The Journal of the Astronautical Sciences, Vol. 56, No. 1, January–March 2008, pp. 41–70

Inclusion of Higher Order Harmonics in the Modeling of Optimal Low-Thrust Orbit Transfer[1]

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Abstract

The higher fidelity modeling of minimum-time transfers using continuous constant acceleration low-thrust is depicted by including the higher zonal harmonics J_3 and J_4 for the Earth gravity model. The inclusion of these higher order harmonics is of great benefit in carrying out accurate transfer simulations, especially for long duration flights dwelling in low altitudes where the effects of these zonals are greatest. The analysis presented here can also be coded in the flight guidance computer of spacecraft for autonomous operations and on ground computers for solution uploads and resetting during low-thrust transfers. Equinoctial elements are used to avoid singularities when orbits are circular or equatorial and the applicability of the theory is of a general nature regardless of the size, shape and spatial orientation of the orbits provided they are not of the parabolic or hyperbolic types. To this end, two sets of dynamic and adjoint differential equations in terms of nonsingular orbital elements are derived by further considering a more accurate perturbation model in the form of the higher order Earth zonal harmonics J_3 and J_4. Previous analyses involved only the first-order J_2 term in order to model optimal low-thrust transfers between any two given circular or elliptic orbits. The first formulation uses the eccentric longitude as the sixth element of an equinoctial set of elements while describing the thrust as well as the zonal accelerations in the so called direct equinoctial frame. The second formulation makes use of the true longitude as the sixth element instead while resolving the thrust and the zonal accelerations in the rotating Euler-Hill frame simplifying considerably the algebraic derivations leading to the generation of the nonsingular differential equations that are also free of any singularity for the important zero eccentricity and zero inclination cases often encountered in Earth orbit transfer problems. The derivations of both nonsingular formulations are mutually validated by generating an optimal transfer example that achieves the same target conditions regardless of which formulation is used.

[1]Paper presented at the 2008 AAS/AIAA Space Flight Mechanics Meeting, Galveston, Texas, January 27–31, 2008.
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Introduction

This paper is a direct extension of several previous contributions made by this author in the field of optimal orbit transfer using unaveraged precision-integrated dynamics. The theory of orbital dynamics and optimal transfer has greatly benefited from the pioneering contributions of Broucke, Cefola, and Edelbaum [1]–[3] who introduced the use of nonsingular equinoctial orbital elements that are immune to the nasty singularities inherent in circular and/or equatorial orbits. These elements were also adopted by Betts [4] and Walker [5] with a slight modification validating their applicability for parabolic and hyperbolic orbits as well. Battin [6] in his classical masterpiece discusses and defines equinoctial-based differential equations to integrate spacecraft trajectories. This author provided a series of contributions [7]–[13] directly applicable for optimal low-thrust orbit transfer work using a variety of equinoctial elements sets while also considering the perturbative effects of the oblateness of the Earth such as J_2. More recently, Feistel [14] made use of a form for the various zonal accelerations resolved in the inertial system and given in terms of the radius vector components along the inertial directions [15] and derived the partial derivatives of these acceleration vectors with respect to the radius vector itself, which in turn allowed him to generate the partial derivatives of these accelerations with respect to the equinoctial elements in order to finally produce the adjoint differential equations needed to solve the two point boundary value problem orbit transfer. Besides J_2, Feistel also considered the J_3, J_4, J_5 and J_6 terms while obtaining a quasi-perfect numerical agreement for J_2 with this author's results, the latter based on the Gaussian formulation.

In this paper, two fundamental sets of equinoctial elements have been used to validate the mathematical derivations that lead to the generation of the corresponding nonsingular state and adjoint differential equations for direct use in operational guidance applications. The elements used are a, $h = e\sin(\Omega + \omega)$, $k = e\cos(\Omega + \omega)$, $p = \tan(i/2)\sin\Omega$, $q = \tan(i/2)\cos\Omega$ and either $F = E + \Omega + \omega$, the eccentric longitude, or $L = \theta^* + \Omega + \omega$ the true longitude as the sixth element of the six-element sets. Furthermore, the direct equinoctial frame $(\hat{\mathbf{f}}, \hat{\mathbf{g}}, \hat{\mathbf{w}})$ which has been historically used first such as in [1]–[3] is adopted for the a, h, k, p, q, F formulation to resolve the various accelerations. Because the right-hand sides of the differential equations can also be cast in terms of F, there is no need to solve Kepler's transcendental equation at each integration step to extract F itself because it is being directly integrated. The $\hat{\mathbf{f}}$, $\hat{\mathbf{g}}$, $\hat{\mathbf{w}}$ unit vectors are such that $\hat{\mathbf{f}}$ and $\hat{\mathbf{g}}$ are in the current orbit plane with $\hat{\mathbf{f}}$ rotated clockwise through the angle Ω from the direction of the ascending node, $\hat{\mathbf{g}}$ is 90 degrees ahead of $\hat{\mathbf{f}}$ in the direction of the motion, while $\hat{\mathbf{w}}$ is along the instantaneous normal to the orbit plane.

The second more compact formulation adopts the set a, h, k, p, q, L and resolves the various accelerations along the rotating $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\mathbf{h}}$ Euler-Hill orbital frame with $\hat{\mathbf{r}}$ along the radius vector $\mathbf{r}$, $\hat{\boldsymbol{\theta}}$ in the instantaneous orbit plane and 90 degrees ahead of $\hat{\mathbf{r}}$ along the direction of motion, and with $\hat{\mathbf{h}}$ along the out-of-plane direction. Contrary to the F formulation, there is no need to rotate the accelerations due to J_2, J_3 and J_4 from the $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\mathbf{h}}$ frame to the $\hat{\mathbf{f}}$, $\hat{\mathbf{g}}$, $\hat{\mathbf{w}}$ frame, thus simplifying to a large extent the algebraic derivations required to produce the adjoint differential equations used in the steering of the thrust vector, when the L formulation is used instead. The subsequent sections deal with the development of the acceleration components due to the zonals both in terms of F and L, before deriving the Euler-Lagrange equations for the various multipliers. A numerical comparison is shown at the end to show that both formulations lead to the exact same transfers.

A short duration transfer example that was used before [7]–[13] is adopted here for illustration purposes and in order to avoid very long integrations. The results presented at the end of this paper show only small differences between the solutions using J_2 only and those more elaborate ones using additionally higher harmonics. However these differences become considerable for long duration transfers that spend a long time at lower altitudes before spiraling out to the higher orbits where the perturbations' effects decrease very quickly. Additional discussions are provided in the Results section to make the case for the adoption of the present higher-fidelity modeling.

Zonal Harmonics Perturbation Acceleration Components in Euler-Hill Frame

Assuming that the Earth is symmetrical about its polar axis, its potential U can then be written in terms of the zonal harmonics J_n as

$$U = \frac{\mu}{r}\left[1 - \sum_{n=2}^{\infty} J_n\left(\frac{R}{r}\right)^n P_n(S_\delta)\right] \tag{1}$$

where r is the magnitude of the spacecraft position vector with respect to the center of the Earth, R, the radius of the Earth at the equator, δ, the spacecraft declination with respect to the equator, $P_n(s_\delta)$, the Legendre polynomials of order n in $\sin\delta$, and μ is the Earth gravity constant. The disturbing potential is then given by

$$F' = U - U_0 = U - \frac{\mu}{r} = -\frac{\mu}{r}\sum_{n=2}^{\infty} J_n\left(\frac{R}{r}\right)^n P_n(S_\delta) \tag{2}$$

Neglecting the J_5 and higher zonals, F takes the form

$$\begin{aligned} F' = -\frac{\mu}{r}\Bigg[& J_2\left(\frac{R}{r}\right)^2\left(\frac{3}{2}s_\delta^2 - \frac{1}{2}\right) + J_3\left(\frac{R}{r}\right)^3\left(\frac{5}{2}s_\delta^3 - \frac{3}{2}s_\delta\right) \\ & + J_4\left(\frac{R}{r}\right)^4\left(\frac{35}{8}s_\delta^4 - \frac{30}{8}s_\delta^2 + \frac{3}{8}\right)\Bigg] \end{aligned} \tag{3}$$

Replacing s_δ, by $\frac{z}{r}$, the perturbing acceleration $\mathbf{f}_{J_i}$ is then given by the gradient of the disturbing potential

$$\begin{aligned} \mathbf{f}_{J_i} = \nabla F' = & -\mu J_2 R^2 \nabla\left(\frac{3}{2}\frac{z^2}{r^5} - \frac{1}{2r^3}\right) - \mu J_3 R^3 \nabla\left(\frac{5}{2}\frac{z^3}{r^7} - \frac{3}{2}\frac{z}{r^5}\right) \\ & -\mu J_4 R^4 \nabla\left(\frac{35}{8}\frac{z^4}{r^9} - \frac{30}{8}\frac{z^2}{r^7} + \frac{3}{8r^5}\right) \end{aligned} \tag{4}$$

Because $\nabla r = \hat{\mathbf{r}}$ and $\nabla z = \hat{\mathbf{z}}$, where $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$ are unit vectors along $\mathbf{r}$ and $\mathbf{z}$, $\mathbf{f}_{J_i}$ can then be expressed in the Euler-Hill frame $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{h}})$ as

$$\begin{aligned} \mathbf{f}_{J_i} = & -\mu J_2 R^2\left[\frac{3z}{r^5}\hat{\mathbf{z}} + \left(\frac{3}{2r^4} - \frac{15}{2}\frac{z^2}{r^6}\right)\hat{\mathbf{r}}\right] \\ & - \mu J_3 R^3\left[\left(\frac{15}{2}\frac{z^2}{r^7} - \frac{3}{2r^5}\right)\hat{\mathbf{z}} + \left(\frac{15}{2}\frac{z}{r^6} - \frac{35}{2}\frac{z^3}{r^8}\right)\hat{\mathbf{r}}\right] \\ & - \mu J_4 R^4\left[\left(\frac{35}{2}\frac{z^3}{r^9} - \frac{15}{2}\frac{z}{r^7}\right)\hat{\mathbf{z}} + \left(\frac{105}{4}\frac{z^2}{r^8} - \frac{315}{8}\frac{z^4}{r^{10}} - \frac{15}{8r^6}\right)\hat{\mathbf{r}}\right] \end{aligned} \tag{5}$$

The transformation from the Euler-Hill frame to the inertial equatorial frame $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ is obtained through the three rotations involving the Eulerian angles Ω, i and $\theta = \omega + \theta^*$ where θ^* stands for the true anomaly

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} c_\Omega c_\theta - s_\Omega c_i s_\theta & -c_\Omega s_\theta - s_\Omega c_i c_\theta & s_\Omega s_i \\ s_\Omega c_\theta + c_\Omega c_i s_\theta & -s_\Omega s_\theta + c_\Omega c_i c_\theta & -c_\Omega s_i \\ s_i s_\theta & s_i c_\theta & c_i \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\mathbf{h}} \end{pmatrix} \tag{6}$$

Because $\hat{\mathbf{z}} = s_i s_\theta \hat{\mathbf{r}} + s_i c_\theta \hat{\boldsymbol{\theta}} + c_i \hat{\mathbf{h}}$ and $z = \hat{\mathbf{z}} \cdot \mathbf{r} = r s_i s_\theta$ where $\mathbf{r} = r\hat{\mathbf{r}}$, the acceleration $\mathbf{f}_{J_i}$ can then be cast into the final form

$$\begin{aligned} \mathbf{f}_{J_i} = &-\frac{3\mu J_2 R^2}{r^4}\left[\left(\frac{1}{2} - \frac{3}{2}s_i^2 s_\theta^2\right)\hat{\mathbf{r}} + s_i^2 s_\theta c_\theta \hat{\boldsymbol{\theta}} + s_i c_i s_\theta \hat{\mathbf{h}}\right] \\ &- \frac{\mu J_3 R^3}{r^5}\left[(6 s_i s_\theta - 10 s_i^3 s_\theta^3)\hat{\mathbf{r}} + \frac{1}{2}(15 s_i^3 s_\theta^2 c_\theta - 3 s_i c_\theta)\hat{\boldsymbol{\theta}} + \frac{1}{2}(15 s_i^2 c_i s_\theta^2 - 3c_i)\hat{\mathbf{h}}\right] \\ &- \frac{5\mu J_4 R^4}{2r^6}\left[\left(-\frac{35}{4}s_i^4 s_\theta^4 + \frac{15}{2}s_i^2 s_\theta^2 - \frac{3}{4}\right)\hat{\mathbf{r}}\right. \\ &\left. + (7 s_i^4 s_\theta^3 c_\theta - 3 s_i^2 s_\theta c_\theta)\hat{\boldsymbol{\theta}} + (7 s_i^3 c_i s_\theta^3 - 3 s_i c_i s_\theta)\hat{\mathbf{h}}\right] \end{aligned} \tag{7}$$

Using the trigonometric power relations to express s_θ^2, s_θ^3, s_θ^4 in terms of the multiple-angle expressions involving only first powers, and rearranging terms, the r, θ, h components of $\mathbf{f}_{J_i}$ due to J_2 can be written as

$$(f_r)_{J_2} = -\frac{3\mu J_2 R^2}{2r^4}(1 - 3 s_i^2 s_\theta^2) \tag{8}$$

$$(f_\theta)_{J_2} = -\frac{3\mu J_2 R^2}{r^4} s_i^2 s_\theta c_\theta \tag{9}$$

$$(f_h)_{J_2} = -\frac{3\mu J_2 R^2}{r^4} s_i c_i s_\theta \tag{10}$$

For the J_3 zonal,

$$(f_r)_{J_3} = -\frac{\mu J_3 R^3}{r^5}\left[6 s_i s_\theta - \frac{5}{2}s_i^3(3 s_\theta - s_{3\theta})\right] \tag{11}$$

$$(f_\theta)_{J_3} = -\frac{\mu J_3 R^3}{2r^5}\left[\frac{15}{4}s_i^3(c_\theta - c_{3\theta}) - 3 s_i c_\theta\right] \tag{12}$$

$$(f_h)_{J_3} = -\frac{\mu J_3 R^3}{2r^5}\left[\frac{15}{2}s_i^2 c_i(1 - c_{2\theta}) - 3c_i\right] \tag{13}$$

And finally for the J_4 zonal,

$$(f_r)_{J_4} = -\frac{5\mu J_4 R^4}{8r^6}\left[15 s_i^2(1 - c_{2\theta}) - \frac{35}{8}s_i^4(3 - 4c_{2\theta} + c_{4\theta}) - 3\right] \tag{14}$$

$$(f_\theta)_{J_4} = -\frac{5\mu J_4 R^4}{2r^6}\left[\frac{7}{4}s_i^4 c_\theta(3 s_\theta - s_{3\theta}) - 3 s_i^2 s_\theta c_\theta\right] \tag{15}$$

$$(f_h)_{J_4} = -\frac{5\mu J_4 R^4}{2r^6}\left[\frac{7}{4}s_i^3 c_i(3 s_\theta - s_{3\theta}) - 3 s_i c_i s_\theta\right] \tag{16}$$

These accelerations components can also be obtained directly from their inertial representations as shown in the Appendix. The above derivations are thus validated.

