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Sharp Bohr Type Real Part Estimates

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Abstract. We consider analytic functions f in the unit disk $\mathbb{D}$ with Taylor coefficients $c_0, c_1, \dots$ and derive estimates with sharp constants for the l_q-norm (quasi-norm for $0 < q < 1$) of the remainder of their Taylor series, where $q \in (0, \infty]$. As the main result, we show that given a function f with $\operatorname{Re} f$ in the Hardy space $h_1(\mathbb{D})$ of harmonic functions on $\mathbb{D}$, the inequality

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q \right)^{1/q} \leq \frac{2r^m}{(1-r^q)^{1/q}} \| \operatorname{Re} f \|_{h_1}$$

holds with the sharp constant, where $r = |z| < 1$, $m \geq 1$. This estimate implies sharp inequalities for l_q-norms of the Taylor series remainder for bounded analytic functions, analytic functions with bounded $\operatorname{Re} f$, analytic functions with $\operatorname{Re} f$ bounded from above, as well as for analytic functions with $\operatorname{Re} f > 0$. In particular, we prove that

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q \right)^{1/q} \leq \frac{2r^m}{(1-r^q)^{1/q}} \sup_{|\zeta|<1} \operatorname{Re}(f(\zeta) - f(0)).$$

As corollary of the above estimate with $\| \operatorname{Re} f \|_{h_1}$ in the right-hand side, we obtain some sharp Bohr type modulus and real part inequalities.

Keywords. Taylor series, Bohr's inequality, Hadamard's Real Part Theorem.

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1. Introduction

The present article is connected with two classical assertions of analytic function theory, namely with Hadamard's Real Part Theorem and Bohr's Theorem on the majorant of a Taylor series.

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Hadamard's Real Part Theorem states that the inequality

$$(1.1)\qquad |f(z)-f(0)| \le \frac{C|z|}{1-|z|}\sup_{|\zeta|<1}\mathrm{Re}(f(\zeta)-f(0))$$

holds for analytic functions in the disk $\mathbb{D} := \{z : |z| < 1\}$. This inequality for analytic functions vanishing at $z = 0$ was first obtained by Hadamard with $C = 4$ in 1892 [28]. The sharp constant $C = 2$ in (1.1) was found by Borel [17, 18] and Carathéodory (see Landau [31, 32]). A detailed historical survey on these and other related inequalities for analytic functions can be found in the paper by Jensen [29].

A refined Bohr's Theorem [14], as stated by M. Riesz, I. Schur, F. Wiener (see Landau [33]), claims that any function

$$(1.2)\qquad f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

analytic and bounded in $\mathbb{D}$, satisfies the inequality

$$(1.3)\qquad \sum_{n=0}^{\infty} |c_n z^n| \le \sup_{|\zeta|<1} |f(\zeta)|,$$

where $|z| \le 1/3$. Moreover, the value $1/3$ of the radius cannot be improved.

In the present paper we deal with, as do Aizenberg, Grossman and Korobeinik [7], Bénéteau, Dahlner and Khavinson [11], Djakov and Ramanujan [25], the value of l_q-norm (quasi-norm, for $0 < q < 1$) of the remainder of the Taylor series (1.2). The particular case $q = \infty$ in all subsequent inequalities can be obtained by passage to the limit as $q \to \infty$.

In Section 2, we prove the inequality

$$(1.4)\qquad \left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \le \frac{2r^m}{(1-r^q)^{1/q}} \|\,\mathrm{Re}\, f\|_{h_1},$$

with the sharp constant, where $r = |z| < 1$, $m \ge 1$, $0 < q \le \infty$, and

$$\|g\|_{h_1} := \sup_{\varrho<1} \frac{1}{2\pi\varrho}\int_{|\zeta|=\varrho} |g(\zeta)|\,|d\zeta|$$

is the norm in the Hardy space $h_1(\mathbb{D})$ of harmonic functions on $\mathbb{D}$.

