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Hamiltonian Mechanics and Relative Equilibria of Orbiting Gyrostats[1]

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Abstract

We use a noncanonical Hamiltonian approach to study the equilibrium attitudes of a rigid satellite with N rotors in a central gravitational field. The relative equilibria of this system of equations represent steady motions of the body as seen in the body frame, and correspond to stationary points of the Hamiltonian constrained by Casimir functions. This approach leads to an algorithm for computing the equilibria, and simultaneously providing direct stability information, since the calculations required to solve the constrained minimization problem are also involved in computing the positive definiteness of the constrained Hamiltonian for use as a Lyapunov function.

Introduction

In this paper we study a subset of the equilibrium attitudes of a rigid satellite with N rotors in a central gravitational field. The work presented herein is an extension of similar results for a rigid body, presented in references [1] and [2], with the added complexity of the flywheels or rotors representing reaction or momentum wheels. The model of a rigid body with axisymmetric wheels is termed a gyrostat. As a result of numerous studies from Volterra [3] to Krishnaprasad and Berenstein [4], the global torque-free motion of a gyrostat is understood in cases with freely spinning rotors or with rotors constrained to spin at a constant speed relative to the platform. The basic results are presented in Hughes [5].

There are also many reports relevant to orbiting gyrostats, where the gravity gradient torque is included. Important work has been done by Kane and Mingori [6], White and Likins [7], Roberson, Longman and others [8–15], Anchev [16], and Hughes [17]. These papers characterize the relative equilibrium motions of gyrostats in circular orbits; as a result, the steady motions of orbiting gyrostats, the subject of this paper, are fairly well understood. Many of the results are given in

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more detail in the lecture notes by Roberson et al. [12] and in Hughes [5], and are generally presented as a set of special cases. Note that the gravitational moment used in all these studies is obtained by truncating the gravitational potential in a way that has recently been shown to be inconsistent (cf. Wang et al. [18]). The significance of the inconsistency has been shown to be negligible for "ordinary" asymmetric, rigid, gravity-gradient spacecraft by Beck and Hall [1]. Wang et al. [19] investigated the gyrostat case and obtained some stability criteria for a variety of cases. However, their analysis was based on the constant-rotor speed case, and is not directly applicable to the problem of performing rotational maneuvers.

Most of the results that have been reported are for the motion of spacecraft with free or constant-speed rotors. The problem of performing rotational maneuvers using flywheels has been investigated by numerous authors, and a brief literature review can be found in reference [20]. During rotational maneuvers, the rotors satisfy neither the free nor the constant-speed rotor condition. Although many researchers have studied problems of maneuvering gyrostats, to our knowledge, only Anchev [16] has used information about equilibrium motions to develop reorientation control laws. Specifically, he showed how to reorient a three-rotor gyrostat from a rigid body gravity gradient equilibrium to one of the known gyrostat equilibria.

Most papers that do not include rotational maneuvers parameterize a rotor's motion by its constant angular velocity relative to the body frame. In contrast, the rotor's absolute angular velocity about its spin axis is more important when one is developing the control torques to perform rotational maneuvers.

Here we first develop the general equations of motion for an N-rotor gyrostat in a central gravitational field and show how to put these equations into a noncanonical Hamiltonian form. We then specialize the equations of motion to the problem of a Keplerian circular orbit, and develop a new noncanonical Hamiltonian formulation similar to that developed in reference [2] for a rigid body. This formulation is equivalent to the equations that have been used by others to study equilibria of orbiting gyrostats, but has the advantage that standard methods can be used to obtain equilibria and to characterize their stability. We then develop the stability criteria for the simplest case, namely the cylindrical equilibria.

System and Equations of Motion

The system model considered here is a gyrostat, or rigid body with a set of N axisymmetric flywheels, whose spin axes are constant in the body frame, $\mathcal{F}_b$ (see Fig. 1). The system moment of inertia, including the wheels, is denoted I, expressed in a body-fixed reference frame. The wheels have axial moments of inertia, I_{sj}, $j = 1, \ldots, N$, which are collected into a diagonal matrix $I_s = \mathrm{diag}[I_{s1} \ldots I_{sN}]$, and their spin axes are described by the vectors $\mathbf{a}_j$, $j = 1, \ldots, N$, expressed in a body-fixed frame. These axis vectors are collected into a $3 \times N$ matrix as $A = [\mathbf{a}_1 \ldots \mathbf{a}_N]$. We also define an inertia-like matrix $J = I - AI_sA^{\mathsf{T}}$, which is easily seen to be symmetric and positive definite.

