X}_n^{0,\dagger,*}, \mathbf{X}_n^{m,\dagger,*}, \mathbf{Y}_n^{m,\dagger,*} : m = 1,\dots,n+1\right\}$$

forms an orthonormal basis in the subspace $\mathcal{M}^+(B;\mathcal{A};n)$ with respect to the real-valued inner product (4). Consequently,

$$\left\{\mathbf{X}_n^{0,\dagger,*}, \mathbf{X}_n^{m,\dagger,*}, \mathbf{Y}_n^{m,\dagger,*} : m = 1,\dots,n+1,\ n = 0,1,\dots\right\}$$

is an orthonormal basis in $\mathcal{M}^+(B;\mathcal{A})$.

The previous theorem makes it possible to define the Fourier expansion of a square integrable $\mathcal{A}$-valued monogenic function. Moreover, according to the fact that the polynomials $\mathbf{X}_n^{n+1,\dagger,*}$ and $\mathbf{Y}_n^{n+1,\dagger,*}$ are hyperholomorphic constants it becomes clear that each monogenic function can be decomposed in an orthogonal sum of a monogenic "main part" ($\mathbf{g}$) of the function and a hyperholomorphic constant ($\mathbf{h}$). More precisely, the following is true:

Lemma 7. *A function* $\mathbf{f} \in \mathcal{M}^+(B;\mathcal{A})$ *can be decomposed into*

$$\mathbf{f} = \mathbf{f}(0) + \mathbf{g} + \mathbf{h}, \tag{20}$$

where the functions $\mathbf{g}$ *and* $\mathbf{h}$ *have the Fourier series*

$$\mathbf{g}(x) = \sum_{n=1}^{\infty}\left(\mathbf{X}_n^{0,\dagger,*}(x)\alpha_n^0 + \sum_{m=1}^{n}\left[\mathbf{X}_n^{m,\dagger,*}(x)\alpha_n^m + \mathbf{Y}_n^{m,\dagger,*}(x)\beta_n^m\right]\right)$$
$$\mathbf{h}(x) = \sum_{n=1}^{\infty}\left[\mathbf{X}_n^{n+1,\dagger,*}(x)\alpha_n^{n+1} + \mathbf{Y}_n^{n+1,\dagger,*}(x)\beta_n^{n+1}\right].$$

The associated Fourier coefficients $\alpha_n^0, \alpha_n^m, \beta_n^m$ $(m = 1, \ldots, n+1)$ *are real-valued.*

Remark 8. Obviously, $\mathbf{h}$ is a hyperholomorphic constant. We still have to show that the set $\{\mathbf{X}_n^{n+1,\dagger,*}, \mathbf{Y}_n^{n+1,\dagger,*}\}_{n=1}^{\infty}$ is an orthonormal basis of the subspace of hyperholomorphic constants with $\mathbf{h}(0) = 0$ (non-trivial constants) and no further hyperholomorphic constants are "hidden" in $\mathbf{g}$. Because of the orthogonal decomposition

$$\mathcal{M}^+(B;\mathcal{A}) = \oplus\sum_{n=0}^{\infty}\mathcal{M}^+(B;\mathcal{A};n)$$

it is enough to answer the question in each subspace $\mathcal{M}^+(B;\mathcal{A};n)$, $n > 0$, separately. Suppose that there is another hyperholomorphic constant in the subspace $\mathcal{M}^+(B;\mathcal{A};n)$ different from linear combinations of $\mathbf{X}_n^{n+1,\dagger,*}$ and $\mathbf{Y}_n^{n+1,\dagger,*}$. Then this hyperholomorphic constant can be represented by linear combinations of $\mathbf{X}_n^{0,\dagger,*}, \mathbf{X}_n^{m,\dagger,*}$ and $\mathbf{Y}_n^{m,\dagger,*}$, where $n > 0$ and $m = 1, \ldots, n$. It is easy to prove that an $\mathcal{A}$-valued non-trivial hyperholomorphic constant can have values only in $\mathrm{span}_{\mathbb{R}}\{\mathbf{e_1}, \mathbf{e_2}\}$. This means that the scalar parts of the above mentioned linear combination of $\mathbf{X}_n^{0,\dagger,*}, \mathbf{X}_n^{m,\dagger,*}$ and $\mathbf{Y}_n^{m,\dagger,*}$ are linearly dependent. This contradicts the orthogonality of these scalar parts which was proved in Theorem 4.

