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# Connectedness of Julia Sets of Rational Functions

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**Abstract.** For a polynomial P it is well known that its Julia set $\mathcal{J}_P$ is connected if and only if the orbits of the finite critical points are bounded. But there is no such simple criteria for the connectedness of the Julia set of a rational function. Indeed, up to the very nice result of Shishikura that any rational function which has one repelling fixed point only has a connected Julia set almost nothing is known on the connectivity. In the first part of the paper we give constructive sufficient conditions for a basin of attraction to be completely invariant and the Julia set to be connected. Then it is shown that the connectedness of a basin of attraction depends heavily on the fact whether the critical points from the basin tend to the attracting fixed point $z_0$ via a preimage of $z_0$ or not. As a consequence we obtain for instance that rational functions with a finite postcritical set or with a Fatou set which contains no Herman rings and each component of which contains at most one critical point, counted without multiplicity, have a connected Julia set.

**Keywords.** Julia set, Fatou set, basin of attraction, connectedness.

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## 1. Introduction and notation

In the following let $R$ be a rational function of the form $R(z) = P(z)/Q(z) = \sum_{j=0}^{d_1} a_j z^j / \sum_{j=0}^{d_2} b_j z^j$, where we assume that $P$ and $Q$ do not have a common factor and where $d = \deg R = \max\{\deg P, \deg Q\} = \max\{d_1, d_2\} \geq 2$ is the degree of $R$. As usual $\mathcal{F}_R$ denotes the Fatou set, that is the set where the family $\{R^n\}_{n=0}^{\infty}$ is normal; $R^n(z) = R(R^{n-1}(z))$, $n \in \mathbb{N}$, with $R^0(z) = z$ denotes the $n$-th composition of $R$. The complement of the Fatou-set is the Julia set $\mathcal{J}_R = \bar{\mathbb{C}} \setminus \mathcal{F}_R$. In this paper we are concerned with the question of the connectivity of $\mathcal{J}_R$. It is not difficult to prove (see e.g. [1, Thm.5.1.6]) that $\mathcal{J}_R$ is connected if and only if all components of $\mathcal{F}_R$ are simply connected. However in general this characterization is difficult to apply directly because $\mathcal{F}_R$ has either at most two components or infinitely many. If $R$ has an attracting fixed point $z_0$ then it is well known (see e.g. [1]) that the Julia set can be expressed as $\mathcal{J}_R = \partial A(z_0)$, where

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$A(z_0)$ is the basin of attraction of $z_0$ defined by $A(z_0) = \{z \in \overline{\mathbb{C}} \mid R^n(z) \xrightarrow{n\to\infty} z_0\}$; the immediate basin of attraction of $z_0$, denoted by $A^*(z_0)$, is the component of $A(z_0)$ that contains $z_0$. Now if $A(z_0) = A^*(z_0)$ then $A(z_0)$ is completely invariant and thus (see [1, Thm.5.2.1]) in this case $\mathcal{J}_R$ is connected if and only if $A(z_0)$ is simply connected. To decide whether an immediate basin of attraction is simply connected one can use the so-called exhaustive procedure (compare [12]) which will be discussed briefly at the beginning of the next Section. In this second Section we derive sufficient constructive conditions such that $A(z_0) = A^*(z_0)$ and that $A^*(z_0)$ is simply connected.

If the Fatou set does not contain a completely invariant component other concepts have to be developed. As we will demonstrate in Section 3 the connectedness of a basin of attraction with $A(z_0) \neq A^*(z_0)$ depends heavily on the fact how critical points tend to an attracting fixed point via a preimage of $z_0$, not contained in the immediate basin of attraction, or not. With the help of this concept sufficient conditions are given such that the Julia set is connected if all critical points are in basins of attractions or strictly preperiodic. As one consequence it turns out that the Julia set is connected if all critical points are preperiodic. Let us recall in this connection, that the Julia set is the whole Riemann sphere if all critical points are strictly preperiodic, see e.g. [1]. By the way, other families of rational functions whose Julia set is the whole Riemann sphere, a dendrite or a Jordan arc are studied in [3, 4], see also [7]. Another consequence is that the Julia set is connected if the Fatou set contains no Herman rings and each of its components contains at most one critical point counted without multiplicity. With the help of all these results the connectedness of the Julia sets of a family of cubic rational maps is studied in Section 4 finally.

An important tool in the investigations of the above questions is the so-called Riemann-Hurwitz Formula. Let $B \subset \overline{\mathbb{C}}$ be an $m_B$ connected domain and let $R\colon A \to B$ be a $k_A$-fold map of some $m_A$ connected component $A$ of $R^{-1}(B)$ onto $B$, i.e., $R\colon A \to B$ is proper on $A$ of degree $k_A$, then Riemann-Hurwitz's Formula says (see e.g. [12]) that

$$m_A - 2 = k_A(m_B - 2) + r_A,$$

where $r_A$ is the number of critical points of $R$ in $A$.

## 2. Complete invariant component — connectivity of the Julia set

Let $D_0(z_0) = \mathbb{D}_{\tilde{r}}(z_0) \subset A^*(z_0)$, where as usual $\mathbb{D}_{\tilde{r}}(z_0) = \{z \bigm| |z - z_0| < \tilde{r}\}$, $\tilde{r} > 0$, and in the case $z_0 = \infty$ we have $D_0(\infty) = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}_{\tilde{r}}} = \{z \bigm| |z| > \tilde{r}\}$. Then we denote by $D_{\kappa+1}(z_0), \kappa \in \mathbb{N}$, the component of $R^{-1}(D_\kappa(z_0))$ that contains $z_0$. Since $D_0(z_0) \subset A^*(z_0)$ it is forward invariant; hence $D_\kappa(z_0) \subset D_{\kappa+1}(z_0)$ and $(D_\kappa(z_0))_{\kappa=0}^{\infty}$ is a regular exhaustion of $A^*(z_0)$, see [12, p.63]. Let us mention that in many cases conditions on the $D_\nu$'s are more constructive than conditions on $A^*(z_0)$ since we do not know $A^*(z_0)$ explicitly. The connection between the connectivity of $A^*(z_0)$ for some attracting fixed point $z_0$ and the sets $D_\kappa(z_0)$ is given in the following lemma, which is known essentially.

**Lemma 2.1.** *Let $z_0$ be an attracting fixed point of $R$. Then the following statements hold:*

a) *The connectivity numbers $m_{D_\nu}$ satisfy the following relations:*
   1. $m_{D_{\nu+1}} \geq m_{D_\nu}$ *for* $\nu \in \mathbb{N}$,
   2. *If* $m_{D_\kappa} \geq 2$ *then* $m_{D_{\nu+1}} > m_{D_\nu}$ *for all* $\nu \geq \kappa$.

b) *The following three assertions are equivalent:*
   (1) $A^*(z_0)$ *is simply connected;*
   (2) *each* $D_\nu(z_0)$ *is simply connected;*
   (3) $k_{D_\nu} = r_{D_\nu} + 1$ *for* $\nu = 1, 2, \ldots$.

