Analysis in Theory and Applications **22:2, 2006, 172-182**

ON STRONG APPROXIMATION APPLIED TO POST-WIDDER OPERATORS

Lucyna Rempulska and Mariola Skorupka
(*Poznań University of Technology, Poland*)

Received Dec. 29, 2005

Abstract

We introduce modified Post-Widder operators in polynomial weighted spaces of differentiable functions and we study strong approximation for them.

Key words *Post-Widder operator, strong approximation, degree of approximation*

AMS(2000) subject classification 41A36, 41A25

1 Introduction

1.1. Approximation properties of Post-Widder operators

$$P_n(f;x) := \int_0^\infty p_n(x,t)\, f(t)\mathrm{d}t, \qquad x \in I, \;\; n \in N, \tag{1}$$

$$p_n(x,t) := \frac{\left(\frac{n}{x}\right)^n t^{n-1}}{(n-1)!}\, \exp\left(-\frac{nt}{x}\right), \tag{2}$$

$I = (0,\infty)$, $N = \{1,2,\cdots\}$, for real-valued functions f continuous and bounded on I were examined in [2] (Chapter 9).

It is known that P_n are well defined also for $f_r(x) = x^r$, $r \in N$, and

$$P_n(1;x) = 1, \qquad P_n(t;x) = x \qquad \text{for} \quad x \in I, \;\; n \in N. \tag{3}$$

Generally, for $n, r \in N$ and $x \in I$, we have

$$P_n\left(t^r;x\right) = \frac{n(n+1)\cdots(n+r-1)x^r}{n^r}\,, \tag{4}$$

which implies that

$$P_n\left((t-x)^2;x\right) = \frac{x^2}{n} \tag{5}$$

and for every $2 \le r \in N$ there exists a positive constant $M_1(r)$ depending only on r such that

$$P_n\left((t-x)^{2r};x\right) \ \le\ M_1(r)\, n^{-r}\, x^{2r} \tag{6}$$

for $x \in I$ and $n \in N$[2]. Moreover, from theorems given in [2], Chapter 9, we deduce that

$$|\,P_n(f;x) - f(x)| \ \le\ M_2\, \omega_2\left(f; \frac{x}{\sqrt{n}}\right), \qquad x \in I,\ \ n \in N \tag{7}$$

for every f continuous and bounded on I, where $\omega_2(f)$ is the second modulus of continuity of f and M_2 =const.> 0.

1.2. The problem of strong approximation related with Fourier series was investigated in many papers, e.g. [4].

For example, if $S_n(f;\cdot)$ and $\sigma_n(f;\cdot)$ are the n-th sum and $(C,1)$-mean of trigonometric Fourier series of f, respectively, i.e.

$$\sigma_n(f;x) \ :=\ \frac{1}{n+1}\sum_{k=0}^{n} S_k(f;x),$$

then

$$\sigma_n(f;x) - f(x) \ =\ \frac{1}{n+1}\sum_{k=0}^{n}\left(S_k(f;x) - f(x)\right), \quad x \in R,\ \ n \in N_0 = N \cup \{0\}.$$

The strong approximation of f by $(C,1)$-means of Fourier series is connected with the following strong differences

$$H_n^q(f;x) \ :=\ \left(\frac{1}{n+1}\sum_{k=0}^{n} |S_k(f;x) - f(x)|^q\right)^{1/q}, \quad x \in R,\ \ n \in N_0,$$

where $q > 0$ is a fixed number. It is easily verified that

$$|\,\sigma_n(f;x) - f(x)| \ \le\ H_n^1(f;x)$$

and

$$H_n^p(f;x) \ \le\ H_n^q(f;x), \qquad 0 < p < q < \infty$$

for $x \in R$ and $n \in N_0$. These inequalities show that the consideration of strong differences $H_n^q(f)$ for Fourier series of f is useful.

1.3. The purpose of this note is to investigate the strong approximation of functions by their Post-Widder operators. We shall define certain modified Post-Widder operators in polynomial weighted space of r times differentiable functions and we shall examine their strong differences.