The 3×1 system angular momentum vector may be written in $\mathcal{F}_b$ as

$$\mathbf{h} = I\boldsymbol{\omega} + AI_s\boldsymbol{\omega}_s \tag{1}$$

where $\boldsymbol{\omega}$ is the angular velocity of $\mathcal{F}_b$, and $\boldsymbol{\omega}_s$ is an $N \times 1$ matrix containing the axial angular velocities of the rotors relative to $\mathcal{F}_b$. The $N \times 1$ matrix $\mathbf{h}_a$ contains the absolute axial angular momenta of the wheels and may be written as

$$\mathbf{h}_a = I_sA^{\mathrm{T}}\boldsymbol{\omega} + I_s\boldsymbol{\omega}_s \tag{2}$$

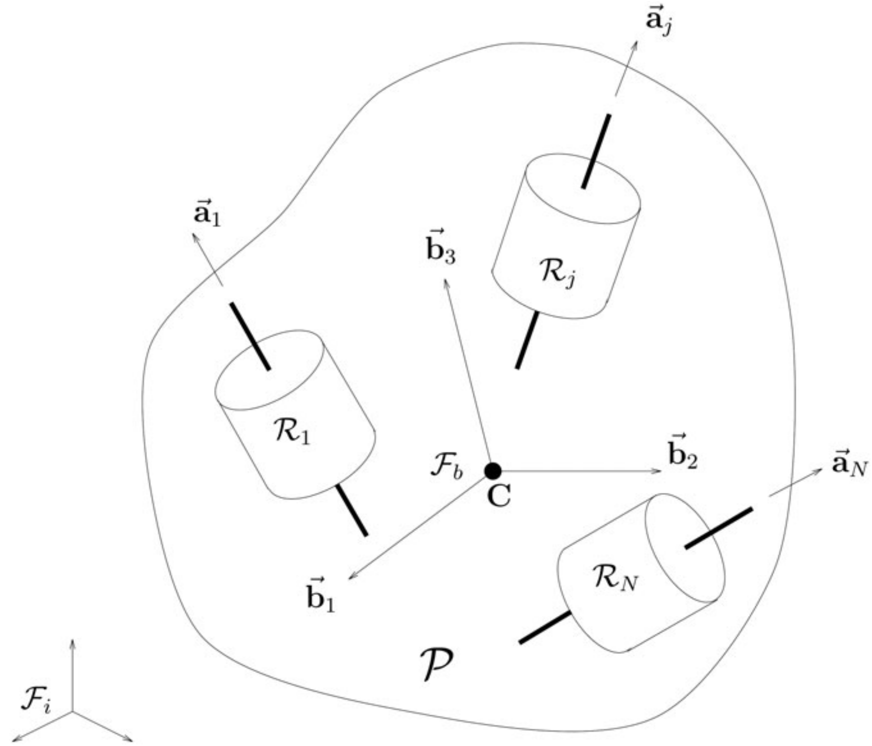

FIG. 1. Gyrostat with *N* Momentum Wheels.

Using equations (1) and (2), and the definition of J, we eliminate $\boldsymbol{\omega}_s$ and write $\boldsymbol{\omega}$ as

$$\boldsymbol{\omega} = J^{-1}(\mathbf{h} - A\mathbf{h}_a) \tag{3}$$

The gyrostat rotational equations of motion may then be written as

$$\dot{\mathbf{h}} = \mathbf{h}^{\times}J^{-1}(\mathbf{h} - A\mathbf{h}_a) + \mathbf{g}_e \tag{4}$$

$$\dot{\mathbf{h}}_a = \mathbf{g}_a \tag{5}$$

where "$\times$" denotes the skew-symmetric matrix form of the appropriate vector [5], $\mathbf{g}_e$ is the 3×1 vector of external torques, and $\mathbf{g}_a$ is the $N \times 1$ matrix containing the axial torques applied by the platform on the rotors. We set $\mathbf{g}_a = \mathbf{0}$ to study equilibrium motions, and to study rotational maneuvers, we choose a suitable control law for the rotor torques. In this paper, we deal exclusively with the $\mathbf{g}_a = \mathbf{0}$ case.

The dynamics described by equations (4–5) must be complemented by appropriate kinematic differential equations if the external or internal torques depend on the orientation of the body frame. For example, the gravity gradient torque is [5]

$$\mathbf{g}_e = \mathbf{r}^{\times}\nabla_{\mathbf{r}}V(\mathbf{r}) \tag{6}$$

where $\mathbf{r}$ is the position vector from the center of the central body to the center of mass of the gyrostat, the potential $V(\mathbf{r})$ is

$$V(\mathbf{r}) = -\int_{\mathcal{B}} \frac{\mu}{\|\mathbf{r} + \rho\|}\,\mathrm{d}m \tag{7}$$

and the force acting on the body is $-\nabla_{\mathbf{r}}V(\mathbf{r})$. Here μ is the gravitational constant for the central body, and ρ is the position vector from the mass center of $\mathcal{B}$ to a mass element dm. Clearly this integral depends on the orientation of the body. A standard approximation for equation (6), assuming $\|\rho\| \ll \|\mathbf{r}\|$ is

$$\mathbf{g}_e = 3\frac{\mu}{\|\mathbf{r}\|^3}\mathbf{o}_3^{\times}I\mathbf{o}_3 \tag{8}$$

where $\mathbf{o}_3$ is the nadir vector; *i.e.* $\mathbf{o}_3 = -\mathbf{r}/\|\mathbf{r}\|$ (see Fig. 2). The notation $\mathbf{o}_3$ denotes the third column of the rotation matrix R^{bo} that takes vectors from the orbital frame, $\mathcal{F}_o$, to the body frame $\mathcal{F}_b$. The convention used here is that the orbital frame's unit vectors are arranged so that $\mathbf{o}_3$ points at the Earth, $\mathbf{o}_2$ is in the negative orbit normal direction, and $\mathbf{o}_1 = \mathbf{o}_2 \times \mathbf{o}_3$. For circular orbits, $\mathbf{o}_1$ is in the velocity direction. Furthermore, for circular orbits, the mean motion $\mu/\|\mathbf{r}\|^3$ is constant, and is usually denoted by ω_c. In the circular case, we can append $\dot{\mathbf{o}}_3 = \mathbf{o}_3^{\times}J^{-1}(\mathbf{h} - A\mathbf{h}_a)$ so that the current state of $\mathbf{o}_3$ is available for computing the gravity gradient torque; otherwise, the translational equations of motion are required to describe the variable radius vector $\mathbf{r}$.

