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Derivatives of Meromorphic Functions with Multiple Zeros and Rational Functions

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To Walter Hayman on his 80th birthday, with great admiration

Abstract. Let f be a transcendental meromorphic function on $\mathbb{C}$, all but finitely many of whose zeros are multiple, and let R be a rational function, $R \not\equiv 0$. Then $f' - R$ has infinitely many zeros.

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1. Introduction

This paper continues our study of the value distribution of transcendental meromorphic functions, all but finitely many of whose zeros are multiple. In [6] (cf. [5]), we showed that the derivative of such a function must take on every nonzero complex value infinitely often. Here we extend that result as follows.

Theorem 1. *Let f be a transcendental meromorphic function on $\mathbb{C}$, all but finitely many of whose zeros are multiple, and let $R \not\equiv 0$ be a rational function. Then $f' - R$ has infinitely many zeros.*

Theorem 1 thus extends the main result of [1], where it was shown that the same conclusion holds under the additional hypothesis that all but finitely many of the poles of f are multiple.

The proof of Theorem 1 is based, quite naturally, on a combination of ideas from [1] and [6] and, in particular, makes use of quasinormal families [2]. Recall that a family $\mathcal{F}$ of functions meromorphic on a plane domain D is said to be quasinormal on D if from each sequence $\{f_n\} \subset \mathcal{F}$ one can extract a subsequence

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$\{f_{n_k}\}$ which converges locally uniformly (with respect to the spherical metric) on $D \setminus E$, where the set E (which may depend on $\{f_{n_k}\}$) has no accumulation point in D. If E can always be chosen to contain no more than m points, $\mathcal{F}$ is said to be quasinormal of order m on D.

We use the following notation. For f meromorphic on $\mathbb{C}$ and D a domain in $\mathbb{C}$,

$$S(D,f) = \frac{1}{\pi} \iint_D [f^{\#}(z)]^2 \, dxdy,$$

where

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

denotes the spherical derivative. We write $\Delta(z_0, \delta) = \{z\colon |z - z_0| < \delta\}$ and $D_{\rho,r} = \{z\colon \rho < |z| < r\}$ and set $S(r,f) = S(\Delta(0,r), f)$ and $S(\rho, r, f) = S(D_{\rho,r}, f)$. We assume the standard results of Nevanlinna theory (for example, see [3]) and use the Ahlfors-Shimizu form of the Nevanlinna characteristic function, given by

$$T(r,f) = \int_0^r \frac{S(t,f)}{t} \, dt. \tag{1}$$

2. Auxiliary results

Our point of departure is the following result, proved in [6].

Theorem A. *Let $D \subset \mathbb{C}$ be a domain and $\{h_n\}$ a sequence of holomorphic functions on D such that $h_n \to h = H'$ locally uniformly on D, where H is univalent on D. Let $\{f_n\}$ be a sequence of functions meromorphic on D such that for each n,*

(i) *all zeros of f_n are multiple,*
(ii) *$f_n'(z) \neq h_n(z)$ for all $z \in D$.*

Then $\{f_n\}$ is quasinormal of order 1 on D. If, moreover, no subsequence of $\{f_n\}$ is normal at $z_0 \in D$, then $f_n \to H - H(z_0)$ locally uniformly on $D \setminus \{z_0\}$ and there exists $\delta > 0$ such that $S(\Delta(z_0, \delta), f_n) \leq 2$ for all n.

Remark. Since Theorem A is not stated explicitly in [6], let us indicate how it follows from the results of that paper. The proof that $\{f_n\}$ is quasinormal of order 1 is essentially identical to that of Theorem 1 (with $k = 1$) of [6]. That proof also shows that condition (b) of Lemma 7 in [6] holds for $a_1 = z_0$. It then follows from Lemma 7 that $f_n \to H - H(z_0)$ locally uniformly on $D \setminus \{z_0\}$. The bound on $S(\Delta(z_0, \delta), f_n)$ follows from Lemma 9 of [6].

Recall that a meromorphic function f is Julia exceptional if $f^{\#}(z) = \mathcal{O}(1/|z|)$ as $z \to \infty$. It follows from (1) that if f is a Julia exceptional function, then $T(r,f) = \mathcal{O}((\log r)^2)$ as $r \to +\infty$.