Analogous to [1] let $r \in N_0$,

$$w_0(x) \;:=\; 1, \quad w_r(x) \;:=\; (1+x^r)^{-1} \quad \text{if} \quad r \ge 1, \;\; x \in I, \tag{8}$$

and let C_r be the set of all real-valued functions f defined on I for which $w_r f$ is uniformly continuous and bounded on I and the norm is given by

$$\|f\|_r \;\equiv\; \|f(\cdot)\|_r \;:=\; \sup_{x\in I} w_r(x)\, |f(x)|\,. \tag{9}$$

Moreover, denote by C^r, $r \in N_0$, the class of all f, r times differentiable on I with the derivatives $f^{(k)} \in C_{r-k}$ for $0 \le k \le r$. Obviously $C^0 \equiv C_0$, $C_r \subset C_s$ if $r < s$ and $\|f\|_s \le \|f\|_r$ for $f \in C_r$.

Definition. Let $r \in N_0$ be a fixed number. For $f \in C^r$ we define the following modified Post-Widder operators:

$$P_{n;r}(f;x) \;:=\; \int_0^\infty p_n(x,t)\, F_r(x,t)\mathrm{d}t, \qquad x \in I, \;\; n \in N, \tag{10}$$

where $p_n(x,t)$ is defined by (2) and

$$F_r(x,t) \;:=\; \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}\,(x-t)^j, \qquad x,t \in I. \tag{11}$$

If $r = 0$ and $f \in C^0$, then we have

$$P_{n;\,0}(f;x) \;=\; \int_0^\infty p_n(x,t)\; f(t)\,dt \;=\; P_n(f;x), \qquad x \in I, \;\; n \in N. \tag{12}$$

In Section 2, we shall prove that P_n is a positive linear operator acting from the space C_r into C_r and $P_{n;r}$ is a linear operator from C^r into C_r.

The formulas (1)$-$(3) and (10)$-$(12) imply that

$$P_{n;r}(f;x) \;=\; P_n\left(F_r(x,t);x\right)$$

and

$$P_{n;r}(f;x) - f(x) \;=\; P_n\left(F_r(x,t) - f(x);x\right) \tag{13}$$

for every $f \in C^r$, $r \in N_0$, $x \in I$ and $n \in N$.

1.4. Let $q > 0$ and $r \in N_0$ be fixed numbers. For $f \in C_r$ and $P_n(f)$ we introduce the following strong difference with the power q:

$$\begin{aligned} H_n^q(f;x) \;&:=\; \left(\int_0^\infty p_n(x,t)\; |f(t)-f(x)|^q\,\mathrm{d}t\right)^{1/q} \\ &\equiv\; \left(P_n\left(|f(t)-f(x)|^q;x\right)\right)^{1/q}, \qquad x \in I, \;\; n \in N. \end{aligned} \tag{14}$$

Analogously, the strong difference of $f \in C^r$, $r \in N$, and $P_{n;r}(f)$ is defined by

$$\begin{aligned} H^q_{n;r}(f;x) &:= \left(\int_0^\infty p_n(x,t)\, |F_r(x,t) - f(x)|^q \,\mathrm{d}t\right)^{1/q} \\ &\equiv \left(P_n\left(|F_r(x,t) - f(x)|^q; x\right)\right)^{1/q}, \qquad x \in I,\ \ n \in N,\ \ q > 0. \end{aligned} \tag{15}$$

From (1)−(3) and (13)−(15) it follows that

$$|P_n(f;x) - f(x)| \;\le\; H^1_n(f;x) \tag{16}$$

and

$$H^q_n(f;x) \;\le\; H^s_n(f;x), \qquad 0 < q < s < \infty \tag{17}$$

for every $f \in C_r$, $r \in N_0$, $x \in I$ and $n \in N$. Moreover, if $f \in C^r$, $r \in N$, then

$$|\, P_{n;r}(f;x) - f(x)| \;\le\; H^1_{n;r}(f;x), \tag{18}$$

$$H^q_{n;r}(f;x) \;\le\; H^s_{n;r}(f;x), \qquad 0 < q < s < \infty, \tag{19}$$

for $x \in I$, $n \in N$ and $q > 0$.

In Section 3, we shall give theorems on strong differences $H^q_n(f)$ and $H^q_{n;r}(f)$, using the modulus of continuity of $f \in C_r$ defined by

$$\omega(f; C_r; t) \;:=\; \sup_{0 \le h \le t} \|\Delta_h f(\cdot)\|_r\,, \qquad t \ge 0, \tag{20}$$

where $\Delta_h f(x) = f(x+h) - f(x)$ for $x \in I$ and $h \ge 0$.

In this paper we shall denote by $M_k(a,b)$, $k \in N$, suitable positive constants depending only on indicated parameters a, b.