Letting $\mathbf{p} = m(\dot{\mathbf{r}} + \boldsymbol{\omega}^{\times}\mathbf{r})$ denote the linear momentum expressed in $\mathcal{F}_b$, the translational equations of motion are written as

$$\dot{\mathbf{r}} = \mathbf{r}^{\times}J^{-1}(\mathbf{h} - A\mathbf{h}_a) + \mathbf{p}/m \tag{9}$$

$$\dot{\mathbf{p}} = \mathbf{p}^{\times}J^{-1}(\mathbf{h} - A\mathbf{h}_a) - \nabla_{\mathbf{r}}V(\mathbf{r}) \tag{10}$$

where $-\nabla_{\mathbf{r}}V(\mathbf{r})$ is the gravitational force acting on the gyrostat.

Equations (4), (9), and (10) are collected into a matrix system of equations as

$$\begin{bmatrix}\dot{\mathbf{h}}\\ \dot{\mathbf{r}}\\ \dot{\mathbf{p}}\end{bmatrix} = \begin{bmatrix}\mathbf{h}^{\times} & \mathbf{r}^{\times} & \mathbf{p}^{\times}\\ \mathbf{r}^{\times} & 0 & 1\\ \mathbf{p}^{\times} & -1 & 0\end{bmatrix}\begin{bmatrix}J^{-1}(\mathbf{h} - A\mathbf{h}_a)\\ \nabla_{\mathbf{r}}V(\mathbf{r})\\ \mathbf{p}/m\end{bmatrix} \tag{11}$$

These equations can be recognized as a noncanonical Hamiltonian system

$$\dot{\mathbf{z}} = \mathcal{J}(\mathbf{z})\nabla H(\mathbf{z}) \tag{12}$$

where $\mathbf{z} = [\mathbf{h}, \mathbf{r}, \mathbf{p}]^{\mathsf{T}}$ is the nine-dimensional vector of states, $\mathcal{J}(\mathbf{z})$ is the skew-symmetric Poisson tensor or structure matrix, $H(\mathbf{z})$ is the Hamiltonian, and

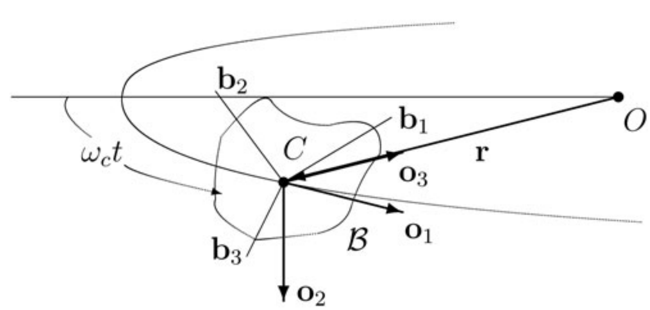

FIG. 2. Configuration of Keplerian Orbit and Orbital Frame.

∇ represents the gradient of H with respect to $\mathbf{z}$. The structure matrix is evident in equation (11), and the Hamiltonian can be written as

$$H = \frac{1}{2}\mathbf{h}^{\mathsf{T}}J^{-1}\mathbf{h} - \mathbf{h}^{\mathsf{T}}J^{-1}A\mathbf{h}_a + \frac{1}{2m}\mathbf{p}^{\mathsf{T}}\mathbf{p} + V(\mathbf{r}) \tag{13}$$

Note that this Hamiltonian is not the same quantity as the total energy, as is usually the case in natural canonical systems. However, it can be made so by addition of a term involving only the rotor momenta, which therefore contributes nothing to ∇H. For further details regarding noncanonical Hamiltonian systems, we refer you to Maddocks [21], Olver [22], Marsden [23], and Beck and Hall [2].

In systems of the form of equation (12), there are special first integrals, known as Casimir functions [22], whose gradients span the nullspace of the structure matrix, $\mathcal{J}(\mathbf{z})$. For equation (11), the nullspace of $\mathcal{J}(\mathbf{z})$ is one dimensional and is spanned by the vector

$$\nabla C = \begin{bmatrix} \mathbf{h} + \mathbf{r}^{\times}\mathbf{p} \\ \mathbf{p}^{\times}\mathbf{h} - \mathbf{p}^{\times}\mathbf{p}^{\times}\mathbf{r} \\ -\mathbf{r}^{\times}\mathbf{h} - \mathbf{r}^{\times}\mathbf{r}^{\times}\mathbf{p} \end{bmatrix} \tag{14}$$

and the Casimir function is

$$C = \frac{1}{2}(\mathbf{h} + \mathbf{r}^{\times}\mathbf{p})^{\mathsf{T}}(\mathbf{h} + \mathbf{r}^{\times}\mathbf{p}) \tag{15}$$

which is the total angular momentum of the gyrostat about the center of the attracting body. This conserved quantity is the only Casimir-type first integral for this system.