Lemma 1 ([1, Lem. 2.2]). *Let f be a meromorphic function which is not Julia exceptional. Then there exists a sequence $\{a_n\}$ in $\mathbb{C}$ such that $a_n \to \infty$, $f(a_n) \to 0$, and $a_n f'(a_n) \to \infty$.*

Indeed, since f is not Julia exceptional, there is a sequence $\{b_k\}$ in $\mathbb{C}$ such that $b_k \to \infty$ and $b_k f^{\#}(b_k) \to \infty$. The proof in [1] gives a procedure for finding a subsequence $\{b_{k_n}\}$ and points $a_n \sim b_{k_n}$ such that the stated conditions hold.

If we assume slightly more than in Lemma 1, we obtain a correspondingly stronger result.

Lemma 2. *Let f be a meromorphic function on $\mathbb{C}$. If*

$$\limsup_{r\to+\infty} \frac{T(r,f)}{(\log r)^2} = +\infty, \tag{2}$$

then there exists $r_n \to +\infty$ such that

$$\lim_{n\to\infty} S(r_n/2, 2r_n, f) = +\infty$$

and a sequence $\{a_n\}$ in $\mathbb{C}$ satisfying $r_n/2 < |a_n| < 2r_n$ such that $f(a_n) \to 0$ and $a_n f'(a_n) \to \infty$.

Proof. First we show that for each $b > 1$, there exist $r_n \to +\infty$ such that $S(r_n, br_n, f) \to +\infty$. Otherwise, there exists a constant $M = M(f,b)$ such that $S(r, br, f) \le M$ for all $r \ge 0$. Thus

$$S(1, b^n, f) = \sum_{n=1}^{n} S(b^{k-1}, b^k, f) \le nM.$$

For each $r > 1$, there exists a positive integer N such that $b^{N-1} \le r < b^N$. Then

$$S(1, r, f) \le S(1, b^N, f) \le NM \le \left(1 + \frac{\log r}{\log b}\right) M$$

and

$$S(r, f) = S(1, f) + S(1, r, f) \le C + \frac{M}{\log b} \log r,$$

where $C = S(1, f) + M$. But then

$$\begin{aligned} T(r,f) &= \int_0^r \frac{S(t,f)}{t}\,dt = \int_0^1 \frac{S(t,f)}{t}\,dt + \int_1^r \frac{S(t,f)}{t}\,dt \\ &\le \int_0^1 \frac{S(t,f)}{t}\,dt + C\log r + \frac{M}{2\log b}(\log r)^2, \end{aligned}$$

which contradicts (2). Thus there exists a sequence of radii $r_n \to +\infty$ such that $S(r_n, 3r_n/2, f) \to +\infty$ and hence complex numbers c_n with $r_n \le |c_n| \le 3r_n/2$ such that $|c_n| f^{\#}(c_n) \to +\infty$. Applying the argument used in [1] to prove Lemma 1, taking subsequences, and renumbering, we obtain a sequence $\{a_n\}$ with $a_n \sim c_n$ as $n \to \infty$ such that $f(a_n) \to 0$ and $a_n f'(a_n) \to \infty$. Since $a_n \sim c_n$, clearly $r_n/2 < |a_n| < 2r_n$ for sufficiently large n. ∎

Lemma 3 ([4, p. 7]). *A transcendental Julia exceptional function has no asymptotic values.*

Lemma 4 ([1, Lem. 2.5]). *Let f be a transcendental meromorphic function and let R be a rational function satisfying $R(z) \sim cz^d$ as $z \to \infty$, where $c \in \mathbb{C} \setminus \{0\}$ and $d \in \mathbb{Z}$. Suppose that $f' - R$ has only finitely many zeros and that $T(r, f) = \mathcal{O}((\log r)^2)$ as $r \to +\infty$. Set $g(z) := f(z)/z^{d+1}$, with $g := f$ if $d = -1$. Then g has an asymptotic value, and there exist $\theta_0 \in [0, 2\pi)$ and $a \in \mathbb{C}$ such that as $r \to +\infty$,*

$$f(re^{i\theta_0}) \sim \frac{c}{d+1}\left(re^{i\theta_0}\right)^{d+1} \qquad \text{if } d \geq 0;$$
$$f(re^{i\theta_0}) = a + \frac{c}{d+1}\left(re^{i\theta_0}\right)^{d+1} + \mathcal{O}(r^d) \qquad \text{if } d \leq -2.$$