Section 3 contains corollaries of (1.4) for analytic functions f in $\mathbb{D}$ with bounded $\mathrm{Re}\, f$, with $\mathrm{Re}\, f$ bounded from above, with $\mathrm{Re}\, f > 0$, as well as for bounded analytic functions. In particular, we obtain the estimate

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \le \frac{2r^m}{(1-r^q)^{1/q}} \sup_{|\zeta|<1} \mathrm{Re}(f(\zeta)-f(0)),$$

with the best possible constant. This estimate, taken with $q = 1$, $m = 1$, is a refinement of (1.1) with $C = 2$. Other inequalities, which follow from (1.4), contain the supremum of $|\operatorname{Re} f(\zeta)| - |\operatorname{Re} f(0)|$ or $|f(\zeta)| - |f(0)|$ in $\mathbb{D}$, as well as $\operatorname{Re} f(0)$ in the case $\operatorname{Re} f > 0$ on $\mathbb{D}$. Each of these estimates, for $q = 1$ and $m = 1$, refines a certain Hadamard type inequality with a sharp constant.

Note that a sharp estimate of the full majorant series by the supremum modulus of f was obtained by Bombieri [15] for $r \in [1/3, 1/\sqrt{2}]$.

In Section 4 we give modifications of Bohr's Theorem as consequences of our inequalities with sharp constants derived in Section 3. For example, if a function (1.2) is analytic on $\mathbb{D}$, then for any $q \in (0, \infty]$, integer $m \geq 1$, and $|z| \leq r_{m,q}$ the inequality

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q \right)^{1/q} \leq \sup_{|\zeta|<1} \operatorname{Re}\left(e^{-i \arg f(0)} f(\zeta)\right) - |f(0)| \tag{1.5}$$

holds, where $r_{m,q}$ is the root of the equation

$$2^q r^{mq} + r^q - 1 = 0$$

in the interval $(0, 1)$ if $0 < q < \infty$, and $r_{m,\infty} := 2^{-1/m}$. In particular,

$$\begin{aligned} r_{1,q} &= (1 + 2^q)^{-1/q}, \\ r_{2,q} &= 2^{1/q} \left(1 + \sqrt{1 + 2^{q+2}}\right)^{-1/q}. \end{aligned} \tag{1.6}$$

Note that $r_{m,q}$ is the radius of the largest disk centered at $z = 0$ in which (1.5) takes place.

Some of the inequalities presented in Section 4 contain known analogues of Bohr's Theorem with $\operatorname{Re} f$ in the right-hand side (see Aizenberg, Aytuna and Djakov [4], Paulsen, Popescu and Singh [36], Sidon [39], Tomić [41]). Multi-dimensional analogues and other generalizations of Bohr's Theorem are treated in the papers by Aizenberg [1, 2, 3], Aizenberg, Aytuna and Djakov [4, 5], Aizenberg and Tarkhanov [6], Aizenberg, Liflyand and Vidras [8], Aizenberg and Vidras [9], Boas and Khavinson [12], Boas [13], Dineen and Timoney [22, 23], Djakov and Ramanujan [25]. Various interesting recent results related to Bohr's inequalities were obtained by Aizenberg, Grossman and Korobeinik [7], Bénéteau, Dahlner and Khavinson [11], Bombieri and Bourgain [16], Guadarrama [27], Defant and Frerick [21]. Certain problems of the operator theory connected with Bohr's Theorem are examined by Defant, Garcia and Maestre [20], Dixon [24], Glazman and Ljubič [26], Nikolski [35], Paulsen, Popescu and Singh [36].

2. Sharp estimate of l_q-norm for the remainder of Taylor series by the norm of $\operatorname{Re} f$ in the Hardy space $h_1(\mathbb{D})$

In what follows, we use the notation $r := |z|$ and $\mathbb{D}_\varrho := \{z \in \mathbb{C} : |z| < \varrho\}$.

We start with a sharp inequality for an analytic function f in the disk $\mathbb{D} := \mathbb{D}_1$. The right-hand side of the inequality contains the norm in the Hardy space $h_1(\mathbb{D})$ of harmonic functions on the disk

$$\|g\|_{h_1} := \sup_{\varrho<1} M_1(g, \varrho),$$

where $M_1(g, \varrho)$ is the integral mean value of g on the circle $|\zeta| = \varrho < 1$ defined by

$$M_1(g, \varrho) := \frac{1}{2\pi\varrho} \int_{|\zeta|=\varrho} |g(\zeta)|\, |d\zeta|.$$

Proposition 1. *Let the function* (1.2) *be analytic on* $\mathbb{D}$ *with* $\operatorname{Re} f \in h_1(\mathbb{D})$, *and let* $q > 0$, $m \geq 1$, $|z| = r < 1$. *Then the inequality*

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \leq \frac{2r^m}{(1-r^q)^{1/q}} \|\operatorname{Re} f\|_{h_1} \tag{2.1}$$

holds and is sharp.