2 Auxiliary Results

2.1. First we shall give some properties of operators P_n in the space C_r.

We shall use the following inequalities derived from (8):

$$(w_r(x))^q \;\le\; w_{rq}(x), \qquad (w_r(x))^{-q} \;\le\; 2^q\,(w_{rq}(x))^{-1}, \tag{21}$$

for $x > 0$ and $r, q \in N_0$.

Applying the Hölder inequality, (3) and (6), we immediately obtain

Lemma 1. *For every $r \in N$ there exists $M_3(r) =$const.> 0 such that*

$$P_n\left(|t - x|^r; x\right) \;\le\; M_3(r)\, n^{-r/2} x^r\,, \quad x \in I,\ \ n \in N.$$

Lemma 2. *Let $q > 0$ and $r \in N_0$ be fixed numbers. Then there exists $M_4(q,r) = \text{const.} > 0$ such that*

$$\left\| \left(P_n \left((w_r(t))^{-q}; \cdot \right) \right)^{1/q} \right\|_r \leq M_4(q,r) \quad \text{for} \quad n \in N. \tag{22}$$

Proof. The inequality (22) is obvious for $r = 0$ and $q > 0$.

Now let $r \in N$. If $q = 1$, then by (1)-(4) and (8) it follows that

$$\begin{aligned} w_r(x) P_n \left((w_r(t))^{-1}; x \right) &= \frac{1}{1+x^r} P_n \left(1 + t^r; x \right) \\ &\leq 1 + \frac{n(n+1)\ldots(n+r-1)}{n^r} \leq 1 + r! \quad \text{for} \quad x \in I, \ \ n \in N. \end{aligned} \tag{23}$$

If $2 \leq q \in N$, then by (1),(2),(21) and (23) we get

$$\begin{aligned} w_r(x) \left(P_n \left((w_r(t))^{-q}; x \right) \right)^{1/q} &\leq 2 \left(w_{rq}(x) P_n \left((w_{rq}(t))^{-1}; x \right) \right)^{1/q} \\ &\leq 2 \left(1 + (rq)! \right)^{1/q}, \qquad x > 0, \ \ n \in N. \end{aligned} \tag{24}$$

Let $0 < q \notin N$ and let $[q]$ be the integral part of q. Then $s := [q] + 1$ belongs to N and $q < s$. Applying the Hölder inequality, (3) and (4), we get

$$\begin{aligned} w_r(x) \left(P_n \left((w_r(t))^{-q}; x \right) \right)^{1/q} &\leq w_r(x) \left(P_n \left((w_r(t))^{-s}; x \right) \right)^{1/s} \\ &\leq 2(1 + (rs)!)^{1/s} \quad \text{for} \quad x > 0, \ \ n \in N. \end{aligned} \tag{25}$$

From (23)−(25) and (9) the inequality (22) follows.

Lemma 3. *Let q and r satisfy the assumptions of Lemma 2. Then for every $f \in C_r$ and $n \in N$, we have*

$$\left\| \left(P_n \left(|f|^q; \cdot \right) \right)^{1/q} \right\|_r \leq M_4(q,r) \left\| f \right\|_r \tag{26}$$

and

$$\left\| P_n(f; \cdot) \right\|_r \leq M_4(1,r) \left\| f \right\|_r, \tag{27}$$

where $M_4(q,r)$ is given by Lemma 2.

The formulas (1), (2) and (27) show that P_n, $n \in N$, is a positive linear operator from the space C_r into C_r, $r \in N_0$.

Proof. By (1),(2) and (9) we get

$$\| \left(P_n \left(|f(t)|^q; \cdot \right) \right)^{1/q} \|_r \leq \|f\|_r \| \left(P_n \left((w_r(t))^{-q}; \cdot \right) \right)^{1/q} \|_r$$

and

$$\left\| P_n(f) \right\|_r \leq \left\| P_n(|f|) \right\|_r,$$

for every $f \in C_r$, $q > 0$ and $n \in N$. Applying (22), we obtain (26) and then (27).