Lemma 5. *Let f be a meromorphic function on $\mathbb{C}$, all but finitely many of whose zeros are multiple and such that $f' - R$ has only finitely many zeros, where R is as in Lemma 4. Let $\{a_n\}$ be a sequence of complex numbers such that $a_n \to \infty$ and*

$$\frac{f(a_n)}{{a_n}^{d+1}} \to 0 \qquad \text{and} \qquad \frac{f'(a_n)}{{a_n}^{d}} \to \infty. \tag{3}$$

Set

$$f_n(z) = \frac{f(a_n z)}{{a_n}^{d+1}}.$$

Then $\{f_n\}$ is quasinormal of order $|d+1|$ in

$$D := \left\{z \colon \frac{1}{b} < |z| < b\right\}$$

for each $b > 1$. Moreover, $d \neq -1$; and there exist points $z_1 (= 1), z_2, \dots z_t$ in D, $1 \leq t \leq |d+1|$, and a subsequence $\{n_k\}$ such that as $k \to \infty$,

$$f_{n_k}(z) \sim \frac{c}{d+1}(z^{d+1} - 1) \qquad \text{in } D \setminus \{z_1, z_2, \dots z_t\}.$$

Proof. Fix $b > 1$. By (3), no subsequence of $\{f_n\}$ is normal at $z = 1$ and, by assumption,

$$f_n'(z) \neq \frac{R(a_n z)}{{a_n}^d}, \qquad z \in D,$$

for large n. Since $R(z) \sim cz^d$ as $z \to \infty$,

$$R(z) = cz^d[1 + r(z)]$$

where $r(z) = o(1)$ as $z \to \infty$. Thus, writing $r_n(z) = r(a_n z)$, we have

$$f_n'(z) \neq cz^d[1 + r_n(z)], \qquad z \in D, \tag{4}$$

for all large n.

We claim that $d \neq -1$. Indeed, suppose to the contrary that $d = -1$. Then

$$f_n'(z) \neq \frac{c}{z}[1 + r_n(z)], \qquad z \in D,$$

as $n \to \infty$. Let $I_1 = (-\pi, \pi)$ and $I_2 = (-\pi/2, 3\pi/2)$. For $j = 1, 2$, we consider the functions

$$H_j(z) = c(\log z)_j$$

defined, respectively, on the simply connected domains

$$D_j = \{z \in D\colon \arg z \in I_j\},$$

where the imaginary part of the branch $(\log z)_j$ takes values in I_j. Then for $j = 1, 2$,

$$H_j'(z) = \frac{c}{z}, \qquad z \in D_j;$$

and the functions

$$h_n(z) = \frac{c}{z}[1 + r_n(z)]$$

converge uniformly to $H_j'(z)$ on D_j. Since $H_j(1) = 0$, we have by Theorem A,

$$f_n(z) \to H_j(z) \tag{5}$$

locally uniformly on $D_j \setminus \{1\}$ for $j = 1, 2$. But this is impossible, since $H_1(z) \neq H_2(z)$ on $D \cap \{z \in \mathbb{C}\colon \operatorname{Re} z < 0, \operatorname{Im} z < 0\}$. This contradiction shows that $d \neq -1$.

Now set $\mu = 1/(d + 1)$. Fix $0 < \alpha < \pi$. Then the function $z(\zeta) = \zeta^\mu$ has $|d + 1|$ distinct branches $z_s(\zeta)$, $s = 1, 2, \dots, |d + 1|$ on the domain

$$\tilde{D}_\alpha = \{\zeta\colon b^{-|d+1|} < |\zeta| < b^{|d+1|}, -\pi + \alpha < \arg z < \pi + \alpha\},$$

each of which maps $\tilde{D}_\alpha$ univalently onto an open sector $D_{\alpha,s}$ of D having angle $2\pi/|d + 1|$.

