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On the Zeros of $af(f^{(k)})^n - 1$ for $n \geq 2$

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Abstract. Let f be a transcendental meromorphic function and n, k be two positive integers. Then $af(f^{(k)})^n - 1$, $n \geq 2$, has infinitely many zeros, where $a(z) \not\equiv 0$ is a meromorphic function such that $T(r, a) = S(r, f)$.

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1. Introduction

In this paper, we are going to use mainly the basic notation of Nevanlinna theory [3] such as $T(r, f)$, $N(r, f)$, $m(r, f)$ and $S(r, f) = o(T(r, f))$. Unless otherwise stated, *meromorphic* will mean meropmorphic in the plane. Hayman [2] proved the following theorem in 1959.

Theorem 1. *Suppose that f is a transcendental meromorphic function and n is a positive integer. Then $f'f^n$ assumes every finite non-zero value infinitely often when $n \geq 3$.*

In 1995, Bergweiler and Eremenko [1] showed that the theorem is true for the remaining cases too, i.e. when $n \geq 1$.

Theorem 2. *Suppose that f is a transcendental meromorphic function and $m > l$ are two positive integers. Then $(f^m)^{(l)}$ assumes every finite non-zero value infinitely often.*

In 1993, C. Yang, L. Yang and Y. Wang [5] conjectured the following.

Conjecture. *Suppose that f is a transcendental entire function and that n, k are two positive integers. Then $f(f^{(k)})^n$ assumes every finite non-zero value infinitely often when $n \geq 2$.*

In 1998, Zhang and Song [6] proved the following result.

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Theorem 3. *Suppose that f is a transcendental meromorphic function and that n, k are two positive integers. Then $f(f^{(k)})^n - A(z)$, $n \geq 2$, has infinitely many zeros where $A(z) \not\equiv 0$ is a small function such that $T(r, A) = S(r, f)$.*

In fact, the proof of Theorem 3 is very complicated and there appear to be some gaps in it. We give a much simpler proof, with some generalisations, by proving Theorem 4.

Before stating this theorem, we make some assumptions which we need throughout this paper. Suppose that f is a transcendental meromorphic function in the plane and $a(z) \not\equiv 0$ is a meromorphic function such that

$$T(r, a) = S(r, f).$$

Let

$$L(w) := w^{(k)} + b_{k-1}w^{(k-1)} + \cdots + b_0 w, \quad k \in \mathbb{N}, \tag{1}$$

where each $b_j(z)$, $j = 0, 1, \ldots, k-1$, is a meromorphic function such that

$$T(r, b_j) = S(r, f). \tag{2}$$

Let $n \in \mathbb{N}$, $n \geq 2$, and set

$$g := L(f), \tag{3}$$

$$\psi := afg^n - 1. \tag{4}$$

Theorem 4. *Suppose that f is a transcendental meromorphic function in the plane. Suppose that L, g and the b_j are given by* (1), (2), (3), *and that*

$$T(r, g) \neq S(r, f), \tag{5}$$

$$T(r, \phi) = S(r, f) \tag{6}$$

for every solution ϕ of $L(w) = 0$ which is meromorphic in the plane. Then

$$T(r, f) \leq \frac{1}{1 - \delta_k} \frac{n(k+1)}{n-1} \bar{N}\left(r, \frac{1}{\psi}\right) + S(r, f),$$

where

$$\delta_k = \frac{n(k+1)}{n-1} \frac{1}{1 + n(k+1)}, \qquad 0 < \delta_k < \frac{1}{n-1} \leq 1. \tag{7}$$

2. Proof of Theorem 4

We first prove a lemma needed in the proof of Theorem 4.

Lemma 1.

$$T(r, f) \leq T(r, \psi) + S(r, f), \qquad \textit{whenever } g \not\equiv 0.$$

In particular, $afL(f)^n$ is non-constant.

