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Unbounded Solutions of the Stationary Schrödinger Equation on Riemannian Manifolds

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Abstract. We study questions of existence and membership to a given functional class of unbounded solutions of the stationary Schrödinger equation $\Delta u - cu = 0$, where c is a smooth non-negative function on a non-compact Riemannian manifold M without boundary. We establish some interrelation between problems of existence of solutions of this equation on M and off some compact $B \subset M$ with the same growth "at infinity".

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1. Introduction and main results

This article is devoted to the investigation of the behavior of solutions of the stationary Schrödinger equation in relation to the geometry of the manifold in question. Such problems originate in the classification theory of non-compact Riemannian surfaces and manifolds. For a non-compact Riemann surface, the well-known problem of conformal type identification can be stated as follows: does a non-trivial positive superharmonic function exist on this surface?

Many questions of this kind fit into the pattern of a Liouville type theorem stating that the space of bounded solutions of some elliptic equation is trivial. In a number of papers conditions ensuring the validity of the Liouville property on non-compact Riemannian manifolds have been derived in terms of volume growth, or isoperimetric inequalities, and so on (see [5, 7, 8, 11, 12, 13]). However, the class of manifolds admitting non-trivial solutions of some elliptic equations is large. For example, conditions ensuring the solvability of the Dirichlet problem with continuous boundary conditions "at infinity" for several non-compact manifolds have been given in many papers (see, e.g. [1, 2, 3, 7, 8]). Notice that the

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very statement of the Dirichlet problem on such manifolds could turn out to be non-trivial, since it is unclear how we should interpret the boundary data.

We study questions of existence and membership to given functional class of unbounded solutions of the stationary Schrödinger equation

$$Lu \equiv \Delta u - cu = 0, \tag{1}$$

where c is a smooth non-negative function on a non-compact Riemannian manifold M without boundary.

Of keen interest is the interrelation between problems of existence of solutions of equation (1) on M and off some compact $B \subset M$ with the same growth "at infinity". For example, in [6] a connection was established between the validity of Liouville type theorems for bounded solutions of equation (1) and the unique solvability of exterior boundary problems on arbitrary Riemannian manifolds. In this article we compare the behavior of unbounded functions "at infinity". In our investigation we use a new approach based on the consideration of equivalence classes of functions on M (for bounded solutions this approach has been realized in [9]).

Let M be an arbitrary smooth connected non-compact Riemannian manifold without boundary and let $\{B_k\}_{k=1}^{\infty}$ be an exhaustion of M, i.e. a sequence of precompact open subsets of M such that $\overline{B_k} \subset B_{k+1}$ and $M = \cup_{k=1}^{\infty} B_k$. Throughout the sequel, we assume that boundaries ∂B_k are $\mathcal{C}^1$-smooth submanifolds.

Let f_1 and f_2 be arbitrary continuous functions on M. We say that f_1 and f_2 are *equivalent on* M and write $f_1 \sim f_2$ if for some exhaustion $\{B_k\}_{k=1}^{\infty}$ of M we have

$$\lim_{k\to\infty} \sup_{M\setminus B_k} |f_1 - f_2| = 0.$$

It is easy to verify that the relation "$\sim$" is an equivalence relation which does not depend on the choice of the exhaustion of the manifold and so partitions the set of all continuous functions on M into equivalence classes. Denote the equivalence class of a function f by $[f]$.

Let $B \subset M$ be an arbitrary connected compact subset and let the boundary of B be a $\mathcal{C}^1$-smooth submanifold. Assume that the interior of B is non-empty and $B \subset B_k$ for all k.

Observe that if the manifold M has compact boundary or there is a natural geometric compactification of M (for example, on manifolds of negative sectional curvature or spherically symmetric manifolds) which adds the boundary at infinity, then this approach leads naturally to the classical statement of the Dirichlet problem (see, e.g. [1, 2, 3, 7, 8]).

Denote by v_k the solution of equation (1) in $B_k \setminus B$ which satisfies the conditions

$$v_k\big|_{\partial B} = 1, \qquad v_k\big|_{\partial B_k} = 0.$$

Using the Maximum Principle, we can easily verify that the sequence v_k is uniformly bounded on $M \setminus B$ and so is compact in the class of twice continuously differentiable functions over every compact subset $G \subset M \setminus B$. Moreover, as k tends to infinity this sequence increases monotonically and converges on $M \setminus B$ to a solution of equation (1)

$$v = \lim_{k\to\infty} v_k, \qquad 0 < v \le 1, \qquad v\big|_{\partial B} = 1.$$

Also, note that the function v is independent of the choice of exhaustion $\{B_k\}_{k=1}^{\infty}$.