2.3. Now we shall prove the analogues of (26) and (27) for $P_{n;r}$.

Lemma 4. *Suppose that $r \in N$ and $q > 0$. Then there exists $M_5(q,r) = \text{const.} > 0$ such that for every $f \in C^r$ and $n \in N$ we have*

$$\| \left(P_n\left(|F_r(\cdot,t)|^q\,;\cdot\right)\right)^{1/q} \|_r \;\leq\; M_5(q,r)\,\left(\|f\|_r + \|f^{(r)}\|_0\right) \tag{28}$$

and

$$\|P_{n;r}(f)\|_r \;\leq\; M_5(1,r)\,\left(\|f\|_r + \|f^{(r)}\|_0\right). \tag{29}$$

The formulas (10), (11) *and* (29) *show that $P_{n;r}$, $n, r \in N$, is a linear operator from the space C^r into C_r.*

Proof. First let $q \in N$. Analogous to [3] and [5] we apply the following modified Taylor formula of $f \in C^r$ at a fixed point $t > 0$:

$$f(x) = \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(x-t)^j + \frac{(x-t)^r}{(r-1)!}\,I_r(x,t), \qquad x > 0, \tag{30}$$

where

$$I_r(x,t) := \int_0^1 (1-u)^{r-1}\left(f^{(r)}(t+u(x-t)) - f^{(r)}(t)\right)\mathrm{d}u. \tag{31}$$

By (11),(30) and (31) we have

$$F_r(x,t) \;=\; f(x) - \frac{(x-t)^r}{(r-1)!}\,I_r(x,t), \qquad x,t > 0,$$

and next by (1)-(3) and the Minkowski inequality we get

$$\begin{aligned} w_r(x)\left(P_n\left(|F_r(x,t)|^q\,;x\right)\right)^{1/q} &= \\ &= w_r(x)\left(P_n\left(\left|f(x) - \frac{(x-t)^r}{(r-1)!}I_r(x,t)\right|^q;x\right)\right)^{1/q} \\ &\leq \|f\|_r + \frac{w_r(x)}{(r-1)!}\left(P_n\left(|t-x|^{rq}\,|\,I_r(x,t)|^q\,;x\right)\right)^{1/q} \end{aligned}$$

for $x > 0$ and $n \in N$. But for $f \in C^r$, $r \in N$, we have $f^{(r)} \in C_0$ and from (31) we deduce that

$$|I_r(x,t)| \;\leq\; 2\left\|f^{(r)}\right\|_0 \int_0^1 (1-u)^{r-1}\mathrm{d}u \;=\; \frac{2}{r}\,\|f^{(r)}\|_0 \qquad \text{for} \quad x,t > 0.$$

These inequalities, (21) and Lemma 1 imply that

$$\begin{aligned} w_r(x)\left(P_n\left(|F_r(x,t)|^q\,;x\right)\right)^{1/q} &\leq \|f\|_r + \frac{2}{r!}\left\|f^{(r)}\right\|_0\left(w_{rq}(x)P_n\left(|t-x|^{rq};x\right)\right)^{1/q} \\ &\leq \|f\|_r + M_6(q,r)\|f^{(r)}\|_0\,n^{-r/2} \end{aligned}$$

for $x > 0$ and $n \in N$. From the above estimate and (9), (28) follows for $q \in N$.

If $0 < q \notin N$, then arguing as in the proof of Lemma 2, we have

$$w_r(x)\left(P_n\left(|F_r(x,t)|^q;x\right)\right)^{1/q} \leq w_r(x)\left(P_n\left(|F_r(x,t)|^s;x\right)\right)^{1/s}$$

for $x > 0$, $n \in N$ and $s = [q]+1$. Using (28) with the power s, we obtain (28) for $0 < q \notin N$.

Moreover, the formulas (10),(11) and (1) imply that

$$\begin{aligned} w_r(x)\left|P_{n;r}(f;x)\right| &\leq w_r(x)P_n\left(|F_r(x,t)|;x\right) \\ &\leq \left\|P_n\left(|F_r(\cdot,t)|;\cdot\right)\right\|_r, \end{aligned}$$

for every $f \in C^r$, $x > 0$ and $n \in N$. Now using (28) with $q = 1$, we obtain (29) and complete the proof.