**Proof.** ad a). Point 1. follows by the defintion of $D_{\nu+1}$. To prove 2. let us assume that $m_{D_\nu} \geq 2$ and consider $R\colon D_{\nu+1} \to D_\nu$. Riemann-Hurwitz shows that

$$m_{D_{\nu+1}} - 2 = k_{D_{\nu+1}}(m_{D_\nu} - 2) + r_{D_{\nu+1}}.$$

First observe that $m_{D_\nu} \geq 2$ is only possible if $D_\nu$ contains at least two critical points. (This follows easily by inductively applying Riemann-Hurwitz.) Hence, $r_{D_{\nu+1}} \geq 2$ and $k_{D_{\nu+1}} \geq 2$. We now claim that $m_{D_{\nu+1}} > m_{D_\nu}$, i.e.,

$$(m_{D_\nu} - 2)(k_{D_{\nu+1}} - 1) + r_{D_{\nu+1}} > 0,$$

which is fulfilled since $r_{D_{\nu+1}} \geq 2$, $k_{D_{\nu+1}} \geq 2$, and $m_{D_\nu} \geq 2$

ad b). The first equivalence follows from [12, Lemma 1, p. 63]. The second one is a direct consequence of the Riemann-Hurwitz Formula. ■

Note that either $m_{D_\nu} = 1$ for all $\nu$ or $m_{D_\nu} \to \infty$ for $\nu \to \infty$, that is, the immediate basin of attraction of an attracting fixed point is either simply or infinitely connected. Furthermore it follows that $A^*(z_0)$ is infinitely connected and thus $\mathcal{J}_R$ is disconnected if there exists a $\kappa \geq 1$ such that $D_\kappa(z_0)$ contains at least as many critical points as preimages of $z_0$, each counted with multiplicities.

In case that the immediate basin of attraction is completely invariant the following theorem is often useful.

**Theorem 2.2.** *Let $R$ be a rational function of degree $d \geq 2$ with an attracting fixed point $z_0$. Suppose that there is a simply connected $D_{\tilde\nu}(z_0)$, $\tilde\nu \in \mathbb{N}$, containing all $d$ preimages of $z_0$. Then $\mathcal{J}_R$ is connected if and only if $A(z_0)$ contains exactly $d-1$ critical points.*

**Proof.** Obviously if $D_{\tilde\nu}(z_0) \subset A^*(z_0)$ contains all $d$ preimages of $z_0$ then $A^*(z_0) = A(z_0)$ is a completely invariant component of $\mathcal{F}_R$. Hence, all other components of $\mathcal{F}_R$ are simply connected (compare Theorem [1, Thm. 5.2.1]), so we only have to check whether or not $A(z_0)$ is simply connected. Since $D_{\tilde{\nu}}$ is simply connected Riemann-Hurwitz shows that there are exactly $d-1$ critical points in $D_{\tilde{\nu}}$. If no additional critical point is in $A(z_0)$ then Riemann-Hurwitz shows that all $D_\nu$'s (and thus $A(z_0)$) are simply connected, whereas as soon as there is an additional critical point in some $D_\nu$, $\nu > \tilde{\nu}$, the Julia set is disconnected by what has been said above. ∎

For the polynomial case we obtain the well known fact that $\mathcal{J}_P$ is connected if and only if there is no finite critical point of P in $A(\infty)$. Next let us give the announced sufficient conditions for rational functions such that $A(z_0) = A^*(z_0)$ and that the Julia set is connected. No conditions of this kind seem to be available in the literature even if $R$ has small degree.

**Theorem 2.3.** *Let $R = a_{d_1}\hat{P}/\hat{Q}$, $d_1 \geq d_2 + 2$, where $\hat{P}$ and $\hat{Q}$ are monic polynomials, and suppose that $\hat{P}$ has all its zeros in $\overline{\mathbb{D}_{r_1}}$ and $\hat{Q}$ has all its zeros in $\{r_2 < |z| \leq r_3\}$, where $0 < r_1 < r_2 < r_3$. If the radii satisfy*

$$\text{(1)} \qquad \frac{r_2(r_2+r_3)^{d_2}}{(r_2-r_1)^{d_1}} \leq |a_{d_1}|,$$

*then $A(\infty) = A^*(\infty)$.*

*Furthermore, if in addition all zeros of $\hat{P}$ and $\hat{Q}$ are simple and located in the intervals $(-r_1, r_1)$ and $(r_2, r_3]$, respectively, $a_{d_1} \in \mathbb{R}$, and $R([-r_1, r_1]) \subset [-r_1, r_1]$, then $\mathcal{J}_R$ is connected.*

**Proof.** First we claim that $|R(z)| > |z|$ for $|z| > r_2$ which implies that all poles are in $A^*(\infty)$ and hence $A(\infty) = A^*(\infty)$. Indeed, by the assumption on the zeros of $\hat{P}$ and $\hat{Q}$ it follows that for $|z| > r_2 > r_1$

$$|R(z)| > |z| \qquad \text{if} \qquad \frac{|a_{d_1}|(|z|-r_1)^{d_1}}{|z|(|z|+r_3)^{d_2}} > 1.$$

For real $x > 0$ we define

$$f(x) = \frac{(x-r_1)^{d_1}}{x(x+r_3)^{d_2}}.$$

Then by considering the derivative it follows by straightforward calculations that for $x > r_1$ the function $f$ is strictly increasing. Thus we have for $|z| > r_2$

$$f(|z|) > f(r_2) = \frac{(r_2-r_1)^{d_1}}{r_2(r_2+r_3)^{d_2}} \geq \frac{1}{|a_{d_1}|},$$

from which the claim follows.

Next let us show that $A(\infty)$ contains exactly $d-1$ critical points. Considering the numerator of $R' = a_{d_1}(\hat{P}'\hat{Q} - \hat{Q}'\hat{P})/\hat{Q}^2$, $a_{d_1} \in \mathbb{R}^+$, at the zeros of $\hat{Q}$ and observing that $\hat{P}(x) > 0$ for $x \geq r_1$ and $R'(x) > 0$ for $x$ sufficiently large it follows that $R'$ has a zero between two concecutive zeros of $\hat{Q}$, between the largest zero of $\hat{Q}$ and $\infty$ and at $\infty$ $d_1 - d_2 - 1$ zeros, hence $d_2 + (d_1 - d_2 - 1)$ zeros in $A(\infty)$. Since

$\hat{Q}$ has no zero in $[-r_1, r_1]$ it follows by considering $R'$ at the zeros of $\hat{P}$ that $R'$ has $d_1 - 1$ zeros in $(-r_1, r_1)$. The assertion follows now by the forward invariance of $R$ on $[-r_1, r_1]$ and Theorem 2.2. ∎

## 3. Connectivity of the basin of attraction and of the Julia set

In this section we study the connectivity of the basin of attraction if $A^*(z_0) = A(z_0)$ is not necessarily satisfied. Let us return to Lemma 2.1. If $A^*(z_0)$ is simply connected, then it must necessarily contain exactly $\deg(R, A^*(z_0)) - 1$ critical points (by Riemann-Hurwitz). Now one could hope that this condition is also sufficient (as in Theorem 2.2) for an immediate basin of attraction to be simply connected. Results in this direction were published in [8] and [13]. Surprisingly it turned out that these results were wrong. Przytycki himself discovered that his earlier proofs were erroneous. In fact he was able to prove the following theorem.

**Theorem 3.1** ([9, Thm. D])**.** *There exists a rational function $R$ of arbitrary degree $d \geq 3$ with a completely invariant infinitely connected immediate basin of attraction of an attracting fixed point, and with an arbitrary number $k$, $2 \leq k \leq 2d - 2$, of critical points in the basin.*

Since we are interested in conditions that guarantee a simply connected immediate basin of attraction, Theorem 3.1 forces us to look for additional conditions on the $D_\nu$'s. It turns out that we have to distinguish whether a point enters a $D_\nu$ being first in a preimage or not. This concept seems to be very natural, and therefore it is surprising that it has not been introduced in the literature so far. Let us give the precise definition and some needed notations first.