We call v the *L-potential of the compact set B relative to M*. For the Laplace-Beltrami equation, the function v is nothing but the capacity potential of the compact set B relative to the manifold M (see [5]).

The manifold M is called *L-strict* if for some compact set $B \subset M$ there is an L-potential v of B such that $v \in [0]$ (see [9]).

We now formulate the main result.

Theorem 1. *Let M be an L-strict manifold, $B \subset M$ an arbitrary connected compact subset and ∂B an $\mathcal{C}^1$-smooth submanifold. Then the following three conditions are equivalent.*

(i) *There is a solution u on M of equation* (1) *such that $u \in [f]$.*
(ii) *For every function $\Phi \in \mathcal{C}(\partial B)$ there is a solution h on $M \setminus B$ of equation* (1) *such that $h \in [f]$ and $h|_{\partial B} = \Phi$.*
(iii) *When $c > 0$ on B there is a solution u_0 on $M \setminus B$ of equation* (1) *such that $u_0 \in \mathcal{C}^{2,\alpha}(\overline{M \setminus B})$ and $u_0 \in [f]$.*

2. Proof of Theorem 1

Let $v^+ := \max\{0, v\}$ and $v^- := -\min\{0, v\}$. We first prove some auxiliary assertions.

Lemma 2. *Suppose that $G \Subset M$ is a precompact subset in M. Let a function $u \in \mathcal{C}(\overline{G}) \cap \mathcal{C}^2(G)$ satisfy the equation $Lu = g$ on G, where $g \in \mathcal{C}_0(\overline{G})$, $\Omega := \operatorname{supp} g$ and $\Omega \Subset G$, $c \ge 0$ on $\overline{G}$ and $c \ne 0$ on Ω. Then*

$$\sup_G |u| \le \sup_{\partial G} |u| + \sup_\Omega \frac{|g|}{c}.$$

Proof. Consider the constant

$$v = \sup_{\partial G} u^+ + \sup_\Omega \frac{g^-}{c}.$$

Then, on the set $G \setminus \Omega$, we have

$$L(v-u) = Lv - Lu = Lv = -cv = -c\left(\sup_{\partial G} u^+ + \sup_\Omega \frac{g^-}{c}\right) \le 0,$$

and on Ω we have

$$L(v-u) = Lv - Lu = -c\left(\sup_{\partial G} u^+ + \sup_{\Omega} \frac{g^-}{c}\right) - g$$
$$= -c\left(\sup_{\partial G} u^+ + \sup_{\Omega} \frac{g^-}{c} + \frac{g}{c}\right) \le 0.$$

Thus $L(v-u) \le 0$ everywhere on G. Moreover, $v - u \ge 0$ on ∂G. Then $u \le v$ in G and

$$\sup_G u \le \sup_G v = \sup_{\partial G} u^+ + \sup_{\Omega} \frac{g^-}{c}. \tag{2}$$

Replacing u by $-u$ we obtain

$$\sup_G(-u) \le \sup_{\partial G}(-u)^+ + \sup_{\Omega} \frac{(-g)^-}{c}$$

or

$$\sup_G(-u) \le \sup_{\partial G} u^- + \sup_{\Omega} \frac{g^+}{c}. \tag{3}$$

The assertion now follows by combining (2) and (3). ■

A similar statement for bounded domains in $\mathbb{R}^n$ can be found in [4, p.55].

Now we demonstrate that L-strictness is independent of the choice of the compact set. Let $B' \subset M$ be a compact subset of M different from B and let $\partial B'$ be an $\mathcal{C}^1$-smooth submanifold. Let us assume that $B' \subset B_k$ for all k. Denote by v' the L-potential of the compact set B', i.e. $v' = \lim_{k\to\infty} v'_k$, where $Lv'_k = 0$ in $B_k \setminus B'$, $v'_k = 1$ on $\partial B'$ and $v'_k = 0$ on ∂B_k.