Proof. *Proof of inequality* (2.1). Let a function f, analytic in $\mathbb{D}$ with $\operatorname{Re} f \in h_1(\mathbb{D})$ be given by (1.2). Since by the Schwarz formula

$$f^{(n)}(0) = \frac{n!}{\pi i} \int_{|\zeta|=\varrho} \frac{\operatorname{Re} f(\zeta)}{\zeta^{n+1}}\, d\zeta,$$

$0 < \varrho < 1$, $n = 1, 2, \dots$, we have

$$|f^{(n)}(0)| \leq \frac{n!}{\pi \varrho^{n+1}} \int_{|\zeta|=\varrho} |\operatorname{Re} f(\zeta)|\, |d\zeta| = \frac{2n!}{\varrho^n} M_1(\operatorname{Re} f, \varrho).$$

Hence, by the arbitrariness of $\varrho \in (0, 1)$ we obtain

$$|c_n| \leq 2\, \|\operatorname{Re} f\|_{h_1} \tag{2.2}$$

for any $n \geq 1$. Using (2.2), we find that

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \leq 2 \left(\sum_{n=m}^{\infty} r^{nq}\right)^{1/q} \|\operatorname{Re} f\|_{h_1} = \frac{2r^m}{(1-r^q)^{1/q}} \|\operatorname{Re} f\|_{h_1}$$

for any z with $|z| = r < 1$.

Sharpness of the constant in (2.1). By (2.1), obtained above, the sharp constant $C(r)$ in

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \leq C(r)\, \|\operatorname{Re} f\|_{h_1} \tag{2.3}$$

satisfies

$$C(r) \leq \frac{2r^m}{(1-r^q)^{1/q}}. \tag{2.4}$$

We show that the converse inequality for $C(r)$ holds as well.

Let $\rho > 1$. Consider the families of analytic functions in $\overline{\mathbb{D}}$

$$g_\rho(z) := \frac{\rho}{z-\rho}, \qquad f_\rho(z) := g_\rho(z) - \beta_\rho, \tag{2.5}$$

depending on the parameter ρ, with the real constant β_ρ defined by

$$M_1(\operatorname{Re} g_\rho - \beta_\rho, 1) = \min_{c\in\mathbb{R}} M_1(\operatorname{Re} g_\rho - c, 1).$$

Then, for any real constant c,

$$M_1(\operatorname{Re} f_\rho - c, 1) \ge M_1(\operatorname{Re} f_\rho, 1) = \| \operatorname{Re} f_\rho\|_{h_1}.$$

Setting here

$$c = \mathcal{A}(f_\rho) := \max_{|\zeta|=1} \operatorname{Re} f_\rho(\zeta)$$

and taking into account

$$\begin{aligned} M_1(\operatorname{Re} f_\rho - \mathcal{A}(f_\rho), 1) &= \frac{1}{2\pi}\int_{|\zeta|=1} (\mathcal{A}(f_\rho) - \operatorname{Re} f_\rho(\zeta))\ |d\zeta| \\ &= \mathcal{A}(f_\rho) - \operatorname{Re} f_\rho(0) = \max_{|\zeta|=1} \operatorname{Re}(f_\rho(\zeta) - f_\rho(0)), \end{aligned}$$

we arrive at

$$\max_{|\zeta|=1} \operatorname{Re}(f_\rho(\zeta) - f_\rho(0)) \ge \| \operatorname{Re} f_\rho\|_{h_1}. \tag{2.6}$$

In view of

$$c_n(\rho) = \frac{f_\rho^{(n)}(0)}{n!} = -\frac{1}{\rho^n} \qquad \text{for } n \ge 1,$$

we find

$$\sum_{n=m}^{\infty} |c_n(\rho) z^n|^q = \sum_{n=m}^{\infty} \left(\frac{r}{\rho}\right)^{nq} = \frac{r^{mq}}{\rho^{(m-1)q}(\rho^q - r^q)}. \tag{2.7}$$