The Treatment of the J_3, J_4 Perturbations within the Eccentric Longitude Formulation

Following the analysis presented in [12] and [13], the dynamic equations for the set *a, h, k, p, q, F* are readily augmented to include the higher zonals' contributions as

$$\dot{a} = (\partial a/\partial \dot{\mathbf{r}})^{\mathrm{T}}(\hat{\mathbf{u}} f_t + \mathbf{f}_{J_2} + \mathbf{f}_{J_3} + \mathbf{f}_{J_4}) \quad (17)$$

$$\dot{h} = (\partial h/\partial \dot{\mathbf{r}})^{\mathrm{T}}(\hat{\mathbf{u}} f_t + \mathbf{f}_{J_2} + \mathbf{f}_{J_3} + \mathbf{f}_{J_4}) \quad (18)$$

$$\dot{k} = (\partial k/\partial \dot{\mathbf{r}})^{\mathrm{T}}(\hat{\mathbf{u}} f_t + \mathbf{f}_{J_2} + \mathbf{f}_{J_3} + \mathbf{f}_{J_4}) \quad (19)$$

$$\dot{p} = (\partial p/\partial \dot{\mathbf{r}})^{\mathrm{T}}(\hat{\mathbf{u}} f_t + \mathbf{f}_{J_2} + \mathbf{f}_{J_3} + \mathbf{f}_{J_4}) \quad (20)$$

$$\dot{q} = (\partial q/\partial \dot{\mathbf{r}})^{\mathrm{T}}(\hat{\mathbf{u}} f_t + \mathbf{f}_{J_2} + \mathbf{f}_{J_3} + \mathbf{f}_{J_4}) \quad (21)$$

$$\dot{F} = na/r + (\partial F/\partial \dot{\mathbf{r}})^{\mathrm{T}}(\hat{\mathbf{u}} f_t + \mathbf{f}_{J_2} + \mathbf{f}_{J_3} + \mathbf{f}_{J_4}) \quad (22)$$

The Hamiltonian of this system of equations can be written as

$$H = \boldsymbol{\lambda}_z^{\mathrm{T}} M(\mathbf{z}, F) f_t \hat{\mathbf{u}} + \lambda_F \frac{na}{r} + \boldsymbol{\lambda}_z^{\mathrm{T}} M(\mathbf{z}, F)\mathbf{f}_{J_2} + \boldsymbol{\lambda}_z^{\mathrm{T}} M(\mathbf{z}, F)\mathbf{f}_{J_3} + \boldsymbol{\lambda}_z^{\mathrm{T}} M(\mathbf{z}, F)\mathbf{f}_{J_4} \quad (23)$$

and the Euler-Lagrange or adjoint equations are thus given by

$$\begin{aligned}\dot{\boldsymbol{\lambda}}_z = -\partial H/\partial \mathbf{z} = -\boldsymbol{\lambda}_z^{\mathrm{T}} \frac{\partial M}{\partial \mathbf{z}} f_t \hat{\mathbf{u}} - \lambda_F \frac{\partial}{\partial \mathbf{z}}\left(\frac{na}{r}\right) - \boldsymbol{\lambda}_z^{\mathrm{T}} \frac{\partial M(\mathbf{z}, F)}{\partial \mathbf{z}} \mathbf{f}_{J_2} - \lambda_z^{\mathrm{T}} M(\mathbf{z}, F) \frac{\partial \mathbf{f}_{J_2}}{\partial \mathbf{z}} \\ -\boldsymbol{\lambda}_z^{\mathrm{T}} \frac{\partial M(\mathbf{z}, F)}{\partial \mathbf{z}} \mathbf{f}_{J_3} - \boldsymbol{\lambda}_z^{\mathrm{T}} M(\mathbf{z}, F) \frac{\partial \mathbf{f}_{J_3}}{\partial \mathbf{z}} - \boldsymbol{\lambda}_z^{\mathrm{T}} \frac{\partial M(\mathbf{z}, F)}{\partial \mathbf{z}} \mathbf{f}_{J_4} - \boldsymbol{\lambda}_z^{\mathrm{T}} M(\mathbf{z}, F) \frac{\partial \mathbf{f}_{J_4}}{\partial \mathbf{z}} \quad (24)\end{aligned}$$

The treatment of the J_2 perturbation is given in full detail for the particular set $z = (a, h, k, p, q, F)$ in [12] and [13], with all the necessary details of the derivations leading to the complete description of the dynamic and adjoint equations including the elements of the M matrix and its various partials with respect to the equinoctial elements of interest. In [12] there are some typographical errors concerning the following equations which should read correctly as

$$\frac{\partial \dot{Y}_1}{\partial a} = \frac{na}{2r}[hk\beta s_F - (1 - k^2\beta)c_F] = -\frac{\dot{Y}_1}{2a} \qquad \text{eq. (80) of [7]}$$

$$\begin{aligned}\frac{\partial M_{62}^F}{\partial k} = \frac{c_F}{nr^2}\left[-2Y_1 + G(h\beta - s_F)\frac{\partial Y_1}{\partial h} + G(k\beta - c_F)\frac{\partial Y_1}{\partial k} - \beta G(ks_F - hc_F)\frac{\dot{Y}_1}{n}\right] \\ + \frac{1}{nar}\Bigg[-2\left(\frac{\partial Y_1}{\partial k}\right)_F + \frac{k\beta}{1-\beta}(s_F - h\beta^2)\frac{\partial Y_1}{\partial h} + G(h\beta - s_F)\frac{\partial^2 Y_1}{\partial k \partial h} \\ + \left\{\frac{k\beta}{1-\beta}(c_F - k\beta^2) + (1-\beta)\right\}\frac{\partial Y_1}{\partial k} + G(k\beta - c_F)\frac{\partial^2 Y_1}{\partial k^2} \\ + \left\{(ks_F - hc_F)\frac{k\beta^3}{1-\beta} - (1-\beta)s_F\right\}\frac{\dot{Y}_1}{n} \\ - (1-\beta)(ks_F - hc_F)\frac{1}{n}\frac{\partial \dot{Y}_1}{\partial k}\Bigg] \qquad \text{eq. (96) of [7]}\end{aligned}$$

$$f_\theta = -12\mu J_2 R^2 r^{-6}(1 + p^2 + q^2)^{-2}(qY_1 - pX_1)(qX_1 + pY_1) \qquad \text{eq. (127) of [7]}$$

and finally

$$\frac{\partial f_\theta}{\partial q} = -12\mu J_2 R^2 r^{-6}[-4qK^{-3}(qY_1 - pX_1)(qX_1 + pY_1) + K^{-2}\{Y_1(qX_1 + pY_1) + X_1(qY_1 - pX_1)\}] \qquad \text{eq. (176) of [7]}$$

The journal version [13] of the original work [12] is however free of these typographic errors.

It has been shown in [8] that the classical elements i and θ are related to p, q, L by the expressions

$$s_i = \frac{2(p^2 + q^2)^{1/2}}{1 + p^2 + q^2} \tag{25}$$

$$c_i = \frac{1 - p^2 - q^2}{1 + p^2 + q^2} \tag{26}$$

$$s_\theta = \frac{qs_L - pc_L}{(p^2 + q^2)^{1/2}} \tag{27}$$

$$c_\theta = \frac{qc_L + ps_L}{(p^2 + q^2)^{1/2}} \tag{28}$$

We also have $3s_\theta - s_{3\theta} = 4s_\theta^3$, $c_\theta - c_{3\theta} = 4c_\theta s_\theta^2$ such that

$$s_i s_\theta = 2(qs_L - pc_L)/(1 + p^2 + q^2) \tag{29}$$

$$s_i c_\theta = 2(qc_L + ps_L)/(1 + p^2 + q^2) \tag{30}$$

$$s_i^3(3s_\theta - s_{3\theta}) = \frac{32(qs_L - pc_L)^3}{(1 + p^2 + q^2)^3} \tag{31}$$

$$s_i^3(c_\theta - c_{3\theta}) = \frac{32(qc_L + ps_L)(qs_L - pc_L)^2}{(1 + p^2 + q^2)^3} \tag{32}$$

$$s_i^2 c_i(1 - c_{2\theta}) = \frac{8(qs_L - pc_L)^2(1 - p^2 - q^2)}{(1 + p^2 + q^2)^3} \tag{33}$$

Expressions (11), (12) and (13) can now be written directly in terms of the nonsingular elements as

$$(f_r)_{J_3} = -\frac{\mu J_3 R^3}{r^5}\frac{(qs_L - pc_L)}{(1 + p^2 + q^2)}\left[12 - \frac{80(qs_L - pc_L)^2}{(1 + p^2 + q^2)^2}\right] \tag{34}$$

$$(f_\theta)_{J_3} = -\frac{\mu J_3 R^3}{2r^5}\frac{(qc_L + ps_L)}{(1 + p^2 + q^2)}\left[120\frac{(qs_L - pc_L)^2}{(1 + p^2 + q^2)^2} - 6\right] \tag{35}$$

$$(f_h)_{J_3} = -\frac{\mu J_3 R^3}{2r^5}\frac{(1 - p^2 - q^2)}{(1 + p^2 + q^2)}\left[60\frac{(qs_L - pc_L)^2}{(1 + p^2 + q^2)^2} - 3\right] \tag{36}$$

The above three acceleration components are readily converted to a form involving F instead of L through the use of the relations [8]

$$qs_L - pc_L = \frac{1}{r}(qY_1 - pX_1) \tag{37}$$

$$qc_L + ps_L = \frac{1}{r}(qX_1 + pY_1) \tag{38}$$

They read as

$$(f_r)_{J_3} = -12\mu J_3 R^3 r^{-6}(qY_1 - pX_1)(1 + p^2 + q^2)^{-1} + 80\mu J_3 R^3 r^{-8}(qY_1 - pX_1)^3(1 + p^2 + q^2)^{-3} \quad (39)$$

$$(f_\theta)_{J_3} = -60\mu J_3 R^3 r^{-8}(qX_1 + pY_1)(qY_1 - pX_1)^2(1 + p^2 + q^2)^{-3} + 3\mu J_3 R^3 r^{-6}(qX_1 + pY_1)(1 + p^2 + q^2)^{-1} \quad (40)$$

$$(f_h)_{J_3} = -30\mu J_3 R^3 r^{-7}(1 - p^2 - q^2)(1 + p^2 + q^2)^{-3}(qY_1 - pX_1)^2 + \frac{3}{2}\mu J_3 R^3 r^{-5}(1 - p^2 - q^2)(1 + p^2 + q^2)^{-1} \quad (41)$$

We have $f_f = c_L f_r - s_L f_\theta, f_g = s_L f_r + c_L f_\theta, f_\omega = f_h$ or in terms of F, $f_f = (X_1/r)f_r - (Y_1/r)f_\theta, f_g = (Y_1/r)f_r + (X_1/r)f_\theta, f_\omega = f_h$.

The partial derivatives $\partial \mathbf{f}_{J_3}/\partial \mathbf{z}$ have the same structure as in [8], namely with $\mathbf{f}_{J_3} = (f_f)_{J_3}\hat{\mathbf{f}} + (f_g)_{J_3}\hat{\mathbf{g}} + (f_\omega)_{J_3}\hat{\mathbf{w}}$. The partials of f_f are given by

$$\frac{\partial(f_f)_{J_3}}{\partial a} = \frac{\partial(X_1/r)}{\partial a}(f_r)_{J_3} + \frac{X_1}{r}\frac{\partial(f_r)_{J_3}}{\partial a} - \frac{\partial(Y_1/r)}{\partial a}(f_\theta)_{J_3} - \frac{Y_1}{r}\frac{\partial(f_\theta)_{J_3}}{\partial a} \quad (42)$$

$$\frac{\partial(f_f)_{J_3}}{\partial h} = \frac{\partial(X_1/r)}{\partial h}(f_r)_{J_3} + \frac{X_1}{r}\frac{\partial(f_r)_{J_3}}{\partial h} - \frac{\partial(Y_1/r)}{\partial h}(f_\theta)_{J_3} - \frac{Y_1}{r}\frac{\partial(f_\theta)_{J_3}}{\partial h} \quad (43)$$

$$\frac{\partial(f_f)_{J_3}}{\partial k} = \frac{\partial(X_1/r)}{\partial k}(f_r)_{J_3} + \frac{X_1}{r}\frac{\partial(f_r)_{J_3}}{\partial k} - \frac{\partial(Y_1/r)}{\partial k}(f_\theta)_{J_3} - \frac{Y_1}{r}\frac{\partial(f_\theta)_{J_3}}{\partial k} \quad (44)$$

$$\frac{\partial(f_f)_{J_3}}{\partial p} = \frac{X_1}{r}\frac{\partial(f_r)_{J_3}}{\partial p} - \frac{Y_1}{r}\frac{\partial(f_\theta)_{J_3}}{\partial p} \quad (45)$$

$$\frac{\partial(f_f)_{J_3}}{\partial q} = \frac{X_1}{r}\frac{\partial(f_r)_{J_3}}{\partial q} - \frac{Y_1}{r}\frac{\partial(f_\theta)_{J_3}}{\partial q} \quad (46)$$

$$\frac{\partial(f_f)_{J_3}}{\partial F} = \frac{\partial(X_1/r)}{\partial F}(f_r)_{J_3} + \frac{X_1}{r}\frac{\partial(f_r)_{J_3}}{\partial F} - \frac{\partial(Y_1/r)}{\partial F}(f_\theta)_{J_3} - \frac{Y_1}{r}\frac{\partial(f_\theta)_{J_3}}{\partial F} \quad (47)$$

The partial derivatives of X_1/r and Y_1/r with respect to a, h, k and F appearing above, are given in equations (146)–(153) of [8]. They involve the partials $\partial X_1/\partial a = X_1/a$ and $\partial Y_1/\partial a = Y_1/a$ given also in [8] as equations (77)–(78), and the partials $\partial X_1/\partial F$ and $\partial Y_1/\partial F$ given in equations (117)–(118) in [8]. As in [8], $\partial r/\partial a = r/a$, $\partial r/\partial h = -as_F$, $\partial r/\partial k = -ac_F$ and $\partial r/\partial F = a(ks_F - hc_F)$ directly derived from the orbit equation $r = a(1 - kc_F - hs_F)$