Proof. Using (4), we have $f = (\psi + 1)/(ag^n)$. So,

$$N\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{\psi+1}\right) + N(r, a) + n\sum_{j=0}^{k-1} N(r, b_j) = N\left(r, \frac{1}{\psi+1}\right) + S(r, f).$$

Also, using (1), (3) and (4) we have

$$\frac{1}{f} = \frac{ag^n}{\psi+1}, \qquad \frac{1}{f^{n+1}} = \frac{a}{\psi+1}\left(\frac{f^{(k)}}{f} + b_{k-1}\frac{f^{(k-1)}}{f} + \cdots + b_0\right)^n.$$

Therefore,

$$\begin{aligned} m\left(r, \frac{1}{f}\right) &\leq (n+1)m\left(r, \frac{1}{f}\right) = m\left(r, \frac{1}{f^{n+1}}\right) \\ &\leq m\left(r, \frac{1}{\psi+1}\right) + m(r, a) + S(r, f) = m\left(r, \frac{1}{\psi+1}\right) + S(r, f). \end{aligned}$$

Hence,

$$\begin{aligned} T(r, f) &= T\left(r, \frac{1}{f}\right) + \mathcal{O}(1) = m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + \mathcal{O}(1) \\ &\leq m\left(r, \frac{1}{\psi+1}\right) + N\left(r, \frac{1}{\psi+1}\right) + S(r, f) \\ &= T\left(r, \frac{1}{\psi+1}\right) + S(r, f) = T(r, \psi) + S(r, f). \end{aligned}$$

■

Proof of Theorem 4. A zero of g of multiplicity p with $a \neq 0, \infty$ and with $b_j \neq \infty$ is a zero of ψ' of multiplicity at least $np-1 \geq (n-1)p$. Also, $\psi = -1 \neq 0$ at such a zero of g. Thus

$$\begin{aligned} \bar{N}\left(r, \frac{1}{g}\right) &\leq \frac{1}{n-1}N\left(r, \frac{\psi}{\psi'}\right) + \bar{N}(r, a) + \bar{N}\left(r, \frac{1}{a}\right) + \sum_{j=0}^{k-1}\bar{N}(r, b_j) \\ &\leq \frac{1}{n-1}N\left(r, \frac{\psi'}{\psi}\right) + S(r, f) \\ &= \frac{1}{n-1}\left[\bar{N}(r, \psi) + \bar{N}\left(r, \frac{1}{\psi}\right)\right] + S(r, f) \qquad (8) \\ &\leq \frac{1}{n-1}\left[\bar{N}(r, f) + \bar{N}(r, a) + \sum_{j=0}^{k-1}\bar{N}(r, b_j) + \bar{N}\left(r, \frac{1}{\psi}\right)\right] + S(r, f) \\ &= \frac{1}{n-1}\left[\bar{N}(r, f) + \bar{N}\left(r, \frac{1}{\psi}\right)\right] + S(r, f). \end{aligned}$$

Put $b_k = 1$, $b_{-1} = b_{k+1} = 0$. So,

$$g = \sum_{j=0}^{k} b_j f^{(j)} = \sum_{j=0}^{k+1} b_j f^{(j)}.$$

Using the fact that $b'_{k+1} = 0$ and $b_{-1} = 0$, we have

$$g' = \sum_{j=0}^{k} \left(b'_j f^{(j)} + b_j f^{(j+1)} \right) = \sum_{j=0}^{k} b'_j f^{(j)} + \sum_{j=1}^{k+1} b_{j-1} f^{(j)} = \sum_{j=0}^{k+1} \left(b'_j + b_{j-1} \right) f^{(j)}.$$

Since $g = \sum_{j=0}^{k+1} b_j f^{(j)}$, we find that $w = f$ solves

$$\sum_{j=0}^{k+1} c_j w^{(j)} = 0, \tag{9}$$

where

$$c_j = b'_j + b_{j-1} - \frac{g'}{g} b_j, \quad c_{k+1} = 1. \tag{10}$$

Let

$$w = uv, \qquad v = \frac{1}{ag^n}. \tag{11}$$

Using Leibnitz' rule in (9), we get

$$0 = \sum_{j=0}^{k+1} c_j (uv)^{(j)} = \sum_{j=0}^{k+1} c_j \sum_{m=0}^{j} \binom{j}{m} u^{(m)} v^{(j-m)} = \sum_{j=0}^{k+1} c_j \sum_{m=0}^{k+1} \binom{j}{m} u^{(m)} v^{(j-m)},$$

using the convention that $\binom{j}{m} = 0$ for $m > j$. Dividing the right hand side by v, we get

$$0 = \sum_{m=0}^{k+1} u^{(m)} \sum_{j=0}^{k+1} \binom{j}{m} c_j \frac{v^{(j-m)}}{v} = \sum_{m=0}^{k+1} u^{(m)} A_m, \tag{12}$$

where, again since $\binom{j}{m} = 0$ for $m > j$,

$$A_m = \sum_{j=m}^{k+1} \binom{j}{m} c_j \frac{v^{(j-m)}}{v}. \tag{13}$$

In particular, this gives, using (10),

$$A_{k+1} = c_{k+1} = 1$$

and

$$A_0 = \sum_{j=0}^{k+1} c_j \frac{v^{(j)}}{v} = \sum_{j=0}^{k+1} \left(b'_j + b_{j-1} - \frac{g'}{g} b_j \right) \frac{v^{(j)}}{v} = \frac{L(v)' - \frac{g'}{g} L(v)}{v}. \tag{14}$$