Putting $z = \zeta = e^{i\vartheta}$ in (2.5), we obtain

$$\max_{|\zeta|=1} \operatorname{Re}(f_\rho(\zeta) - f_\rho(0)) = \frac{1}{2} \max_{\vartheta}\left(1 - \frac{\rho^2-1}{\rho^2 - 2\rho\cos\vartheta + 1}\right) = \frac{1}{\rho+1}. \tag{2.8}$$

It follows from (2.3), (2.6), (2.7) and (2.8) that

$$C(r) \ge \frac{r^m(\rho+1)}{\rho^{m-1}(\rho^q - r^q)^{1/q}}. \tag{2.9}$$

Passing to the limit as $\rho \downarrow 1$ in the last inequality, we obtain

$$C(r) \ge \frac{2r^m}{(1-r^q)^{1/q}}, \tag{2.10}$$

which together with (2.4) proves the sharpness of the constant in (2.1). ∎

Remark 1. In [30] we found an explicit formula for the constant $\mathcal{C}_p$ in the inequality

$$|c_n| \le \mathcal{C}_p \| \operatorname{Re} f \|_p$$

with $1 \le p \le \infty$. In particular, we showed that $\mathcal{C}_1 = 1/\pi$, $\mathcal{C}_2 = 1/\sqrt{\pi}$ and $\mathcal{C}_\infty = 4/\pi$.

3. Sharp estimates of l_q -norm for the remainder of Taylor series for certain classes of analytic functions

In this section we obtain estimates with sharp constants for the l_q -norm (quasi-norm for $0 < q < 1$) of the Taylor series remainder for bounded analytic functions and analytic functions whose real part is bounded or one-side bounded.

We start with a theorem concerning analytic functions with real part bounded from above which refines Hadamard's real part estimate (1.1).

Theorem 1. *Let the function* (1.2) *be analytic on* $\mathbb{D}$ *with*

$$\sup_{|\zeta|<1} \operatorname{Re} f(\zeta) < \infty,$$

and let $q > 0$, $m \ge 1$, $|z| = r < 1$. *Then the inequality*

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q \right)^{1/q} \le \frac{2r^m}{(1-r^q)^{1/q}} \sup_{|\zeta|<1} \operatorname{Re}(f(\zeta) - f(0)) \tag{3.1}$$

holds and is sharp.

Proof. By dilation, we write (2.1) for the disk $\mathbb{D}_\varrho$, $r < \varrho < 1$, with f replaced by $f - \omega$, where ω is an arbitrary real constant. Then

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q \right)^{1/q} \le \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \| \operatorname{Re} f - \omega \|_{h_1(\mathbb{D}_\varrho)}. \tag{3.2}$$

Putting here

$$\omega = \mathcal{A}(f) := \sup_{|\zeta|<1} \operatorname{Re} f(\zeta)$$

and taking into account that

$$\| \operatorname{Re} f - \mathcal{A}(f) \|_{h_1(\mathbb{D}_\varrho)} = M_1(\operatorname{Re} f - \mathcal{A}(f), \varrho) = \sup_{|\zeta|<1} \operatorname{Re}(f(\zeta) - f(0)),$$

we find

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q \right)^{1/q} \le \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \sup_{|\zeta|<1} \operatorname{Re}(f(\zeta) - f(0)),$$

which implies (3.1) after the passage to the limit as $\varrho \uparrow 1$.

Hence, the sharp constant $C(r)$ in

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \le C(r) \sup_{|\zeta|<1} \mathrm{Re}(f(\zeta) - f(0)) \tag{3.3}$$

satisfies (2.4).