In a similar way, the partials of f_g are given by

$$\frac{\partial(f_g)_{J_3}}{\partial a} = \frac{\partial(Y_1/r)}{\partial a}(f_r)_{J_3} + \frac{Y_1}{r}\frac{\partial(f_r)_{J_3}}{\partial a} + \frac{\partial(X_1/r)}{\partial a}(f_\theta)_{J_3} + \frac{X_1}{r}\frac{\partial(f_\theta)_{J_3}}{\partial a} \quad (48)$$

$$\frac{\partial(f_g)_{J_3}}{\partial h} = \frac{\partial(Y_1/r)}{\partial h}(f_r)_{J_3} + \frac{Y_1}{r}\frac{\partial(f_r)_{J_3}}{\partial h} + \frac{\partial(X_1/r)}{\partial h}(f_\theta)_{J_3} + \frac{X_1}{r}\frac{\partial(f_\theta)_{J_3}}{\partial h} \quad (49)$$

$$\frac{\partial(f_g)_{J_3}}{\partial k} = \frac{\partial(Y_1/r)}{\partial k}(f_r)_{J_3} + \frac{Y_1}{r}\frac{\partial(f_r)_{J_3}}{\partial k} + \frac{\partial(X_1/r)}{\partial k}(f_\theta)_{J_3} + \frac{X_1}{r}\frac{\partial(f_\theta)_{J_3}}{\partial k} \quad (50)$$

$$\frac{\partial(f_g)_{J_3}}{\partial p} = \frac{Y_1}{r}\frac{\partial(f_r)_{J_3}}{\partial p} + \frac{X_1}{r}\frac{\partial(f_\theta)_{J_3}}{\partial p} \quad (51)$$

$$\frac{\partial (f_g)_{J_3}}{\partial q} = \frac{Y_1}{r}\frac{\partial (f_r)_{J_3}}{\partial q} + \frac{X_1}{r}\frac{\partial (f_\theta)_{J_3}}{\partial q} \tag{52}$$

$$\frac{\partial (f_g)_{J_3}}{\partial F} = \frac{\partial (Y_1/r)}{\partial F}(f_r)_{J_3} + \frac{Y_1}{r}\frac{\partial (f_r)_{J_3}}{\partial F} + \frac{\partial (X_1/r)}{\partial F}(f_\theta)_{J_3} + \frac{X_1}{r}\frac{\partial (f_\theta)_{J_3}}{\partial F} \tag{53}$$

And finally for f_w,

$$\partial (f_w)_{J_3}/\partial a = \partial (f_h)_{J_3}/\partial a \tag{54}$$

$$\partial (f_w)_{J_3}/\partial h = \partial (f_h)_{J_3}/\partial h \tag{55}$$

$$\partial (f_w)_{J_3}/\partial k = \partial (f_h)_{J_3}/\partial k \tag{56}$$

$$\partial (f_w)_{J_3}/\partial p = \partial (f_h)_{J_3}/\partial p \tag{57}$$

$$\partial (f_w)_{J_3}/\partial q = \partial (f_h)_{J_3}/\partial q \tag{58}$$

$$\partial (f_w)_{J_3}/\partial F = \partial (f_h)_{J_3}/\partial F \tag{59}$$

Note that the partials of X_1/r and Y_1/r with respect to h and k involve the partials $(\partial X_1/\partial h)_F$, $(\partial X_1/\partial k)_F$, $(\partial Y_1/\partial h)_F$, $(\partial Y_1/\partial k)_F$, given in [8]. We can now generate the partials of $(f_r)_{J_3}$, $(f_\theta)_{J_3}$, and $(f_h)_{J_3}$ with respect to *a, h, k, p, q* and *F* directly from equations (39)–(41) of this paper.

For $(f_r)_{J_3}$, we have the partials

$$\begin{aligned}\frac{\partial (f_r)_{J_3}}{\partial a} &= 72\mu J_3 R^3 r^{-7}\frac{\partial r}{\partial a}(qY_1 - pX_1)(1+p^2+q^2)^{-1}\\ &- 12\mu J_3 R^3 r^{-6}\left(q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right)(1+p^2+q^2)^{-1}\\ &- 640\mu J_3 R^3 r^{-9}\frac{\partial r}{\partial a}(qY_1 - pX_1)^3(1+p^2+q^2)^{-3}\\ &+ 240\mu J_3 R^3 r^{-8}(qY_1 - pX_1)^2\left(q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right)(1+p^2+q^2)^{-3}\end{aligned} \tag{60}$$

$$\begin{aligned}\frac{\partial (f_r)_{J_3}}{\partial h} &= 72\mu J_3 R^3 r^{-7}\left(\frac{\partial r}{\partial h}\right)_F(qY_1 - pX_1)(1+p^2+q^2)^{-1}\\ &- 12\mu J_3 R^3 r^{-6}\left[q\left(\frac{\partial Y_1}{\partial h}\right)_F - p\left(\frac{\partial X_1}{\partial h}\right)_F\right](1+p^2+q^2)^{-1}\\ &- 640\mu J_3 R^3 r^{-9}\left(\frac{\partial r}{\partial h}\right)_F(qY_1 - pX_1)^3(1+p^2+q^2)^{-3}\\ &+ 240\mu J_3 R^3 r^{-8}(qY_1 - pX_1)^2\left[q\left(\frac{\partial Y_1}{\partial h}\right)_F - p\left(\frac{\partial X_1}{\partial h}\right)_F\right](1+p^2+q^2)^{-3}\end{aligned} \tag{61}$$

$$\begin{aligned}\frac{\partial (f_r)_{J_3}}{\partial k} &= 72\mu J_3 R^3 r^{-7}\left(\frac{\partial r}{\partial k}\right)_F(qY_1 - pX_1)(1+p^2+q^2)^{-1}\\ &- 12\mu J_3 R^3 r^{-6}\left[q\left(\frac{\partial Y_1}{\partial k}\right)_F - p\left(\frac{\partial X_1}{\partial k}\right)_F\right](1+p^2+q^2)^{-1}\end{aligned}$$

$$- 640\mu J_3R^3r^{-9}\left(\frac{\partial r}{\partial k}\right)_F(qY_1 - pX_1)^3(1 + p^2 + q^2)^{-3}$$
$$+ 240\mu J_3R^3r^{-8}(qY_1 - pX_1)^2\left[q\left(\frac{\partial Y_1}{\partial k}\right)_F - p\left(\frac{\partial X_1}{\partial k}\right)_F\right](1 + p^2 + q^2)^{-3} \tag{62}$$

$$\frac{\partial(f_r)_{J_3}}{\partial p} = 12\mu J_3R^3r^{-6}X_1(1 + p^2 + q^2)^{-1}$$
$$+ 12\mu J_3R^3r^{-6}(qY_1 - pX_1)(1 + p^2 + q^2)^{-2}2p$$
$$- 240\mu J_3R^3r^{-8}(qY_1 - pX_1)^2X_1(1 + p^2 + q^2)^{-3}$$
$$- 240\mu J_3R^3r^{-8}(qY_1 - pX_1)^3(1 + p^2 + q^2)^{-4}2p \tag{63}$$

$$\frac{\partial(f_r)_{J_3}}{\partial q} = -12\mu J_3R^3r^{-6}Y_1(1 + p^2 + q^2)^{-1}$$
$$+ 12\mu J_3R^3r^{-6}(qY_1 - pX_1)(1 + p^2 + q^2)^{-2}2q$$
$$+ 240\mu J_3R^3r^{-8}(qY_1 - pX_1)^2Y_1(1 + p^2 + q^2)^{-3}$$
$$- 240\mu J_3R^3r^{-8}(qY_1 - pX_1)^3(1 + p^2 + q^2)^{-4}2q \tag{64}$$

$$\frac{\partial(f_r)_{J_3}}{\partial F} = 72\mu J_3R^3r^{-7}\frac{\partial r}{\partial F}(qY_1 - pX_1)(1 + p^2 + q^2)^{-1}$$
$$- 12\mu J_3R^3r^{-6}\left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F}\right)(1 + p^2 + q^2)^{-1}$$
$$- 640\mu J_3R^3r^{-9}\frac{\partial r}{\partial F}(qY_1 - pX_1)^3(1 + p^2 + q^2)^{-3}$$
$$+ 240\mu J_3R^3r^{-8}(qY_1 - pX_1)^2\left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F}\right)(1 + p^2 + q^2)^{-3} \tag{65}$$

For $(f_\theta)_{J_3}$, we have

$$\frac{\partial(f_\theta)_{J_3}}{\partial a} = 480\mu J_3R^3r^{-9}\frac{\partial r}{\partial a}(qX_1 + pY_1)(qY_1 - pX_1)^2(1 + p^2 + q^2)^{-3}$$
$$- 60\mu J_3R^3r^{-8}\left(q\frac{\partial X_1}{\partial a} + p\frac{\partial Y_1}{\partial a}\right)(qY_1 - pX_1)^2(1 + p^2 + q^2)^{-3}$$
$$- 120\mu J_3R^3r^{-8}(qX_1 + pY_1)(qY_1 - pX_1)\left(q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right)(1 + p^2 + q^2)^{-3}$$
$$- 18\mu J_3R^3r^{-7}\frac{\partial r}{\partial a}(qX_1 + pY_1)(1 + p^2 + q^2)^{-1}$$
$$+ 3\mu J_3R^3r^{-6}\left(q\frac{\partial X_1}{\partial a} + p\frac{\partial Y_1}{\partial a}\right)(1 + p^2 + q^2)^{-1} \tag{66}$$

$$\frac{\partial(f_\theta)_{J_3}}{\partial h} = 480\mu J_3R^3r^{-9}\left(\frac{\partial r}{\partial h}\right)_F(qX_1 + pY_1)(qY_1 - pX_1)^2(1 + p^2 + q^2)^{-3}$$
$$- 60\mu J_3R^3r^{-8}\left[q\left(\frac{\partial X_1}{\partial h}\right)_F + p\left(\frac{\partial Y_1}{\partial h}\right)_F\right](qY_1 - pX_1)^2(1 + p^2 + q^2)^{-3}$$

$$- 120\mu J_3 R^3 r^{-8}(qX_1 + pY_1)(qY_1 - pX_1)\left[q\left(\frac{\partial Y_1}{\partial h}\right)_F - p\left(\frac{\partial X_1}{\partial h}\right)_F\right]$$
$$\times (1 + p^2 + q^2)^{-3} - 18\mu J_3 R^3 r^{-7}\left(\frac{\partial r}{\partial h}\right)_F (qX_1 + pY_1)(1 + p^2 + q^2)^{-1}$$
$$+ 3\mu J_3 R^3 r^{-6}\left[q\left(\frac{\partial X_1}{\partial h}\right)_F + p\left(\frac{\partial Y_1}{\partial h}\right)_F\right](1 + p^2 + q^2)^{-1} \tag{67}$$

$$\frac{\partial (f_\theta)_{J_3}}{\partial k} = 480\mu J_3 R^3 r^{-9}\left(\frac{\partial r}{\partial k}\right)_F (qX_1 + pY_1)(qY_1 - pX_1)^2 (1 + p^2 + q^2)^{-3}$$
$$- 60\mu J_3 R^3 r^{-8}\left[q\left(\frac{\partial X_1}{\partial k}\right)_F + p\left(\frac{\partial Y_1}{\partial k}\right)_F\right](qY_1 - pX_1)^2 (1 + p^2 + q^2)^{-3}$$
$$- 120\mu J_3 R^3 r^{-8}(qX_1 + pY_1)(qY_1 - pX_1)\left[q\left(\frac{\partial Y_1}{\partial k}\right)_F - p\left(\frac{\partial X_1}{\partial k}\right)_F\right]$$
$$\times (1 + p^2 + q^2)^{-3} - 18\mu J_3 R^3 r^{-7}\left(\frac{\partial r}{\partial k}\right)_F (qX_1 + pY_1)(1 + p^2 + q^2)^{-1}$$
$$+ 3\mu J_3 R^3 r^{-6}\left[q\left(\frac{\partial X_1}{\partial k}\right)_F + p\left(\frac{\partial Y_1}{\partial k}\right)_F\right](1 + p^2 + q^2)^{-1} \tag{68}$$

$$\frac{\partial (f_\theta)_{J_3}}{\partial p} = -60\mu J_3 R^3 r^{-8} Y_1 (qY_1 - pX_1)^2 (1 + p^2 + q^2)^{-3}$$
$$+ 120\mu J_3 R^3 r^{-8}(qX_1 + pY_1)(qY_1 - pX_1) X_1 (1 + p^2 + q^2)^{-3}$$
$$+ 180\mu J_3 R^3 r^{-8}(qX_1 + pY_1)(qY_1 - pX_1)^2 (1 + p^2 + q^2)^{-4}\, 2p$$
$$+ 3\mu J_3 R^3 r^{-6} Y_1 (1 + p^2 + q^2)^{-1}$$
$$- 3\mu J_3 R^3 r^{-6}(qX_1 + pY_1)(1 + p^2 + q^2)^{-2}\, 2p \tag{69}$$

$$\frac{\partial (f_\theta)_{J_3}}{\partial q} = -60\mu J_3 R^3 r^{-8} X_1 (qY_1 - pX_1)^2 (1 + p^2 + q^2)^{-3}$$
$$- 120\mu J_3 R^3 r^{-8}(qX_1 + pY_1)(qY_1 - pX_1) Y_1 (1 + p^2 + q^2)^{-3}$$
$$+ 180\mu J_3 R^3 r^{-8}(qX_1 + pY_1)(qY_1 - pX_1)^2 (1 + p^2 + q^2)^{-4}\, 2q$$
$$+ 3\mu J_3 R^3 r^{-6} X_1 (1 + p^2 + q^2)^{-1}$$
$$- 3\mu J_3 R^3 r^{-6}(qX_1 + pY_1)(1 + p^2 + q^2)^{-2}\, 2q \tag{70}$$

$$\frac{\partial (f_\theta)_{J_3}}{\partial F} = 480\mu J_3 R^3 r^{-9}\frac{\partial r}{\partial F}(qX_1 + pY_1)(qY_1 - pX_1)^2 (1 + p^2 + q^2)^{-3}$$
$$- 60\mu J_3 R^3 r^{-8}\left[q\left(\frac{\partial X_1}{\partial F}\right) + p\left(\frac{\partial Y_1}{\partial F}\right)\right](qY_1 - pX_1)^2 (1 + p^2 + q^2)^{-3}$$
$$- 120\mu J_3 R^3 r^{-8}(qX_1 + pY_1)(qY_1 - pX_1)\left[q\left(\frac{\partial Y_1}{\partial F}\right) - p\left(\frac{\partial X_1}{\partial F}\right)\right]$$
$$\times (1 + p^2 + q^2)^{-3} - 18\mu J_3 R^3 r^{-7}\frac{\partial r}{\partial F}(qX_1 + pY_1)(1 + p^2 + q^2)^{-1}$$
$$+ 3\mu J_3 R^3 r^{-6}\left[q\left(\frac{\partial X_1}{\partial F}\right) + p\left(\frac{\partial Y_1}{\partial F}\right)\right](1 + p^2 + q^2)^{-1} \tag{71}$$