Denote by $z_1(\zeta)$ the branch for which $z_1(1) = 1$. Put

$$h_{s,n}(\zeta) = \frac{d+1}{c} f_n(z_s(\zeta)) = \frac{d+1}{c{a_n}^{d+1}} f(a_n z_s(\zeta)) \tag{6}$$

for $s = 1, 2, \dots, |d + 1|$ and $n = 1, 2, \dots$. Then $h_{s,n}$ is a meromorphic function on $\tilde{D}_\alpha$, all of whose zeros are multiple if n is sufficiently large. Also, since

$$h_{s,n}'(\zeta) = \frac{d+1}{c} f_n'(z_s(\zeta)) z_s'(\zeta) = \frac{1}{c} f_n'(z_s(\zeta))[z_s(\zeta)]^{-d} = \frac{1}{c}\,\frac{f'(a_n z_s(\zeta))}{{a_n}^d [z_s(\zeta)]^d}, \tag{7}$$

we have from (4)

$$h_{s,n}'(\zeta) \neq 1 + r_n(z_s(\zeta))$$

on $\tilde{D}_\alpha$ for large n. Thus, by Theorem A, for each $s = 1, 2, \dots, |d + 1|$, $\{h_{s,n}\}$ is quasinormal of order 1 on $\tilde{D}_\alpha$. It follows that $\{f_n\}$ is quasinormal of order 1 on each $D_{\alpha,s}$, $s = 1, 2, \dots, |d + 1|$. Since α can be chosen freely, this means that $\{f_n\}$ is quasinormal of order 1 on any open sector of D of opening $2\pi/|d+1|$. It follows that $\{f_n\}$ is quasinormal of order $|d + 1|$ on D.

Now by (6) and (3),

$$h_{1,n}(1) = \frac{d+1}{c}\,\frac{f(a_n)}{{a_n}^{d+1}} \to 0;$$

and by (7) and (3),

$$h'_{1,n}(1) = \frac{1}{c}\,\frac{f'(a_n)}{{a_n}^{d}} \to \infty.$$

Thus no subsequence of $\{h_{1,n}\}$ is normal at $\zeta = 1$. It follows from Theorem A that

$$h_{1,n}(\zeta) \to \zeta - 1$$

and hence

$$f_n(z_1(\zeta)) = \frac{f(a_n z_1(\zeta))}{{a_n}^{d+1}} \to \frac{c}{d+1}(\zeta - 1)$$

locally uniformly on $\tilde{D}_\alpha \setminus \{1\}$. Accordingly,

$$f_n(z) \to \frac{c}{d+1}(z^{d+1} - 1) \qquad \text{on } D_{\alpha,1} \setminus \{1\}, \tag{8}$$

where $D_{\alpha,1} = z_1(\tilde{D}_\alpha)$, the convergence being locally uniform. Since $\{f_n\}$ is quasinormal of order $|d+1|$, there exist points $z_1(=1), z_2, \ldots, z_t$ $(1 \le t \le |d+1|)$ and a subsequence $\{f_{n_k}\}$ such that

$$f_{n_k}(z) \to f(z) \qquad \text{on } D \setminus \{z_1, \ldots, z_t\}$$

locally uniformly, where f is a meromorphic function on $D \setminus \{z_1, \ldots, z_t\}$. By (8),

$$f(z) = \frac{c}{d+1}(z^{d+1} - 1)$$

on $D_{\alpha,1} \setminus \{1\}$ and hence on $D \setminus \{z_1, \ldots, z_t\}$. Thus

$$f_{n_k}(z) \to \frac{c}{d+1}(z^{d+1} - 1)$$

locally uniformly on $D \setminus \{z_1, \ldots, z_t\}$. ■

3. Proof of Theorem 1

We assume that $f' - R$ has at most finitely many zeros and derive a contradiction. Let $R(z) \sim cz^d$ as $z \to \infty$, where $c \in \mathbb{C} \setminus \{0\}$ and $d \in \mathbb{Z}$. Set $g(z) = f(z)/z^{d+1}$.