After the above extensive preparations, we are now ready to study Bohr's phenomena using the previous Fourier expansion. For simplicity we begin with the simple case of functions such that $\mathbf{f}(0) = \mathbf{0}$. This first version of a quaternionic Bohr type inequality was considered in [19] and it is given in the next theorem.

Theorem 9. *Let* $\mathbf{f}$ *be a square integrable* $\mathcal{A}$*-valued monogenic function with* $\mathbf{f}(0) = \mathbf{0}$ *and* $|\mathbf{f}(x)| < 1$ *in* B *and let*

$$\sum_{n=1}^{\infty} \left[\mathbf{X}_n^{0,\dagger,*} \alpha_n^0 + \sum_{m=1}^{n+1} \left(\mathbf{X}_n^{m,\dagger,*} \alpha_n^m + \mathbf{Y}_n^{m,\dagger,*} \beta_n^m \right) \right]$$

be its Fourier expansion. Then

$$\sum_{n=1}^{\infty} \left[|\mathbf{X}_n^{0,\dagger,*}||\alpha_n^0| + \sum_{m=1}^{n+1} \left(|\mathbf{X}_n^{m,\dagger,*}||\alpha_n^m| + |\mathbf{Y}_n^{m,\dagger,*}||\beta_n^m| \right) \right] < 1$$

holds in the ball $\{x \colon |x| = r < 0.047\}$.

As a consequence of the previous theorem, we obtain the weaker result.

Corollary 10. *Let* $\mathbf{f}$ *be a square integrable* $\mathcal{A}$*-valued monogenic function with* $\mathbf{f}(0) = \mathbf{0}$ *and* $|\mathbf{f}(x)| < 1$ *in* B *and let*

$$\sum_{n=1}^{\infty} \left[\mathbf{X}_n^{0,\dagger,*} \alpha_n^0 + \sum_{m=1}^{n+1} \left(\mathbf{X}_n^{m,\dagger,*} \alpha_n^m + \mathbf{Y}_n^{m,\dagger,*} \beta_n^m \right) \right]$$

be its Fourier expansion. Then

$$\sum_{n=1}^{\infty} \left| \mathbf{X}_n^{0,\dagger,*} \alpha_n^0 + \sum_{m=1}^{n+1} \left(\mathbf{X}_n^{m,\dagger,*} \alpha_n^m + \mathbf{Y}_n^{m,\dagger,*} \beta_n^m \right) \right| < 1$$

holds in the ball $\{x \colon |x| = r < 0.047\}$.

This last result is very well adapted to the complex situation. The absolute value is taken from all summands of the same degree n. In the complex case this is also an important first step. All the functions considered with $f(0) = 0$ are orthogonal to the constants. This is used later on to estimate all Fourier coefficients of a general holomorphic function with $|f(z)| \leq 1$ in terms of the first Fourier coefficient (see, e.g. [25]).

However, in the quaternionic case the set of "constants" is much bigger. If we understand constants as monogenic functions which have an identically vanishing hypercomplex derivative, then it is immediately clear that the constant function and all monogenic functions which depend only on x_1 and x_2 are the hyperholomorphic constants. Moreover, if we, as in this paper, consider only $\mathcal{A}$-valued functions then a non-trivial hyperholomorphic constant can have values only in $\mathrm{span}_{\mathbb{R}}\{\mathbf{e_1}, \mathbf{e_2}\}$. With these observations it seems to be natural to start by extending the Bohr type theorem to all monogenic functions with $|\mathbf{f}(x)| < 1$ in B which are orthogonal to the subspace of the non-trivial hyperholomorphic constants in $L_2(B; \mathcal{A}; \mathbb{R})$. This approach is also supported by the fact that in Lemma 7 it is shown that each monogenic function can be decomposed in an orthogonal sum of a monogenic "main part" of the function and a hyperholomorphic constant.

