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# Students' Interpretations of Mathematical Statements Involving Quantification

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Mathematical statements involving both universal and existential quantifiers occur frequently in advanced mathematics. Despite their prevalence, mathematics students often have difficulties interpreting and proving quantified statements. Through task-based interviews, this study took a qualitative look at undergraduate mathematics students' interpretations and proof-attempts for mathematical statements involving multiple quantifiers. The findings of this study suggest that statements of the form "There exists . . . for all . . ." (which can be referred to as EA statements) evoked a larger variety of interpretations than statements of the form "For all . . . there exists . . ." (AE statements). Furthermore, students' proof techniques for such statements, at times, unintentionally altered the students' interpretations of these statements. The results of this study suggest that being confronted with both the EA and AE versions of a statement may help some students determine the correct mathematical meanings of such statements. Moreover, knowledge of the structure of the mathematical language and the use of formal logic may be useful tools for students in proving such mathematical statements.

Quantification is an important component of the mathematical language. Two commonly used quantifiers in mathematics are the universal quantifier ($\forall$) and the existential quantifier ($\exists$). Common phrases used to express the universal quantifier are "for all", "for every", and "for each", such as in the example, "For all $x$ in the real numbers, $x^2 > 0$". Phrases frequently used to represent the existential quantifier are "there exists", "there is", and "there is at least one", such as in the example, "There exists a real number $x$, such that $x^2 = 5$".

Mathematical statements involving both universal and existential quantifiers occur frequently in advanced mathematics. Calculus students often face mathematical statements involving multiple quantifiers when studying limits and continuity. Students continuing to study mathematics beyond calculus will see such statements again in nearly all their further mathematics classes, for example the division algorithm in number theory, the definition of a group in abstract algebra, the definition of open and closed sets in topology, and convergence of functions in analysis, just to name a few. Although some textbook authors write such definitions and theorems in ways that avoid using multiple quantifiers explicitly, the underlying structure remains the same. Despite the prevalence of quantified statements in undergraduate mathematics, students continue to have difficulties interpreting and proving such statements (Epp, 1999). Therefore, it is critical to learn more about students' interpretations of quantification and what guides these interpretations. Dubinsky, Elterman, and Gong (1988) go as far as to suggest that because of the abundance of quantified statements in undergraduate mathematics, "finding something out about understanding quantification, how it is learned, and what we as teachers can do to help might contribute to the goal of improving all students' understanding of advanced mathematical ideas" (p. 44).

Although many scholars have written about the nuances of the mathematical language (Bagchi & Wells, 1998a, 1998b; Bullock, 1994; Epp, 1999, 2003; Hersh, 1997; Piatek-Jimenez, 2004; Pimm, 1988; Wells, 2003) and students' difficulties interpreting mathematical statements (Burton, 1988; Ferrari, 2002; Selden & Selden, 1995) very little work has focused on exploring students' understanding of quantification. Tall and Chin (2002) found that in the context of equivalence relations, students often overlook the role of the universal quantifier in the definition of the reflexive property. Dubinsky and others (Dubinsky, 1997; Dubinsky et al., 1988) explored students' understanding of complex English sentences involving multi-level quantification. These scholars found that students, in order to determine the truth-values of such statements, had difficulties negating the statements. Furthermore, students who attempted to parse the statements by simply negating the *meaning* of the statements were less successful than the students who either used symbolic rules of negation or negated the statements in a recursive format, which involves negating each piece of the statement separately and then compiling the pieces back together. Dubinsky (1997) did find that the use of the computer program ISETL in the instruction of quantification aided students in developing "some understanding of quantification and the ability to work with it" (p. 335).