Suppose first that

$$T(r, g) = \mathcal{O}((\log r)^2) \qquad \text{as } r \to +\infty.$$

Then, since $T(r, g) = T(r, f) + \mathcal{O}(\log r)$, we have $T(r, f) = \mathcal{O}((\log r)^2)$. By Lemma 4, g has an asymptotic value. Hence, by Lemma 3, g is not a Julia exceptional function. Thus, by Lemma 1, there exists a sequence of complex numbers $\{a_n\}$ such that $a_n \to \infty$, $g(a_n) \to 0$, and $a_n g'(a_n) \to \infty$. Hence $f(a_n)/{a_n}^{d+1} = g(a_n) \to 0$ and $f'(a_n)/{a_n}^{d} = a_n g'(a_n) + (d+1)g(a_n) \to \infty$ as $n \to \infty$. It follows from Lemma 5 that $d \ne -1$. Taking an appropriate subsequence (which we continue to call $\{a_n\}$), we see also from Lemma 5 that for each

$b > 1$, there exist points $z_1(= 1), z_2, \ldots, z_t$, $1 \le t \le |d+1|$, in $D = \{1/b < |z| < b\}$ such that

$$\frac{f(a_n z)}{{a_n}^{d+1}} \sim \frac{c}{d+1}(z^{d+1} - 1), \qquad z \in D \setminus \{z_1, \ldots, z_t\}$$

as $n \to \infty$. Taking r_1 and r_2 such that $1/b < r_1 < r_2 < b$ and $|z_j| \notin [r_1, r_2]$ for $1 \le j \le t$, we have as $n \to \infty$,

$$\frac{f(a_n z)}{{a_n}^{d+1}} \sim \frac{c}{d+1}(z^{d+1} - 1), \qquad r_1 \le |z| \le r_2.$$

Set $z = re^{i\theta}$, $a_n = |a_n|e^{i\varphi_n}$. Then for $r_1 \le r \le r_2$,

$$f(|a_n|re^{i(\theta+\varphi_n)}) \sim \frac{c}{d+1}\left(|a_n|re^{i(\theta+\varphi_n)}\right)^{d+1}\left(1 - \frac{1}{(re^{i\theta})^{d+1}}\right). \tag{9}$$

By Lemma 4, there exist $\theta_0 \in [0, 2\pi)$ and $a \in \mathbb{C}$ such that

$$f(\rho e^{i\theta_0}) = \begin{cases} \dfrac{c}{d+1}(\rho e^{i\theta_0})^{d+1} + o(\rho^{d+1}) & \text{if } d \ge 0, \\ a + \dfrac{c}{d+1}(\rho e^{i\theta_0})^{d+1} + \mathcal{O}(\rho^d) & \text{if } d \le -2, \end{cases} \tag{10}$$

as $\rho \to +\infty$. In particular, (10) holds for $|a_n|r_1 \le \rho \le |a_n|r_2$ as $n \to \infty$. Fixing n, putting $\theta = \theta_0 - \varphi_n$ in (9), and comparing (9) with (10), we obtain for $r_1 \le r \le r_2$

$$\begin{aligned} &\frac{c}{d+1}\left(|a_n|re^{i\theta_0}\right)^{d+1}\left(1 - \frac{1}{(re^{i(\theta_0-\varphi_n)})^{d+1}}\right) \\ &= \begin{cases} \dfrac{c}{d+1}(|a_n|re^{i\theta_0})^{d+1} + o(|a_n|^{d+1}) & \text{if } d \ge 0, \\ a + \dfrac{c}{d+1}(|a_n|re^{i\theta_0})^{d+1} + \mathcal{O}(|a_n|^d) & \text{if } d \le -2. \end{cases} \end{aligned} \tag{11}$$

Letting $n \to \infty$ in (11) now yields a contradiction.