For $(f_h)_{J_3}$, we have the partials

$$\frac{\partial (f_h)_{J_3}}{\partial a} = 210\mu J_3 R^3 r^{-8}\frac{\partial r}{\partial a}(1-p^2-q^2)(1+p^2+q^2)^{-3}(qY_1-pX_1)^2 - 60\mu J_3 R^3 r^{-7}(1-p^2-q^2)(1+p^2+q^2)^{-3}(qY_1-pX_1) \times\left[q\frac{\partial Y_1}{\partial a}-p\frac{\partial X_1}{\partial a}\right] - \frac{15}{2}\mu J_3 R^3 r^{-6}\frac{\partial r}{\partial a}(1-p^2-q^2)(1+p^2+q^2)^{-1} \tag{72}$$

$$\frac{\partial (f_h)_{J_3}}{\partial h} = 210\mu J_3 R^3 r^{-8}\left(\frac{\partial r}{\partial h}\right)_F(1-p^2-q^2)(1+p^2+q^2)^{-3}(qY_1-pX_1)^2 - 60\mu J_3 R^3 r^{-7}(1-p^2-q^2)(1+p^2+q^2)^{-3}(qY_1-pX_1) \times\left[q\left(\frac{\partial Y_1}{\partial h}\right)_F-p\left(\frac{\partial X_1}{\partial h}\right)_F\right] - \frac{15}{2}\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial h}\right)_F \times(1-p^2-q^2)(1+p^2+q^2)^{-1} \tag{73}$$

$$\frac{\partial (f_h)_{J_3}}{\partial k} = 210\mu J_3 R^3 r^{-8}\left(\frac{\partial r}{\partial k}\right)_F(1-p^2-q^2)(1+p^2+q^2)^{-3}(qY_1-pX_1)^2 - 60\mu J_3 R^3 r^{-7}(1-p^2-q^2)(1+p^2+q^2)^{-3}(qY_1-pX_1) \times\left[q\left(\frac{\partial Y_1}{\partial k}\right)_F-p\left(\frac{\partial X_1}{\partial k}\right)_F\right] - \frac{15}{2}\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial k}\right)_F \times(1-p^2-q^2)(1+p^2+q^2)^{-1} \tag{74}$$

$$\frac{\partial (f_h)_{J_3}}{\partial p} = 60\mu J_3 R^3 r^{-7}p(1+p^2+q^2)^{-3}(qY_1-pX_1)^2 + 180\mu J_3 R^3 r^{-7}(1-p^2-q^2)(1+p^2+q^2)^{-4}p(qY_1-pX_1)^2 + 60\mu J_3 R^3 r^{-7}(1-p^2-q^2)(1+p^2+q^2)^{-3}(qY_1-pX_1)X_1 - 3\mu J_3 R^3 r^{-5}p(1+p^2+q^2)^{-1} - 3\mu J_3 R^3 r^{-5}(1-p^2-q^2)(1+p^2+q^2)^{-2}p \tag{75}$$

$$\frac{\partial (f_h)_{J_3}}{\partial q} = 60\mu J_3 R^3 r^{-7}q(1+p^2+q^2)^{-3}(qY_1-pX_1)^2 + 180\mu J_3 R^3 r^{-7}(1-p^2-q^2)(1+p^2+q^2)^{-4}q(qY_1-pX_1)^2 - 60\mu J_3 R^3 r^{-7}(1-p^2-q^2)(1+p^2+q^2)^{-3}(qY_1-pX_1)Y_1 - 3\mu J_3 R^3 r^{-5}q(1+p^2+q^2)^{-1} - 3\mu J_3 R^3 r^{-5}(1-p^2-q^2)(1+p^2+q^2)^{-2}q \tag{76}$$

$$\frac{\partial (f_h)_{J_3}}{\partial F} = 210\mu J_3 R^3 r^{-8}\frac{\partial r}{\partial F}(1-p^2-q^2)(1+p^2+q^2)^{-3}(qY_1-pX_1)^2 - 60\mu J_3 R^3 r^{-7}(1-p^2-q^2)(1+p^2+q^2)^{-3}(qY_1-pX_1) \times\left[q\left(\frac{\partial Y_1}{\partial F}\right)-p\left(\frac{\partial X_1}{\partial F}\right)\right] - \frac{15}{2}\mu J_3 R^3 r^{-6}\frac{\partial r}{\partial F}(1-p^2-q^2)(1+p^2+q^2)^{-1} \tag{77}$$

The J_4 accelerations given in equations (14)–(16) in terms of the angles i and θ are converted to a form involving the equinoctial elements by using the equivalencies

$$s_i^2(1 - c_{2\theta}) = 8(qs_L - pc_L)^2/(1 + p^2 + q^2)^2 \tag{78}$$

$$s_i^2 s_\theta c_\theta = 4(qs_L - pc_L)(qc_L + ps_L)/(1 + p^2 + q^2)^2 \tag{79}$$

$$s_i c_i s_\theta = 2(1 - p^2 - q^2)(qs_L - pc_L)/(1 + p^2 + q^2)^2 \tag{80}$$

$$s_i^4 c_\theta(3s_\theta - s_{3\theta}) = 64(qs_L - pc_L)^3(qc_L + ps_L)/(1 + p^2 + q^2)^4 \tag{81}$$

$$s_i^3 c_i(3s_\theta - s_{3\theta}) = 32(qs_L - pc_L)^3(1 - p^2 - q^2)/(1 + p^2 + q^2)^4 \tag{82}$$

$$s_i^4(3 - 4c_{2\theta} + c_{4\theta}) = 128(qs_L - pc_L)^4/(1 + p^2 + q^2)^4 \tag{83}$$

Therefore, the J_4 acceleration components can be written as

$$\begin{aligned}(f_r)_{J_4} = &-75\mu J_4 R^4 r^{-6}(qs_L - pc_L)^2(1 + p^2 + q^2)^{-2} \\ &+ 350\mu J_4 R^4 r^{-6}(qs_L - pc_L)^4(1 + p^2 + q^2)^{-4} + \frac{15}{8}\mu J_4 R^4 r^{-6}\end{aligned} \tag{84}$$

$$\begin{aligned}(f_\theta)_{J_4} = &-280\mu J_4 R^4 r^{-6}(qs_L - pc_L)^3(qc_L + ps_L)(1 + p^2 + q^2)^{-4} \\ &+ 30\mu J_4 R^4 r^{-6}(qs_L - pc_L)(qc_L + ps_L)(1 + p^2 + q^2)^{-2}\end{aligned} \tag{85}$$

$$\begin{aligned}(f_h)_{J_4} = &-140\mu J_4 R^4 r^{-6}(qs_L - pc_L)^3(1 - p^2 - q^2)(1 + p^2 + q^2)^{-4} \\ &+ 15\mu J_4 R^4 r^{-6}(qs_L - pc_L)(1 - p^2 - q^2)(1 + p^2 + q^2)^{-2}\end{aligned} \tag{86}$$

And in terms of the fast variable F, the J_4 acceleration components take the form

$$\begin{aligned}(f_r)_{J_4} = &-75\mu J_4 R^4 r^{-8}(qY_1 - pX_1)^2(1 + p^2 + q^2)^{-2} \\ &+ 350\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^4(1 + p^2 + q^2)^{-4} + \frac{15}{8}\mu J_4 R^4 r^{-6}\end{aligned} \tag{87}$$

$$\begin{aligned}(f_\theta)_{J_4} = &-280\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^3(qX_1 + pY_1)(1 + p^2 + q^2)^{-4} \\ &+ 30\mu J_4 R^4 r^{-8}(qY_1 - pX_1)(qX_1 + pY_1)(1 + p^2 + q^2)^{-2}\end{aligned} \tag{88}$$

$$\begin{aligned}(f_h)_{J_4} = &-140\mu J_4 R^4 r^{-9}(qY_1 - pX_1)^3(1 - p^2 - q^2)(1 + p^2 + q^2)^{-4} \\ &+ 15\mu J_4 R^4 r^{-7}(qY_1 - pX_1)(1 - p^2 - q^2)(1 + p^2 + q^2)^{-2}\end{aligned} \tag{89}$$

The partial derivatives $\partial \mathbf{f}_{J_4}/\partial \mathbf{z}$ with $\mathbf{f}_{J_4} = (f_f)_{J_4}\hat{\mathbf{f}} + (f_g)_{J_4}\hat{\mathbf{g}} + (f_h)_{J_4}\hat{\mathbf{h}}$ are identical to equations (43)–(59) except that the J_3 accelerations are replaced by the J_4 counterparts shown in equations (87)–(89) above. These last three equations are used to generate the partials of $(f_r)_{J_4}$, $(f_\theta)_{J_4}$, and $(f_h)_{J_4}$ with respect to a, h, k, p, q and F. For $(f_r)_{J_4}$, the six partial derivatives can be written as

$$\begin{aligned}\frac{\partial (f_r)_{J_4}}{\partial a} = &\ 600\mu J_4 R^4 r^{-9}\frac{\partial r}{\partial a}(qY_1 - pX_1)^2(1 + p^2 + q^2)^{-2} \\ &- 150\mu J_4 R^4 r^{-8}(qY_1 - pX_1)\left(q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right)(1 + p^2 + q^2)^{-2} \\ &- 3500\mu J_4 R^4 r^{-11}\frac{\partial r}{\partial a}(qY_1 - pX_1)^4(1 + p^2 + q^2)^{-4}\end{aligned}$$

$$+\,1400\mu J_4R^4r^{-10}(qY_1-pX_1)^3\left(q\frac{\partial Y_1}{\partial a}-p\frac{\partial X_1}{\partial a}\right)(1+p^2+q^2)^{-4}$$
$$-\,\frac{45}{4}\,\mu J_4R^4r^{-7}\,\frac{\partial r}{\partial a} \tag{90}$$

$$\begin{aligned}\frac{\partial(f_r)_{J_4}}{\partial h}&=600\mu J_4R^4r^{-9}\left(\frac{\partial r}{\partial h}\right)_F(qY_1-pX_1)^2(1+p^2+q^2)^{-2}\\&-150\mu J_4R^4r^{-8}(qY_1-pX_1)\left[q\left(\frac{\partial Y_1}{\partial h}\right)_F-p\left(\frac{\partial X_1}{\partial h}\right)_F\right](1+p^2+q^2)^{-2}\\&-3500\mu J_4R^4r^{-11}\left(\frac{\partial r}{\partial h}\right)_F(qY_1-pX_1)^4(1+p^2+q^2)^{-4}\\&+1400\mu J_4R^4r^{-10}(qY_1-pX_1)^3\left[q\left(\frac{\partial Y_1}{\partial h}\right)_F-p\left(\frac{\partial X_1}{\partial h}\right)_F\right](1+p^2+q^2)^{-4}\\&-\frac{45}{4}\,\mu J_4R^4r^{-7}\left(\frac{\partial r}{\partial h}\right)_F\end{aligned} \tag{91}$$

$$\begin{aligned}\frac{\partial(f_r)_{J_4}}{\partial k}&=600\mu J_4R^4r^{-9}\left(\frac{\partial r}{\partial k}\right)_F(qY_1-pX_1)^2(1+p^2+q^2)^{-2}\\&-150\mu J_4R^4r^{-8}(qY_1-pX_1)\left[q\left(\frac{\partial Y_1}{\partial k}\right)_F-p\left(\frac{\partial X_1}{\partial k}\right)_F\right](1+p^2+q^2)^{-2}\\&-3500\mu J_4R^4r^{-11}\left(\frac{\partial r}{\partial k}\right)_F(qY_1-pX_1)^4(1+p^2+q^2)^{-4}\\&+1400\mu J_4R^4r^{-10}(qY_1-pX_1)^3\left[q\left(\frac{\partial Y_1}{\partial k}\right)_F-p\left(\frac{\partial X_1}{\partial k}\right)_F\right](1+p^2+q^2)^{-4}\\&-\frac{45}{4}\,\mu J_4R^4r^{-7}\left(\frac{\partial r}{\partial k}\right)_F\end{aligned} \tag{92}$$

$$\begin{aligned}\frac{\partial(f_r)_{J_4}}{\partial p}&=150\mu J_4R^4r^{-8}(qY_1-pX_1)X_1(1+p^2+q^2)^{-2}\\&+300\mu J_4R^4r^{-8}(qY_1-pX_1)^2(1+p^2+q^2)^{-3}p\\&-1400\mu J_4R^4r^{-10}(qY_1-pX_1)^3X_1(1+p^2+q^2)^{-4}\\&-2800\mu J_4R^4r^{-10}(qY_1-pX_1)^4(1+p^2+q^2)^{-5}p\end{aligned} \tag{93}$$

$$\begin{aligned}\frac{\partial(f_r)_{J_4}}{\partial q}&=-150\mu J_4R^4r^{-8}(qY_1-pX_1)Y_1(1+p^2+q^2)^{-2}\\&+300\mu J_4R^4r^{-8}(qY_1-pX_1)^2(1+p^2+q^2)^{-3}q\\&+1400\mu J_4R^4r^{-10}(qY_1-pX_1)^3Y_1(1+p^2+q^2)^{-4}\\&-2800\mu J_4R^4r^{-10}(qY_1-pX_1)^4(1+p^2+q^2)^{-5}q\end{aligned} \tag{94}$$

$$\begin{aligned}\frac{\partial(f_r)_{J_4}}{\partial F}&=600\mu J_4R^4r^{-9}\,\frac{\partial r}{\partial F}(qY_1-pX_1)^2(1+p^2+q^2)^{-2}\\&-150\mu J_4R^4r^{-8}(qY_1-pX_1)\left[q\frac{\partial Y_1}{\partial F}-p\frac{\partial X_1}{\partial F}\right](1+p^2+q^2)^{-2}\end{aligned}$$

$$
\begin{aligned}
&- 3500\mu J_4 R^4 r^{-11} \frac{\partial r}{\partial F}(qY_1 - pX_1)^4 (1 + p^2 + q^2)^{-4} \\
&+ 1400\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^3 \left[q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F} \right](1 + p^2 + q^2)^{-4} \\
&- \frac{45}{4}\mu J_4 R^4 r^{-7} \frac{\partial r}{\partial F}
\end{aligned} \tag{95}
$$