Influenced by this prior work, Dubinsky and Yiparaki (2000) studied students' interpretations of bi-level quantified statements involving both universal and existential quantifiers. They coded statements in which the universal quantifier came before the existential quantifier as AE statements and statements in which the existential quantifier preceded the universal one as EA statements. Nine of the statements that they gave students were everyday language statements (such as "Every pot has a cover" and "There is a fertiliser for all plants") while two of the statements were mathematics statements (such as "For every positive number $a$ there exists a positive number $b$ such that $b < a$"). The intent of this study was to investigate how people make sense of quantified statements in everyday discourse for the purpose of utilising these modes of thinking in teaching quantification in mathematics. Their results, however, suggest that students do not have a strong understanding of quantification in everyday language, particularly for EA statements. Furthermore, what understanding students do have does not appear to transfer into the mathematical realm. These authors also noted that conventions in the mathematical language with respect to quantification do not appear to be commonly accepted as conventions in everyday language. As a result, Dubinsky and Yiparaki argue that the usage of everyday statements as examples when teaching quantification "may not be a powerful resource for helping students understand quantified statements in mathematical contexts" (p. 240).

These prior studies on quantification, by and large, focused on students' understanding of quantification in everyday language statements. Although some of these studies also included a few mathematical statements in their tasks, these statements were sparse and were not the focus of the analysis. In light of Dubinsky and Yiparaki's (2000) results, however, in order to assist students in a better understanding of quantification in mathematics, it is critical to analyse students' learning and understanding of quantification *in mathematics*.

This study set out to do just that. The purpose of this study was to explore the following questions regarding undergraduate students' understanding of mathematical statements involving multiple quantifiers: 1) What interpretations do students hold of mathematical statements involving multiple quantifiers? 2) What influences these students' interpretations? and 3) Does the pairing of two similar statements in which nothing but the order of the quantifiers are switched cause mental conflict for students and help students interpret the statements? Furthermore, by analysing students who were successful at interpreting and/or proving such statements, I intend to contribute to the knowledge base of how understanding quantification in mathematics is gained.

## Method

### *The Participants*

The participants in this study consisted of six undergraduate students enrolled in a mathematical reasoning and writing course at a large, public university in the United States. The participants had each obtained junior class-standing. Five of the participants had chosen mathematics as a major; the sixth participant was minoring in mathematics with a major in biochemistry. Three of the six participants were studying to become secondary mathematics teachers.

These participants, though volunteers, represented average students within the course. None of them were at the very top of the class, yet none of them were at the bottom either. These students did frequent office hours more often than the average student however, suggesting that they were all serious and hardworking students in this course.

## *The Course and the Instructor*

The mathematical reasoning and writing course in which the participants were enrolled at the time of the study plays the role of a "transitional course" or "bridge course" at this university. The intent of the course is to introduce students to abstract mathematics and the use of the mathematical language. The emphasis of the course is on learning to prove theorems; the mathematical content of the course is considered to be secondary. The course is taken by mathematics students who have completed most, if not all, of their lower-division mathematics classes, and is a prerequisite for many upper-division classes in the department.

The course met three times a week for 50-minute sessions. There were approximately 30 students enrolled in the course. The textbook used was Steven R. Lay's *Analysis with an Introduction to Proof, Third Edition* (2001). During the semester of the study, topics were covered from the following units: 1) Logic and Proofs, 2) Sets and Functions, 3) The Real Numbers, 4) Sequences, and 5) Limits and Continuity. The author of the textbook assumes no prior proof-writing knowledge from the students.

The instructor of the course was an experienced and well-respected professor in the department. He had taught this course on many occasions and enjoys teaching the course. Students generally rate him highly as well. The instructor taught the course using a lecture format. This means that during a typical class, the professor lectured from the front of the classroom, presenting definitions, theorems, and proofs to the students. He often encouraged students in the course to participate during class by asking the class questions about the material being covered. Homework was assigned and collected on a weekly basis by the instructor.