Lemma 3. *If $v \sim 0$ then $v' \sim 0$.*

Proof. First we consider the case $B \subset B' \subset B_k$ for all k, where $\{B_k\}_{k=1}^{\infty}$ is an arbitrary exhaustion of the manifold M with smooth boundaries ∂B_k. Since $v > 0$ and $v \sim 0$ we have $\inf_{M\setminus B} v = 0$ and for some constant $c > 0$ we have $c\cdot v|_{\partial B'} > 1$. Applying the Comparison Principle we have $0 \le v'_k < c\cdot v$ on $B_k\setminus B'$ for all k, and hence we have $0 < v' \le c \cdot v$. Since $v \sim 0$ we get $v' \sim 0$.

Now we consider the case $B' \subset B \subset B_k$ for all k. Then $v'_k|_{\partial B} \le 1 = v|_{\partial B}$, $v'_k|_{\partial B_k} = 0 < v|_{\partial B_k}$ and by the Maximum Principle we have $v'_k \le v$ on $B_k \setminus B$ for all k. This implies $0 < v' \le v$. Since $v \sim 0$ again we get $v' \sim 0$.

Finally let B and B' be arbitrary compact subsets of M. We consider an arbitrary compact subset $B'' \subset M$ ($\partial B''$ is a $\mathcal{C}^1$-smooth submanifold) such that $B \cup B' \subset B'' \subset B_k$. It is possible to take $\overline{B_1}$ as B'' and to take the sequence of sets $\{B_k\}_{k=2}^{\infty}$ as exhaustion of M. Let v'' denote the L-potential of the compact set B''.

Then, as was proved above, $v'' \sim v \sim 0$ since $B \subset B''$ and $v' \sim v'' \sim 0$ since $B' \subset B''$. Therefore, $v' \sim 0$ for an arbitrary compact subset B'. ■

Next we turn to the proof of the main result.

Proof of Theorem 1. (i)⇒(ii). First we show that for every continuous function Φ on ∂B there is a solution w of equation (1) on $M \setminus B$ such that $w|_{\partial B} = \Phi$ and $w \in [0]$. Consider the sequence of functions w_k that are solutions to the boundary value problems

$$\begin{cases} Lw_k = 0 \quad \text{in } B_k \setminus B, \\ w_k\big|_{\partial B} = \Phi, \\ w_k\big|_{\partial B_k} = 0. \end{cases}$$

By the Maximum Principle, for every k we have

$$|w_k| \le \sup_{\partial(B_k \setminus B)} |w_k| = \sup_{\partial B} |\Phi|,$$

i.e. the sequence $\{w_k\}_{k=1}^{\infty}$ is uniformly bounded on M and so compact in the class of twice continuously differentiable functions on every compact subset of M. Let $w(x)$ be a limit function. First we show $w|_{\partial B} = \Phi$.

Put $U = \max_{\partial B} |\Phi|$ and consider the functions w^- and w^+ on $B_1 \setminus B$ which are solutions to the boundary value problems

$$\begin{cases} Lw^- = 0 \quad \text{in } B_1 \setminus B, \\ w^-\big|_{\partial B} = \Phi, \\ w^-\big|_{\partial B_1} = -(U+1), \end{cases} \qquad \begin{cases} Lw^+ = 0 \quad \text{in } B_1 \setminus B, \\ w^+\big|_{\partial B} = \Phi, \\ w^+\big|_{\partial B_1} = U+1. \end{cases}$$

Then for all k we have

$$w_k\big|_{\partial B} = w^-\big|_{\partial B} = w^+\big|_{\partial B},$$
$$w^-\big|_{\partial B_1} \le w_k\big|_{\partial B_1} \le w^+\big|_{\partial B_1}$$

and hence, by the Comparison Principle, we have $w^- \le w_k \le w^+$ on $B_1 \setminus B$ for all k. Taking the limit as k tends to infinity we obtain $w^- \le w \le w^+$. Since $w^-|_{\partial B} = w^+|_{\partial B} = \Phi$ it follows that $w|_{\partial B} = \Phi$.