3 Theorems

3.1. First we shall prove two theorems on $H_n^q(f)$ (defined by (14)) for $f \in C_r$.

Theorem 1. *Suppose that $q > 0$ and $r \in N_0$ are fixed numbers. Then there exists $M_7(q,r) = const. > 0$ such that for every $f \in C_r$ having the derivative $f' \in C_r$ we have*

$$w_r(x)H_n^q(f;x) \leq M_7(q,r)\,\|f'\|_r\,\frac{x}{\sqrt{n}}, \qquad x \in I, \quad n \in N. \tag{32}$$

Proof. Let $q \in N$. Then for f satisfying our assumptions we can write

$$\begin{aligned} |f(t)-f(x)| &= \left|\int_x^t f'(u)\mathrm{d}u\right| \leq \|f'\|_r\left|\int_x^t \frac{du}{w_r(u)}\right| \\ &\leq \|f'\|_r\left(\frac{1}{w_r(t)}+\frac{1}{w_r(x)}\right)|t-x|, \qquad x,t \in I. \end{aligned}$$

Using the above inequality to (14) and next by Minkowski and Hölder inequalities, we get

$$\begin{aligned} w_r(x)\,H_n^q(f;x) &\leq \|f'\|_r\left\{w_r(x)\left(P_n\left(\left(\frac{|t-x|}{w_r(t)}\right)^q;x\right)\right)^{1/q}\right. \\ &\quad \left. + \left(P_n\left(|t-x|^q;x\right)\right)^{1/q}\right\} \\ &\leq \|f'\|_r\left\{w_r(x)\left(P_n\left((w_r(t))^{-2q};x\right)\right)^{1/2q}\left(P_n\left((t-x)^{2q};x\right)\right)^{1/2q}\right. \\ &\quad \left. + \left(P_n\left((t-x)^{2q};x\right)\right)^{1/2q}\left(P_n(1;x)\right)^{1/2q}\right\} \end{aligned}$$

which by (3),(5),(6) and Lemma 2 gives the immediately (32) for $q \in N$.

If $0 < q \notin N$, then by (17) we have

$$H_n^q(f;x) \leq H_n^{[q]+1}(f;x), \qquad x > 0, \quad n \in N,$$

and by (32) with the power $[q]+1$ we get (32) for $0 < q \notin N$.

Thus the proof is completed.

Theorem 2. *Let $q > 0$ and $r \in N_0$ be given numbers. Then there exists $M_8(q,r) =$ const. > 0 such that for every $f \in C_r$ we have*

$$w_r(x)H_n^q(f;x) \;\leq\; M_8(q,r)\,\omega\left(f;C_r;\frac{x}{\sqrt{n}}\right), \qquad x \in I,\;\; n \in N \tag{33}$$

and

$$\|H_n^q(f)\|_{r+1} \;\leq\; 4M_8(q,r)\,\omega\left(f;C_r;\frac{1}{\sqrt{n}}\right), \qquad n \in N. \tag{34}$$

Proof. First let $q \geq 1$. Similar to [5] we apply the Stieklov function f_h of $f \in C_r$:

$$f_h(x) \;:=\; \frac{1}{h}\int_0^h f(x+t)\mathrm{d}t, \qquad x, h > 0.$$

This formula and (20) imply that

$$\|f - f_h\|_r \;\leq\; \omega(f;C_r;h), \tag{35}$$

$$\|f_h'\|_r \;\leq\; h^{-1}\,\omega(f;C_r;h), \tag{36}$$

for $h > 0$. Thus f_h and f_h' belong to C_r if $f \in C_r$. Now, using the inequality

$$|f(t) - f(x)| \leq |f(t) - f_h(t)| + |f_h(t) - f_h(x)| + |f_h(x) - f(x)|,$$

the Minkowski inequality and (3) we get from (14)

$$\begin{aligned} H_n^q(f;x) \;&\leq\; \left(P_n\left(|f(t) - f_h(x)|^q\,;x\right)\right)^{1/q} \\ &\quad + \left(P_n\left(|f_h(t) - f_h(x)|^q\,;x\right)\right)^{1/q} \;+\; |f_h(x) - f(x)| \\ &:= \sum_{k=1}^{3} T_{n,k}(x), \qquad x, h > 0,\;\; n \in N. \end{aligned}$$

Next, by (26) and (35) we have

$$\|\,T_{n,1}\|_r \;\leq\; M_4(q,r)\;\|f - f_h\|_r \;\leq\; M_4(q,r)\,\omega(f;C_r;h)$$

and

$$\|\,T_{n,3}\|_r \;\leq\; \omega(f;C_r;h) \qquad \text{for} \quad h > 0,\;\; n \in N.$$