A second integral of the motion is the Hamiltonian (for $\mathbf{g}_a = \mathbf{0}$), which because of the skew symmetry of the structure matrix, is easily seen to be constant

$$\dot{H} = \nabla H^{\mathsf{T}}\dot{\mathbf{z}} = \nabla H^{\mathsf{T}}\mathcal{J}(\mathbf{z})\nabla H = 0$$

Thus the system of equations is a ninth-order system with two first integrals. If the internal wheel torques, $\mathbf{g}_a$, are not all zero, then equation (5) must be used with equation (11). In this case, the Casimir functions are still first integrals, since they are independent of the particular form of the Hamiltonian; however, the Hamiltonian is not conserved, but satisfies

$$\dot{H} = [\partial H/\partial \mathbf{h}_a]\dot{\mathbf{h}}_a = -\mathbf{h}^{\mathsf{T}}J^{-1}A\mathbf{g}_a \tag{16}$$

Elsewhere, we have used this relationship with the method of averaging to reduce the spinup problem of torque-free gyrostats from five dimensions to two [20]. We have not yet investigated a similar application to the present problem. For the work presented here, the $N \times 1$ vector of rotor momenta is considered as a vector of parameters.

Equation (11) is a general form of the equations of motion for a gyrostat in a central gravitational field. The potential $V(\mathbf{r})$ that appears in the Hamiltonian is difficult to evaluate in its general form (equation (7)). Three types of approximations are usually necessary to obtain useful results; in reference [2], a $3 \times 3 \times \infty$ "matrix" of approximations is described, where the three dimensions are approximations of the potential, restriction of the mass center motion, and material symmetry of the body. The potential can be expanded as a Taylor series for small $\|\rho\|/\|\mathbf{r}\|$, leading to an infinity of approximations of the potential, $V_n(\mathbf{r})$, $n = 0, \ldots, \infty$. We can also restrict the motion of the mass center, either to a Keplerian orbit or to a fixed point. Including the "free" problem, there are thus three possibilities. Finally, we can consider an arbitrary shape of body, or a body with axial or spherical symmetry, leading to three possibilities. In this paper, we consider the case of a second-order potential approximation for an arbitrary body moving in a circular Keplerian orbit.

Second-Order Keplerian Approximation

We now assume the gyrostat is in a circular Keplerian orbit, and approximate the potential with a second-order expansion, obtaining results that are equivalent to the classic results that have been previously reported. However, our approach is distinct from the classic approach in that we develop a new noncanonical formulation similar to that developed in reference [2] for the rigid body problem. We begin by defining the relative angular velocity and angular momentum of the gyrostat with respect to the rotating orbital reference frame, $\mathcal{F}_0$

$$\boldsymbol{\omega}_r = \boldsymbol{\omega} + \omega_c \mathbf{o}_2 \tag{17}$$

$$\mathbf{h}_r = \mathbf{h} + \omega_c J \mathbf{o}_2 \tag{18}$$

One may easily show that the relative angular velocity, relative angular momentum, and wheel momenta are related by

$$\mathbf{h}_r = J\boldsymbol{\omega}_r + A\mathbf{h}_a \tag{19}$$

$$\boldsymbol{\omega}_r = J^{-1}(\mathbf{h}_r - A\mathbf{h}_a) \tag{20}$$

We also use a second-order approximation of the potential as

$$V_2(\mathbf{o}_3) = \frac{3}{2}\omega_c^2 \mathbf{o}_3^{\mathsf{T}} I \mathbf{o}_3 \tag{21}$$

Using these definitions, we establish the noncanonical system of equations

$$\begin{bmatrix} \dot{\bar{\mathbf{h}}}_r \\ \dot{\mathbf{o}}_2 \\ \dot{\mathbf{o}}_3 \end{bmatrix} = \begin{bmatrix} [\bar{\mathbf{h}}_r + \bar{\omega}_c\{(\operatorname{tr} \bar{J} 1 - 2\bar{J})\mathbf{o}_2\}]^\times & \mathbf{o}_2^\times & \mathbf{o}_3^\times \\ \mathbf{o}_2^\times & 0 & 0 \\ \mathbf{o}_3^\times & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{J}^{-1}(\bar{\mathbf{h}}_r - A\bar{\mathbf{h}}_a) \\ -\bar{\omega}_c^2 \bar{J}\mathbf{o}_2 + \bar{\omega}_c A\bar{\mathbf{h}}_a \\ 3\bar{\omega}_c^2 \bar{I}\mathbf{o}_3 \end{bmatrix} \tag{22}$$

The overbar has been added to denote dimensional variables. To nondimensionalize the system, we choose the mass, length, and time scales

$$m = \bar{m} = \int_{\mathcal{B}} \mathrm{d}\bar{m} \qquad l = \left(\frac{\operatorname{tr} \bar{J}}{\bar{m}}\right)^2 \qquad t = \bar{\omega}_c^{-1} \tag{23}$$