The Fourier representation in the hypothesis of the Theorem 11 describes the general form of these parts. The non-trivial hyperholomorphic constants in the decomposition do not influence the real part of the function at the origin because their image belongs to $\mathrm{span}_{\mathbb{R}}\{\mathbf{e_1}, \mathbf{e_2}\}$.

Theorem 11. *Let* $\mathbf{f}$ *be an* $\mathcal{A}$*-valued monogenic function such that* $\mathbf{f}(x) - \mathbf{f}(0)$ *is orthogonal to the hyperholomorphic constants with respect to the inner product* (4) *with* $|\mathbf{f}(x)| < 1$ *in* B *and let*

$$\sum_{n=0}^{\infty} \left[\mathbf{X}_n^{0,\dagger,*} \alpha_n^0 + \sum_{m=1}^{n} \left(\mathbf{X}_n^{m,\dagger,*} \alpha_n^m + \mathbf{Y}_n^{m,\dagger,*} \beta_n^m \right) \right]$$

be its Fourier expansion. Then

$$\sum_{n=0}^{\infty} \left[|\mathbf{X}_n^{0,\dagger,*}||\alpha_n^0| + \sum_{m=1}^{n} \left(|\mathbf{X}_n^{m,\dagger,*}||\alpha_n^m| + |\mathbf{Y}_n^{m,\dagger,*}||\beta_n^m| \right) \right] < 1$$

holds in the ball of radius r*, with* $0 \leq r < 0.0186$.

Proof. Let $\mathbf{f}$ be written as Fourier series

$$\mathbf{f} = \sum_{n=0}^{\infty} \left[\mathbf{X}_n^{0,\dagger,*} \alpha_n^0 + \sum_{m=1}^{n} \left(\mathbf{X}_n^{m,\dagger,*} \alpha_n^m + \mathbf{Y}_n^{m,\dagger,*} \beta_n^m \right) \right],$$

where for each $n \in \mathbb{N}_0$, $\alpha_n^0, \alpha_n^m, \beta_n^m \in \mathbb{R}$ $(m = 1, \ldots, n)$ are the associated Fourier coefficients. First, notice that in the previous series the sum which contains the variable m runs now only from 1 to n. This fact expresses the supposed orthogonality to the hyperholomorphic constants $\mathbf{X}_n^{n+1,\dagger}$ and $\mathbf{Y}_n^{n+1,\dagger}$.

We recognize from the structure of the series, that $\mathbf{f}(0)$ is real. In fact, since the basis polynomials are homogeneous, the value of $\mathbf{f}$ at the origin is given by

$$\mathbf{f}(0) = \sum_{n=0}^{\infty} \left(\mathbf{X}_n^{0,\dagger,*}(0) \alpha_n^0 + \sum_{m=1}^{n} \left[\mathbf{X}_n^{m,\dagger,*}(0) \alpha_n^m + \mathbf{Y}_n^{m,\dagger,*}(0) \beta_n^m \right] \right) = \frac{1}{2}\sqrt{\frac{3}{\pi}} \alpha_0^0.$$

Without loss of generality we assume that $\mathbf{f}(0)$ is positive (otherwise we work with $-\mathbf{f}$). Since the associated Fourier coefficients are real-valued, the real part of $\mathbf{f}$ is given by

$$\mathbf{Sc}(\mathbf{f}) = \sum_{n=0}^{\infty} \left\{ \mathbf{Sc}(\mathbf{X}_n^{0,\dagger,*}) \alpha_n^0 + \sum_{m=1}^{n} \left[\mathbf{Sc}(\mathbf{X}_n^{m,\dagger,*}) \alpha_n^m + \mathbf{Sc}(\mathbf{Y}_n^{m,\dagger,*}) \beta_n^m \right] \right\}.$$

The main idea of the proof is to find relations between the general Fourier coefficients and the coefficient of the zeroth term, i.e. α_0^0. Multiplying both sides of the expression

$$\mathbf{Sc}(1 - \mathbf{f}) = 1 - \mathbf{Sc}(\mathbf{f}) \tag{21}$$

by the homogeneous harmonic polynomials

$$\left\{ \mathbf{Sc}(\mathbf{X}_n^{0,\dagger,*}), \mathbf{Sc}(\mathbf{X}_n^{m,\dagger,*}), \mathbf{Sc}(\mathbf{Y}_n^{m,\dagger,*}) \colon m = 1, \ldots, n \right\} \tag{22}$$

and integrating over the ball, we get the desired relations. For simplicity we just present the proof for the coefficients of $\mathbf{X}_n^{0,\dagger,*}$, i.e. α_n^0.