Applying Theorem 1 and (36), we get

$$\begin{aligned} w_r(x)\,T_{n,2}(x) \;&\leq\; M_7(q,r)\;\|f_h'\|_r\;\frac{x}{\sqrt{n}} \\ &\leq\; M_7(q,r)\,\frac{x}{h\sqrt{n}}\,\omega(f;C_r;h), \end{aligned}$$

for $x, h > 0$ and $n \in N$. Summarizing, we obtain

$$w_r(x)\, H_n^q(f;x) \;\le\; M_8(q,r)\, \omega(f; C_r; h) \left(1 + \frac{x}{h\sqrt{n}}\right) \tag{37}$$

for $x, h > 0$ and $n \in N$. Setting $h = \dfrac{x}{\sqrt{n}}$ to (37), we obtain (33) for $q \ge 1$.

If $0 < q < 1$, then by (17) we have

$$H_n^q(f;x) \;\le\; H_n^1(f;x), \qquad x > 0, \;\; n \in N,$$

which by (33) with $q = 1$ gives (33) for $0 < q < 1$.

Choosing $h = \dfrac{1}{\sqrt{n}}$, we get from (37):

$$\begin{aligned} w_{r+1}(x)\, H_n^q(f;x) \;&\le\; M_8(q,r)\, \omega\left(f; C_r; \frac{1}{\sqrt{n}}\right) \frac{(1+x)w_{r+1}(x)}{w_r(x)} \\ &\le\; 4M_8(q,r)\, \omega\left(f; C_r; \frac{1}{\sqrt{n}}\right) \end{aligned}$$

for $x > 0$ and $n \in N$, which by (9) implies (34) for $q \ge 1$. The proof of (34) for $0 < q < 1$ is analogous to that of (33).

3.2. Now we shall give the analogues of (33) and (34) for $H_{n;r}^q(f)$ defined by (15).

Theorem 3. *Assume that $q > 0$ and $r \in N$. Then there exists $M_9(q,r) = const. > 0$ such that for every $f \in C^r$ we have*

$$w_r(x)\, H_{n;r}^q(f;x) \;\le\; M_9(q,r)\, n^{-r/2}\, \omega\left(f^{(r)}; C_0; \frac{x}{\sqrt{n}}\right), \;\; x > 0, \; n \in N, \tag{38}$$

and

$$\left\|H_{n;r}^q(f)\right\|_{r+1} \;\le\; 4M_9(q,r)\, n^{-r/2}\, \omega\left(f^{(r)}; C_0; \frac{1}{\sqrt{n}}\right), \qquad n \in N. \tag{39}$$

Proof. Let $q \in N$. Analogous to the proof of Lemma 4 we apply the Taylor formula (30) of $f \in C^r$. From (15),(11),(30) and (31) it follows that

$$H_{n;r}^q(f;x) \;\le\; \frac{1}{(r-1)!}\left(P_n\left(|t-x|^{rq}\, |\, I_r(x,t)|^q\, ; x\right)\right)^{1/q}, \qquad x > 0, \;\; n \in N.$$

Next, by (20) and properties of $\omega(f^{(r)}; C_0; \cdot)$, we get from (31)

$$\begin{aligned} |\, I_r(x,t)| \;&:=\; \int_0^1 (1-u)^{r-1}\, \omega\left(f^{(r)}; C_0; u\,|\,x-t|\right) \mathrm{d}u \\ &\le\; \omega\left(f^{(r)}; C_0; |\,t-x|\right) \int_0^1 (1-u)^{r-1} \mathrm{d}u \\ &\le\; \frac{1}{r}\, \omega\left(f^{(r)}; C_0; \frac{x}{\sqrt{n}}\right) \left(\frac{\sqrt{n}}{x}|\,t-x| + 1\right) \end{aligned}$$

for $x, t > 0$ and $n \in N$. Applying the above results, Minkowski inequality and Lemma 1, we can write

$$\begin{aligned} H^q_{n;r}(f;x) &\le \frac{1}{r!}\,\omega\left(f^{(r)};C_0;\frac{x}{\sqrt{n}}\right)\left\{\frac{\sqrt{n}}{x}\left(P_n\left(|\,t-x|^{q(r+1)};x\right)\right)^{1/q}\right. \\ &\qquad\qquad\qquad\qquad\left. +\ \left(P_n\left(|\,t-x|^{qr};x\right)\right)^{1/q}\right\} \\ &\le M_{10}(q,r)\,\omega\left(f^{(r)};C_0;\frac{x}{\sqrt{n}}\right)\,n^{-r/2}\,x^r, \quad x>0,\ \ n\in N. \end{aligned}$$

This inequality, (8) and (9) immediately yield (38) for $q \in N$.