## *Data Collection and Analysis*

The primary source of data for this study was individual task-based interviews.[1] The interviews took place during the last month of the semester and each interview lasted between 45 minutes and an hour and a half. During each interview, the participant was presented with five mathematical statements involving multiple quantifiers and was asked to

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[1] I also collected data through field notes taken during class sessions and through informal interactions with the instructor and the students throughout the semester. References throughout this paper describing in-class activity are based on this data.

first decide whether he or she thought the statement was true or false and then was asked to either prove or disprove the statement, depending on the participant's prior decision. The participants were asked to "think out loud" while working on the items. Each item was placed on a separate sheet of paper. Throughout the interview, each student worked on an item until the student either determined that he or she had completed what was being asked or until giving up on trying to answer the item. If at any point a student wanted to go back to a previous item, I allowed them to do so. At times, I even encouraged this. Throughout the interviews, I did not attempt to assist the students with the mathematics; I simply gathered information about what they were thinking and how they were interpreting the mathematics. If I did not understand what a student was trying to express or wanted a student to expand on one of their thoughts, however, I generally asked the student to explain further. Simply participating in such interactions could have encouraged the students to think more critically than they may have otherwise.

All interviews were audio and video recorded and fully transcribed. The transcripts were read and each of the students' interpretations of the statements was coded with respect to the order of the quantifiers.[2] This coding scheme was based on the one designed by Dubinsky and Yiparaki (2000). For example, when a student interpreted a statement as a "For all . . . , there exists . . ." statement, it was given the code "AE". When the student interpreted the statement as a "There exists . . . , for all . . ." statement, it was given the code "EA". When a student interpreted a statement as a "For all . . . , for all . . ." statement, it was given the code "AA". These codes can also be extended to describe mathematical statements involving more than two quantifiers. For example, a statement of the form "For all . . . , there exists . . . , for all . . ." would be coded as "AEA" as in item 5 from the interviews. After each interpretation was coded, the students' interpretations were analysed to determine what guided them to have each interpretation. It is important to note, however, that it is only possible to code for the interpretations that the students expressed verbally or through writing.

## *Mathematical Tasks*

During the interviews described above, the students were asked to determine if each of the following statements was either true or false. If they believed the statement to be true, they were asked to prove it true. If they believed the statement was false, they were asked to prove it false. The

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[2] I am not claiming that the students realised that they were interpreting the statements each of these ways. Rather I coded for what I understood to be the students' interpretations based on what they had said and written.

statements were as follows:

**Item 1:** For every natural number $n$, there exists a natural number $K$ such that

$2n < K$.

**Item 2:** There is a natural number $M$ such that for all positive real numbers $k$,

$\frac{1}{k} < M$.

**Item 3:** For all positive real numbers $k$, there is a natural number $M$ such that

$\frac{1}{k} < M$.

**Item 4:** Let $x$ and $y$ be real numbers. There exists an $x$ such that for every $y$,

$x + y = 0$.

**Item 5:** Let $x$, $y$, and $z$ be real numbers. For every $x$ there exists a $y$ such that for every $z$, we have $x + y = z$.

Using the coding scheme described above, the mathematically correct interpretation of each of these items could be described as follows: Items 1 and 3 are AE statements, items 2 and 4 are EA statements, and item 5 is an AEA statement.

Item 1 was chosen to be a statement somewhat familiar to the students in order to build confidence at the beginning of the interview. Items 2 and 3 were chosen to determine if the students acknowledged that the order of the quantifiers was important to the meaning of the statements. It was intentional that item 2 was given to the students prior to item 3, based on Dubinsky and Yiparaki's (2000) work that suggests that EA statements are more difficult to interpret than AE statements, and are often assigned AE meanings. I wanted the students to reflect on the EA statement first, prior to having the AE version to compare it to. Item 4 was intended to gather more data on the students' interpretation of EA statements after they had the opportunity to struggle with the order of the quantifiers in items 2 and 3. Item 5, involving three quantifiers, was intended to determine if the students could extend their reasoning to statements involving more than two quantifiers. Furthermore, the statements were chosen to have a relatively simple structure in that none of them involved implications. Dubinsky (1997) suggests that statements involving quantifiers and implications add an additional level of complexity, and I wanted to focus on the students' understanding of quantifiers in this study.

# Results

## *Brief Overview of the Students' Interpretations*

Table 1 summarises the students' interpretations of the five interview items. If a student verbalised more than one interpretation of a statement, each interpretation is listed in chronological order. When a student had switched which quantifier modified which variable, the variables are added into the table entry in order to distinguish this interpretation from the others. For HF, "None" means that the student was ignoring the existence of quantifiers altogether, claiming that the statement was both true and false at the same time. All correct interpretations are denoted in bold.