For $(f_\theta)_{J_4}$, in a similar way, the partial derivatives are obtained as

$$
\begin{aligned}
\frac{\partial (f_\theta)_{J_4}}{\partial a} &= 2800\mu J_4 R^4 r^{-11} \frac{\partial r}{\partial a}(qY_1 - pX_1)^3 (qX_1 + pY_1)(1 + p^2 + q^2)^{-4} \\
&- 840\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^2 \left(q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a} \right) \\
&\times (qX_1 + pY_1)(1 + p^2 + q^2)^{-4} \\
&- 280\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^3 \left[q\frac{\partial X_1}{\partial a} + p\frac{\partial Y_1}{\partial a} \right](1 + p^2 + q^2)^{-4} \\
&- 240\mu J_4 R^4 r^{-9} \frac{\partial r}{\partial a}(qY_1 - pX_1)(qX_1 + pY_1)(1 + p^2 + q^2)^{-2} \\
&+ 30\mu J_4 R^4 r^{-8} \left(q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a} \right)(qX_1 + pY_1)(1 + p^2 + q^2)^{-2} \\
&+ 30\mu J_4 R^4 r^{-8}(qY_1 - pX_1)\left(q\frac{\partial X_1}{\partial a} + p\frac{\partial Y_1}{\partial a} \right)(1 + p^2 + q^2)^{-2}
\end{aligned} \tag{96}
$$

$$
\begin{aligned}
\frac{\partial (f_\theta)_{J_4}}{\partial h} &= 2800\mu J_4 R^4 r^{-11} \left(\frac{\partial r}{\partial h}\right)_F (qY_1 - pX_1)^3 (qX_1 + pY_1)(1 + p^2 + q^2)^{-4} \\
&- 840\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^2 \left[q\left(\frac{\partial Y_1}{\partial h}\right)_F - p\left(\frac{\partial X_1}{\partial h}\right)_F \right] \\
&\times (qX_1 + pY_1)(1 + p^2 + q^2)^{-4} \\
&- 280\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^3 \left[q\left(\frac{\partial X_1}{\partial h}\right)_F + p\left(\frac{\partial Y_1}{\partial h}\right)_F \right](1 + p^2 + q^2)^{-4} \\
&- 240\mu J_4 R^4 r^{-9} \left(\frac{\partial r}{\partial h}\right)_F (qY_1 - pX_1)(qX_1 + pY_1)(1 + p^2 + q^2)^{-2} \\
&+ 30\mu J_4 R^4 r^{-8} \left[q\left(\frac{\partial Y_1}{\partial h}\right)_F - p\left(\frac{\partial X_1}{\partial h}\right)_F \right](qX_1 + pY_1)(1 + p^2 + q^2)^{-2} \\
&+ 30\mu J_4 R^4 r^{-8}(qY_1 - pX_1)\left[q\left(\frac{\partial X_1}{\partial h}\right)_F + p\left(\frac{\partial Y_1}{\partial h}\right)_F \right](1 + p^2 + q^2)^{-2}
\end{aligned} \tag{97}
$$

$$
\begin{aligned}
\frac{\partial (f_\theta)_{J_4}}{\partial k} &= 2800\mu J_4 R^4 r^{-11} \left(\frac{\partial r}{\partial k}\right)_F (qY_1 - pX_1)^3 (qX_1 + pY_1)(1 + p^2 + q^2)^{-4} \\
&- 840\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^2 \left[q\left(\frac{\partial Y_1}{\partial k}\right)_F - p\left(\frac{\partial X_1}{\partial k}\right)_F \right]
\end{aligned}
$$

$$
\begin{aligned}
&\times (qX_1 + pY_1)(1 + p^2 + q^2)^{-4} \\
&- 280\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^3 \left[q\left(\frac{\partial X_1}{\partial k}\right)_F + p\left(\frac{\partial Y_1}{\partial k}\right)_F \right](1 + p^2 + q^2)^{-4} \\
&- 240\mu J_4 R^4 r^{-9}\left(\frac{\partial r}{\partial k}\right)_F (qY_1 - pX_1)(qX_1 + pY_1)(1 + p^2 + q^2)^{-2} \\
&+ 30\mu J_4 R^4 r^{-8}\left[q\left(\frac{\partial Y_1}{\partial k}\right)_F - p\left(\frac{\partial X_1}{\partial k}\right)_F \right](qX_1 + pY_1)(1 + p^2 + q^2)^{-2} \\
&+ 30\mu J_4 R^4 r^{-8}(qY_1 - pX_1)\left[q\left(\frac{\partial X_1}{\partial k}\right)_F + p\left(\frac{\partial Y_1}{\partial k}\right)_F \right](1 + p^2 + q^2)^{-2}
\end{aligned}
\tag{98}
$$

$$
\begin{aligned}
\frac{\partial (f_\theta)_{J_4}}{\partial p} = {}& 840\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^2 X_1(qX_1 + pY_1)(1 + p^2 + q^2)^{-4} \\
&- 280\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^3 Y_1(1 + p^2 + q^2)^{-4} \\
&+ 2240\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^3(qX_1 + pY_1)(1 + p^2 + q^2)^{-5} p \\
&- 30\mu J_4 R^4 r^{-8} X_1(qX_1 + pY_1)(1 + p^2 + q^2)^{-2} \\
&+ 30\mu J_4 R^4 r^{-8}(qY_1 - pX_1) Y_1(1 + p^2 + q^2)^{-2} \\
&- 120\mu J_4 R^4 r^{-8}(qY_1 - pX_1)(qX_1 + pY_1)(1 + p^2 + q^2)^{-3} p
\end{aligned}
\tag{99}
$$

$$
\begin{aligned}
\frac{\partial (f_\theta)_{J_4}}{\partial q} = {}& -840\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^2 Y_1(qX_1 + pY_1)(1 + p^2 + q^2)^{-4} \\
&- 280\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^3 X_1(1 + p^2 + q^2)^{-4} \\
&+ 2240\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^3(qX_1 + pY_1)(1 + p^2 + q^2)^{-5} q \\
&+ 30\mu J_4 R^4 r^{-8} Y_1(qX_1 + pY_1)(1 + p^2 + q^2)^{-2} \\
&+ 30\mu J_4 R^4 r^{-8}(qY_1 - pX_1) X_1(1 + p^2 + q^2)^{-2} \\
&- 120\mu J_4 R^4 r^{-8}(qY_1 - pX_1)(qX_1 + pY_1)(1 + p^2 + q^2)^{-3} q
\end{aligned}
\tag{100}
$$

$$
\begin{aligned}
\frac{\partial (f_\theta)_{J_4}}{\partial F} = {}& 2800\mu J_4 R^4 r^{-11}\frac{\partial r}{\partial F}(qY_1 - pX_1)^3(qX_1 + pY_1)(1 + p^2 + q^2)^{-4} \\
&- 840\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^2\left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F} \right) \\
&\times (qX_1 + pY_1)(1 + p^2 + q^2)^{-4} \\
&- 280\mu J_4 R^4 r^{-10}(qY_1 - pX_1)^3\left[q\frac{\partial X_1}{\partial F} + p\frac{\partial Y_1}{\partial F} \right](1 + p^2 + q^2)^{-4} \\
&- 240\mu J_4 R^4 r^{-9}\frac{\partial r}{\partial F}(qY_1 - pX_1)(qX_1 + pY_1)(1 + p^2 + q^2)^{-2} \\
&+ 30\mu J_4 R^4 r^{-8}\left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F} \right)(qX_1 + pY_1)(1 + p^2 + q^2)^{-2} \\
&+ 30\mu J_4 R^4 r^{-8}(qY_1 - pX_1)\left(q\frac{\partial X_1}{\partial F} + p\frac{\partial Y_1}{\partial F} \right)(1 + p^2 + q^2)^{-2}
\end{aligned}
\tag{101}
$$

And finally for $(f_h)_{J_4}$, the relevant partials can be cast in the form

$$\begin{aligned}\frac{\partial (f_h)_{J_4}}{\partial a} &= 1260\mu J_4 R^4 r^{-10}\frac{\partial r}{\partial a}(qY_1 - pX_1)^3(1-p^2-q^2)(1+p^2+q^2)^{-4}\\ &- 420\mu J_4 R^4 r^{-9}(qY_1 - pX_1)^2\left(q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right)\\ &\times (1-p^2-q^2)(1+p^2+q^2)^{-4}\\ &- 105\mu J_4 R^4 r^{-8}\frac{\partial r}{\partial a}(qY_1 - pX_1)(1-p^2-q^2)(1+p^2+q^2)^{-2}\\ &+ 15\mu J_4 R^4 r^{-7}\left(q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right)(1-p^2-q^2)(1+p^2+q^2)^{-2}\end{aligned} \tag{102}$$

$$\begin{aligned}\frac{\partial (f_h)_{J_4}}{\partial h} &= 1260\mu J_4 R^4 r^{-10}\left(\frac{\partial r}{\partial h}\right)_F(qY_1 - pX_1)^3(1-p^2-q^2)(1+p^2+q^2)^{-4}\\ &- 420\mu J_4 R^4 r^{-9}(qY_1 - pX_1)^2\left[q\left(\frac{\partial Y_1}{\partial h}\right)_F - p\left(\frac{\partial X_1}{\partial h}\right)_F\right]\\ &\times (1-p^2-q^2)(1+p^2+q^2)^{-4}\\ &- 105\mu J_4 R^4 r^{-8}\left(\frac{\partial r}{\partial h}\right)_F(qY_1 - pX_1)(1-p^2-q^2)(1+p^2+q^2)^{-2}\\ &+ 15\mu J_4 R^4 r^{-7}\left[q\left(\frac{\partial Y_1}{\partial h}\right)_F - p\left(\frac{\partial X_1}{\partial h}\right)_F\right](1-p^2-q^2)(1+p^2+q^2)^{-2}\end{aligned} \tag{103}$$

$$\begin{aligned}\frac{\partial (f_h)_{J_4}}{\partial k} &= 1260\mu J_4 R^4 r^{-10}\left(\frac{\partial r}{\partial k}\right)_F(qY_1 - pX_1)^3(1-p^2-q^2)(1+p^2+q^2)^{-4}\\ &- 420\mu J_4 R^4 r^{-9}(qY_1 - pX_1)^2\left[q\left(\frac{\partial Y_1}{\partial k}\right)_F - p\left(\frac{\partial X_1}{\partial k}\right)_F\right]\\ &\times (1-p^2-q^2)(1+p^2+q^2)^{-4}\\ &- 105\mu J_4 R^4 r^{-8}\left(\frac{\partial r}{\partial k}\right)_F(qY_1 - pX_1)(1-p^2-q^2)(1+p^2+q^2)^{-2}\\ &+ 15\mu J_4 R^4 r^{-7}\left[q\left(\frac{\partial Y_1}{\partial k}\right)_F - p\left(\frac{\partial X_1}{\partial k}\right)_F\right](1-p^2-q^2)(1+p^2+q^2)^{-2}\end{aligned} \tag{104}$$

$$\begin{aligned}\frac{\partial (f_h)_{J_4}}{\partial p} &= 420\mu J_4 R^4 r^{-9}(qY_1 - pX_1)^2 X_1(1-p^2-q^2)(1+p^2+q^2)^{-4}\\ &+ 280\mu J_4 R^4 r^{-9}(qY_1 - pX_1)^3 p(1+p^2+q^2)^{-4}\\ &+ 1120\mu J_4 R^4 r^{-9}(qY_1 - pX_1)^3(1-p^2-q^2)(1+p^2+q^2)^{-5}p\\ &- 15\mu J_4 R^4 r^{-7}X_1(1-p^2-q^2)(1+p^2+q^2)^{-2}\\ &- 30\mu J_4 R^4 r^{-7}(qY_1 - pX_1)p(1+p^2+q^2)^{-2}\\ &- 60\mu J_4 R^4 r^{-7}(qY_1 - pX_1)(1-p^2-q^2)(1+p^2+q^2)^{-3}p\end{aligned} \tag{105}$$

$$\frac{\partial(f_h)_{J_4}}{\partial q} = -420\mu J_4 R^4 r^{-9}(qY_1 - pX_1)^2 Y_1(1 - p^2 - q^2)(1 + p^2 + q^2)^{-4}$$
$$+ 280\mu J_4 R^4 r^{-9}(qY_1 - pX_1)^3 q(1 + p^2 + q^2)^{-4}$$
$$+ 1120\mu J_4 R^4 r^{-9}(qY_1 - pX_1)^3(1 - p^2 - q^2)(1 + p^2 + q^2)^{-5} q$$
$$+ 15\mu J_4 R^4 r^{-7} Y_1(1 - p^2 - q^2)(1 + p^2 + q^2)^{-2}$$
$$- 30\mu J_4 R^4 r^{-7}(qY_1 - pX_1) q(1 + p^2 + q^2)^{-2}$$
$$- 60\mu J_4 R^4 r^{-7}(qY_1 - pX_1)(1 - p^2 - q^2)(1 + p^2 + q^2)^{-3} q \tag{106}$$

$$\frac{\partial(f_h)_{J_4}}{\partial F} = 1260\mu J_4 R^4 r^{-10}\frac{\partial r}{\partial F}(qY_1 - pX_1)^3(1 - p^2 - q^2)(1 + p^2 + q^2)^{-4}$$
$$- 420\mu J_4 R^4 r^{-9}(qY_1 - pX_1)^2\left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F}\right)$$
$$\times (1 - p^2 - q^2)(1 + p^2 + q^2)^{-4}$$
$$- 105\mu J_4 R^4 r^{-8}\frac{\partial r}{\partial F}(qY_1 - pX_1)(1 - p^2 - q^2)(1 + p^2 + q^2)^{-2}$$
$$+ 15\mu J_4 R^4 r^{-7}\left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F}\right)(1 - p^2 - q^2)(1 + p^2 + q^2)^{-2} \tag{107}$$

The Treatment of the J_3, J_4 Perturbations within the True Longitude Formulation

The dynamic and adjoint equations for the true longitude formulation were given in references [7] and [11]. The dynamic equations are given by

$$\begin{bmatrix} \dot{a} \\ \dot{h} \\ \dot{k} \\ \dot{p} \\ \dot{q} \\ \dot{L} \end{bmatrix} = \begin{bmatrix} B^L_{11} & B^L_{12} & B^L_{13} \\ B^L_{21} & B^L_{22} & B^L_{23} \\ B^L_{31} & B^L_{32} & B^L_{33} \\ B^L_{41} & B^L_{42} & B^L_{43} \\ B^L_{51} & B^L_{52} & B^L_{53} \\ B^L_{61} & B^L_{62} & B^L_{63} \end{bmatrix} \left[\begin{pmatrix} u_r \\ u_\theta \\ u_h \end{pmatrix} (f_t) + \mathbf{f}_{J_2} + \mathbf{f}_{J_3} + \mathbf{f}_{J_4} \right] + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{na^2(1 - h^2 - k^2)^{1/2}}{r^2} \end{bmatrix} \tag{108}$$