To get the lower estimate for $C(r)$, we shall use the functions g_ρ given by (2.5). Taking into account the equalities

$$g^{(n)}(0) = f^{(n)}(0), \qquad \mathrm{Re}(g_\rho(\zeta) - g_\rho(0)) = \mathrm{Re}(f_\rho(\zeta) - f_\rho(0))$$

as well as (3.3), (2.7) and (2.8), we arrive at (2.9). Passing to the limit in (2.9) as $\rho \downarrow 1$, we obtain (2.10), which together with (2.4) proves the sharpness of the constant in (3.1). ∎

Remark 2. Inequality (3.1) for $q = m = 1$ is well known (see, e.g. Pólya and Szegö [37, III, Ch. 5, § 2]). Adding $|c_0|$ and $|f(0)|$ to the left- and right-hand sides of (3.1) with $q = m = 1$, respectively, and replacing $-\mathrm{Re}\, f(0)$ by $|f(0)|$ in the resulting relation, we arrive at

$$\sum_{n=0}^{\infty} |c_n z|^n \le \frac{1+r}{1-r}|f(0)| + \frac{2r}{1-r} \sup_{|\zeta|<1} \mathrm{Re}\, f(\zeta),$$

which is a refinement of the Hadamard-Borel-Carathéodory inequality

$$|f(z)| \le \frac{1+r}{1-r}|f(0)| + \frac{2r}{1-r} \sup_{|\zeta|<1} \mathrm{Re}\, f(\zeta)$$

(cf. Burckel [19, Ch. 6] and references therein, Titchmarsh [40, Ch. 5]).

The next assertion contains a sharp estimate for analytic functions with real part in the Hardy space $h_\infty(\mathbb{D})$ of harmonic functions bounded in $\mathbb{D}$. It is a refinement of the inequality

$$|f(z) - f(0)| \le \frac{2r}{1-r} \sup_{|\zeta|<1} [|\mathrm{Re}\, f(\zeta)| - |\mathrm{Re}\, f(0)|]$$

which follows from (1.1) with $C = 2$.

Theorem 2. *Let the function* (1.2) *be analytic on* $\mathbb{D}$ *with* $\mathrm{Re}\, f \in h_\infty(\mathbb{D})$, *and let* $q > 0$, $m \ge 1$, $|z| = r < 1$. *Then the inequality*

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \le \frac{2r^m}{(1-r^q)^{1/q}} \sup_{|\zeta|<1} [|\mathrm{Re}\, f(\zeta)| - |\mathrm{Re}\, f(0)|] \tag{3.4}$$

holds and is sharp.

Proof. Setting

$$\omega = \mathcal{R}(f) := \sup_{|\zeta|<1} |\operatorname{Re} f(\zeta)|$$

in (3.2) and making use of the equalities

$$\| \operatorname{Re} f - \mathcal{R}(f)\|_{h_1(\mathbb{D}_\varrho)} = M_1(\operatorname{Re} f - \mathcal{R}(f), \varrho) = \sup_{|\zeta|<1} [|\operatorname{Re} f(\zeta)| - \operatorname{Re} f(0)],$$

we arrive at

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q \right)^{1/q} \leq \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \sup_{|\zeta|<1} [|\operatorname{Re} f(\zeta)| - \operatorname{Re} f(0)].$$

This estimate leads to

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q \right)^{1/q} \leq \frac{2r^m}{(1 - r^q)^{1/q}} \sup_{|\zeta|<1} [|\operatorname{Re} f(\zeta)| - \operatorname{Re} f(0)] \tag{3.5}$$

after the passage to the limit as $\varrho \uparrow 1$. Replacing f by $-f$ in the last inequality, we obtain

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q \right)^{1/q} \leq \frac{2r^m}{(1 - r^q)^{1/q}} \sup_{|\zeta|<1} [|\operatorname{Re} f(\zeta)| + \operatorname{Re} f(0)],$$

which together with (3.5) results in (3.4).

Let us show that the constant in (3.4) is sharp. By $C(r)$ we denote the best constant in

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q \right)^{1/q} \leq C(r) \sup_{|\zeta|<1} [|\operatorname{Re} f(\zeta)| - |\operatorname{Re} f(0)|]. \tag{3.6}$$

As shown above, $C(r)$ satisfies (2.4).