We now claim that $A_0 \not\equiv 0$. To prove this claim, suppose that $A_0 \equiv 0$. Using (14), we get

$$L(v)' = \frac{g'}{g} L(v). \tag{15}$$

We consider two cases.

Case 1, $L(v) \not\equiv 0$*:* Using (15), we have

$$\frac{L(v)'}{L(v)} = \frac{g'}{g}, \qquad L(v) = cg,$$

where $0 \neq c \in \mathbb{C}$ since $L(v) \not\equiv 0$. Using (3), we have $L(v) = cL(f)$. Solving this, we get

$$v = cf + h, \quad L(h) = 0, \qquad v = c(f + H), \quad L(H) = 0.$$

Let $F = f + H$. This gives

$$v = cF, \qquad L(f) = L(f + H) = L(F).$$

Since $v = cF$ and $v = 1/(ag^n)$, we have

$$\frac{1}{ag^n} = cF, \qquad 1 = acL(f)^n F = acL(F)^n F.$$

Since $L(H) = 0$ and $H = (v - cf)/c$ is meromorphic in the plane, we get $T(r, H) = S(r, f)$ by (6). Hence

$$T(r, f) = T(r, F) + S(r, F), \qquad T(r, a) + \sum_{j=0}^{k-1} T(r, b_j) = S(r, f) = S(r, F).$$

This contradicts Lemma 1, applied to $F = f + H$.

Case 2, $L(v) \equiv 0$*:* Using (6), we get $T(r, v) = S(r, f)$. Using (11), we have $v = 1/(ag^n)$. Thus, $T(r, g) = S(r, f)$ which is a contradiction to the first hypothesis given by equation (5). So, the claim is proved.

Returning to the proof of Theorem 4, we have, using (4) and (11),

$$u = \frac{w}{v} = afg^n = \psi + 1.$$

So $\psi + 1$ solves (12), which gives

$$\begin{aligned}
(\psi + 1)^{(k+1)} + A_k(\psi + 1)^{(k)} + \cdots + A_1(\psi + 1)' + A_0(\psi + 1) &= 0, \\
\psi^{(k+1)} + A_k \psi^{(k)} + \cdots + A_1 \psi' + A_0(\psi + 1) &= 0, \\
\frac{\psi^{(k+1)}}{A_0} + \frac{A_k}{A_0}\psi^{(k)} + \cdots + \frac{A_1}{A_0}\psi' + \psi + 1 &= 0, \\
\frac{1}{A_0}\left(\frac{\psi^{(k+1)}}{\psi} + A_k \frac{\psi^{(k)}}{\psi} + \cdots + A_1 \frac{\psi'}{\psi}\right) + 1 + \frac{1}{\psi} &= 0.
\end{aligned}$$

Thus,

$$\frac{1}{\psi} = \frac{-1}{A_0}\left(\frac{\psi^{(k+1)}}{\psi} + A_k\frac{\psi^{(k)}}{\psi} + \cdots + A_1\frac{\psi'}{\psi}\right) - 1. \tag{16}$$

Using (13) we note that

$$A_m = \binom{k+1}{m}\frac{v^{(k+1-m)}}{v} + \sum_{j=m}^{k}\binom{j}{m}c_j\frac{v^{(j-m)}}{v}.$$

Hence the contribution to $n(r, A_m)$ from the terms $v^{(j-m)}/v$ and g'/g is at most $k+1-m$, and the contribution to $n(r, A_0)$ and $n(r, A_m\psi^{(m)}/\psi)$ from these terms is at most $k+1$. Furthermore, using (14) we see that any pole of A_0 can only occur at poles or zeros of g, poles or zeros of $a(z)$, or poles of $b_j(z)$. So

$$\begin{aligned} N(r, A_0) &\leq (k+1)\left[\bar{N}(r,g) + \bar{N}\left(r, \frac{1}{g}\right) + \bar{N}(r,a) + \bar{N}\left(r,\frac{1}{a}\right)\right] + 2\sum_{j=0}^{k-1}N(r,b_j) \\ &\leq (k+1)\left[\bar{N}(r,f) + \bar{N}\left(r,\frac{1}{g}\right)\right] + S(r,f). \end{aligned}$$