We now use the result of Theorem 4 and get immediately that the harmonic extensions (22) to the unit ball of the spherical harmonics (18) are orthogonal to each other with respect to the inner product (4). Multiplying both sides of (21) by $\mathbf{Sc}(\mathbf{X}_k^{0,\dagger,*})$, $k = 1, \ldots$, and integrating over the ball, we obtain

$$-\sum_{n=0}^{\infty} \alpha_n^0 \int_B \mathbf{Sc}(\mathbf{X}_n^{0,\dagger,*})\mathbf{Sc}(\mathbf{X}_k^{0,\dagger,*})\, dV = \int_B \mathbf{Sc}(1-\mathbf{f})\mathbf{Sc}(\mathbf{X}_k^{0,\dagger,*})\, dV$$

$$\Updownarrow$$

$$-\alpha_k^0 = \frac{1}{\|\mathbf{Sc}(\mathbf{X}_k^{0,\dagger,*})\|_{L_2(B)}^2} \int_B \mathbf{Sc}(1-\mathbf{f})\mathbf{Sc}(\mathbf{X}_k^{0,\dagger,*})\, dV.$$

Following the idea of our proof we shall now relate the integral on the right-hand side of the previous equality with the coefficient α_0^0.

Applying the modulus in the previous equation it follows that

$$\begin{aligned}
|\alpha_k^0| &\leq \frac{1}{\|\mathbf{Sc}(\mathbf{X}_k^{0,\dagger,*})\|_{L_2(B)}^2} \int_B |\mathbf{Sc}(1-\mathbf{f})||\mathbf{Sc}(\mathbf{X}_k^{0,\dagger,*})|\, dV \\
&\leq \frac{1}{\|\mathbf{Sc}(\mathbf{X}_k^{0,\dagger,*})\|_{L_2(B)}^2} \max_B |\mathbf{Sc}(\mathbf{X}_k^{0,\dagger,*})| \int_B \mathbf{Sc}(1-\mathbf{f})\, dV \\
&= \max_B |\mathbf{Sc}(\mathbf{X}_k^{0,\dagger})| \frac{\|\mathbf{X}_k^{0,\dagger}\|_{L_2(B;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(\mathbf{X}_k^{0,\dagger})\|_{L_2(B)}^2}\, 2\sqrt{\frac{\pi}{3}} \left(2\sqrt{\frac{\pi}{3}} - \alpha_0^0\right).
\end{aligned}$$

The last step follows from the relation

$$\begin{aligned}
2\sqrt{\frac{\pi}{3}} - \alpha_0^0 &= 2\sqrt{\frac{\pi}{3}} - \langle \mathbf{X}_0^{0,\dagger,*}, \mathbf{f}\rangle_{L_2(B;\mathcal{A};\mathbb{R})} \\
&= 2\sqrt{\frac{\pi}{3}} - \int_B \frac{1}{2}\sqrt{\frac{3}{\pi}}\mathbf{Sc}(\mathbf{f})\, dV \\
&= \frac{1}{2}\sqrt{\frac{3}{\pi}} \int_B \mathbf{Sc}(1-\mathbf{f})\, dV.
\end{aligned}$$

Accordingly, we have found the desired relations for the coefficients α_k^0. The remaining coefficients α_k^m and β_k^m, $m = 1, \ldots, k$, are obtained in a similar way.