Unlike the eccentric longitude formulation of the previous section, where the thrust acceleration, the components of the rows of matrix M, and the J_2, J_3, and J_4 accelerations were resolved along the equinoctial frame $\hat{\mathbf{f}}$, $\hat{\mathbf{g}}$, $\hat{\mathbf{w}}$, this true longitude formulation uses the Euler-Hill $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\mathbf{h}}$ frame such that the rows of matrix B^L above as well as the thrust and zonal accelerations are resolved in this Hill frame. The Hamiltonian and the adjoint differential equations for the combined thrust and J_2 perturbations given in reference [11] are completed below by the addition of the higher order J_3 and J_4 terms such that with the unit vector $\hat{\boldsymbol{u}} = (u_r, u_\theta, u_h)$

$$H = \boldsymbol{\lambda}_z^{\mathrm{T}} B^L(\mathbf{z}) f_t \hat{\mathbf{u}} + \lambda_L[na^2(1 - h^2 - k^2)^{1/2}/r^2] + \boldsymbol{\lambda}_z^{\mathrm{T}} B^L(\mathbf{z})(\mathbf{f}_{J_2} + \mathbf{f}_{J_3} + \mathbf{f}_{J_4}) \tag{109}$$

$$\dot{\boldsymbol{\lambda}}_z = -\frac{\partial H}{\partial \mathbf{z}} = -\boldsymbol{\lambda}_z^{\mathrm{T}}\frac{\partial B^L}{\partial \mathbf{z}} f_t \hat{\mathbf{u}} - \lambda_L \frac{\partial}{\partial \mathbf{z}}\left[\frac{na^2(1 - h^2 - k^2)^{1/2}}{r^2}\right]$$
$$- \boldsymbol{\lambda}_z^{\mathrm{T}}\frac{\partial B^L}{\partial \mathbf{z}}(\mathbf{f}_{J_2} + \mathbf{f}_{J_3} + \mathbf{f}_{J_4}) - \boldsymbol{\lambda}_z^{\mathrm{T}} B^L\left(\frac{\partial \mathbf{f}_{J_2}}{\partial \mathbf{z}} + \frac{\partial \mathbf{f}_{J_3}}{\partial \mathbf{z}} + \frac{\partial \mathbf{f}_{J_4}}{\partial \mathbf{z}}\right) \tag{110}$$

In reference [11], J_3 and J_4 were not included in the analysis, and there $\mathbf{f}_{J_2}$ was simply written as $\mathbf{f}$. The $\partial \mathbf{f}_{J_3}/\partial \mathbf{z}$ and $\partial \mathbf{f}_{J_4}/\partial \mathbf{z}$ partials are needed to complete this analysis. Note that the partial derivatives of matrix B^L with respect to $\mathbf{z} = (a, h, k, p, q, L)$ as given in the appendix of reference [7] have two typographic errors in equations (A65) and (A74) that should read as

$$B^L_{63} = n^{-1}a^{-2}rG^{-1}(qs_L - pc_L) \qquad \text{(A65) of [2]}$$

$$\frac{\partial}{\partial k}\left[\frac{na^2(1 - h^2 - k^2)^{1/2}}{r^2}\right] = -2na^2r^{-3}G\frac{\partial r}{\partial k} - na^2r^{-2}kG^{-1} \qquad \text{(A74) of [2]}$$

Now using the expressions for $(f_r)_{J_3}$, $(f_\theta)_{J_3}$ and $(f_h)_{J_3}$, in equations (34)–(36), directly in terms of *a, h, k, p, q* and *L*, we have for $(f_r)_{J_3}$, the partials with respect to the elements *a, h, k, p, q* and *L* as

$$\begin{aligned}\frac{\partial(f_r)_{J_3}}{\partial a} = {} & 60\mu J_3R^3r^{-6}\left(\frac{\partial r}{\partial a}\right)_L(qs_L - pc_L)(1 + p^2 + q^2)^{-1} \\ & - 400\mu J_3R^3r^{-6}\left(\frac{\partial r}{\partial a}\right)_L(qs_L - pc_L)^3(1 + p^2 + q^2)^{-3}\end{aligned} \qquad (111)$$

$$\begin{aligned}\frac{\partial(f_r)_{J_3}}{\partial h} = {} & 60\mu J_3R^3r^{-6}\left(\frac{\partial r}{\partial h}\right)_L(qs_L - pc_L)(1 + p^2 + q^2)^{-1} \\ & - 400\mu J_3R^3r^{-6}\left(\frac{\partial r}{\partial h}\right)_L(qs_L - pc_L)^3(1 + p^2 + q^2)^{-3}\end{aligned} \qquad (112)$$

$$\begin{aligned}\frac{\partial(f_r)_{J_3}}{\partial k} = {} & 60\mu J_3R^3r^{-6}\left(\frac{\partial r}{\partial k}\right)_L(qs_L - pc_L)(1 + p^2 + q^2)^{-1} \\ & - 400\mu J_3R^3r^{-6}\left(\frac{\partial r}{\partial k}\right)_L(qs_L - pc_L)^3(1 + p^2 + q^2)^{-3}\end{aligned} \qquad (113)$$

$$\begin{aligned}\frac{\partial(f_r)_{J_3}}{\partial p} = {} & 12\mu J_3R^3r^{-5}c_L(1 + p^2 + q^2)^{-1} \\ & + 24\mu J_3R^3r^{-5}(qs_L - pc_L)(1 + p^2 + q^2)^{-2}p \\ & - 240\mu J_3R^3r^{-5}(qs_L - pc_L)^2c_L(1 + p^2 + q^2)^{-3} \\ & - 480\mu J_3R^3r^{-5}(qs_L - pc_L)^3(1 + p^2 + q^2)^{-4}p\end{aligned} \qquad (114)$$

$$\begin{aligned}\frac{\partial(f_r)_{J_3}}{\partial q} = {} & -12\mu J_3R^3r^{-5}s_L(1 + p^2 + q^2)^{-1} \\ & + 24\mu J_3R^3r^{-5}(qs_L - pc_L)(1 + p^2 + q^2)^{-2}q \\ & + 240\mu J_3R^3r^{-5}(qs_L - pc_L)^2s_L(1 + p^2 + q^2)^{-3} \\ & - 480\mu J_3R^3r^{-5}(qs_L - pc_L)^3(1 + p^2 + q^2)^{-4}q\end{aligned} \qquad (115)$$

$$\begin{aligned}\frac{\partial(f_r)_{J_3}}{\partial L} = {} & 60\mu J_3R^3r^{-6}\left(\frac{\partial r}{\partial L}\right)_L(qs_L - pc_L)(1 + p^2 + q^2)^{-1} \\ & - 12\mu J_3R^3r^{-5}(qc_L + ps_L)(1 + p^2 + q^2)^{-1} \\ & - 400\mu J_3R^3r^{-6}\left(\frac{\partial r}{\partial L}\right)_L(qs_L - pc_L)^3(1 + p^2 + q^2)^{-3} \\ & + 240\mu J_3R^3r^{-5}(qs_L - pc_L)^2(qc_L + ps_L)(1 + p^2 + q^2)^{-3}\end{aligned} \qquad (116)$$

For the $(f_\theta)_{J_3}$ partials, we have in a similar way

$$\frac{\partial (f_\theta)_{J_3}}{\partial a} = 300\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial a}\right)_L (qc_L + ps_L)(qs_L - pc_L)^2(1 + p^2 + q^2)^{-3} - 15\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial a}\right)_L (qc_L + ps_L)(1 + p^2 + q^2)^{-1} \tag{117}$$

$$\frac{\partial (f_\theta)_{J_3}}{\partial h} = 300\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial h}\right)_L (qc_L + ps_L)(qs_L - pc_L)^2(1 + p^2 + q^2)^{-3} - 15\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial h}\right)_L (qc_L + ps_L)(1 + p^2 + q^2)^{-1} \tag{118}$$

$$\frac{\partial (f_\theta)_{J_3}}{\partial k} = 300\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial k}\right)_L (qc_L + ps_L)(qs_L - pc_L)^2(1 + p^2 + q^2)^{-3} - 15\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial k}\right)_L (qc_L + ps_L)(1 + p^2 + q^2)^{-1} \tag{119}$$

$$\begin{aligned}\frac{\partial (f_\theta)_{J_3}}{\partial p} = &-60\mu J_3 R^3 r^{-5} s_L(qs_L - pc_L)^2(1 + p^2 + q^2)^{-3} \\ &+ 120\mu J_3 R^3 r^{-5}(qc_L + ps_L)(qs_L - pc_L)c_L(1 + p^2 + q^2)^{-3} \\ &+ 360\mu J_3 R^3 r^{-5}(qc_L + ps_L)(qs_L - pc_L)^2(1 + p^2 + q^2)^{-4} p \\ &+ 3\mu J_3 R^3 r^{-5} s_L(1 + p^2 + q^2)^{-1} \\ &- 6\mu J_3 R^3 r^{-5}(qc_L + ps_L)(1 + p^2 + q^2)^{-2} p\end{aligned} \tag{120}$$

$$\begin{aligned}\frac{\partial (f_\theta)_{J_3}}{\partial q} = &-60\mu J_3 R^3 r^{-5} c_L(qs_L - pc_L)^2(1 + p^2 + q^2)^{-3} \\ &- 120\mu J_3 R^3 r^{-5}(qc_L + ps_L)(qs_L - pc_L)s_L(1 + p^2 + q^2)^{-3} \\ &+ 360\mu J_3 R^3 r^{-5}(qc_L + ps_L)(qs_L - pc_L)^2(1 + p^2 + q^2)^{-4} q \\ &+ 3\mu J_3 R^3 r^{-5} c_L(1 + p^2 + q^2)^{-1} \\ &- 6\mu J_3 R^3 r^{-5}(qc_L + ps_L)(1 + p^2 + q^2)^{-2} q\end{aligned} \tag{121}$$

$$\begin{aligned}\frac{\partial (f_\theta)_{J_3}}{\partial L} = &\, 300\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial L}\right)_L (qc_L + ps_L)(qs_L - pc_L)^2(1 + p^2 + q^2)^{-3} \\ &+ 60\mu J_3 R^3 r^{-5}(qs_L - pc_L)^3(1 + p^2 + q^2)^{-3} \\ &- 120\mu J_3 R^3 r^{-5}(qs_L - pc_L)(qc_L + ps_L)^2(1 + p^2 + q^2)^{-3} \\ &- 15\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial L}\right)_L (qc_L + ps_L)(1 + p^2 + q^2)^{-1} \\ &- 3\mu J_3 R^3 r^{-5}(qs_L - pc_L)(1 + p^2 + q^2)^{-1}\end{aligned} \tag{122}$$

For $(f_h)_{J_3}$, the relevant partial derivatives take the form

$$\frac{\partial (f_h)_{J_3}}{\partial a} = 150\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial a}\right)_L (qs_L - pc_L)^2(1 - p^2 - q^2)(1 + p^2 + q^2)^{-3} - \frac{15}{2}\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial a}\right)_L (1 - p^2 - q^2)(1 + p^2 + q^2)^{-1} \tag{123}$$

$$\frac{\partial (f_h)_{J_3}}{\partial h} = 150\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial h}\right)_L (qs_L - pc_L)^2(1 - p^2 - q^2)(1 + p^2 + q^2)^{-3} - \frac{15}{2}\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial h}\right)_L (1 - p^2 - q^2)(1 + p^2 + q^2)^{-1} \quad (124)$$

$$\frac{\partial (f_h)_{J_3}}{\partial k} = 150\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial k}\right)_L (qs_L - pc_L)^2(1 - p^2 - q^2)(1 + p^2 + q^2)^{-3} - \frac{15}{2}\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial k}\right)_L (1 - p^2 - q^2)(1 + p^2 + q^2)^{-1} \quad (125)$$

$$\begin{aligned}\frac{\partial (f_h)_{J_3}}{\partial p} = {} & 60\mu J_3 R^3 r^{-5}(qs_L - pc_L)c_L(1 - p^2 - q^2)(1 + p^2 + q^2)^{-3} \\ & + 60\mu J_3 R^3 r^{-5}(qs_L - pc_L)^2 p(1 + p^2 + q^2)^{-3} \\ & + 180\mu J_3 R^3 r^{-5}(qs_L - pc_L)^2(1 - p^2 - q^2)(1 + p^2 + q^2)^{-4} p \\ & - 3\mu J_3 R^3 r^{-5} p(1 + p^2 + q^2)^{-1} \\ & - 3\mu J_3 R^3 r^{-5}(1 - p^2 - q^2)(1 + p^2 + q^2)^{-2} p \quad (126)\end{aligned}$$

$$\begin{aligned}\frac{\partial (f_h)_{J_3}}{\partial q} = {} & -60\mu J_3 R^3 r^{-5}(qs_L - pc_L)s_L(1 - p^2 - q^2)(1 + p^2 + q^2)^{-3} \\ & + 60\mu J_3 R^3 r^{-5}(qs_L - pc_L)^2 q(1 + p^2 + q^2)^{-3} \\ & + 180\mu J_3 R^3 r^{-5}(qs_L - pc_L)^2(1 - p^2 - q^2)(1 + p^2 + q^2)^{-4} q \\ & - 3\mu J_3 R^3 r^{-5} q(1 + p^2 + q^2)^{-1} \\ & - 3\mu J_3 R^3 r^{-5}(1 - p^2 - q^2)(1 + p^2 + q^2)^{-2} q \quad (127)\end{aligned}$$

$$\begin{aligned}\frac{\partial (f_h)_{J_3}}{\partial L} = {} & 150\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial L}\right)_L (qs_L - pc_L)^2(1 - p^2 - q^2)(1 + p^2 + q^2)^{-3} \\ & - 60\mu J_3 R^3 r^{-5}(qs_L - pc_L)(qc_L + ps_L)(1 - p^2 - q^2)(1 + p^2 + q^2)^{-3} \\ & - \frac{15}{2}\mu J_3 R^3 r^{-6}\left(\frac{\partial r}{\partial L}\right)_L (1 - p^2 - q^2)(1 + p^2 + q^2)^{-1} \quad (128)\end{aligned}$$