Next, we show that $w \in [0]$. It is obvious that

$$-(U+1) \le \Phi \le U+1,$$
$$-(U+1) \le w\big|_{\partial B} \le U+1$$

and for every k

$$-(U+1) \le w_k\big|_{\partial B} \le U+1.$$

Consider the functions $u_1 = -(U+1) \cdot v$ and $u_2 = (U+1) \cdot v$ on $M \setminus B$, where v is the L-potential of the compact set B, $v \in [0]$. The functions u_1 and u_2 are solutions of (1) and satisfy the conditions

$$u_1\big|_{\partial B} = -(U+1), \qquad -(U+1) \le u_1 \le 0, \qquad u_1 \in [0],$$
$$u_2\big|_{\partial B} = U+1, \qquad 0 \le u_2 \le U+1, \qquad u_2 \in [0].$$

Then $u_1 \le u_2$ on $M \setminus B$ and, by the Comparison Principle, we have

$$u_1 \le w_k \le u_2$$

on $B_k \setminus B$ for all k. Taking the limit as k tends to infinity we obtain $u_1 \le w \le u_2$. Since $u_1 \sim u_2 \sim 0$, we have $w \in [0]$.

Now, let $u \in [f]$ be a solution to the boundary value problem on M and let Φ be an arbitrary continuous function on ∂B. As shown above, there is a solution w of (1) on $M \setminus B$ such that $w|_{\partial B} = u|_{\partial B} - \Phi$ and $w \in [0]$. Then the function $h = u - w$ is a solution to the exterior boundary value problem on $M \setminus B$ such that $h \in [f]$ and $h|_{\partial B} = \Phi$.

We note that a similar statement for the class $[f]$ of bounded continuous functions on M has been proved in [9].

(ii)⇒(iii). This implication is obvious.

(iii)⇒(i). Consider the function $U = u_0 \varphi$, where φ is a smooth function on M such that $\varphi = 0$ on a precompact set $B' \Subset B$ and $\varphi = 1$ outside B. Then $U \in \mathcal{C}^{2,\alpha}(M)$ and

$$LU = \Delta(u_0\varphi) - c \cdot u_0\varphi = g,$$

where the function $g \in \mathcal{C}^{\alpha}(M)$ and satisfies $g = 0$ on the compact set B' and outside the compact set B, respectively $g \neq 0$ on $B \setminus B'$. Thus, $\Omega := \operatorname{supp} g$ is compact and $\Omega \subset B \setminus B'$.

Consider now the sequence of functions φ_k solving the problems

$$\begin{cases} L\varphi_k = 0 \quad \text{in } B_k, \\ \varphi_k\big|_{\partial B_k} = u_0\big|_{\partial B_k} \end{cases}$$

and the sequence of functions $\psi_k := \varphi_k - U$. Obviously, ψ_k solves

$$\begin{cases} L\psi_k = -g \quad \text{in } B_k, \\ \psi_k\big|_{\partial B_k} = 0. \end{cases}$$

By Lemma 2 for any k and any $x \in B_k$ we have

$$|\psi_k| \le \sup_{B_k} |\psi_k| \le \sup_{\partial B_k} |\psi_k| + \sup_{\Omega} \frac{|g|}{c} = \sup_{\Omega} \frac{|g|}{c}$$

showing the uniform boundedness of the family of functions $\{\psi_k\}_{k=1}^{\infty}$ on M. Hence, we obtain compactness of this family in the class $\mathcal{C}^2(G)$ for an arbitrary compact subset $G \subset M$. This in turn implies the existence of a limit function $\psi := \lim_{k\to\infty} \psi_k$ on M such that $L\psi = -g$ on M.

Now we shall show that $\psi \in [0]$. The function ψ is a solution of the equation $L\psi = 0$ on $M \setminus B$. Since ∂B is compact by continuity of the function ψ there exists $A := \max_{\partial B} |\psi|$ and we have

$$-A \le \psi\big|_{\partial B} \le A.$$

Hence we obtain

$$-(A+1) \leq \psi_k\big|_{\partial B} \leq A+1$$

for sufficiently large values k.

Consider the functions

$$\overline{\psi} := (A+1)\cdot v \qquad \text{and} \qquad \underline{\psi} := -(A+1)\cdot v$$

on $M \setminus B$, where v is the L-potential of the compact set B and $v \in [0]$. The functions $\overline{\psi}$ and $\underline{\psi}$ are solutions of (1) and satisfy

$$\begin{aligned} &\overline{\psi}\big|_{\partial B} = A+1, && 0 \leq \overline{\psi} \leq A+1, && \overline{\psi} \in [0], \\ &\underline{\psi}\big|_{\partial B} = -(A+1), && -(A+1) \leq \underline{\psi} \leq 0, && \underline{\psi} \in [0]. \end{aligned}$$