Using (8), we have

$$\begin{aligned} N(r, A_0) &\leq (k+1)\left[\bar{N}(r,f) + \frac{1}{n-1}\bar{N}(r,f) + \frac{1}{n-1}\bar{N}\left(r,\frac{1}{\psi}\right)\right] + S(r,f) \\ &= (k+1)\left(1+\frac{1}{n-1}\right)\bar{N}(r,f) + \frac{k+1}{n-1}\bar{N}\left(r,\frac{1}{\psi}\right) + S(r,f) \qquad (17) \\ &= \frac{n(k+1)}{n-1}\bar{N}(r,f) + \frac{k+1}{n-1}\bar{N}\left(r,\frac{1}{\psi}\right) + S(r,f). \end{aligned}$$

Using (1), (3), and (4), we have

$$\begin{aligned} \frac{\psi+1}{a} &= fg^n = f\left(f^{(k)} + b_{k-1}f^{(k-1)} + \cdots + b_0 f\right)^n \\ &= f(f^{(k)})^n\left(1 + b_{k-1}\frac{f^{(k-1)}}{f^{(k)}} + \cdots + b_0\frac{f}{f^{(k)}}\right)^n. \end{aligned}$$

So, a pole of f of multiplicity p with $b_j \neq \infty$ is a pole of $(\psi+1)/a$ of multiplicity $p + (p+k)n \geq 1 + (1+k)n$. Thus,

$$\begin{aligned} \bar{N}(r,f) &\leq \frac{1}{1+n(1+k)}N\left(r,\frac{\psi+1}{a}\right) + \sum_{j=0}^{k-1}\bar{N}(r,b_j) \\ &\leq \frac{1}{1+n(1+k)}N(r,\psi+1) + S(r,f) \qquad (18) \\ &\leq \frac{1}{1+n(1+k)}T(r,\psi) + S(r,f). \end{aligned}$$

Using (7), (17), and (18), we get

$$\begin{aligned} N(r, A_0) &\leq \frac{n(k+1)}{n-1} \frac{1}{1+n(1+k)} T(r,\psi) + \frac{k+1}{n-1} \bar{N}\left(r, \frac{1}{\psi}\right) + S(r,f) \\ &= \delta_k T(r,\psi) + \frac{k+1}{n-1} \bar{N}\left(r, \frac{1}{\psi}\right) + S(r,f). \end{aligned} \tag{19}$$

Using (13) and (16), we have

$$N\left(r, \frac{1}{\psi}\right) \leq N\left(r, \frac{1}{A_0}\right) + (k+1)\bar{N}\left(r, \frac{1}{\psi}\right) + S(r,f), \tag{20}$$

$$m\left(r, \frac{1}{\psi}\right) \leq m\left(r, \frac{1}{A_0}\right) + S(r,f). \tag{21}$$

Using (19), (20), and (21), we get

$$\begin{aligned} T(r,\psi) &= T\left(r, \frac{1}{\psi}\right) + \mathcal{O}(1) = m\left(r, \frac{1}{\psi}\right) + N\left(r, \frac{1}{\psi}\right) + \mathcal{O}(1) \\ &\leq m\left(r, \frac{1}{A_0}\right) + N(r, \frac{1}{A_0}) + (k+1)\bar{N}\left(r, \frac{1}{\psi}\right) + S(r,f) \\ &= T\left(r, \frac{1}{A_0}\right) + (k+1)\bar{N}\left(r, \frac{1}{\psi}\right) + S(r,f) \\ &= T(r, A_0) + (k+1)\bar{N}\left(r, \frac{1}{\psi}\right) + S(r,f) \\ &= N(r, A_0) + (k+1)\bar{N}\left(r, \frac{1}{\psi}\right) + S(r,f) \\ &\leq \delta_k T(r,\psi) + \frac{k+1}{n-1}\bar{N}\left(r, \frac{1}{\psi}\right) + (k+1)\bar{N}\left(r, \frac{1}{\psi}\right) + S(r,f) \\ &= \delta_k T(r,\psi) + \frac{n(k+1)}{n-1}\bar{N}\left(r, \frac{1}{\psi}\right) + S(r,f). \end{aligned}$$