We now turn to the case in which

$$\limsup_{r\to+\infty} \frac{T(r, g)}{(\log r)^2} = +\infty.$$

By Lemma 2, there exist radii $r_n \to +\infty$ and complex numbers a_n satisfying $r_n/2 < |a_n| < 2r_n$ such that

$$S(r_n/2, 2r_n, g) \to \infty \tag{12}$$

and

$$g(a_n) \to 0, \qquad a_n g'(a_n) \to \infty.$$

As before, we have

$$\frac{f(a_n)}{{a_n}^{d+1}} \to 0 \qquad \text{and} \qquad \frac{f'(a_n)}{{a_n}^{d}} \to \infty.$$

Set

$$f_n(z) = \frac{f(a_n z)}{{a_n}^{d+1}}$$

and put $D = \{z\colon 1/8 < |z| < 8\}$. Then clearly $\{f_n\}$ is not normal at $z = 1$, and it follows from Lemma 5 that $\{f_n\}$ is quasinormal of order $|d+1|$ on D. Taking subsequences, we may assume that no subsequence is normal at the points $z_1(=1), z_2, \dots, z_t$, $1 \le t \le |d+1|$, and that $\{f_n\}$ is normal on $D \setminus \{z_1, z_2, \dots, z_t\}$. By Theorem A (applied to $h_n(z) = R(a_n z)/a_n^d$ on a small disc about each z_j, $1 \le j \le t$), there exists $\delta > 0$ such that

$$S(\Delta(z_j, \delta), f_n) \le 2, \qquad j = 1, 2, \dots, t.$$

Now let

$$D' = \left\{ \frac{1}{7} < |z| < 7 \right\} \qquad \text{and} \qquad K = \bigcup_{j=1}^{t} \Delta(z_j, \delta).$$

Since $\{f_n\}$ is normal on $D \setminus \{z_1, z_2, \dots, z_t\}$, by Marty's Theorem there exists $M > 0$ such that $f_n^{\#}(z) \le M$ for $z \in D' \setminus K$. It follows that

$$S(1/7, 7, f_n) \le S(D' \setminus K, f_n) + S(K, f_n) \le 49M^2 + 2|d+1| := M_1. \tag{13}$$

Let

$$g_n(z) = g(a_n z) = \frac{f_n(z)}{z^{d+1}}.$$

Then

$$g_n^{\#}(z) = \frac{|z^{d+1} f_n'(z) - (d+1) z^d f_n(z)|}{|z|^{2(d+1)} + |f_n(z)|^2},$$

so

$$[g_n^{\#}(z)]^2 \le \frac{2|z^{d+1} f_n'(z)|^2}{(|z|^{2(d+1)} + |f_n(z)|^2)^2} + \frac{2|(d+1) z^d f_n(z)|^2}{(|z|^{2(d+1)} + |f_n(z)|^2)^2}. \tag{14}$$

Using the simple inequality

$$\frac{C}{C^2 + x^2} \le \max(C, 1/C) \frac{1}{1 + x^2}$$

for $C > 0$, we have

$$\frac{2|z^{d+1} f_n'(z)|^2}{(|z|^{2(d+1)} + |f_n(z)|^2)^2} \le 2 \max\left(|z|^{2(d+1)}, \frac{1}{|z|^{2(d+1)}} \right) [f_n^{\#}(z)]^2. \tag{15}$$

The second term on the right of (14) is

$$\frac{1}{2} \frac{(d+1)^2}{|z|^2} \left(\frac{2|z|^{d+1} |f_n(z)|}{|z|^{2(d+1)} + |f_n(z)|^2} \right)^2 \le \frac{(d+1)^2}{2|z|^2}. \tag{16}$$

Putting (14), (15), and (16) together, we have for $1/7 \le |z| \le 7$,

$$[g_n^{\#}(z)]^2 \le 2 \cdot 7^{2|d|+2} \cdot |f_n^{\#}(z)|^2 + 7^2 \cdot (d+1)^2.$$

It follows from (13) that

$$S(1/7, 7, g_n) \le 2 \cdot 7^{2|d|+2} M_1 + 7^4 (d+1)^2 := M_2.$$

But since $r_n/2 < |a_n| < 2r_n$,

$$\frac{|a_n|}{7} < \frac{r_n}{2} < 2r_n < 7|a_n|,$$

so that

$$S(r_n/2, 2r_n, g) \le S(|a_n|/7, 7|a_n|, g) = S(1/7, 7, g_n) \le M_2,$$

which contradicts (12).

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