Therefore, $\underline{\psi} \leq \overline{\psi}$ on $M \setminus B$. But since

$$L\underline{\psi} = L\psi_k = L\overline{\psi} = 0,$$
$$\underline{\psi}\big|_{\partial B_k} \leq \psi_k\big|_{\partial B_k} \leq \overline{\psi}\big|_{\partial B_k}$$

on $B_k \setminus B$ and

$$\underline{\psi}\big|_{\partial B} \leq \psi_k\big|_{\partial B} \leq \overline{\psi}\big|_{\partial B},$$

the Comparison Principle implies $\underline{\psi} \leq \psi_k \leq \overline{\psi}$ on $B_k \setminus B$ for sufficiently large values k. Passing to the limit as $k \to \infty$ we obtain $\underline{\psi} \leq \psi \leq \overline{\psi}$. Since $\underline{\psi} \sim \overline{\psi} \sim 0$ we have $\psi \in [0]$.

Finally, the existence of $\psi := \lim_{k\to\infty} \psi_k$ implies the existence of a limit function $u := \lim_{k\to\infty} \varphi_k$ such that $Lu = 0$ on M and $u \sim u_0$. ∎

3. The case of harmonic functions

The fact that $c \neq 0$ on some compact set $B \subset M$ had been crucial in the proof of Theorem 1. However, the condition $c \equiv 0$ on M does not violate the equivalence of conditions (i) and (ii) of Theorem 1 for harmonic functions.

Similarly as the concept of L-strictness introduced above we can define the concept of Δ-strictness of a manifold M. A manifold M is called Δ-*strict* if there exists a non-trivial capacity potential $v \in [0]$ for some compact set $B \subset M$. Note that the Δ-strictness of the manifold M implies the non-parabolicity of M (cf. [10]). The following result holds.

Theorem 4. *Let M be a Δ-strict manifold, $B \subset M$ an arbitrary connected compact subset and ∂B a $\mathcal{C}^1$-smooth submanifold. Then the following two conditions are equivalent.*

(i) *There is a harmonic function u on M such that $u \in [f]$.*
(ii) *For every function $\Phi \in \mathcal{C}(\partial B)$ there is a harmonic function v on $M \setminus B$ such that $v \in [f]$ and $v\big|_{\partial B} = \Phi$.*

Proof. (i)⇒(ii). The proof of this implication is exactly the same as the proof of the analogous statement in Theorem 1.

(ii)⇒(i). As in the proof of Theorem 1 consider the function $U := v \cdot \varphi$, where φ is a smooth function on M such that $\varphi = 0$ on a precompact set $B' \Subset B$ and $\varphi = 1$ outside B. Then $U \in \mathcal{C}^\infty(M)$ and $\Delta U = \Delta(v \cdot \varphi) = g$, where $g \in \mathcal{C}^\infty(M)$. Obviously, $\Omega := \operatorname{supp} g$ is a compact set and $\Omega \subset B \setminus B'$.

Consider now the sequence of functions φ_k solving the problems

$$\begin{cases} \Delta\varphi_k = 0 \quad \text{in } B_k, \\ \varphi_k\big|_{\partial B_k} = v\big|_{\partial B_k} \end{cases}$$

and the sequence of functions $\psi_k := \varphi_k - U$. For these functions we have

$$\begin{cases} \Delta\psi_k = -g \quad \text{in } B_k, \\ \psi_k.\big|_{\partial B_k} = 0. \end{cases}$$

Since the manifold M is non-parabolic, there is a Green function on every set B_k, i.e. a function with

$$\Delta_x G_k(x,y) = -\delta_y(x), \qquad G_k\big|_{x\in\partial B_k} = 0$$

for any $y \in B_k$, where $\delta_y(x)$ is the Dirac δ-function. Therefore,

$$\psi_k(x) = -\int\limits_{B_k} G_k(x,y) g(y)\,dy.$$

The existence of the limit of the Green functions implies the existence of the limit of the sequence $\{\psi_k\}$. Let $\psi := \lim_{k\to\infty}\psi_k$. Then $\Delta\psi = -g$ on M. As in Theorem 1, we can show $\psi \in [0]$. Hence, the limit function $u := \lim_{k\to\infty}\varphi_k$ of the sequence $\{\varphi_k\}$ exists and satisfies $u = \psi + U$, $\Delta u = 0$ on M and $u \sim v$. ∎

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