**Notation 3.2.** *If $w$ is a preimage of $z_0$ under $R$ then we define $U_{\kappa+1}(z_0, w)$ to be the component of $R^{-1}(D_\kappa(z_0))$ that contains $w$. Obviously $U_\kappa(z_0, z_0) = D_\kappa(z_0)$. If $w_1, \ldots, w_l$ are preimages of $z_0$, and if $U_n(z_0, w_1), \ldots, U_n(z_0, w_l)$ are disjoint and $U_{n+1}(z_0, w_1) = \cdots = U_{n+1}(z_0, w_l)$ then we say that these sets melt together. Finally we denote by $W_j^\kappa(z_0)$ the $j$-th component of $R^{-\kappa}(D_0(z_0))$ with the convention that $W_0^\kappa(z_0) := D_\kappa(z_0)$.*

*If it is clear which fixed point is considered, we will only write $D_\kappa$ instead of $D_\kappa(z_0)$, $U_\kappa(w)$ instead of $U_\kappa(z_0, w)$ and $W_j^\kappa$ instead of $W_j^\kappa(z_0)$.*

**Definition 3.3.** *Let $z_0$ be an attracting fixed point of $R$. We say a point $z$ tends to $z_0$ via a preimage if there is an iterate of $z$ that is in a $U_\kappa(w)$ before it is in a $D_\kappa$, i.e.,*

$$\exists_{j \in \mathbb{N}_0} \quad \exists_{\substack{w \neq z_0 \\ R(w) = z_0}} \quad \exists_\kappa \quad R^j(z) \in U_\kappa(z_0, w) \neq D_\kappa(z_0). \tag{2}$$

*In the case $z_0 = \infty$ we also say that $z$ tends to $\infty$ via a pole.*

To derive the desired result we first need a lemma concerning the "holes" of the $W_j^n(z_0)$'s and another one on the connectivity numbers.

**Lemma 3.4.** *Let $z_0$ be an attracting fixed point, and let $A$ be a component of $\overline{\mathbb{C}} \setminus \bigcup_j W_j^n(z_0)$ for some $n \in \mathbb{N}$. Then $R^n(A \cap \mathcal{J}_R) = \mathcal{J}_R$.*

**Proof.** Let $W^n = \bigcup_j W_j^n(z_0)$. Suppose that there is a point $z_1 \in \mathcal{J}$ such that $z_1 \notin R^n(A \cap \mathcal{J})$. We may assume that $z_0 = 0$ and $z_1 = \infty$ (otherwise we move $z_0$ to 0 and $z_1$ to $\infty$ by a Möbius transformation). By the definition of the $W_j^n$'s, recall also that $D_0 = \mathbb{D}_{\tilde{r}}(0)$, it follows that $|R^n(z)| < \tilde{r}$ for any $z \in W^n$. Further we see that $\infty$ and all its preimages are in $\mathcal{J}$. By our assumption $\infty \notin R^n(A \cap \mathcal{J})$ and hence $\infty \notin R^n(A)$. Thus $R^n$ is holomorphic on $W^n \cup A$. (Note that $W^n \cup A = \overset{\circ}{\overline{W^n \cup A}}$ since $W^n$ is an open set and $A$ is a component of its complement and thus closed.) Using the maximum modulus principle we see that $R^n$ assumes its maximum value on $\partial(W^n \cup A)$, i.e., $|R^n(z)| \le \tilde{r}$ for all $z \in \overline{W^n \cup A}$ in particular $|R^n(z)| \le \tilde{r}$ for all $z \in A$ from which we conclude that each $z \in A$ is also in $W^n$, which of course is a contradiction. ∎

**Lemma 3.5.** *Let $A_1, \dots, A_r \subset \mathcal{F}_R$ be disjoint domains such that in each component of $\overline{\mathbb{C}} \setminus \bigcup_{j=1}^r A_j$ there is at least one point of $\mathcal{J}_R$. Let $A$ be a domain with $A \subset \mathcal{F}_R$ and $A \supset \bigcup_{j=1}^r A_j$, then*

$$m_A \ge \Big(\sum_{j=1}^r m_{A_j}\Big) - (r-1).$$

**Proof.** First we inductively construct a (not necessarily connected) set $B_r$ that is a superset of $\overline{\mathbb{C}} \setminus A$. Let $n_c(B)$ denote the number of components of a set $B$. We start with $B_0 = \overline{\mathbb{C}}$, such that $n_c(B_0) = 1$. Then inductively we choose $B_j = B_{j-1} \setminus A_j$. Note that the sets $B_j$ are closed sets. Since $A_j$ must always be in one single component of $B_{j-1}$ we see that

$$n_c(B_j) = n_c(B_{j-1}) + m_{A_j} - 1.$$

Thus

$$n_c(B_r) = \Big(\sum_{j=1}^r m_{A_j}\Big) - (r-1).$$

Since each component of $B_r = \overline{\mathbb{C}} \setminus \bigcup_{j=1}^r A_j$ contains elements of $\mathcal{J}_R$ we conclude that each component of $B_r$ also contains points of $\overline{\mathbb{C}} \setminus A$. Since $\overline{\mathbb{C}} \setminus A \subset B_r$ we thus see that $\overline{\mathbb{C}} \setminus A$ consists of at least $n_c(B_r)$ components such that

$$m_A \ge n_c(B_r) = \Big(\sum_{j=1}^r m_{A_j}\Big) - (r-1). \tag{3}$$

∎

With Definition 3.3 we can now provide a condition that ensures that the immediate basin of attraction is simply connected.

**Theorem 3.6.** *Let $R$ be a rational function with attracting fixed point $z_0$. Then $A^*(z_0)$ is simply connected if and only if there is some simply connected $D_{\tilde{\nu}}(z_0)$, $\tilde{\nu} \in \mathbb{N}$, such that all critical points that are not in $D_{\tilde{\nu}}(z_0)$ either never enter $A^*(z_0)$ or tend to $z_0$ via a preimage that is not in $D_{\tilde{\nu}}(z_0)$. In this case* $\deg(R, A^*(z_0)) = \deg(R, D_{\tilde{\nu}}(z_0)) = r_{D_{\tilde{\nu}}} - 1$.

**Proof.** First suppose that $A^*(z_0)$ is simply connected. Then by Lemma 2.1 all $D_n$'s are simply connected. We choose $\tilde{\nu} \in \mathbb{N}$ such that $D_{\tilde{\nu}}$ contains all critical points of $R$ that are in $A^*(z_0)$. Then all remaining critical points either never enter $A^*(z_0)$ or tend to $z_0$ via a preimage that is not in $A^*(z_0)$ and thus not in $D_{\tilde{\nu}}$. Since all $D_n$'s are simply connected and $D_{\tilde{\nu}}$ already contains all critical points in $A^*(z_0)$ Riemann-Hurwitz shows that $\deg(R, D_{\tilde{\nu}}) = \deg(R, A^*(z_0)) = r_{D_{\tilde{\nu}}} - 1$.

The idea of the proof of the other direction is to show that the critical points that are not in $D_{\tilde{\nu}}$ cannot be in $A^*(z_0)$. Let $c_1, \ldots, c_l$ be the critical points that tend to $z_0$ via the preimages $w_{c_i} \notin D_{\tilde{\nu}}$. (The notation "$w_{c_i}$" should simply denote that each critical point can tend to $z_0$ via its "own" preimage.) Let $z_c \in \{c_1, \ldots, c_l\}$; by our assumptions we have that if $z_c \in D_\mu$ for some $\mu > \tilde{\nu}$ then $D_\mu = U_\mu(w)$ for some $w \notin D_{\tilde{\nu}}$. Thus as long as $D_n$ does not melt with some $U_n$, $z_c$ cannot be in $D_n$ and thus $D_n$ remains simply connected.