Now let $0 < q \notin N$. Then by (19) we have

$$H^q_{n;r}(f;x) \ \le\ H^{[q]+1}_{n;r}(f;x), \qquad x>0,\ \ n\in N,$$

and by (38) with the power $[q]+1$, (38) follows for $0 < q \notin N$.

The inequality (39) for $f \in C^r$ is easily t obtain from (38), by applying the inequalities: $(1+x)w_{r+1}(x)(w_r(x))^{-1} \le 4$ and

$$\omega\left(f^{(r)};C_0;\frac{x}{\sqrt{n}}\right) \ \le\ (x+1)\omega\left(f^{(r)};C_0;\frac{1}{\sqrt{n}}\right)$$

for $x > 0$ and $n \in N$.

3.3. From Theorem 2, Theorem 3, (16) and (18) we derive the following corollaries:

Corollary 1. *Let $f \in C_r$ with $r \in N_0$. Then for every $q > 0$ we have*

$$\lim_{n\to\infty} \|H^q_n(f)\|_{r+1} \ =\ 0\,,$$

which implies

$$\lim_{n\to\infty} H^q_n(f;x) \ =\ 0 \quad \text{at every}\ \ x>0\,.$$

Corollary 2. *Let $f \in C^r$, $r \in N$. Then for every $q > 0$*

$$\lim_{n\to\infty} n^{r/2}\ \left\|H^q_{n;r}(f)\right\|_{r+1} \ =\ 0\,.$$

Consequently, we have

$$\lim_{n\to\infty} n^{r/2}\,H^q_{n;r}(f;x) \ =\ 0 \quad \text{at every}\ \ x>0.$$

Corollary 3. *Let f and r satisfy the assumptions of Theorem 2. Then*

$$w_r(x)\ |P_n(f;x)-f(x)| \ \le\ M_8(1,r)\,\omega\left(f;C_r;\frac{x}{\sqrt{n}}\right) \quad \text{for} \quad x>0,\ \ n\in N.$$

If f and r satisfy the assumptions of Theorem 3, then

$$w_r(x)\ |P_{n;r}(f;x)-f(x)| \ \le\ M_9(1,r)\,n^{-r/2}\,\omega\left(f^{(r)};C_0;\frac{x}{\sqrt{n}}\right)$$

for $x > 0$ *and* $n \in N$.

Remark. By (16)−(18) we see that the results concerning strong differences $H_n^q(f)$ and $H_{n;r}^q(f)$ imply direct approximation theorems for operators $P_n(f)$ and $P_{n;r}(f)$ and functions f belonging to spaces C_r and C^r respectively.

The above corollaries and (7) show that the operators $P_{n;r}(f)$ for $f \in C^r$, $r \geq 2$, have approximation properties better than the classical Post-Widder operators P_n defined by (1).

Finally we mention that similar theorems can be obtained for the Stancu beta operator

$$B_n(f;x) \ := \ \frac{1}{B(nx, n+1)} \int_0^\infty \frac{t^{nx-1}}{(1+t)^{n+1}} \, f(t) \mathrm{d}t$$

introduced in [6].

References

[1] Becker, M., Global Approximation Theorems for Szász-Mirakyan and Baskakov Operators in Polynomial Weight Spaces, Indiana Univ. Math. J., 27:(1978), 127-142.

[2] Ditzian, Z. and Totik, V., Moduli of Smoothness, Springer-Verlag, New York, 1987.

[3] Kirov, G. H., A Generalization of the Bernstein Polynomials, Math. Balkanica, 6:2(1992), 147-153.

[4] Leindler, L., Strong Approximation by Fourier Series, Akad. Kiado, Budapest, 1985.

[5] Rempulska, L. and Skorupka, M., On Strong Approximation of Functions by Certain Linear Operators, J. Okayama Univ., 46(2004), 153-161.

[6] Stancu, D. D., On the Beta Approximating Operators of Second Kind, Rev. Anal. Numer. Theor. Approx., 24:1-2(1995), 231-239.

Institute of Mathematics

Poznań University of Technology

ul. Piotrowo 3A

60-965 Poznań, Poland

E-mail: mariolas@math.put.poznan.pl; lrempuls@math.put.poznan.pl