Table 1
*Students' Interpretations of the Interview Items*

| Item | Correct categorisation | RC | HF | LG | CJ | DP | MK |
|---|---|---|---|---|---|---|---|
| 1 | **AE** | **AE** | **AE** | AA<br>**AE** | AA<br>**AE** | **AE** | **AE** |
| 2 | **EA** | **EA**<br>AE | AE | AA<br>**EA**<br>AE | AE<br>AA<br>AE<br>AA | AE<br>**EA**<br>AA | AE<br>**EA**<br>AA<br>**EA** |
| 3 | **AE** | **AE** | **AE** | **AE** | **AE** | **AE** | **AE** |
| 4 | **EA** | AxEy | None<br>AE | AE | AE<br>AA | **EA**<br>AA<br>**EA** | **EA**<br>AA<br>**EA** |
| 5 | **AEA** | AAE | AxAyEz | AAE | AAE | AAE | AAE |

As can be seen from the chart, the students had a larger variety of interpretations for EA statements (items 2 and 4) than AE statements (items 1 and 3). For items 1 and 3, all six participants were able to come up with the correct AE interpretation of the statements. For item 3, this was the only interpretation the students had. Fewer students, however, were able to determine the correct mathematical meaning of the EA statements. For item 2, although four of the six students at some point during the interview read the statement correctly as an EA statement, only one student concluded with this interpretation. None of the students in the study was able to correctly interpret item 5, the statement involving three quantifiers.

## *Frustration with EA Statements*

The students in this study expressed more confidence and felt more comfortable working with AE statements than EA statements. They claimed that the EA statements were confusing and argued that they did not know what the statement was asking and did not know what they could assume. Many of the students expressed great frustration with these statements and claimed that they simply did not like the way the statement was worded.

One student, LG, explained that one reason she found EA statements to be confusing was because she was used to statements being written as AE statements:

> Cause usually it's "For all this, then there exists". And so that's why it's confusing, because you know, you see it a certain way and you almost then don't see it that way and you want to force it to see it the way you've seen it all your life.

LG is correct in saying that "usually" one sees statements in the form of AE rather than EA. Besides the fact that AE statements occur more often than EA statements in everyday conversation (Dubinsky & Yiparaki, 2000), throughout the entire semester, the instructor of these students' transitional course only provided two examples of mathematical EA statements to the class. These statements were "There exists a 0 in the integers such that for every $n$ in the integers, $0 + n = n$" and "There exists an $i$ such that for every $a$, $a \cdot i = a$", both of which were mentioned when talking about ordered fields. The students had only two homework problems dealing with EA statements, both being given in the final assignment of the semester, and they were not collected or assessed by the instructor. As a result, these students had less experience working with statements of this type.

## *Determining the Significance of the Order of the Quantifiers*

The students had not spent much time thinking about EA statements prior to the interviews and had not considered the difference between AE and EA statements. Though the students were exposed to both AE and EA statements in class, the professor did not specifically talk about how the two types of statements differ from each other. The partnership of items 2 and 3, however, caused the students to face the question of whether the order of the quantified phrases matters to the meaning of the statements. Prior to being confronted with item 3, five of the six students had determined an AE interpretation of item 2. (The only student who did not was LG, who later changed to an AE interpretation.) By the end of the interviews, three of the participants had determined that the order of the quantifiers was irrelevant to the meaning of the statement whereas the other three participants decided that the order of the quantifiers do affect the statements' meanings. I have classified the six students into three different categories: 1) Students who decided that order does not matter, 2) Students who determined order does matter but could not consistently produce correct arguments for EA statements, and 3) Students who determined order does matter and produced correct arguments for EA statements. I will discuss the students in each category below.

*Students who decided that order does not matter.* After considering the statements in items 2 and 3, RC, LG, and HF all determined that the order of the quantifiers does not change the meaning of the statements. RC nearly immediately decided that the statements in items 2 and 3 mean the same thing and claimed that he did not believe that he could determine otherwise based on the syntax of the statement:

> I think they are the same statement. I think it's just semantics at this point. . . . I don't think there are any tidbits in this sentence that I could like pick at and analyse it and come to a different conclusion.