Suppose that some $D_n$ does melt with some $U_n$'s, i.e., there is some $n$, $n \geq \tilde{\nu}$, such that $\deg(R, D_{\tilde{\nu}}) = \deg(R, D_n)$ (which implies that $D_n$ is still simply connected), $D_n, U_n(w_1), \ldots, U_n(w_\lambda)$ are disjoint, but $D_{n+1} = U_{n+1}(w_1) = \cdots = U_{n+1}(w_\lambda)$ and the remaining $U_{n+1}$'s are disjoint from $D_{n+1}$. By our assumptions $R$ has no critical point in $D_{n+1} \setminus \left(D_n \cup \bigcup_{j=1}^{\lambda} U_n(w_j)\right)$. From this we obtain

$$r_{D_{n+1}} = r_{D_n} + \sum_{j=1}^{\lambda} r_{U_n(w_j)}. \tag{4}$$

Obviously we have

$$k_{D_{n+1}} = k_{D_n} + \sum_{j=1}^{\lambda} k_{U_n(w_j)}. \tag{5}$$

Now we consider the Riemann-Hurwitz Formula for various maps:

$$R\colon U_n(w_j) \to D_{n-1}: \qquad m_{U_n(w_j)} - 2 = -k_{U_n(w_j)} + r_{U_n(w_j)}, \tag{6}$$

$$R\colon D_n \to D_{n-1}: \qquad -1 = -k_{D_n} + r_{D_n}, \tag{7}$$

and with the help of (4) (5), (6) and (7) we obtain

$$
\begin{aligned}
R\colon D_{n+1} \to D_n : \qquad m_{D_{n+1}} - 2 &= -k_{D_{n+1}} + r_{D_{n+1}} \\
&= -\Big(k_{D_n} + \sum_{j=1}^{\lambda} k_{U_n(w_j)}\Big) + r_{D_n} + \sum_{j=1}^{\lambda} r_{U_n(w_j)} \\
&= \sum_{j=1}^{\lambda}(m_{U_n(w_j)} - 2) - 1,
\end{aligned}
$$

from which we conclude that

$$m_{D_{n+1}} = \Big(\sum_{j=1}^{\lambda} m_{U_n(w_j)}\Big) - (2\lambda - 1) < \Big(\sum_{j=1}^{\lambda} m_{U_n(w_j)}\Big) - (\lambda - 1).$$

But by Lemma 3.5 and Lemma 3.4 we have that

$$m_{D_{n+1}} \geq \Big(\sum_{j=1}^{\lambda} m_{U_n(w_j)}\Big) - (\lambda - 1),$$

which gives the desired contradiction. Thus for every $n > \tilde{\nu}$ $D_n$ does not melt with any $U_n$ and therefore $D_n$ contains the same number of critical points as $D_{\tilde{\nu}}$. Hence, each $D_n$ and consequently $A^*(z_0)$ is simply connected and $\deg(R, D_{\tilde{\nu}}) = \deg(R, A^*(z_0)) = r_{D_{\tilde{\nu}}} - 1$. ■

As it was already hinted at in the proof of Theorem 3.6 each of the critical points that is not in $D_{\tilde{\nu}}$ but tends to $z_0$ via a preimage can (but does not need to) tend to $z_0$ via its "own" preimage.

**Remark 3.7.** The conditions of Theorem 3.6 only guarantee that $A^*(z_0)$ is simply connected, but they do not imply that all components of $A(z_0)$ are simply connected. As a counterexample consider the example given in [1, p. 263], where (if we conjugate the origin to infinity by a Möbius transformation) it is proven that $A^*(\infty)$ is simply connected for sufficiently small $t > 0$ and yet $A(\infty)$ has components of connectivity 3 or 4.

The following theorem provides sufficient conditions such that not only the immediate basin of attraction $A^*(z_0)$ but the whole basin of attraction $A(z_0)$ is simply connected.

**Theorem 3.8.** *Let $R$ be a rational function with attracting fixed point $z_0$. Suppose that $D_{\tilde{\nu}}(z_0)$, $\tilde{\nu} \in \mathbb{N}$, is simply connected and the remaining critical points $c_1, \ldots, c_l$ in $A(z_0)$ tend to $z_0$ via a preimage not in $D_{\tilde{\nu}}(z_0)$. Suppose that there exist $W^{\kappa}_{j_\mu}(z_0)$, $\mu = 1, \ldots, n_l$, with the following properties:*

1. *The $W^{\kappa}_{j_\mu}(z_0)$'s are disjoint.*
2. *Each critical point $c_1, \ldots, c_l$ has an iterate in $\bigcup_{\mu=1}^{n_l} W^{\kappa}_{j_\mu}(z_0)$.*

3. *The $W^{\kappa}_{j_\mu}(z_0)$'s are mapped at equal speed into $A^*(z_0)$, or more precisely*

$$\exists_{1\le\kappa_1\le\kappa} \quad \forall_{1\le\mu\le n_l} \qquad \begin{array}{c} R^{\kappa_1}(W^{\kappa}_{j_\mu}(z_0)) = D_{\kappa-\kappa_1}(z_0) \quad \wedge \\ R^{\kappa_1-1}(W^{\kappa}_{j_\mu}(z_0)) \cap A^*(z_0) = \emptyset. \end{array} \tag{8}$$

4. *Each $W^{\kappa}_{j_\mu}(z_0)$ contains at most* $\deg(R, W^{\kappa}_{j_\mu}(z_0)) - 1$ *critical points. No two other critical points enter $W^{\kappa}_{j_\mu}$ in the same number of iterations.*

*Then $A(z_0)$ is simply connected.*

**Proof.** The idea of the proof is to show that $A^*(z_0)$ is simply connected and in each component $U$ of $A(z_0)$ there are exactly $\deg(R, U) - 1$ critical points.

Using Theorem 3.6 we know that $A^*(z_0)$ and all $D_n$'s are simply connected and none of the remaining critical points is in $A^*(z_0)$. From 2. and 3. we derive that all critical points in $R^{-\kappa_1}(A^*(z_0)) \setminus A^*(z_0)$ are in $\bigcup_{\mu=1}^{n_l} W^{\kappa}_{j_\mu}$ and there are no critical points in $R^{-(\kappa_1-1)}(A^*(z_0)) \setminus A^*(z_0)$. Since all $D_n$'s are simply connected and $W^{\kappa}_{j_\mu}$ contains at most $\deg(R, W^{\kappa}_{j_\mu}) - 1$ critical points, we can inductively apply Riemann-Hurwitz to preimages of $A^*(z_0)$ and of $D_{\kappa-\kappa_1}$ and obtain that all components of $R^{-(\kappa_1-1)}(A^*(z_0))$ and all $W^{\kappa}_{j_\mu}$'s, $\mu = 1, \ldots, n_l$, are simply connected, too. Moreover, Riemann-Hurwitz yields that on each component of $R^{-(\kappa_1-1)}(A^*(z_0)) \setminus A^*(z_0)$ $R$ must be a one-to-one map such that in such a component there is only one preimage $W^{\kappa-1}_j$ of $D_{\kappa-\kappa_1}$.