The dimensionless equations of motion then become

$$\begin{bmatrix} \dot{\mathbf{h}}_r \\ \dot{\mathbf{o}}_2 \\ \dot{\mathbf{o}}_3 \end{bmatrix} = \begin{bmatrix} [\mathbf{h}_r + \{(1 - 2J)\mathbf{o}_2\}]^\times & \mathbf{o}_2^\times & \mathbf{o}_3^\times \\ \mathbf{o}_2^\times & 0 & 0 \\ \mathbf{o}_3^\times & 0 & 0 \end{bmatrix} \begin{bmatrix} J^{-1}(\mathbf{h}_r - A\mathbf{h}_a) \\ -J\mathbf{o}_2 + A\mathbf{h}_a \\ 3I\mathbf{o}_3 \end{bmatrix} \tag{24}$$

This system is in the form of equation (12), with $\mathbf{z} = (\mathbf{h}_r, \mathbf{o}_2, \mathbf{o}_3)^{\mathsf{T}}$, and Hamiltonian

$$H = \frac{1}{2}\mathbf{h}_r^{\mathsf{T}} J^{-1}\mathbf{h}_r - \mathbf{h}_r^{\mathsf{T}} J^{-1} A\mathbf{h}_a - \frac{1}{2}\mathbf{o}_2^{\mathsf{T}} J\mathbf{o}_2 + \mathbf{o}_2^{\mathsf{T}} A\mathbf{h}_a + V(\mathbf{o}_3) \tag{25}$$

This system admits three Casimir functions, as the nullspace of the structure matrix is spanned by the three vectors.

$$\mathcal{N}[J(\mathbf{z})] = \text{span}\left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{o}_2 \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{o}_3 \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{o}_3 \\ \mathbf{o}_2 \end{pmatrix} \right\} \tag{26}$$

From the spanning vectors, we identify three independent Casimir functions as

$$C_1(\mathbf{z}) = \frac{1}{2}\,\mathbf{o}_2^{\mathsf{T}}\mathbf{o}_2 \qquad C_2(\mathbf{z}) = \frac{1}{2}\,\mathbf{o}_3^{\mathsf{T}}\mathbf{o}_3 \qquad C_3(\mathbf{z}) = \mathbf{o}_2^{\mathsf{T}}\mathbf{o}_3 \tag{27}$$

In this problem, all three Casimir functions are trivial since $\mathbf{o}_2$ and $\mathbf{o}_3$ are columns of the rotation matrix R^{bo} so that

$$\mathbf{o}_2^{\mathsf{T}}\mathbf{o}_2 \equiv 1 \qquad \mathbf{o}_3^{\mathsf{T}}\mathbf{o}_3 \equiv 1 \qquad \mathbf{o}_2^{\mathsf{T}}\mathbf{o}_3 \equiv 0 \tag{28}$$

The Hamiltonian is also constant (if $\mathbf{g}_a = \mathbf{0}$). We thus have a ninth-order system with four known first integrals. Roberson [11] presented equivalent statements but used the approximation of the gravity gradient torque stated here as equation (8). Since the potential does not affect the structure matrix, the results presented here are independent of the level of approximation of the potential; however, the second-order approximation is most appropriate, since it gives the same gravity gradient torque as is usually used when the translational and rotational motion are decoupled [5].

Equilibria

In canonical Hamiltonian systems, equilibria are found as the critical points of the Hamiltonian; i.e., by setting $\nabla H = \mathbf{0}$, and computing $\mathbf{q}_e$ and $\mathbf{p}_e$. In the noncanonical case, the structure matrix can be singular, so that equilibria can also satisfy $\nabla H \in \mathcal{N}[J(\mathbf{z})]$. Since the gradients of the Casimir functions lie in the nullspace of the structure matrix, equilibria may be expressed as the critical points of a “variational Lagrangian”

$$F(\mathbf{z}, \boldsymbol{\mu}) = H(\mathbf{z}) - \mu_1 C_1(\mathbf{z}) - \mu_2 C_2(\mathbf{z}) - \mu_3 C_3(\mathbf{z}) \tag{29}$$

subject to the constraints that the Casimir functions are constant. For the system represented by equation (24), setting $\nabla F = \mathbf{0}$ leads to the conditions for equilibrium as

$$J^{-1}(\mathbf{h}_r - A\mathbf{h}_a) = \mathbf{0} \tag{30}$$

$$-J\mathbf{o}_2 + A\mathbf{h}_a - \mu_1\mathbf{o}_2 - \mu_3\mathbf{o}_3 = \mathbf{0} \tag{31}$$

$$3I\mathbf{o}_3 - \mu_2\mathbf{o}_3 - \mu_3\mathbf{o}_2 = \mathbf{0} \tag{32}$$

$$1 - \mathbf{o}_2^{\mathsf{T}}\mathbf{o}_2 = 0 \tag{33}$$

$$1 - \mathbf{o}_3^{\mathsf{T}}\mathbf{o}_3 = 0 \tag{34}$$

$$-\mathbf{o}_2^{\mathsf{T}}\mathbf{o}_3 = 0 \tag{35}$$

Equations (30–35) comprise a nonlinear algebraic system of 12 equations in 12 unknowns: $(\mathbf{z}, \boldsymbol{\mu}) = (\mathbf{h}_r, \mathbf{o}_2, \mathbf{o}_3, \mu_1, \mu_2, \mu_3)$, and the gradient is with respect to this 12-dimensional vector including the states and the Lagrange multipliers.