Thus,

$$\begin{aligned} (1-\delta_k)T(r,\psi) &\leq \frac{n(k+1)}{n-1}\bar{N}\left(r, \frac{1}{\psi}\right) + S(r,f), \\ T(r,\psi) &\leq \frac{1}{1-\delta_k}\frac{n(k+1)}{n-1}\bar{N}\left(r, \frac{1}{\psi}\right) + S(r,f). \end{aligned}$$

Using Lemma 1, we get

$$T(r,f) \leq \frac{1}{1-\delta_k}\frac{n(k+1)}{n-1}\bar{N}\left(r, \frac{1}{\psi}\right) + S(r,f).$$

This completes the proof of Theorem 4. ∎

3. Examples and corollaries

Now, we will give two examples to show that without the hypotheses stated in Theorem 4 we can find $\psi = afg^n - 1$ with no zeros. In the first example, we do not have the first hypothesis, i.e. equation (5), and in the second example we do not have the second hypothesis, i.e. equation (6).

Example 1. Suppose that

$$f(z) = e^z + z, \qquad L(w) = w'' - 2w' + w.$$

This gives

$$g = L(f) = z - 2, \qquad T(r, g) = S(r, f), \qquad fg^2 = (e^z + z)(z - 2)^2.$$

Let $a = 1/[z(z-2)^2]$. So, $T(r, a) = S(r, f)$. From all of these, we see that

$$\psi = afg^2 - 1 = \frac{1}{z(z-2)^2}(e^z + z)(z-2)^2 - 1 = \frac{e^z + z}{z} - 1$$
$$= \frac{e^z}{z} + 1 - 1 = \frac{e^z}{z} \neq 0.$$

Example 2. Suppose that

$$L(w) = w' - w, \qquad \psi = fg^2 - 1,$$

where

$$f = e^z - \frac{2}{3}e^{-z/2}.$$

So we have

$$g = f' - f = e^z + \frac{1}{3}e^{-z/2} - e^z + \frac{2}{3}e^{-z/2} = e^{-z/2}.$$

Thus,

$$\psi = fg^2 - 1 = (e^z - \frac{2}{3}e^{-z/2})(e^{-z/2})^2 - 1 = (e^z - \frac{2}{3}e^{-z/2})e^{-z} - 1$$
$$= 1 - \frac{2}{3}e^{-3z/2} - 1 = -\frac{2}{3}e^{-3z/2} \neq 0.$$

Here, $\phi = e^z$ solves $L(w) = 0$, but $T(r, \phi) \neq S(r, f)$.

Corollary 1. *Suppose that f is a transcendental meromorphic function in the plane. Let*

$$\psi = afg^n - 1, \qquad n \geq 2,\ n, k \in \mathbb{N},$$

where $a \not\equiv 0$ is a meromorphic function,

$$T(r, a) = S(r, f), \qquad T(r, g) \neq S(r, f), \qquad T(r, \phi) = S(r, f),$$

for every solution ϕ of $L(w) = 0$ which is meromorphic in the plane. Then, $\psi = afg^n - 1$ has infinitely many zeros and the function afg^n assumes every non-zero value infinitely often.

Corollary 2. *Suppose that f is a transcendental meromorphic function in the plane. Let $a \not\equiv 0$ be meromorphic in $\mathbb{C}$ with $T(r,a) = S(r,f)$ and let*

$$\psi = af(f^{(k)})^n - 1, \qquad n \geq 2,\ n,k \in \mathbb{N}.$$

Then,

$$T(r,f) \leq \frac{1}{1-\delta_k}\frac{n(k+1)}{n-1}\bar{N}\left(r,\frac{1}{\psi}\right) + S(r,f).$$

Proof. We have here $g = f^{(k)}$. So, using the Hayman-Miles Theorem [4], we have $T(r,g) \neq S(r,f)$. Also, we have here $L(w) = w^{(k)}$. Thus, every ϕ which is a solution of $L(w) = 0$ would be a polynomial. So, $T(r,\phi) = S(r,f)$. ■

Theorem 3 follows at once from Corollary 2, using $A = 1/a$.

Corollary 3. *Suppose that f is a transcendental meromorphic function in the plane. Let $a \not\equiv 0$ be meromorphic in $\mathbb{C}$ with $T(r,a) = S(r,f)$ and let*

$$\psi = af(f^{(k)})^n - 1, \qquad n \geq 2,\ n,k \in \mathbb{N}.$$

Then, $\psi = af(f^{(k)})^n - 1$ has infinitely many zeros and the function $af(f^{(k)})^n$ assumes every non-zero value infinitely often.

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