Next let us show that all $W^{\kappa}_{j_\mu}$'s, $\mu = 1, \ldots, n_l$, are in different components of $A(z_0)$. To see this we assume the opposite, namely that w.l.o.g. $W^{\kappa}_{j_1}, \ldots, W^{\kappa}_{j_\sigma}$, $\sigma \le n_l$, are in the same component $U$ of $A(z_0)$. Since each $W^{\kappa}_{j_\mu}$ maps under $R$ into $R^{-(\kappa_1-1)}(A^*(z_0))$ (compare (8)) and since—as we have seen above—all components of $R^{-(\kappa_1-1)}(A^*(z_0))$ are simply connected and contain only one $W^{\kappa-1}_j$, we have $R(W^{\kappa}_{j_1}) = \cdots = R(W^{\kappa}_{j_\sigma}) = W^{\kappa-1}_j \subset V$, where $V$ is a simply connected component of $A(z_0)$. Thus we have $R\colon U \xrightarrow{k_U:1} V$ with $k_U \ge \sum_{\mu=1}^{\sigma} \deg(R, W^{\kappa}_{j_\mu})$; applying Riemann-Hurwitz yields $m_U - 2 = -k_U + r_U$ with $r_U \le \sum_{\mu=1}^{\sigma} \bigl(\deg(R, W^{\kappa}_{j_\mu}) - 1\bigr)$, from which we obtain $m_U - 2 \le -\sigma$, which is impossible for $\sigma \ge 2$. Thus all $W^{\kappa}_{j_\mu}$'s, $\mu = 1, \ldots, n_l$, are in different components of $A(z_0)$, which implies that $\deg(R, W^{\kappa}_{j_\mu}) - 1 = \deg(R, U) - 1$.

It is obvious that if $R^{\sigma_1}(c_1) \in W^{\kappa}_{j_\alpha}$ and $R^{\sigma_2}(c_2) \in W^{\kappa}_{j_\alpha}$ with $\sigma_1 \ne \sigma_2$ or if $R^{\sigma}(c_1) \in W^{\kappa}_{j_\alpha}$ and $R^{\sigma}(c_2) \in W^{\kappa}_{j_\beta}$, $\alpha \ne \beta$, then $c_1$ and $c_2$ cannot be in the same component of $A(z_0)$ such that $c_1$ is the only critical point in "its" component of $A(z_0)$ and analogous with $c_2$.

Thus we obtain that in each component $U$ of $A(z_0)$ there are either no, exactly one or exactly $\deg(R, W^{\kappa}_{j_\mu}) - 1 = \deg(R, U) - 1$ critical points. In all cases there are exactly $\deg(R, U) - 1$ critical points in $U$. Since $A^*(z_0)$ is simply connected we can inductively apply Riemann-Hurwitz and obtain that $A(z_0)$ consists of simply connected components only. ∎

As we have seen in Theorem 3.6 $\deg(R, A^*(z_0)) = \deg(R, D_\nu(z_0))$. Let $w_1, \dots, w_\rho$ be the preimages of $z_0$ that are not in $D_\nu(z_0)$. Then obviously $w_1, \dots, w_\rho \notin A^*(z_0)$. Thus as long as there is a $w_i$, $i \in \{1, \dots, \rho\}$, or some preimage of such a $w_i$ in a $W_j^n$, this $W_j^n$ cannot be in $A^*(z_0)$. This observation makes it easier to check condition (8).

It is obvious that Theorem 3.8 can be generalized to the case of more attracting fixed points, which leads to the following Theorem:

**Theorem 3.9.** *Let $R$ be a rational function with attracting fixed points $z_1, \dots, z_l$. Suppose that all critical points are either strictly preperiodic or in $\bigcup_{j=1}^l A(z_j)$ and that each $A(z_j)$, $j = 1, \dots, l$ has the following properties:*

1. *There exists a $\nu_j \in \mathbb{N}$ such that $D_{\nu_j}(z_j)$ is simply connected and all critical points in $A(z_j)$ that are not in $D_{\nu_j}(z_j)$ tend to $z_j$ via a preimage that is not in $D_{\nu_j}(z_j)$.*
2. *The conditions 1.–4. of Theorem 3.8 are fulfilled for $z_j$ instead of $z_0$.*

*Then $\mathcal{J}_R$ is connected.*

**Proof.** By Theorem 3.8 each $A(z_j)$ is simply connected. Since any other component of $\mathcal{F}_R$ would require either a periodic critical point or a critical point with infinite orbit it is obvious that $\mathcal{F}_R = \bigcup_{j=1}^l A(z_j)$ and thus consists of simply connected components only such that $\mathcal{J}_R$ is connected. ■

**Example 3.10.** In this example we consider the rational function $R = P/Q$ with

$$\begin{aligned} P(z) &\approx -1.842z^5 - 0.818z^4 + 4.191z^3 - 4.258z^2 + 1.253z + 2.842, \\ Q(z) &\approx -2.968z^3 + 0.125z^2 + 4.307z + 4.237, \end{aligned}$$

where the coefficients were generated randomly with a given bound for the absolute value and vanishing imaginary parts. Here we have one attracting fixed point $z_0 \approx 0.4741$, and obviously $\infty$ is a second attracting fixed point. We now claim that the Fatou set $\mathcal{F}_R$ consists of $A(\infty)$ and $A(z_0)$ and that the conditions of Theorem 3.9 are fulfilled.

First let us consider $A(z_0)$ and check that the conditions of Theorem 3.8 are fulfilled. We choose $\tilde{\nu} = 1$ and see, taking a look at Figure 1 that $D_1(z_0)$ contains exactly 4 preimages of $z_0$ and 3 critical points. Moreover, in our case it is sufficient to have only one $W_{j_\mu}^\kappa$, namely $U_1(z_0, w)$ as shown in the picture ($w$ is the preimage of $z_0$ that is not in $D_1(z_0)$). The only critical point not in $D_1(z_0)$ is $z_c$. Since, taking a look at Figure 1 again, $z_c$ is mapped into $U_1(z_0, w)$ we see that it tends to $z_0$ via a preimage not in $D_1(z_0)$. The conditions 1.–3. of Theorem 3.8 are obviously fulfilled, since $\mu = 1$.

Next let us consider $A(\infty)$: The image clearly shows that $D_1(\infty)$ contains exactly 5 preimages of $\infty$ ($\infty$ twice and the three poles) and 4 critical points ($\infty$

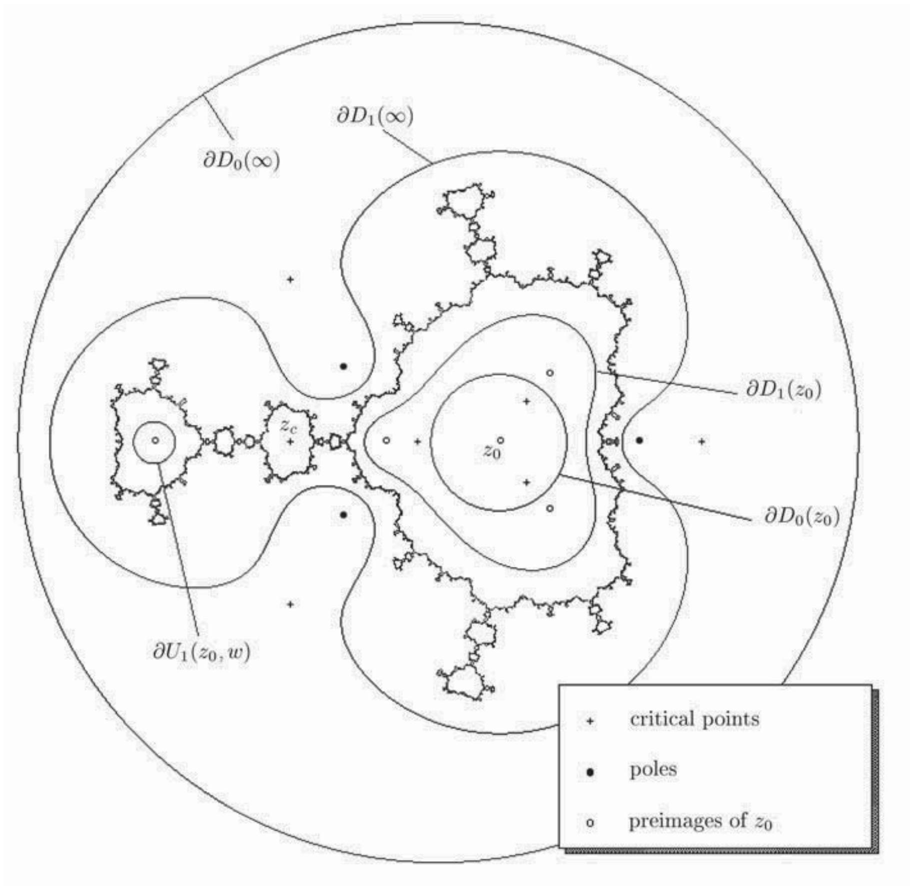


FIGURE 1. The connected Julia set and the sets $D_0(\infty)$, $D_1(\infty)$, $D_0(z_0)$, $D_1(z_0)$ and $U_1(z_0, w)$ of a randomly chosen rational function of degree 5.