A typical problem involves choosing values for the wheel momenta $A\mathbf{h}_a$, and computing the associated equilibria. A well-known solution algorithm is Newton's method [24], where an initial guess, $\mathbf{z}_n$ is improved by the iteration

$$\mathbf{z}_{n+1} = \mathbf{z}_n - [\nabla^2 F(\mathbf{z}_n)]^{-1} \nabla F(\mathbf{z}_n) \tag{36}$$

Here, the Hessian $\nabla^2 F(\mathbf{z}_e, \boldsymbol{\mu}_e)$ is easily shown to be

$$\nabla^2 F = \begin{bmatrix} J^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -J - \mu_1 1 & -\mu_3 1 & -\mathbf{o}_2 & \mathbf{0} & -\mathbf{o}_3 \\ \mathbf{0} & -\mu_3 1 & 3I - \mu_2 1 & \mathbf{0} & -\mathbf{o}_3 & -\mathbf{o}_2 \\ \mathbf{0} & -\mathbf{o}_2^\mathsf{T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{o}_3^\mathsf{T} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{o}_2^\mathsf{T} & -\mathbf{o}_2^\mathsf{T} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{37}$$

The full 12×12 Hessian is required for numerical computation of equilibria, but as we show below, the Hessian also plays a role in computing the stability of equilibria, and it is only the upper left 9×9 block that is required for the stability calculations.

Linearization, Linear Stability, and Nonlinear Stability

Once an equilibrium is found, we wish to determine its stability. As described in reference [2], there are two standard approaches to computing stability in non-canonical Hamiltonian systems: linear or spectral stability, and nonlinear stability. The former is based on linearizing the equations of motion about the equilibrium, and the latter is based on establishing a suitable Lyapunov function.

The linearization about an equilibrium, $\mathbf{z}_e$, of a system in the form of equation (12) may be expressed as

$$A(\mathbf{z}_e) = \mathcal{J}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e) \tag{38}$$

Thus, one can check the linear stability of an equilibrium by computing the eigenvalues of $A(\mathbf{z}_e)$. Because the eigenvalues of a Hamiltonian system occur in pairs that are symmetric about both the real and imaginary axes, this approach only provides conditions for instability of the nonlinear system. Thus the linear stability analysis defines regions of nonlinear instability, whereas nonlinear stability analysis can define regions of stability. Of course, nonlinear stability analysis depends on the identification of a suitable Lyapunov function, and an equilibrium can be stable even if a suitable Lyapunov function is not found.

Since the Hamiltonian and Casimir functions are all constants, the variational Lagrangian, $F(\mathbf{z})$, is a candidate Lyapunov function. Thus, the eigenvalues of $\nabla^2 F(\mathbf{z}_e)$ can also be used to determine nonlinear stability. This Lyapunov function is especially useful since we have already computed the Hessian in computing the equilibria, and in computing the linearization.

Referring to equation (37), only the upper left 9×9 block is needed for computing the positive definiteness of $F(\mathbf{z})$ with respect to the state vector $\mathbf{z}$. Clearly the matrix is block diagonal, and the upper left 3×3 block, J^{-1}, is positive definite. If the eigenvalues of the remaining 6×6 block are also positive, then the equilibrium is stable. Evidently the Lagrange multipliers μ_1, μ_2, and μ_3 are important in determining the stability of an equilibrium.

The Hessian $\nabla^2 F$ may be indefinite in which case F is not immediately useful as a Lyapunov function. However, as described in reference [2], the equilibrium may be viewed as a constrained extremum of the Hamiltonian subject to the constant values of the Casimir functions. Thus, we introduce the orthogonal projection matrix, $P(\mathbf{z})$ onto the range of $A(\mathbf{z})$ as follows. Define

$$K(\mathbf{z}) = [\nabla C_1(\mathbf{z}) \quad \nabla C_2(\mathbf{z}) \quad \nabla C_3(\mathbf{z})] \tag{39}$$

and let $Q(\mathbf{z})$ be the projection onto $\mathcal{N}[A^{\mathsf{T}}(\mathbf{z})]$

$$Q(\mathbf{z}) = K(\mathbf{z})(K^{\mathsf{T}}(\mathbf{z})K(\mathbf{z}))^{-1}K^{\mathsf{T}}(\mathbf{z}) \tag{40}$$

Then the desired projection operator is

$$P(\mathbf{z}) = 1 - Q(\mathbf{z}) \tag{41}$$

The projected Hessian is then given by $P(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)P(\mathbf{z}_e)$. This matrix has three zero eigenvalues associated with the nullspace of $A(\mathbf{z}_e)$, and with the three Casimir functions. If its remaining eigenvalues are all positive, then the equilibrium is a constrained minimum and the relative equilibrium is nonlinearly stable.