We introduce the family of analytic functions in $\overline{\mathbb{D}}$

$$h_\rho(z) := \frac{\rho}{z - \rho} + \frac{\rho^2}{\rho^2 - 1}, \tag{3.7}$$

depending on a parameter $\rho > 1$. After elementary calculations we arrive at

$$\sup_{|\zeta|<1} [|\operatorname{Re} h_\rho(\zeta)| - |\operatorname{Re} h_\rho(0)|] = \frac{1}{\rho + 1}. \tag{3.8}$$

Taking into account the fact that the functions (2.5) and (3.7) differ by a constant, and using (3.6), (2.7) and (3.8), we arrive at (2.9). Passing there to the limit as $\rho \downarrow 1$, we conclude that (2.10) holds, which together with (2.4) proves the sharpness of the constant in (3.4). ∎

The following assertion contains an estimate with the sharp constant for analytic functions in the Hardy space $H_\infty(\mathbb{D})$. It gives a refinement of the estimate

$$|f(z) - f(0)| \le \frac{2r}{1-r} \sup_{|\zeta|<1} [|f(\zeta)| - |f(0)|]$$

which follows from (1.1) with $C = 2$ and is valid for bounded analytic functions in $\mathbb{D}$.

Theorem 3. *Let the function* (1.2) *be analytic and bounded on* $\mathbb{D}$*, and let* $q > 0$*,* $m \ge 1$*,* $|z| = r < 1$*. Then the inequality*

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \le \frac{2r^m}{(1-r^q)^{1/q}} \sup_{|\zeta|<1} [|f(\zeta)| - |f(0)|] \tag{3.9}$$

holds and is sharp.

Proof. Setting

$$\omega = \mathcal{M}(f) := \sup_{|\zeta|<1} |f(\zeta)|$$

in (3.2) and using the equalities

$$\| \operatorname{Re} f - \mathcal{M}(f)\|_{h_1(\mathbb{D}_\varrho)} = M_1(\operatorname{Re} f - \mathcal{M}(f), \varrho) = \sup_{|\zeta|<1} [|f(\zeta)| - \operatorname{Re} f(0)],$$

we obtain

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \le \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \sup_{|\zeta|<1} [|f(\zeta)| - \operatorname{Re} f(0)].$$

Passing here to the limit as $\varrho \uparrow 1$, we find

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \le \frac{2r^m}{(1-r^q)^{1/q}} \sup_{|\zeta|<1} [|f(\zeta)| - \operatorname{Re} f(0)].$$

Replacing f by $fe^{i\alpha}$, we arrive at

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \le \frac{2r^m}{(1-r^q)^{1/q}} \sup_{|\zeta|<1} [|f(\zeta)| - \operatorname{Re}(f(0)e^{i\alpha})],$$

which implies (3.9) by the arbitrariness of α.

Let us show that the constant in (3.9) is sharp. By $C(r)$ we denote the best constant in

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \le C(r) \sup_{|\zeta|<1} [|f(\zeta)| - |f(0)|]. \tag{3.10}$$

As shown above, $C(r)$ satisfies (2.4).

We consider the family h_ρ of analytic functions in $\overline{\mathbb{D}}$, defined by (3.7). After elementary calculations, we arrive at

$$\sup_{|\zeta|<1}[|h_\rho(\zeta)| - |h_\rho(0)|] = \frac{1}{\rho+1}. \tag{3.11}$$

The sharpness of the constant in (3.9) is shown as in Theorem 2 using (3.10) and (3.11) instead of (3.6) and (3.8). ■

Remark 3. The inequality

$$|f^{(n)}(0)| \leq n!\,\frac{\mathcal{M}(f)^2 - |f(0)|^2}{\mathcal{M}(f)}, \tag{3.12}$$

with $f \in H_\infty(\mathbb{D})$ and $\mathcal{M}(f)$ being the supremum of $|f(z)|$ on $\mathbb{D}$, was obtained by Landau (see [31, pp. 305–306]) for $n = 1$ and by F. Wiener (see Bohr [14], Jensen [29]) for all n.

A consequence of (3.12) is the inequality

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \leq \frac{r^m}{(1-r^q)^{1/q}}\,\frac{\mathcal{M}(f)^2 - |f(0)|^2}{\mathcal{M}(f)} \tag{3.13}$$

with the constant factor on the right-hand side smaller by a factor 2 as in (3.9) and sharp which can be checked using the sequence of functions given by (3.7) and the limit passage as $\rho \downarrow 1$.