In a similar way using the $\mathbf{f}_{J_4}$ acceleration components given in equations (84)–(86) in terms of *a, h, k, p, q* and *L*, we have for the $(f_r)_{J_4}$ component, the partials

$$\frac{\partial (f_r)_{J_4}}{\partial a} = 450\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial a}\right)_L (qs_L - pc_L)^2(1 + p^2 + q^2)^{-2} - 2100\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial a}\right)_L (qs_L - pc_L)^4(1 + p^2 + q^2)^{-4} - \frac{45}{4}\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial a}\right)_L \quad (129)$$

$$\frac{\partial (f_r)_{J_4}}{\partial h} = 450\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial h}\right)_L (qs_L - pc_L)^2(1 + p^2 + q^2)^{-2} - 2100\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial h}\right)_L (qs_L - pc_L)^4(1 + p^2 + q^2)^{-4} - \frac{45}{4}\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial h}\right)_L \quad (130)$$

$$\frac{\partial (f_r)_{J_4}}{\partial k} = 450\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial k}\right)_L (qs_L - pc_L)^2 (1 + p^2 + q^2)^{-2} - 2100\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial k}\right)_L (qs_L - pc_L)^4 (1 + p^2 + q^2)^{-4} - \frac{45}{4}\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial k}\right)_L \tag{131}$$

$$\begin{aligned}\frac{\partial (f_r)_{J_4}}{\partial p} = {} & 150\mu J_4 R^4 r^{-6}(qs_L - pc_L)c_L(1 + p^2 + q^2)^{-2} \\ & + 300\mu J_4 R^4 r^{-6}(qs_L - pc_L)^2 (1 + p^2 + q^2)^{-3}p \\ & - 1400\mu J_4 R^4 r^{-6}(qs_L - pc_L)^3 c_L (1 + p^2 + q^2)^{-4} \\ & - 2800\mu J_4 R^4 r^{-6}(qs_L - pc_L)^4 (1 + p^2 + q^2)^{-5} p\end{aligned} \tag{132}$$

$$\begin{aligned}\frac{\partial (f_r)_{J_4}}{\partial q} = {} & -150\mu J_4 R^4 r^{-6}(qs_L - pc_L)s_L(1 + p^2 + q^2)^{-2} \\ & + 300\mu J_4 R^4 r^{-6}(qs_L - pc_L)^2 (1 + p^2 + q^2)^{-3}q \\ & + 1400\mu J_4 R^4 r^{-6}(qs_L - pc_L)^3 s_L (1 + p^2 + q^2)^{-4} \\ & - 2800\mu J_4 R^4 r^{-6}(qs_L - pc_L)^4 (1 + p^2 + q^2)^{-5} q\end{aligned} \tag{133}$$

$$\begin{aligned}\frac{\partial (f_r)_{J_4}}{\partial L} = {} & 450\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial L}\right)_L (qs_L - pc_L)^2 (1 + p^2 + q^2)^{-2} \\ & - 150\mu J_4 R^4 r^{-6} (qs_L - pc_L)(qc_L + ps_L)(1 + p^2 + q^2)^{-2} \\ & - 2100\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial L}\right)_L (qs_L - pc_L)^4 (1 + p^2 + q^2)^{-4} \\ & + 1400\mu J_4 R^4 r^{-6}(qs_L - pc_L)^3 (qc_L + ps_L)(1 + p^2 + q^2)^{-4} \\ & - \frac{45}{4}\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial L}\right)_L\end{aligned} \tag{134}$$

For the $(f_\theta)_{J_4}$ partials, we have

$$\begin{aligned}\frac{\partial (f_\theta)_{J_4}}{\partial a} = {} & 1680\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial a}\right)_L (qs_L - pc_L)^3 (qc_L + ps_L)(1 + p^2 + q^2)^{-4} \\ & - 180\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial a}\right)_L (qs_L - pc_L)(qc_L + ps_L)(1 + p^2 + q^2)^{-2}\end{aligned} \tag{135}$$

$$\begin{aligned}\frac{\partial (f_\theta)_{J_4}}{\partial h} = {} & 1680\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial h}\right)_L (qs_L - pc_L)^3 (qc_L + ps_L)(1 + p^2 + q^2)^{-4} \\ & - 180\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial h}\right)_L (qs_L - pc_L)(qc_L + ps_L)(1 + p^2 + q^2)^{-2}\end{aligned} \tag{136}$$

$$\begin{aligned}\frac{\partial (f_\theta)_{J_4}}{\partial k} = {} & 1680\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial k}\right)_L (qs_L - pc_L)^3 (qc_L + ps_L)(1 + p^2 + q^2)^{-4} \\ & - 180\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial k}\right)_L (qs_L - pc_L)(qc_L + ps_L)(1 + p^2 + q^2)^{-2}\end{aligned} \tag{137}$$

$$\begin{aligned}\frac{\partial(f_\theta)_{J_4}}{\partial p} &= 840\mu J_4R^4r^{-6}(qs_L - pc_L)^2c_L(qc_L + ps_L)(1 + p^2 + q^2)^{-4}\\ &- 280\mu J_4R^4r^{-6}(qs_L - pc_L)^3s_L(1 + p^2 + q^2)^{-4}\\ &+ 2240\mu J_4R^4r^{-6}(qs_L - pc_L)^3(qc_L + ps_L)(1 + p^2 + q^2)^{-5}p\\ &- 30\mu J_4R^4r^{-6}c_L(qc_L + ps_L)(1 + p^2 + q^2)^{-2}\\ &+ 30\mu J_4R^4r^{-6}(qs_L - pc_L)s_L(1 + p^2 + q^2)^{-2}\\ &- 120\mu J_4R^4r^{-6}(qs_L - pc_L)(qc_L + ps_L)(1 + p^2 + q^2)^{-3}p \qquad (138)\end{aligned}$$

$$\begin{aligned}\frac{\partial(f_\theta)_{J_4}}{\partial q} &= -840\mu J_4R^4r^{-6}(qs_L - pc_L)^2s_L(qc_L + ps_L)(1 + p^2 + q^2)^{-4}\\ &- 280\mu J_4R^4r^{-6}(qs_L - pc_L)^3c_L(1 + p^2 + q^2)^{-4}\\ &+ 2240\mu J_4R^4r^{-6}(qs_L - pc_L)^3(qc_L + ps_L)(1 + p^2 + q^2)^{-5}q\\ &+ 30\mu J_4R^4r^{-6}s_L(qc_L + ps_L)(1 + p^2 + q^2)^{-2}\\ &+ 30\mu J_4R^4r^{-6}(qs_L - pc_L)c_L(1 + p^2 + q^2)^{-2}\\ &- 120\mu J_4R^4r^{-6}(qs_L - pc_L)(qc_L + ps_L)(1 + p^2 + q^2)^{-3}q \qquad (139)\end{aligned}$$

$$\begin{aligned}\frac{\partial(f_\theta)_{J_4}}{\partial L} &= 1680\mu J_4R^4r^{-7}\left(\frac{\partial r}{\partial L}\right)_L(qs_L - pc_L)^3(qc_L + ps_L)(1 + p^2 + q^2)^{-4}\\ &- 840\mu J_4R^4r^{-6}(qs_L - pc_L)^2(qc_L + ps_L)^2(1 + p^2 + q^2)^{-4}\\ &+ 280\mu J_4R^4r^{-6}(qs_L - pc_L)^4(1 + p^2 + q^2)^{-4}\\ &- 180\mu J_4R^4r^{-7}\left(\frac{\partial r}{\partial L}\right)_L(qs_L - pc_L)(qc_L + ps_L)(1 + p^2 + q^2)^{-2}\\ &+ 30\mu J_4R^4r^{-6}(qc_L + ps_L)^2(1 + p^2 + q^2)^{-2}\\ &- 30\mu J_4R^4r^{-6}(qs_L - pc_L)^2(1 + p^2 + q^2)^{-2} \qquad (140)\end{aligned}$$

And finally for $(f_h)_{J_4}$, the six partial derivatives are obtained as

$$\begin{aligned}\frac{\partial(f_h)_{J_4}}{\partial a} &= 840\mu J_4R^4r^{-7}\left(\frac{\partial r}{\partial a}\right)_L(qs_L - pc_L)^3(1 - p^2 - q^2)(1 + p^2 + q^2)^{-4}\\ &- 90\mu J_4R^4r^{-7}\left(\frac{\partial r}{\partial a}\right)_L(qs_L - pc_L)(1 - p^2 - q^2)(1 + p^2 + q^2)^{-2} \qquad (141)\end{aligned}$$

$$\begin{aligned}\frac{\partial(f_h)_{J_4}}{\partial h} &= 840\mu J_4R^4r^{-7}\left(\frac{\partial r}{\partial h}\right)_L(qs_L - pc_L)^3(1 - p^2 - q^2)(1 + p^2 + q^2)^{-4}\\ &- 90\mu J_4R^4r^{-7}\left(\frac{\partial r}{\partial h}\right)_L(qs_L - pc_L)(1 - p^2 - q^2)(1 + p^2 + q^2)^{-2} \qquad (142)\end{aligned}$$

$$\begin{aligned}\frac{\partial(f_h)_{J_4}}{\partial k} &= 840\mu J_4R^4r^{-7}\left(\frac{\partial r}{\partial k}\right)_L(qs_L - pc_L)^3(1 - p^2 - q^2)(1 + p^2 + q^2)^{-4}\\ &- 90\mu J_4R^4r^{-7}\left(\frac{\partial r}{\partial k}\right)_L(qs_L - pc_L)(1 - p^2 - q^2)(1 + p^2 + q^2)^{-2} \qquad (143)\end{aligned}$$

$$\begin{aligned}\frac{\partial (f_h)_{J_4}}{\partial p} = {} & 420\mu J_4 R^4 r^{-6}(qs_L - pc_L)^2 c_L (1 - p^2 - q^2)(1 + p^2 + q^2)^{-4} \\ & + 280\mu J_4 R^4 r^{-6}(qs_L - pc_L)^3 p (1 + p^2 + q^2)^{-4} \\ & + 1120\mu J_4 R^4 r^{-6}(qs_L - pc_L)^3 (1 - p^2 - q^2)(1 + p^2 + q^2)^{-5} p \\ & - 15\mu J_4 R^4 r^{-6} c_L (1 - p^2 - q^2)(1 + p^2 + q^2)^{-2} \\ & - 30\mu J_4 R^4 r^{-6}(qs_L - pc_L) p (1 + p^2 + q^2)^{-2} \\ & - 60\mu J_4 R^4 r^{-6}(qs_L - pc_L)(1 - p^2 - q^2)(1 + p^2 + q^2)^{-3} p \end{aligned} \tag{144}$$

$$\begin{aligned}\frac{\partial (f_h)_{J_4}}{\partial q} = {} & -420\mu J_4 R^4 r^{-6}(qs_L - pc_L)^2 s_L (1 - p^2 - q^2)(1 + p^2 + q^2)^{-4} \\ & + 280\mu J_4 R^4 r^{-6}(qs_L - pc_L)^3 q (1 + p^2 + q^2)^{-4} \\ & + 1120\mu J_4 R^4 r^{-6}(qs_L - pc_L)^3 (1 - p^2 - q^2)(1 + p^2 + q^2)^{-5} q \\ & + 15\mu J_4 R^4 r^{-6} s_L (1 - p^2 - q^2)(1 + p^2 + q^2)^{-2} \\ & - 30\mu J_4 R^4 r^{-6}(qs_L - pc_L) q (1 + p^2 + q^2)^{-2} \\ & - 60\mu J_4 R^4 r^{-6}(qs_L - pc_L)(1 - p^2 - q^2)(1 + p^2 + q^2)^{-3} q \end{aligned} \tag{145}$$

$$\begin{aligned}\frac{\partial (f_h)_{J_4}}{\partial L} = {} & 840\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial L}\right)_L (qs_L - pc_L)^3 (1 - p^2 - q^2)(1 + p^2 + q^2)^{-4} \\ & - 420\mu J_4 R^4 r^{-6}(qs_L - pc_L)^2 (qc_L + ps_L)(1 - p^2 - q^2)(1 + p^2 + q^2)^{-4} \\ & - 90\mu J_4 R^4 r^{-7}\left(\frac{\partial r}{\partial L}\right)_L (qs_L - pc_L)(1 - p^2 - q^2)(1 + p^2 + q^2)^{-2} \\ & + 15\mu J_4 R^4 r^{-6}(qc_L + ps_L)(1 - p^2 - q^2)(1 + p^2 + q^2)^{-2} \end{aligned} \tag{146}$$

Note that the $\left(\frac{\partial r}{\partial a}\right)_L, \left(\frac{\partial r}{\partial h}\right)_L, \left(\frac{\partial r}{\partial k}\right)_L, \left(\frac{\partial r}{\partial L}\right)_L$ partials appearing in the above equations, are as in equation (65) of reference [11], namely

$$\left(\frac{\partial r}{\partial a}\right)_L = \frac{r}{a}, \quad \left(\frac{\partial r}{\partial h}\right)_L = -\frac{r}{a(1 - h^2 - k^2)}(2ah + rs_L)$$

$$\left(\frac{\partial r}{\partial k}\right)_L = -\frac{r}{a(1 - h^2 - k^2)}(2ak + rc_L), \quad \left(\frac{\partial r}{\partial L}\right)_L = -\frac{r^2}{a}\frac{hc_L - ks_L}{(1 - h^2 - k^2)}$$

Numerical Results

We first show the need to include the J_3 and J_4 accelerations by running an optimal trajectory generated in [12] and [13] with only J_2 taken into account. The initial and final orbits for the example transfer are shown in Table 1. The overall minimum-time solution is obtained by also searching for the optimal departure point on the initial orbit at time $t = 0$ as well as the optimal arrival location on the final orbit at time t_f, which is itself one of the search parameters. The shooting method is used by starting from guessed values for the five multipliers at time $t = 0$, namely, $(\lambda_a)_0$, $(\lambda_h)_0$, $(\lambda_k)_0$, $(\lambda_p)_0$, $(\lambda_q)_0$ as well as the initial mean longitude $(\lambda)_0$ or rather eccentric longitude $(F)_0$ corresponding to the initial angular position $(\theta)_0$. The value of t_f is also guessed and the dynamic and adjoint differential

TABLE 1. Initial, Target and Achieved Orbit Parameters Using J_2-Compliant Solution, with J_2 only, and with J_2, J_3, J_4 Perturbations

Orbit	a, km	e	i, deg	Ω, deg	ω, deg	M, F, L deg
Initial	7000	0	28.5	0	0	$M_0 = -131.7396776$ (opt.)
Target	42,000	10^{-3}	1	0	0	Free
Achieved [L] J_2	41,999.99992	1.000022×10^{-3}	1.0000013	359.999569	359.999136	$L = 45.4919264$ (opt.)
Achieved [F] J_2	41,999.99992	1.000022×10^{-3}	1.0000013	359.999569	359.999130	$F = 45.45107830$ (opt.)
Achieved [L] J_2, J_3, J_4	41,996.09349	$9.222074532 \times 10^{-4}$	$9.995354342 \times 10^{-1}$	359.9363898	4.3177075	$M = 41.17416016$

equations with J_2 only are integrated forward in time simultaneously starting from $(\lambda_F)_0 = 0$ until t_f while thrusting along the optimal $\hat{\mathbf{u}}$ orientation given by $\hat{\mathbf{u}} = \boldsymbol{\lambda}_z^T M(\mathbf{z}, F)/|\boldsymbol{\lambda}_z^T M(\mathbf{z}, F)|$ that effectively maximizes the Hamiltonian H in equation (23) with f_{J_3} and f_{J_4} turned off. The quantities $(\lambda_a)_0$, $(\lambda_h)_0$, $(\lambda_k)_0$, $(\lambda_p)_0$, $(\lambda_q)_0$ as well as $(F)_0$ and t_f are slowly adjusted until the final orbit parameters $(a)_f$, $(h)_f$, $(k)_f$, $(p)_f$, $(q)_f$ that correspond to the target a, e, i, Ω, ω given in Table 1 are met to within a reasonably small tolerance, with $(F)_f = 0$ and $H_f = 1$ also satisfied.