An Example: Cylindrical Equilibria

The set of equilibria for an orbiting gyrostat is substantially richer than the rigid body case. Most presentations consider the equilibria as a variety of cases, including cylindrical, conical, hyperbolic, and offset hyperbolic [5]. The approach developed here is applicable to all of these cases. Here, however, we examine only the existence of and stability of the "standard" gravity gradient equilibria with the additional variable of a single wheel aligned with the $\mathbf{b}_2$ axis. Thus I and J are diagonal, and the diagonal elements of J are $\{I_1, I_2 - I_s, I_3\}$. Using the principal reference frame, the standard gravity gradient equilibria have the body principal axes aligned with the orbital frame axes, whence $\boldsymbol{\omega}_r = \mathbf{0}$. We assume without loss of generality that $\mathbf{o}_2 = [0, 1, 0]^{\mathsf{T}}$ and $\mathbf{o}_3 = [0, 1, 0]^{\mathsf{T}}$.

From equations (30–35), the Lagrange multipliers are

$$\mu_1 = -J_2 + h_{r2} \qquad \mu_2 = 3I_3 \qquad \mu_3 = 0 \tag{42}$$

Thus the Hessian matrix of F with respect to the state vector simplifies to

$$\nabla^2 F = \begin{bmatrix} J^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -J - \mu_1 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 3I - \mu_2 1 \end{bmatrix} \tag{43}$$

which is diagonal, but is not positive definite, since the last diagonal element is zero. Using the values of the Lagrange multipliers from equation (42), the eigenvalues of the Hessian are

$$\begin{matrix} 1/I_1 & 1/(I_2 - I_s) & 1/I_3 \\ J_2 - I_1 - h_{r2} & -h_{r2} & J_2 - I_3 - h_{r2} \\ 3(I_1 - I_3) & 3(I_2 - I_3) & 0 \end{matrix} \tag{44}$$

The first three eigenvalues are always positive; the fourth through sixth can be made positive by making h_{r2} sufficiently large and negative, and the seventh and eighth are equivalent to two of the conditions that define the Lagrange region in the rigid body problem. The remaining Lagrange condition is contained in the fourth eigenvalue: $J_2 - I_1 - h_{r2} = (I_2 - I_1) - (I_s + h_{r2})$. If $I_s + h_{r2} = 0$, then the remaining Lagrange condition, $I_2 - I_1 > 0$, is recovered. Similarly the sixth eigenvalue leads to $(I_1 - I_3) - (I_s + h_{r2}) > 0$. Introducing the Smelt parameters

$$k_1 = \frac{I_2 - I_3}{I_1}, \qquad k_3 = \frac{I_1 - I_2}{I_3} \tag{45}$$

these conditions can be written as

$$k_1 > k_3, \qquad k_1 > 0, \tag{46}$$

$$\frac{(1 - k_1)k_3}{3 - k_3 - k_1(1 + k_3)} - h > 0, \qquad \frac{(1 - k_3)k_1}{3 - k_3 - k_1(1 + k_3)} - h > 0 \tag{47}$$

where $h = I_s + h_{r2}$. When $h = 0$, the third of these four conditions is equivalent to $k_3 > 0$, and the fourth is equivalent to $k_1 > k_3$. When $h \neq 0$, the third condition becomes $k_3 > h(3 - k_1)/(1 - k_1 + h(1 + k_1))$, and the fourth becomes $k_3 > (h(3 - k_1) - k_1)/(h(1 - k_1) - k_1)$. Notice that the well-known DeBra-Delp region is not predicted by these stability conditions. The criteria provided here are based on using F as a Lyapunov function, and are therefore sufficient conditions only. (However, since $\nabla^2 F$ is only positive semidefinite, strictly speaking, we cannot even make these weak conclusions.)

Figures 3 and 4 illustrate the stability regions in the k_1k_3 plane, with shading used to indicate that the stability conditions are unknown with the given criteria. Figure 3 gives the $h = 0$ conditions using the positive indefinite F as a Lyapunov function. These conditions give the familiar Lagrange region. Figure 4 gives the stability region using the positive indefinite F as a Lyapunov function, with $h = -0.2$. There are two problems with this result. Since $F \geq 0$, it is not actually a valid Lyapunov function, although LaSalle's principle can be invoked to strengthen the result. However, the most important difficulty from a practical point of view is that the region of stability is unnecessarily restrictive, and can be broadened in a rigorous manner.

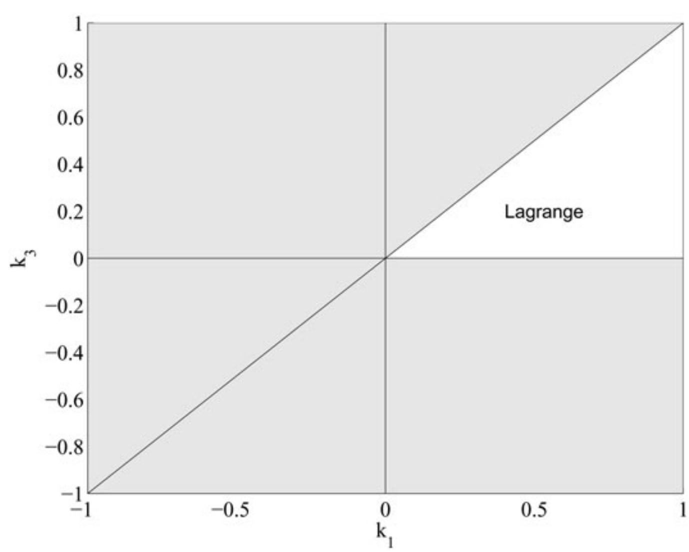

FIG. 3. Smelt Parameter Plane Using F, with $h = 0$.