We can then state the following results:

$$\begin{aligned}
|\alpha_k^p| &\leq \max_B |\mathbf{Sc}(\mathbf{X}_k^{p,\dagger})| \frac{\|\mathbf{X}_k^{p,\dagger}\|_{L_2(B;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(\mathbf{X}_k^{p,\dagger})\|_{L_2(B)}^2} 2\sqrt{\frac{\pi}{3}} \left(2\sqrt{\frac{\pi}{3}} - \alpha_0^0\right), \\
|\beta_k^p| &\leq \max_B |\mathbf{Sc}(\mathbf{Y}_k^{p,\dagger})| \frac{\|\mathbf{Y}_k^{p,\dagger}\|_{L_2(B;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(\mathbf{Y}_k^{p,\dagger})\|_{L_2(B)}^2} 2\sqrt{\frac{\pi}{3}} \left(2\sqrt{\frac{\pi}{3}} - \alpha_0^0\right)
\end{aligned}$$

for $p = 1, \ldots, k$.

With some calculations, using Proposition 5 and the estimate (11), the previous inequalities can be rewritten as follows

$$|\alpha_k^0| \leq \frac{(2k+1)}{\sqrt{2}\sqrt{k+1}} \frac{1}{\sqrt{3}} \left(2\sqrt{\frac{\pi}{3}} - \alpha_0^0 \right),$$
$$|\alpha_k^p| \leq \frac{\sqrt{k+1}\,(2k+1)}{\sqrt{(k+1)^2 - p^2}} \frac{1}{\sqrt{3}} \left(2\sqrt{\frac{\pi}{3}} - \alpha_0^0 \right),$$
$$|\beta_k^p| \leq \frac{\sqrt{k+1}\,(2k+1)}{\sqrt{(k+1)^2 - p^2}} \frac{1}{\sqrt{3}} \left(2\sqrt{\frac{\pi}{3}} - \alpha_0^0 \right)$$

for $p = 1, \ldots, k$.

Using the latter inequalities and Proposition 3 we end with

$$\begin{aligned} &\frac{1}{2}\sqrt{\frac{3}{\pi}}\alpha_0^0 + \sum_{n=1}^{\infty} \left[|\mathbf{X}_n^{0,\dagger,*}||\alpha_n^0| + \sum_{m=1}^{n} \left(|\mathbf{X}_n^{m,\dagger,*}||\alpha_n^m| + |\mathbf{Y}_n^{m,\dagger,*}||\beta_n^m| \right) \right] \\ &\quad \leq \frac{1}{2}\sqrt{\frac{3}{\pi}}\alpha_0^0 + \frac{1}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \sqrt{2n+3}(n+1)(2r)^n \left\{ |\alpha_n^0| + \sum_{m=1}^{n} \left(|\alpha_n^m| + |\beta_n^m| \right) \right\} \\ &\quad \leq \frac{1}{2}\sqrt{\frac{3}{\pi}}\alpha_0^0 + \sqrt{\frac{2}{3\pi}} \left(2\sqrt{\frac{\pi}{3}} - \alpha_0^0 \right) \sum_{n=1}^{\infty} (2r)^n (n+1)^3 (n+2). \end{aligned}$$

Now, note that

$$\frac{1}{2}\sqrt{\frac{3}{\pi}}\alpha_0^0 + \sum_{n=1}^{\infty} \left[|\mathbf{X}_n^{0,\dagger,*}||\alpha_n^0| + \sum_{m=1}^{n} \left(|\mathbf{X}_n^{m,\dagger,*}||\alpha_n^m| + |\mathbf{Y}_n^{m,\dagger,*}||\beta_n^m| \right) \right] < 1$$

if

$$\frac{2\sqrt{2}}{3} \sum_{n=1}^{\infty} (2r)^n (n+1)^3 (n+2) = \frac{32\sqrt{2}r}{3} \frac{(4r^4 - 10r^3 + 10r^2 - 3r + 3)}{(1-2r)^5} < 1.$$

We see that the series on the left-hand side converges for $r < 1/2$, and by an approximative solution of the previous inequality we obtain that the inequality is satisfied for $0 \leq r < 0.0186$. ■

It has to be studied in the future if the Bohr type theorem proved can be extended to all monogenic functions with $|\mathbf{f}(x)| < 1$ in the ball and how the estimate for the Bohr radius can be improved.

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