once and the three critical points marked in the picture). There is no other critical point in $A(\infty)$. Hence, $A^*(\infty)$ is simply connected. Moreover, since there are already all 5 preimages of $\infty$ in $D_1(\infty)$, we have $A(\infty) = A^*(\infty)$.

The considerations above show that all conditions of Theorem 3.9 are fulfilled, such that $\mathcal{J}_R$ is connected.

Another method to prove this result would be to use Theorem 2.2. In our case we know that $D_1(\infty)$ contains all 5 preimages and 4 critical points. Since $D_0(\infty)$ is simply connected this implies that $D_1(\infty)$ is simply connected, too. Moreover, the remaining 4 critical points are in $A(z_0)$, such that $A(\infty)$ contains exactly $d - 1 = 4$ critical points. Theorem 2.2 shows that $\mathcal{J}_R$ is then connected.

## 4. Connectedness under special assumptions

If we have more detailed information on the behavior and location of the critical points or their orbits or on the degree of the rational function on the components of $\mathcal{F}_R$ then it is often possible to immediately determine whether or not the Julia set is connected. We will demonstrate this below. For instance, an application of Theorem 3.6 leads to the following theorem which guarantees the connectedness of the Julia set for rational functions with finite postcritical set.

**Theorem 4.1.** *If $R$ is postcritically finite, then $\mathcal{J}_R$ is connected.*

**Proof.** First we claim that in our case the Fatou set $\mathcal{F}_R$ consists of basins of attraction of superattracting cycles only. This is true since rotation domains, parabolic cycles and attracting but not superattracting cycles all require a critical point with infinite orbit (see Theorem III.2.2 and the following remark in [2] and [6]).

So by choosing an appropriate $R^m$ instead of $R$ we can assume that $\mathcal{F}_R$ consists of basins of attraction of superattracting fixed points only. (Note that the choice of $R^m$ instead of $R$ does not change the property that all critical points are (pre)periodic since the critical points of $R^m$ are the critical points of $R$ and their preimages up to level $m-1$.)

Let $z_0$ be a superattracting fixed point. We will show that $A(z_0)$ is simply connected. Observe that any point in $A(z_0)$ that is not eventually mapped onto $z_0$ has infinite forward orbit. Since $R$ is postcritically finite we see that all critical points in $A(z_0)$ are eventually mapped onto $z_0$. First let us show that $A^*(z_0)$ is simply connected. To that end let us choose $D_0(z_0)$ sufficiently small such that the only preimage of $z_0$ in $D_1$ is $z_0$ itself (which implies that $D_1$ is still simply connected, since $z_0$ is the only critical point in $D_1$). Since all critical points in $A(z_0)$ are preimages of $z_0$ (under some iterate $R^k$) it is clear that all critical points not being in $D_1$ eventually map onto some $w_j \in R^{-1}(z_0) \setminus \{z_0\}$ and thus obviously tend to $z_0$ via a preimage that is not in $D_1$. This enables us to apply Theorem 3.6 from which we obtain that $A^*(z_0)$ is simply connected. Moreover, $z_0$ is the only preimage of $z_0$ in $A^*(z_0)$.

We now have to show that all other components of $A(z_0)$ are simply connected, too. To do this let $V$ be a simply connected component of $A(z_0)$ containing only one preimage of $z_0$ (counted without multiplicities; "preimage" is meant to be any element of any $R^{-n}(z_0)$). Let $U$ be a component of $R^{-1}(V)$. We consider the Riemann-Hurwitz Formula for $R\colon U \xrightarrow{k_U:1} V$, i.e.,

$$m_U - 2 = -k_U + r_U.$$

If $r_U = 0$, then obviously $k_U = 1 = m_U$ such that $U$ is again simply connected and contains only one preimage of $z_0$.

Next suppose there are $\mu \geq 1$ critical points in $U$ of order $r_1, \ldots, r_\mu$, such that $r_U = r_1 + \cdots + r_\mu$. Since these critical points are all eventually mapped onto $z_0$

and since there is only one preimage $\tilde{w}$ of $z_0$ in $V$ we conclude that all $\mu$ critical points are mapped onto $\tilde{w}$. Hence,

$$k_U \geq \sum_{j=1}^{\mu}(r_j + 1) = r_U + \mu$$

such that

$$m_U - 2 \leq -r_U - \mu + r_U = -\mu.$$

This is only possible for $\mu = m_U = 1$ and $k_U = r_U + 1$ such that again there is only one preimage of $z_0$ in $U$ and $U$ is simply connected. By induction we obtain that all components of $A(z_0)$ are simply connected, each containing at most one critical point (counted without multiplicities), such that, by considering all possible basins of attraction, we obtain that all components of $\mathcal{F}_R$ are simply connected and thus $\mathcal{J}_R$ is connected. ■

**Theorem 4.2.** *Let $R$ be a rational function such that $\mathcal{F}_R$ has no Herman rings and in each component of $\mathcal{F}_R$ there is at most one critical point, counted* without *multiplicities. Then $\mathcal{J}_R$ is connected.*

**Proof.** Because there are no Herman rings, we only have to distinguish the following cases by the Classification Theorem:

*Case 1:* Suppose $\mathcal{F}_R$ has a Siegel disk $U$. $U$ is by definition simply connected. If any preimage of $U$ contains a critical point of multiplicity $r$ then $R$ is at least an $(r+1)$-fold map there. Since by assumption there is no other critical point Riemann-Hurwitz shows that $R$ is exactly an $(r+1)$-fold map and the preimage of $U$ is simply connected. Inductively all other and further preimages are simply connected, too.

*Case 2:* Suppose $\mathcal{F}_R$ has a Siegel-cycle. Then each component of this cycle is simply connected. Proceeding as above shows that again all preimages are simply connected.

*Case 3:* Suppose $R$ has an attracting fixed point $z_0$ or a rationally indifferent fixed point $z_0$ with derivative 1. Since each immediate basin of attraction contains at least one critical point and since by our assumption there is at most one critical point in $A^*(z_0)$ we know that in $A^*(z_0)$ there is exactly one critical point. Now $D_0(z_0)$ is simply connected. So suppose that some $D_n$, $n \geq 0$, is simply connected.

*(a)* If $D_{n+1}$ does not contain the critical point, then Riemann-Hurwitz shows that $D_{n+1}$ is simply connected, too.

*(b)* If $D_{n+1}$ contains the critical point with multiplicity $r$, then again $R$ is at least an $(r+1)$-fold map on $D_{n+1}$, such that the application of Riemann-Hurwitz shows that $D_{n+1}$ is simply connected.

Inductively all $D_n$'s and thus $A^*(z_0)$ are simply connected. As above it can be shown that all components of $A(z_0)$ are simply connected, too.