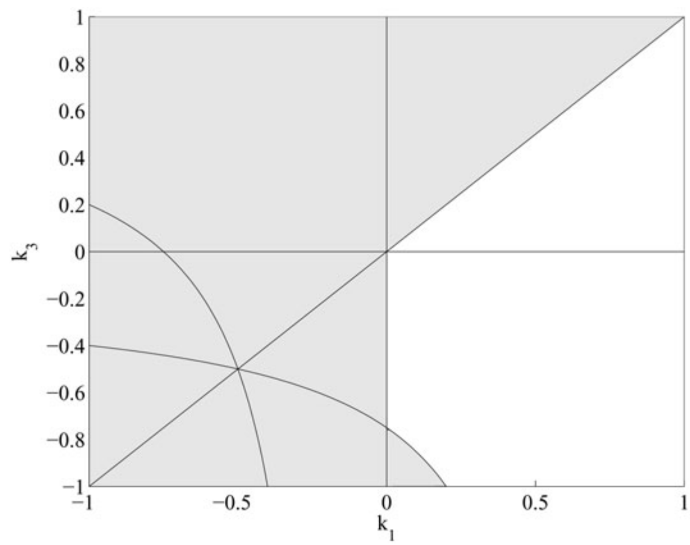

FIG. 4. Smelt Parameter Plane Using F, with $h = -0.2$.

The projected Hessian, $P\nabla^2 FP$ yields sharper stability criteria. For the single-rotor, cylindrical equilibria case, the six nonzero eigenvalues of the projected Hessian are

$$\begin{array}{cccc} 1/I_1 & 1/(I_2 - I_s) & 1/I_3 & 3(I_1 - I_3) \\ I_2 - I_1 - (I_s + h_{r2}) & & \frac{1}{2}(4(I_2 - I_3) - (I_s + h_{r2})) & \end{array} \tag{48}$$

There are three additional zero eigenvalues associated with the Casimir functions. If the remaining six eigenvalues are positive, then the equilibrium is stable (this is a sufficient condition). The first three of the nonzero eigenvalues are always positive. The fourth leads to the condition $I_1 > I_3$, which can also be expressed as

$$k_1 > k_3 \tag{49}$$

The fifth leads to the condition

$$\frac{k_3(1 - k_1)}{3 - k_3 - k_1(1 + k_3)} - h > 0 \tag{50}$$

and the sixth eigenvalue leads to

$$\frac{4k_1(1 - k_3)}{3 - k_3 - k_1(1 + k_3)} - h > 0 \tag{51}$$

As stated above, the eigenvalues of the projected Hessian give sharper stability conditions than those of the Hessian. The corresponding stability regions are shown in Fig. 5. The principal benefit is that the $k_1 > 0$ condition is replaced with a criterion that provides an additional region of nonlinear stability. Furthermore, the conditions derived from the projected Hessian permit stability even in the case of $h_{r2} > 0$, which immediately leads to a negative eigenvalue of the unprojected Hessian.

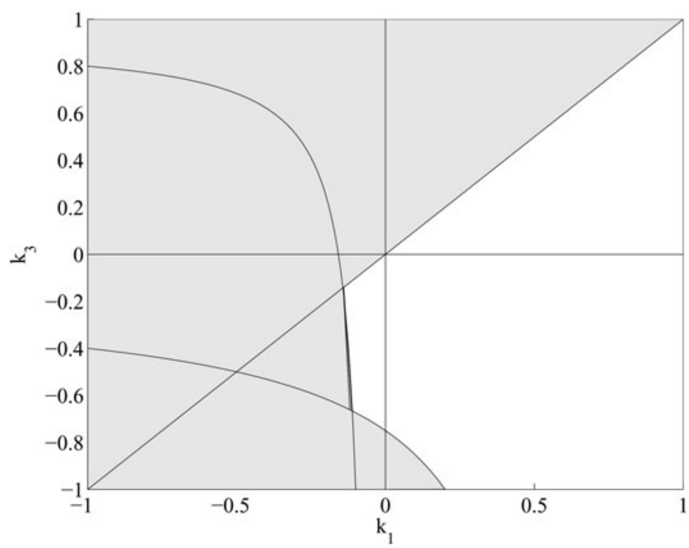

FIG. 5. Smelt Parameter Plane Using Projected Hessian, with $h = -0.2$.

Conclusions

A noncanonical Hamiltonian formulation of the equations of motion for orbiting gyrostats leads to straightforward algorithms for computing relative equilibria and determining their stability. Results that have been obtained previously using a variety of manipulations of the equations of motion and their conserved quantities are obtained in a more straightforward fashion when the equations are put into a noncanonical form. Although we have treated only the simplest case of gyrostat equilibria in this paper, this approach should provide the means to unify the various cases that are usually treated separately. Our goal is to apply this approach to unify the treatment of both rigid body and gyrostat relative equilibria.

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