LG and HF, on the other hand, spent more time analysing the meaning of these two statements. They both felt confident that they understood the statement in item 3 and correctly interpreted it as an AE statement, but struggled with the meaning of item 2. Because they were unable to produce an interpretation other than the AE interpretation for this statement, they determined that these two statements must mean the same thing, however, they felt uncomfortable with this conclusion. Believing that the statements in items 2 and 3 must be written differently for a reason, they both suggested that though the statements mean the same thing, they must have different proofs. After attempting to prove both statements, however, these students determined that their proofs turned out to be the same, both being proofs based on the AE interpretation of the statements. Throughout the rest of the interview, LG continued to be concerned by the wording of the statements. HF was concerned as well, to an extent, but rationalised to herself that the statements must be worded differently simply because different people have different writing styles.

What all three of these students have in common is that they were unable to conceptualise an alternate meaning for these statements. Although LG and HF felt uncomfortable with their decision that the order of the quantifiers is irrelevant to the meaning of the statements, since they were unable to envision another meaning for the statement, they decided that none must exist. Consequently, all three of these students incorrectly interpreted item 4 as an AE statement.

*Students who determined order does matter but could not consistently produce correct arguments for EA statements.* Two students, DP and CJ, determined that the order of the quantifiers does change the meaning of the two statements, but neither was successful at disproving both items 2 and 4. Both of these students initially read item 2 as an AE statement, but changed their interpretations almost immediately after reading item 3. These students argued that the statement in item 3 is much clearer to understand than the statement in item 2, and confidently determined that it has the meaning that they had originally assigned to item 2. Furthermore, they were both confident that the meanings of the two statements are different and that their interpretation of item 2 must be incorrect. CJ reasoned that the statements must have different meanings by first rewriting item 3 as a conditional statement and then saying:

> I'm trying to think, so if you have "If A, then B", then that. And you can't say, "If B, then A". Those are not equivalent. . . . Because these [items 2 and 3] are complete opposites of each other. So, like one is saying, "If A, then B". The other is saying, "If B, then A".

Although CJ incorrectly references the statements in items 2 and 3 as converses of each other, she, like DP, understood the significance of structure in the mathematical language. Using this knowledge, these students were both able to conceptualise different meanings for the statement in item 2, other than their original AE interpretation. Noting that item 4 is of similar structure to item 2, they both dealt with item 4 in a manner similar to how they dealt with item 2.

Even though these students were able to correctly determine both EA statements to be false, they offered counterexamples to disprove the statements. Whereas CJ felt confident that her numerical counterexamples demonstrated that the statements in items 2 and 4 were false, DP felt unease with her counterexample for item 2.[3] To prove that item 2 is false, DP wrote, "Counterexample $\frac{1}{.0001} > 100$" in which she was assigning $M = 100$ and $k = .0001$. Despite the concern DP expressed about using a counterexample to show that this statement is false ("Cause you can't really pick a counterexample . . . because that seems a little too abstract for that."), she

[3] The counterexample DP used for item 4 was "If I pick the $x$ only $-x$ will work." Because she left it in the general case, this makes it a valid argument.

finally determined, though without much confidence, that her counterexample would be a valid argument.

*Students who determined order does matter and produced correct arguments for EA statements.* MK was the only participant in the study who was able to correctly interpret the meaning of statements 2 and 4 and was able to produce correct arguments to disprove them. Interestingly enough, after simply analysing items 2 and 3, she had determined that the two statements mean the same thing and assigned them both the AE meaning. It was not until reading the statement in item 4 before MK was able to conceptualise both the AE interpretation and the EA interpretation of a statement:

> Okay, now this one. The wording makes me think that. Like, at a glance you would say, "Oh yeah, that's true." But it says, "There exists an $x$," like one $x$, such that for *every* $y$, $x$ plus $y$ is zero. Which sounds to me like, okay, pick an $x$. I'll pick 2. Well, 2 plus, let's pick a $y$, 3, is not 0. 2 plus, let's pick a $y$, -2, is 0, but that's not for every $y$. So, if that's the way that's supposed to be read, like if it said, "For every $y$ there exists an $x$, such that $x + y = 0$," then I would say that is true. But since it says, "There exists an $x$, such that for every $y$, $x + y = 0$," then I would have to say that would be false.