The initial and final orbit parameters are given as classical elements in Table 1 but are converted at time $t = 0$ to obtain the equinoctial equivalents to start the integrations, and converted from the equinoctials at t_f to the equivalent classicals for tabulation purposes.

The initial values of the converged multipliers are given by $(\lambda_a)_0 = 4.800100306$ s/km, $(\lambda_h)_0 = 8.060772261 \times 10^2$ s, $(\lambda_k)_0 = -9.150040837 \times 10^3$ s, $(\lambda_p)_0 = 3.281827358 \times 10^1$ s, $(\lambda_q)_0 = -2.254928992 \times 10^4$ s, the optimal initial location was obtained as $(\lambda)_0 = -2.299291130$ rad, and the minimum time t_f was obtained as 58,104.83438 s.

With $(\lambda)_0$ corresponding to the mean anomaly $M_0 = -131.7396776$ deg the state and costate equations are propagated forward in time starting from the converged multipliers above and using equations (108) and (110) by firing along the optimal direction $\hat{\mathbf{u}} = \boldsymbol{\lambda}_z^T B^L(\mathbf{z})/|\boldsymbol{\lambda}_z^T B^L(\mathbf{z})|$ until $t_f = 58{,}104.83438$ s. Both t_f and the converged multipliers as well as the optimal $(\lambda)_0$ correspond to the optimal solution obtained with J_2 only [12], [13]. Thus three runs are made using these converged values, the first two using the original J_2 formulation of [11], [12], and [13] respectively and the third run using equations (108) and (110) of this present paper with J_3 and J_4 turned on in order to fly the trajectory using a more accurate Earth gravity model. The first two runs from time $t = 0$ to t_f will achieve the desired end conditions because the initial converged multiplier values are the optimal ones corresponding to the simplified gravity model that uses J_2 only. Obviously using these converged values to fly with a more accurate gravity model that uses J_2, J_3 and J_4 and terminating the flight at the J_2-optimal flight time t_f will fail to meet the end conditions.

Table 1 shows the achieved final parameters for this LEO to GEO transfer using the Achieved [L] of [11] and Achieved [F] of [12] and [13] formulations using J_2 only, and the present paper Achieved [L] formulation using J_2, J_3 and J_4, i.e., equations (108)–(110). As in all runs, we use $f_t = 9.8 \times 10^{-5}$ km/s^2 and $J_2 = 1.08263 \times 10^{-3}$, $J_3 = -2.56 \times 10^{-6}$, $J_4 = -1.58 \times 10^{-6}$, and a Runge-Kutta 78 integration method as given by the Fehlberg coefficients with relative and absolute error controls set at the 10^{-9} level.

As expected, the Achieved [L]-J_2 and Achieved [F]-J_2 forward runs meet the target conditions effectively as shown in Table 1, while the Achieved [L]-$J_2J_3J_4$ run misses the final semimajor axis by 4 km and the argument of perigee by 4 deg or equivalently the final position by as much. These results show that restricting ourselves to a reduced model with J_2 only and using the corresponding optimal solution to fly in the more realistic and accurate model with J_2, J_3 and J_4 will miss the target even for this very short duration transfer which experiences very little perturbation from the zonal harmonics. These differences will be dramatically more enhanced for long duration transfers that dwell for long periods of time in the gravity well at LEO where the zonal perturbations affect the trajectories considerably.

Note that the integrated value of L at the final time t_f, namely $L = 45.4919264$ deg from Achieved [L], corresponds to $M = 45.411543$ deg which corresponds to the optimal insertion point on the target orbit with J_2 only. As discussed above, the value of M as well as the values of the other parameters from Achieved [L]-$J_2J_3J_4$ are clearly significantly different due to the addition of the J_3 and J_4 terms.

We now generate an optimal minimum-time trajectory using equations (108)–(110) and searching on the initial values of $(\lambda_a)_0$, $(\lambda_h)_0$, $(\lambda_k)_0$, $(\lambda_p)_0$, $(\lambda_q)_0$, and L_0 and starting with $(\lambda_L)_0 = 0$ until the final a, h, k, p, q, $(\lambda_L) = 0$, and $H = 1$ are satisfied. The result is $(\lambda_a)_0 = 4.800362521$ s/km, $(\lambda_h)_0 = 8.064085075 \times 10^2$ s, $(\lambda_k)_0 = -9.149918686 \times 10^3$ s, $(\lambda_p)_0 = 3.284285354 \times 10^1$ s, $(\lambda_q)_0 = -2.254952264 \times 10^4$ s, $(L)_0 = -2.299361237$ rad, and $t_f = 58{,}104.96895$ s.

Because $e_0 = 0$, $\omega_0 = 0$ and $\Omega_0 = 0$, $(L)_0$ is identical to $(M)_0 = -131.7436944$ deg. Using these initial values, the Achieved [L]-$J_2J_3J_4$ [equations (108)–(110)], and Achieved [F]-$J_2J_3J_4$ [equations (17)–(24)] parameters at time t_f are in perfect agreement as seen in Table 2, thus validating both the L and F formulations of the present paper. Note that $(\lambda_L)_0 = (\lambda_F)_0 = 0$.

As stated above, this fast transfer trajectory experiences very little zonal perturbation effects and that is why the J_2-optimized and the present $J_2J_3J_4$-optimized solutions differ very little in total flight time t_f as well as in the final insertion location. These differences can become quite substantial and cannot be neglected both during preliminary designs and actual operations. Even the small differences seen in this present example can be misleading because if the J_3 and J_4 perturbations are not accounted for, and the solutions rely only on the J_2-only model, they cannot be flown accurately as shown in Table 1 due to the presence of the higher order harmonics. In fact, further harmonics such as J_5 and J_6 must also be included extending the present analysis especially for long duration transfers spanning several months with a considerable amount spent in LEO.

However, it is possible to use a cruder J_2-only model and continuously update the transfer solution during an actual flight in order to eventually correct for the effects of the unmodelled higher order perturbations as the spacecraft approaches its higher destination orbit such as GEO. It is however better to use a more refined model such as the one exposed here even if continuous updates are planned for actual flights because more accurate and economical trajectories can then be flown. If low-thrust is used instead to transfer to lower orbits or if the transfers are done entirely in low orbit and of long duration, then higher fidelity models are even more beneficial to implement.

The value of $F = 45.45009839$ deg corresponds to $M = 45.40920885$ deg and the value of $L = 45.49094391$ deg corresponds to $M = 45.40920884$ deg indicating perfect agreement for the optimized insertion location on the target orbit.

Conclusion

A higher-fidelity modeling of optimal low-thrust transfers between any given circular or elliptic orbits using a more complete Earth gravity model in the form of higher-order harmonics has thus been derived based on the Gaussian form of the dynamic system equations in terms of the nonsingular equinoctial elements.

Two previously developed formulations based on the use of the eccentric and true longitude as the sixth or fast orbit element respectively, and fully accounting for the J_2 perturbation both in the dynamic and adjoint differential equations, have been extended in this present paper by the inclusion of the higher zonal harmonics terms J_3 and J_4 in order to generate even higher accuracy optimal low-thrust trajectories than with J_2 alone. A fast LEO to GEO minimum-time transfer using a

TABLE 2. Initial, Target and Achieved Orbit Parameters Using J_2, J_3, J_4 Perturbations-Compliant Solution

Orbit	a, km	e	i, deg	Ω, deg	ω, deg	M, F, L deg
Initial	7000	0	28.5	0	0	$M_0 = -131.7436944$
Target	42,000	10^{-3}	1	0	0	Free
Achieved [L] J_2, J_3, J_4	41,999.99995	$1.000001445 \times 10^{-3}$	1.000000093	359.9999474	$1.107722739 \times 10^{-4}$	$L = 45.49094391$
Achieved [F] J_2, J_3, J_4	41,999.99995	$1.000001445 \times 10^{-3}$	1.000000093	359.9999474	$1.107673521 \times 10^{-4}$	$F = 45.45009831$

relatively high constant thrust acceleration is generated for illustration purposes and the achieved trajectories from both formulations are compared leading to a perfect match, thus validating both formulations at once. The consideration of two different formulations is necessary in order to make sure that the nonsingular state and adjoint differential equations are both free of algebraic and coding errors regardless of which formulation is adopted for operational use. The true longitude formulation which also resolves the various accelerations in the Euler-Hill frame is more concise and easier to derive than the eccentric longitude formulation which resolves the accelerations along the direct equinoctial frame. However, the latter formulation was the one that was historically developed first. All the derivations were done by hand without the use of any symbolic manipulation software. The numerical results and their discussion show the benefits of adopting this more refined theory for both simulations and flight guidance applications. The consideration of additional harmonics such as J_5 and J_6 within the present context of the Gaussian formulation can lead to even higher accuracy modeling of optimal low-thrust transfers.

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Appendix: Transformation of the J_3, J_4 Inertial Accelerations to the Rotating Frame

The perturbation accelerations due to J_3 and J_4 in the Euler-Hill frame can also be obtained directly from the inertial system $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$. Going back to equation (3) for the disturbing potential F', we have with $r = (x^2 + y^2 + z^2)^{1/2}$, $\partial r/\partial x = x/r$, $\partial r/\partial y = y/r$, $\partial r/\partial z = z/r$, such that for the J_3 terms, and in view of $s_\theta = z/r$

$$\ddot{x} = \partial F'/\partial x = -\frac{\mu x}{r^3} J_3 \frac{5}{2}\left(\frac{R}{r}\right)^3\left(3\frac{z}{r} - 7\frac{z^3}{r^3}\right) \tag{A-1}$$

$$\ddot{y} = \partial F'/\partial y = -\frac{\mu y}{r^3} J_3 \frac{5}{2}\left(\frac{R}{r}\right)^3\left(3\frac{z}{r} - 7\frac{z^3}{r^3}\right) \tag{A-2}$$

$$\ddot{z} = \partial F'/\partial z = -\frac{\mu z}{r^3} J_3 \frac{3}{2}\left(\frac{R}{r}\right)^3\left(10\frac{z}{r} - \frac{35}{3}\frac{z^3}{r^3} - \frac{r}{z}\right) \tag{A-3}$$

Replacing x, y and z above by their respective expressions in terms of the Eulerian angles with $x = r(c_\Omega c_\theta - s_\Omega c_i s_\theta)$, $y = r(s_\Omega c_\theta + c_\Omega c_i s_\theta)$ and $z = r s_i s_\theta$ and in view of the rotation matrix in equation (6), and after some algebra,

$$\begin{aligned}(f_r)_{J_3} &= (c_\Omega c_\theta - s_\Omega c_i s_\theta)\ddot{x} + (s_\Omega c_\theta + c_\Omega c_i s_\theta)\ddot{y} + s_i s_\theta \ddot{z} \\ &= -\frac{\mu J_3 R^3}{r^5}(6 s_i s_\theta - 10 s_i^3 s_\theta^3)\end{aligned} \tag{A-4}$$

In a similar way

$$\begin{aligned}(f_\theta)_{J_3} &= (-c_\Omega s_\theta - s_\Omega c_i c_\theta)\ddot{x} + (-s_\Omega s_\theta + c_\Omega c_i c_\theta)\ddot{y} + s_i c_\theta \ddot{z} \\ &= -\frac{\mu J_3 R^3}{2r^5}\left[\frac{15}{4} s_i^3 (c_\theta - c_{3\theta}) - 3 s_i c_\theta\right]\end{aligned} \tag{A-5}$$

$$\begin{aligned}(f_h)_{J_3} &= s_\Omega s_i \ddot{x} - c_\Omega s_i \ddot{y} + c_i \ddot{z} \\ &= -\frac{\mu J_3 R^3}{2r^5}(15 s_i^2 c_i s_\theta^2 - 3 c_i)\end{aligned} \tag{A-6}$$

For the J_4 accelerations

$$\ddot{x} = \partial F'/\partial x = -\frac{\mu x}{r^3} J_4 \left(\frac{R}{r}\right)^4 \frac{5}{8}\left(-63\frac{z^4}{r^4} + 42\frac{z^2}{r^2} - 3\right) \tag{A-7}$$

$$\ddot{y} = \partial F'/\partial y = -\frac{\mu y}{r^3} J_4 \left(\frac{R}{r}\right)^4 \frac{5}{8}\left(-63\frac{z^4}{r^4} + 42\frac{z^2}{r^2} - 3\right) \tag{A-8}$$

$$\ddot{z} = \partial F'/\partial z = -\frac{\mu z}{r^3} J_4 \left(\frac{R}{r}\right)^4 \frac{5}{8}\left(-63\frac{z^4}{r^4} + 70\frac{z^2}{r^2} - 15\right) \tag{A-9}$$

which leads to the expressions

$$(f_r)_{J_4} = (c_\Omega c_\theta - s_\Omega c_i s_\theta)\ddot{x} + (s_\Omega c_\theta + c_\Omega c_i s_\theta)\ddot{y} + s_i s_\theta \ddot{z}$$
$$= -\frac{5\mu J_4 R^4}{8r^6}(-35 s_i^4 s_\theta^4 + 30 s_i^2 s_\theta^2 - 3) \quad \text{(A-10)}$$

where s_θ^4 can be written as $\frac{1}{8}(3 - 4c_{2\theta} + c_{4\theta})$.

$$(f_\theta)_{J_4} = (-c_\Omega s_\theta - s_\Omega c_i c_\theta)\ddot{x} + (-s_\Omega s_\theta + c_\Omega c_i c_\theta)\ddot{y} + s_i c_\theta \ddot{z}$$
$$= -\frac{5\mu J_4 R^4}{2r^6}(7 s_i^4 s_\theta^3 c_\theta - 3 s_i^2 s_\theta c_\theta) \quad \text{(A-11)}$$

with $s_\theta^3 = \frac{1}{4}(3s_\theta - s_{3\theta})$

$$(f_h)_{J_4} = s_\Omega s_i \ddot{x} - c_\Omega s_i \ddot{y} + c_i \ddot{z}$$
$$= -\frac{5\mu J_4 R^4}{2r^6}(7 s_i^3 c_i s_\theta^3 - 3 s_i c_i s_\theta) \quad \text{(A-12)}$$

These expressions are equivalent to those in equations (11)–(16). The inertial accelerations, namely $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, for both the J_3 and J_4 terms can be found in reference [15]. However, the J_2 terms in equations (9.7-4) and (9.7-6) of that reference have a typographical error as they should read $J_2 \frac{3}{2}(\frac{r_e}{r})^2(5\frac{z^2}{r^2} - 1)$ and $J_2 \frac{3}{2}(\frac{r_e}{r})^2(3 - 5\frac{z^2}{r^2})$ respectively. The symbol r_e is used as R in this paper.