A proof of (3.12), different from that one given by Wiener was found by Paulsen, Popescu and Singh [36] who derived (3.13) for $q = 1, m = 1$ and $\mathcal{M}(f) \leq 1$. A generalization of (3.12) is due to Jensen [29]. Sharp estimates for derivatives of f at an arbitrary point of the disk, involving (3.12), are due to Ruscheweyh [38] who applied classical methods. A different approach to these estimates and their generalizations was worked out by Anderson and Rovnyak [10], Bénéteau, Dahlner and Khavinson [11] and MacCluer, Stroethoff and Zhao [34]. Extensions to several variables can be found in the articles by Bénéteau, Dahlner and Khavinson [11] and MacCluer, Stroethoff and Zhao [34].

The next assertion refines the inequality

$$|f(z) - f(0)| \leq \frac{2r}{1-r}\,\mathrm{Re}\, f(0)$$

resulting from (1.1) with $C = 2$ for analytic functions in $\mathbb{D}$ with $\mathrm{Re}\, f > 0$.

Theorem 4. *Let the function* (1.2) *be analytic with positive* $\mathrm{Re}\, f$ *on* $\mathbb{D}$, *and let* $q > 0$, $m \geq 1$, $|z| = r < 1$. *Then the inequality*

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \leq \frac{2r^m}{(1-r^q)^{1/q}}\,\mathrm{Re}\, f(0) \tag{3.14}$$

holds and is sharp.

Proof. Setting $\omega = 0$ in (3.2), with f such that $\operatorname{Re} f > 0$ in $\mathbb{D}$, we obtain

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \le \frac{2r^m}{\varrho^{m-1}(\varrho^q - r^q)^{1/q}} \operatorname{Re} f(0),$$

which leads to (3.14) as $\varrho \uparrow 1$.

Thus, the sharp constant $C(r)$ in

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \le C(r) \operatorname{Re} f(0) \tag{3.15}$$

satisfies the estimate (2.4).

To show the sharpness of the constant in (3.14), consider the family of analytic functions in $\overline{\mathbb{D}}$

$$w_\rho(z) := \frac{\rho}{\rho - z} - \frac{\rho}{\rho + 1}, \tag{3.16}$$

depending on the parameter $\rho > 1$. The real part of w_ρ is positive in $\mathbb{D}$. Taking into account that the functions (2.5) and (3.16) differ by a constant and using (3.15), (2.7) and $\operatorname{Re} w_\rho(0) = (\rho+1)^{-1}$, we arrive at (2.9). Passing there to the limit as $\rho \downarrow 1$, we obtain (2.10), which together with (2.4) proves the sharpness of the constant in (3.14). ∎

4. Bohr type modulus and real part theorems

In this section we collect some corollaries of the theorems in Section 3.

Corollary 1. *Let the function* (1.2) *be analytic on* $\mathbb{D}$, *and let*

$$\sup_{|\zeta|<1} \operatorname{Re}(e^{-i \arg f(0)} f(\zeta)) < \infty,$$

where $\arg f(0)$ *is replaced by zero if* $f(0) = c_0 = 0$.

Then for any $q \in (0, \infty]$, *integer* $m \ge 1$, *and* $|z| \le r_{m,q}$ *the inequality*

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q\right)^{1/q} \le \sup_{|\zeta|<1} \operatorname{Re}(e^{-i \arg f(0)} f(\zeta)) - |f(0)| \tag{4.1}$$

holds, where $r_{m,q}$ *is the root of the equation* $2^q r^{mq} + r^q - 1 = 0$ *in the interval* $(0,1)$ *if* $0 < q < \infty$, *and* $r_{m,\infty} := 2^{-1/m}$. *Moreover,* $r_{m,q}$ *is the radius of the largest disk centered at* $z = 0$ *in which* (4.1) *takes place for all* f. *In particular,* (1.6) *holds.*

Proof. Obviously, the condition

$$\frac{2r^m}{(1 - r^q)^{1/q}} \le 1$$

for the sharp constant in (3.1) holds if $|z| \le r_{m,q}$. Therefore, the disk of radius $r_{m,q}$ centered at $z = 0$ is the largest disk where the inequality

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q \right)^{1/q} \le \sup_{|\zeta|<1} \operatorname{Re} f(\zeta) - \operatorname{Re} f(0) \tag{4.2}$$

holds for all f. The last inequality coincides with (4.1) for $f(0) = c_0 = 0$.