By being able to conceptualise the two different possible interpretations for item 4, MK was able to utilise the language to create two different statements, one for each interpretation. She saw the need to have a way to distinguish between the two interpretations and this allowed her to understand why the structure of the language, that is, the order of the quantifiers, must be how one distinguishes between the two interpretations. This is different from how DP and CJ came upon the conclusion that the order of the quantifiers matters. DP and CJ used their knowledge that one cannot just change the order of mathematical statements without changing their meaning, and then tried to use that knowledge to determine what those different meanings would be. MK conceptualised the different meanings first, and then determined how mathematicians use the language as a tool to create each of the meanings. After coming upon this realisation, MK reconsidered the statements in items 2 and 3 and determined the correct EA and AE interpretations, respectively, for these statements.

To justify that items 2 and 4 are false, MK began by providing what she thought were numerical counterexamples for items 2 and 4, but realised the error she was making. After creating a counterexample for the statement in item 4, MK noted:

> I'm treating this like a "for all" statement and not a "there exists" statement. And so I just picked one and said, "Oh wait, that doesn't work," but what if that one wasn't the one that exists? So I can't do that. . . . So I still think it's false, but I can't prove it's false by using a counterexample because I would have to prove that for every single real number, and that would just take

too long.

MK was saying this last sentence facetiously and recognises the fact that not only would it "take too long" but would be an impossible task.

After a short pause, MK wrote on her paper the statement in item 4 translated into symbolic notation. Directly below it, she then wrote the negation of that statement in symbolic notation. MK explained:

> If I'm saying there doesn't exist an $x$ such that for all $y$, $x + y = 0$, then I'm saying that every $x$, has at least one $y$ such that $x + y$ does not equal 0. Right. Right. So if I'm saying that this statement is false [pointing at the statement in item 4], then I'm saying the opposite of it, the negation of it, is true. So that's what I want to prove.

Being comfortable knowing how to write proofs for AE statements, MK proved the negation of the statement in item 4, demonstrating a valid argument to show that the original statement in item 4 is false. She then went back to the statement in item 2 and correctly followed the same procedure.

## *AA Interpretations of Quantified Statements*

AA interpretations of the statements in this study occurred frequently, but for two separate reasons. The AA interpretations of items 1 and 2 by LG and CJ occur from interpreting that the statements must be true for all $n$ and $K$ and for all $M$ and $k$, respectively. Both of these students claimed that they would rather provide a counterexample for a statement than have to write a proof. For this reason, it appears that they initially interpreted the statements as AA statements, using a "wishful thinking" strategy, so that they could provide a counterexample. Upon rereading the statements, however, these students recognised that they had not been interpreting the statements correctly and came upon different interpretations for these statements.

The other AA interpretations from MK, CJ, and DP occur, however, as a result of intending to provide an EA interpretation of a statement, but not knowing how to prove an EA statement false. These students (at least initially) attempted to provide counterexamples for these statements, which is not a valid means of showing EA statements to be false. In attempting to use numerical counterexamples, these students were unknowingly changing their interpretation of the statements to an AA interpretation. For example, in using $M = 100$ and $k = .0001$ as a counterexample for item 2, DP shows that it is not true that "For every $M$ and for every $k$, $\frac{1}{k} < M$".

# Discussion

None of the participants in this study, all of whom were mathematics students completing a junior-level transitional course, was initially able to correctly interpret and (dis)prove the EA statements presented to them. Furthermore, even after three of the six students had determined that the order of the quantifiers is critical to the meaning of the statement, none of the participants correctly interpreted the statement in item 5 containing three alternating quantifiers. Although some of the participants did focus on the order of the phrases containing $x$ and $y$ for this item, none of them was able to incorporate the phrase with the $z$ correctly. Consequently, for relatively inexperienced students such as these, it appears that statements with three alternating quantifiers are much more difficult than statements with two quantifiers.