*Case 4:* Suppose $\mathcal{F}_R$ contains an attracting or rationally indifferent cycle, where the components of the immediate basin of attraction have period $n$. Let $z_0$ be a periodic point in this cycle. We choose $D_0$ by considering $z_0$ as a fixed point of $R^n$. (In the case of a rationally indifferent cycle we have to consider $R^{\mu n}$ for some $\mu$ as to make $z_0$ a fixed point and to make the appropriate component of $A^*(z_0)$ forward invariant.) $D_0$ is simply connected. Now we successively take preimages (with respect to $R$) of $D_0$ within the cycle obtaining a sequence $(D_0, D_1, \ldots)$ with $D_{kn} \supset D_{(k-1)n}$ for all $k \geq 1$. As above we see that all $D_n$'s are simply connected and thus the immediate basin of attraction and in consequence the whole basin of attraction of this cycle is simply connected.

By the Classification Theorem we have that all components of $\mathcal{F}_R$ are simply connected such that the Julia set is connected. ∎

As an application of Theorem 4.2 we can now settle the problem of the connectedness of the Julia set for quadratic rational functions.

**Corollary 4.3.** *Let $R$ be a rational map of degree two. Then $\mathcal{J}_R$ is connected if and only if each component of the Fatou set $\mathcal{F}_R$ contains at most one critical point.*

**Proof.** First suppose that $\mathcal{J}_R$ is connected. Then all components of $\mathcal{F}_R$ are simply connected. Let $U$ and $V$ be two components of $\mathcal{F}_R$, denote by $r$ the number of critical points of $R$ in $U$ and suppose that $R\colon U \to V$ is a $k$-fold map, where $k \in \{1, 2\}$. The Riemann-Hurwitz Formula then shows that $k = r + 1$. This leaves the following two possibilities:

1. $k = 1$ and $r = 0$ or
2. $k = 2$ and $r = 1$.

In any case there is at most one critical point in $U$.

To prove the other direction we suppose that all components of $\mathcal{F}_R$ contain at most one critical point. Shishikura has proven in [10] that quadratic rational maps cannot have Herman rings. Hence, we can apply Theorem 4.2 and obtain that $\mathcal{J}_R$ is connected. ∎

A detailed investigation of the iteration of quadratic rational functions can be found in [5].

For rational functions that have only one fixed point which is repelling or has multiplier 1, the question of the connectedness of the Julia set is settled by the following very elegant theorem which is due to Shishikura.

**Theorem 4.4** ([11])**.** *If a rational map has only one fixed point which is repelling or has multiplier 1, then the Julia set is connected.*

The proof uses quasiconformal surgeries. A corollary of this theorem is that the Julia set of a Newton's function is connected.

## 5. On the connectedness of a family of cubic rational functions

In this section we will investigate the family $R(z) = c(z^3 - 2)/z$, $c \in \mathbb{R}$. With the help of the results derived above we are able to investigate the connectedness of the Julia sets of this family.

**Proposition 5.1.** *Let* $R(z) = c(z^3 - 2)/z$ *with* $c \in \mathbb{R} \setminus \{0\}$*. If* $-\sqrt[3]{2}/3 \leq c \leq 2/3$ *then* $\mathcal{J}_R$ *is connected, otherwise* $\mathcal{J}_R$ *is a Cantor set.*

**Proof.** First of all we compute the finite fixed points $z_0$, $z_1$, $z_2$ of $R$, and with

$$\alpha = \sqrt[3]{1 + 27c^3 + 3\sqrt{c^3(6 + 81c^3)}}$$

we obtain

$$\begin{aligned} z_0 &= \frac{1 + \frac{1}{\alpha} + \alpha}{3c}, \\ z_1 &= \frac{2 - \frac{1+i\sqrt{3}}{\alpha} - (1 - i\sqrt{3})\alpha}{6c}, \\ z_2 &= \frac{2 - \frac{1-i\sqrt{3}}{\alpha} - (1 + i\sqrt{3})\alpha}{6c} \end{aligned}$$

The derivative of $R$ is given by

$$R'(z) = \frac{2c(z^3 + 1)}{z^2} \tag{9}$$

such that the critical points are

$$z_{c,1} = -1, \quad z_{c,2} = e^{\pi i/3}, \quad z_{c,3} = e^{-\pi i/3}, \quad z_{c,4} = \infty.$$

Note that $R(e^{2\pi ik/3}z) = e^{-2\pi ik/3}R(z), k \in \{1, 2\}$. Therefore it is only necessary to consider the real critical point $-1$, since the other two finite critical points have the same properties due to the symmetry of $R$. Since $R$ has real coefficients and a real critical point and due to the symmetry again we see that $R$ cannot have Siegel disks or Herman rings.

In the following it will be useful to know the derivative of $R$ in a fixed point $\tilde{z} \in \{z_0, z_1, z_2\}$. We have

$$R'(\tilde{z}) = c\frac{3\tilde{z}^3 - (\tilde{z}^3 - 2)}{\tilde{z}^2} = 3c\tilde{z} - 1. \tag{10}$$

*Case 1,* $c < 0$*:* With the help of (9) we see that $R$ is increasing on $(-\infty, -1)$ and decreasing on $(-1, 0)$ and on $(0, \infty)$.

*Subcase 1.1,* $c < -\sqrt[3]{2}/3$*:* There is no real fixed point in $\mathbb{R}^-$ such that $(-\infty, 0] \subset A^*(\infty)$. Hence, all critical points are in $A^*(\infty)$ and thus, see e.g. [1, Thm.9.8.1], $\mathcal{J}_R$ is a Cantor set.

*Subcase 1.2, $c = -\sqrt[3]{2}/3$:* There exists a double fixed point at $z = -\sqrt[3]{4}$ with derivative 1, i.e., a rationally indifferent fixed point.

*Subcase 1.3, $-\sqrt[3]{2}/3 < c < 0$:* There exist two real fixed points $p, q \in \mathbb{R}^-$, where $p < -\sqrt[3]{4} < q < 0$ and $\lim_{c\to 0} p = -\infty$, $\lim_{c\to 0} q = 0$. Moreover we have

$$|cq| = cq \leq |c|\sqrt[3]{4} < \frac{2}{3}$$

With the help of (10) we obtain

$$R'(q) = 3cq - 1 \in (-1, 1),$$

i.e., $q$ is an attracting fixed point! For symmetry-reasons we have that $qe^{\pm i\pi/3}$ is an attracting 2-cycle, and in $A^*(q)$ and in each component of $A^*(\{qe^{i\pi/3}, qe^{-i\pi/3}\})$ there must be a critical point of $R$. Hence, in each component of $\mathcal{F}_R$ there is at most one critical point such that — by Theorem 4.2 — $\mathcal{J}_R$ is connected. The same argumentation holds in the case above where $c = -\sqrt[3]{2}/3$.

*Case 2, $c > 0$:* Again with the help of (9) we see that $R$ is decreasing on $(-\infty, -1)$ and increasing on $(-1, 0)$ and on $(0, \infty)$.

For $c > 0$ it is obvious that $z_0$ is real and $z_1$, $z_2$ are complex conjugate. With (10) we see that

$$R'(z_0) = 3cz_0 - 1 = \frac{1}{\alpha} + \alpha \geq 2,$$

which again shows that $z_0$ is a repelling fixed point. It is not difficult to show that $|R(z)| > |z|$ for $|z| > z_0 + \epsilon$, $\epsilon > 0$, or in other words that $A^*(\infty) \supset \bar{\mathbb{C}} \backslash \mathbb{D}_{z_0+\epsilon}$.