Suppose now that $f(0) \ne 0$. Setting $e^{-i \arg f(0)} f$ in place of f in (4.2) and noting that the coefficients $|c_n|$ in the left-hand side of (4.2) do not change, when $\operatorname{Re} f(0)$ is replaced by $|f(0)| = |c_0|$, we arrive at (4.1). ∎

Inequality (4.1) with $q = 1, m = 1$ becomes

$$\sum_{n=1}^{\infty} |c_n z^n| \le \sup_{|\zeta|<1} \operatorname{Re}(e^{-i \arg f(0)} f(\zeta)) - |f(0)| \tag{4.3}$$

with $|z| \le 1/3$, where $1/3$ is the radius of the largest disk centered at $z = 0$ in which (4.3) takes place. Note that (4.3) is equivalent to a sharp inequality obtained by Sidon [39] in his proof of Bohr's Theorem and to the inequality derived by Paulsen, Popescu and Singh [36].

For $q = 1, m = 2$ the inequality (4.1) is

$$\sum_{n=2}^{\infty} |c_n z^n| \le \sup_{|\zeta|<1} \operatorname{Re}(e^{-i \arg f(0)} f(\zeta)) - |f(0)|, \tag{4.4}$$

where $|z| \le 1/2$ and $1/2$ is the radius of the largest disk about $z = 0$ in which (4.4) takes place.

The next assertion follows from Theorem 3. For $q = 1$, $m = 1$ it contains Bohr's inequality (1.3).

Corollary 2. *Let the function* (1.2) *be analytic and bounded on* $\mathbb{D}$. *Then for any* $q \in (0, \infty]$, *integer* $m \ge 1$, *and* $|z| \le r_{m,q}$ *the inequality*

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q \right)^{1/q} \le \sup_{|\zeta|<1} |f(\zeta)| - |f(0)| \tag{4.5}$$

holds, where $r_{m,q}$ *is defined in Corollary 1. Moreover,* $r_{m,q}$ *is the radius of the largest disk centered at* $z = 0$ *in which* (4.5) *takes place for all* f. *In particular,* (1.6) *holds.*

For $q = 1$, $m = 2$ the inequality (4.5) takes the form

$$|c_0| + \sum_{n=2}^{\infty} |c_n z^n| \le \sup_{|\zeta|<1} |f(\zeta)|, \tag{4.6}$$

where $|z| \le 1/2$. The value 1/2 of the radius of the disk where (4.6) holds cannot be improved. Note that the inequality

$$|c_0|^2 + \sum_{n=1}^{\infty} |c_n z^n| \le 1, \tag{4.7}$$

was obtained by Paulsen, Popescu and Singh [36] for functions (1.2) satisfying the condition $|f(\zeta)| \le 1$ in $\mathbb{D}$ and is valid for $|z| \le 1/2$. The value 1/2 of the radius of the disk where (4.7) holds is sharp. Comparison of (4.6) and (4.7) shows that none of these inequalities is a consequence of the other one.

We conclude this section by an assertion which follows from Theorem 4.

Corollary 3. *Let the function* (1.2) *be analytic, and* $\mathrm{Re}(e^{-i \arg f(0)} f) > 0$ *on* $\mathbb{D}$. *Then for any* $q \in (0, \infty]$, *integer* $m \ge 1$, *and* $|z| \le r_{m,q}$ *the inequality*

$$\left(\sum_{n=m}^{\infty} |c_n z^n|^q \right)^{1/q} \le |f(0)| \tag{4.8}$$

holds, where $r_{m,q}$ *is the same as in Corollary 1. Moreover,* $r_{m,q}$ *is the radius of the largest disk centered at* $z = 0$ *in which* (4.8) *takes place for all* f. *In particular,* (1.6) *holds.*

Note that the inequality (4.8) for $q = 1$, $m = 1$ with $|z| \le 1/3$ was obtained by Aizenberg, Aytuna and Djakov [4] (see also Aizenberg, Grossman and Korobeinik [7]).

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