Though it has been pointed out that EA statements do not occur frequently in everyday language or in mathematics, statements involving three or more alternating quantifiers are integral in many basic concepts in mathematics, such as in the definitions of limits and continuous functions. These two definitions, for example, can be classified as AEA statements. Having the EA construct within them, it seems reasonable to conclude (as was supported by these data) that students unable to conceptualise EA statements would furthermore be unsuccessful at interpreting such statements. Moreover, students who are unsuccessful at interpreting such statements will surely be unable to prove them.

In addition to being able to conceptualise EA statements, it is necessary for students to be alert to the order of quantification in mathematical statements and be able to recognise the intended interpretation. A classic example to illustrate this significance is in comparing the definition of a continuous function to that of a uniformly continuous function. Both definitions are similar and include four quantifiers; the definitions vary only in the order of the quantified portions of the statement. Another example is noted by Walk (2004) in what he refers to as something that appeared to be "an instance of well-meaning but misguided editorial meddling" (p. 363). In a published linear algebra textbook he was using for a course, Walk observed that a blanket universal quantifier that was placed at the beginning of the definition of a vector space unintentionally altered the meaning of one of the axioms, causing the definition of a vector space in the textbook to be incorrect.

In addressing the question of what influences students' interpretations of quantified statements, I noted many different occurrences. Many students tried to reword or reorder a statement to make it similar to statements they had seen frequently before, which they believed made the statement easier to understand. Wanting a statement that was comfortable to them therefore influenced how they interpreted the statement. Some of these students who were astute to the fact that the structure of the mathematical language is very precise displaced their initial thought to reorder the statement, knowing that one cannot simply rearrange the order of a statement in mathematics without possibly changing its meaning. This understanding helped these students recognise that their initial interpretation may not be correct, which led them to consider alternate interpretations.

Some students unknowingly changed their interpretations of a statement when they attempted to either prove or disprove the statement. Beginning proof-writing students, such as the students in this study, who are not efficient at knowing how to prove or disprove certain types of statements can unintentionally change the statement's meaning in their construction of their justification. This occurred most often in this study when the students attempted to disprove EA statements.

Finally, being able to conceptualise both the AE and EA interpretations of a statement and seeing how these two statements have different meanings appeared to be the most successful influence on these students' interpretations. This often occurred when students considered two statements that were the same except for the order of the quantification at the beginning of the statements (items 2 and 3).

Although the study reported in this paper involved only a small sample of participants and more research should be conducted before generalising to larger populations, the results do provide the preliminary suggestion that providing students with two similar statements with only the quantified portions reversed may be useful in helping some students interpret the mathematical meanings of EA statements. When provided with a situation that evoked internal conflict, some students were able to resolve this conflict by conceptualising both the AE and EA meaning of the statements. I would like to note, however, that statements less mathematically complex than items 2 and 3 might be more useful in making this distinction. One participant in this study was unable to conceptualise the EA meaning of the statement in item 2 until she read the statement in item 4. Item 4 is a straightforward statement involving simple arithmetic, whereas to produce a counterargument for the statement in item 2, one needs to consider numbers getting infinitely large and infinitely small. Further research should examine whether such aspects of these items will affect the outcome of student interpretations. Furthermore, Dubinsky and Yiparaki (2000) suggest that students are less comfortable with statements that are false. It may be interesting for further research to explore students' interpretations of statements that are true in both their AE and EA forms to determine if these types of statements provide better teaching tools than those that are false in their EA form.