*Subcase 2.1, $c > 2/3$:* In this case $R(z) > z_0$ for $z \in (-\infty, 0]$ such that $(-\infty, 0] \subset A^*(\infty)$ and all critical points are in $A^*(\infty)$ which shows that $\mathcal{J}_R$ is a Cantor set.

*Subcase 2.2, $c = 2/3$:* $R(-1) = z_0 = 2$. Thus all finite critical points are strictly preperiodic and the set of critical points is finite. By Theorem 4.1 $\mathcal{J}_R$ is connected.

*Subcase 2.3, $0 < c < 2/3$:* Here we have $z_0 > 2$ and $z_0 \xrightarrow{c\to 0} \infty$. Let $\{z_0, a_1, a_2\} = R^{-1}(z_0)$ with $a_2 < a_1 < z_0$ and let $a_3 \in (0, z_0)$ be the real preimage of $a_2$ under $R$ (compare Figure 2). As $z_0 \in \mathcal{J}_R$ we have that $a_1, a_2, a_3 \in \mathcal{J}_R$. Obviously $(a_1, 0] \subset U_1 \subset U_2$ and $(0, a_3) \subset U_2$ such that $(a_1, a_3) \subset U_2$, where $U_i = U_i(0) = U_i(\infty, 0)$. (Note that $a_2 \geq -z_0$, since $a_2 \in \mathcal{J}_R$ and for every $z > z_0$ we have $z \in \mathcal{F}_R$.)

If we take a look at the graph of $R$ we see that $z \in U_2$ or $R(z) \in (a_2, z_0)$ for $z \in (a_2, z_0)$ i.e., all points in $(a_2, z_0)$ either remain bounded or tend to $\infty$ via $U_2$. In order to apply Theorem 3.6 we have to make sure that $U_2 \neq D_2$.

Since $0 < c < 2/3$ we see that $R^2(-1) = (-2 + 27c^3)/3 \in (-2/3, 2)$. Especially, since $z_0 > 2$ we have $|R^2(-1)| < z_0$ such that $-1$ is not in any $W_j^2$ and thus not in $U_2$. With the help of the Riemann-Hurwitz Formula we obtain that $D_0$, $D_1$, $D_2$, $U_1$, $U_2$ are all simply connected. Moreover, $R$ is a two-fold map on $D_1$ and $D_2$ such that $U_1 \neq D_1$ and $U_2 \neq D_2$. The conclusion is that all critical points either remain bounded or tend to $\infty$ via a pole not in $D_1$ such that — by

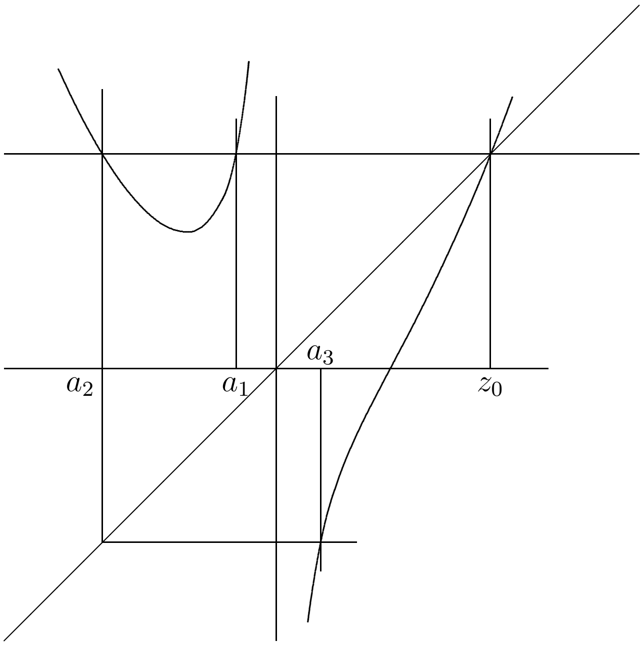


FIGURE 2. Graph of $R(z) = c(z^3 - 2)/z$ for $0 < c < 2/3$.

Theorem 3.6 — $A^*(\infty)$ is simply connected. The fact that $R^2(-1) \in (-2/3, 2)$ shows even more: Observe that $U_2 \cap \mathbb{R} = (a_1, a_3)$ and if $z \in U_2 \cap \mathbb{R}$ then $R(z) > z_0$ or $R(z) < a_2$. In our case we have

$$a_2 < -1 < -\frac{2}{3} < R^2(-1) < 2 < z_0,$$

such that $R(-1) \notin U_2$. This shows that $U_2$ has exactly 3 disjoint preimages $W_1^3$, $W_2^3$, $W_3^3$.

*1.* If $R^n(-1) \xrightarrow{n\to\infty} \infty$, then there must be a $\mu \geq 1$ such that w.l.o.g. $R^\mu(-1) \in W_1^3$, $R^\mu(z_{c,2}) \in W_2^3$, $R^\mu(z_{c,3}) \in W_3^3$. But then the conditions of Theorem 3.9 (with only one fixed point, namely $\infty$) are fulfilled such that $\mathcal{J}_R$ is connected.

*2.* If $R^n(-1)$ does not tend to $\infty$ then at least the conditions of Theorem 3.8 are obviously fulfilled such that $A(\infty)$ is simply connected. We have already seen that $R$ cannot have Siegel disks or Herman rings. Thus $-1$ is either in $\mathcal{J}_R$ (such that $\mathcal{F}_R = A(\infty)$ and $\mathcal{J}_R$ is then connected) or $-1$ is attracted by some real periodic point of period $m$ (either attracting or rationally indifferent). $z_{c,2}$ and $z_{c,3}$ must then necessarily converge to a non-real $2m$-cycle. But in each immediate basin of attraction of such a periodic point there must be a critical point of $R$, such that we can conclude that in each component of $\mathcal{F}_R$ there is again at most one critical point and consequently $\mathcal{J}_R$ is connected. ■

Let us mention, that Proposition 5.1 can also be applied to the family $R_2(z) = (z^3 - c_2)/z$, since $R$ and $R_2$ are obviously linearly conjugate, i.e.,

$$R \circ h = h \circ R_2$$

with $h(z) = z/c$ and $c_2 = 2c^3$.

Finally let us plot the Julia set of $R$ for $c = \sqrt[3]{2}/3$, i.e. for a value $c$ with $0 < c < 2/3$ and for the boundary value $c = 2/3$.

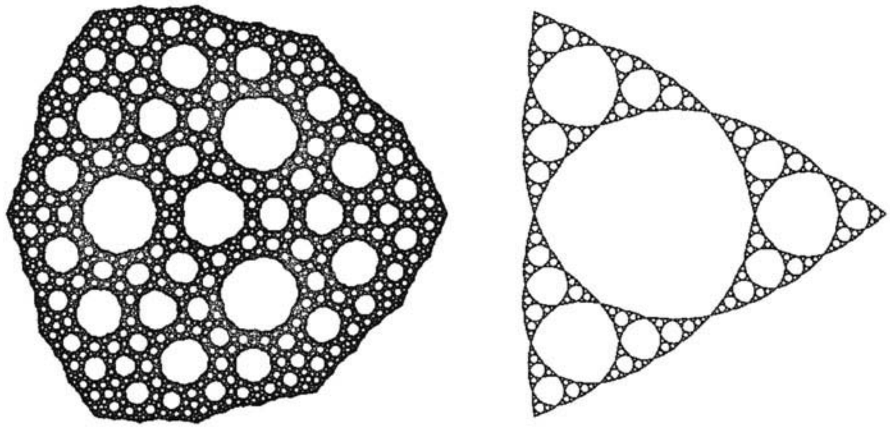

FIGURE 3. The Julia set of $R(z) = c(z^3 - 2)/z$ for $c = \sqrt[3]{2}/3$ and for $c = 2/3$.

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