This study also suggests that while being able to distinguish between the meanings of AE and EA statements may be a necessary condition in order to understand and verify (or disprove) EA statements, this is clearly not a sufficient condition. Whereas three students in this study were able to distinguish between these two types of statements, only one student was able to correctly disprove both EA statements. The student who did so used symbolic logic to translate and negate the statement she was attempting to disprove. This supports Dubinsky et al.'s (1988) work that suggests that negation by rules or by recursion were more effective means of negating statements than negation by *meaning* alone. Epp (2003) recommends having students practise translating back and forth between formal and informal versions of quantified statements. Instruction on the rules of formal logic alone, however, has proven to be relatively ineffective in improving students' abilities to interpret conditional statements. Yet when students were provided with the rules of formal logic, examples, and an explanation tying the abstract rules to the interpretation of the examples, the percentage of errors that students made in interpreting conditional statements was significantly lower than for students who were trained with only abstract rules or with only examples (Cheng, Holyoak, Nisbett, & Oliver, 1986). As is with conditional statements, it is possible that such modes of instruction may be useful in assisting students who are first beginning to analyse the use of quantification in mathematics.

In summary, the results of this study suggest that interpreting statements with multiple quantifiers, in particular those where the existential quantifier precedes the universal quantifier, is not a trivial task, even for undergraduate mathematics students. This difficulty, however, was alleviated for some when the student was prompted to conceptualise more than one possible meaning for the mathematical statement. Furthermore, knowledge that the mathematical language follows rigid rules and the use of symbolic logic were seen to be two very useful tools in assisting students in interpreting and proving quantified statements. Additional research should be conducted to determine whether different aspects of specific quantified statements, such as the truth value of the statement, has an effect on student learning.

## References

Bagchi, A., & Wells, C. (1998a). On the communication of mathematical reasoning. *Primus*, *8*(1), 15-27.

Bagchi, A., & Wells, C. (1998b). Varieties of mathematical prose. *Primus*, *8*(2), 116-136.

Bullock, J. (1994). Literacy in the language of mathematics. *The American Mathematical Monthly*, *101*(8), 735-743.

Burton, M. (1988). A linguistic basis for student difficulties with algebra. *For the*

*Learning of Mathematics*, *8*(1), 2-7.

Cheng, P., Holyoak, K., Nisbett, R., & Oliver, L. (1986). Pragmatic versus syntactic approaches to training deductive reasoning. *Cognitive Psychology*, *18*, 293-328.

Dubinsky, E. (1997). On learning quantification. *Journal of Computers in Mathematics and Science Teaching*, *16*(2/3), 335-362.

Dubinsky, E., Elterman, F., & Gong, C. (1988). The student's construction of quantification. *For the Learning of Mathematics*, *8*(2), 44-51.

Dubinsky, E. & Yiparaki, O. (2000). On student understanding of AE and EA quantification. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *CMBS issues in mathematics education* (pp. 239-289). Providence, RI: American Mathematical Society.

Epp, S. (1999). The language of quantification in mathematics instruction. In L. V. Stiff & F. R. Curcio (Eds.), *Developing mathematical reasoning in grades K-12 (1999 Yearbook)* (pp. 188-197). Reston, VA: National Council of Teachers of Mathematics.

Epp, S. (2003). The role of logic in teaching proof. *The American Mathematical Monthly*, *110*, 886-899.

Ferrari, P. (2002). Understanding elementary number theory at the undergraduate level: A semiotic approach. In S. R. Campbell & R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (pp. 97-115). Westport, CT: Greenwood Publishing Group, Inc.

Hersh, R. (1997). Math lingo vs. plain English: Double entendre. *The American Mathematical Monthly*, *104*(1), 48-51.

Lay, S. (2001). *Analysis with an introduction to proof*. Upper Saddle River, NJ: Prentice Hall.

Piatek-Jimenez, K. (2004). Undergraduate mathematics students' understanding of mathematical statements and proofs. Doctoral dissertation, University of Arizona, Tucson.

Pimm, D. (1988). Mathematical metaphor. *For the Learning of Mathematics*, 8(1), 30-34.

Selden, J. & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*, 123-151.

Tall, D. and Chin, E.T. (2002). University students' embodiment of quantifier. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education*, (pp. 273-280). Norwich, UK.

Walk, S. M. (2004). Mind your ∀'s and ∃'s. *The College Mathematics Journal*, *35*(5), 362-369.

Wells, C. (2003). *A handbook of mathematical discourse*. Haverford, PA: Infinity Publishing.com.

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