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NOTE ON ASYMPTOTIC EXPANSION OF RIEMANN-SIEGEL INTEGRAL

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Abstract

In this note we establish two theorems concerning asymptotic expansion of Riemann-Siegel integrals as well as formula of generating function (double series) of coefficents of that expansion (for computation aims); we also discuss similar results for Dirichlet series ($L(s, f_h)$ *and* $L(s, X)$)*, with* m *odd integer and* $X(n)(mod(m))$ (*even*) *primitive characters* ($inappendixB$).

Key words *Riemann-Siegel integral, asymptotic expansion, asymptotic functional equation, Binet formula, Titchmarsh technique*

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1 Preface

C. L. Siegel found in Riemann's private papers a work relating to asymptotic expansion of Riemann's zeta function on the critical line $\sigma = 1/2$.

With general notations in [1] b, p. 79 (for $\vartheta(t)$, also cf. (1.7), Lemma 2 below), let

$$Z(t) = e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right) = 2\sum_{n\le\sqrt{\frac{t}{2\pi}}}\frac{\cos(\vartheta(t)-t\log(n))}{\sqrt{n}} + R.$$

More detail asymptotic expansion of remainder R is given in [2], p. 154. i. e.

$$R \sim (-1)^{m-1}(\frac{t}{2\pi})^{-\frac{1}{4}}\left[C_0 + C_1(\frac{t}{2\pi})^{-\frac{1}{2}} + C_2(\frac{t}{2\pi})^{-\frac{2}{2}} + C_3(\frac{t}{2\pi})^{-\frac{3}{2}} + C_4(\frac{t}{2\pi})^{-\frac{4}{2}}\right], \qquad (1.1)$$

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where m is the integral part of $x=(t/2\pi)^{1/2}, p$ the fractional part, and

$$\begin{aligned} C_0 &= \Psi(p) = \frac{\cos(2\pi(p^2-p-1/16))}{\cos(2\pi p)} = -e^{\pi i/8-2\pi i p^2}\frac{1}{2\pi i}\int_{u\nearrow 0}\frac{e^{iu^2/4\pi+2pu}du}{e^u-1}, \\ C_1 &= -\frac{\Psi^{(3)}(p)}{2^5\cdot 3\pi^2}, C_2 = \frac{\Psi^{(6)}(p)}{2^{11}\cdot 3^2\pi^4}+\frac{\Psi^{(2)}(p)}{2^6\pi^2}, \\ C_3 &= -\frac{\Psi^{(9)}(p)}{2^{16}\cdot 3^4\pi^6}-\frac{\Psi^{(5)}(p)}{2^8\cdot 15\pi^4}-\frac{\Psi^{(1)}(p)}{2^6\pi^2}, \\ C_4 &= \frac{\Psi^{(12)}(p)}{2^{23}\cdot 3^5\pi^8}+\frac{11\Psi^{(8)}(p)}{2^{17}\cdot 45\pi^6}+\frac{19\Psi^{(4)}(p)}{2^{13}\cdot 3\pi^4}+\frac{\Psi(p)}{2^7\pi^2}. \end{aligned} \tag{1.2}$$

The coefficients in (1.2) have a remarkable property that seems not be proven yet. Rewriting (1.1) as

$$R \sim (-1)^{m-1}(\frac{t}{2\pi})^{-1/4}\sum_{0\le j<\infty} c_{j,k}(\sqrt{t})^{-j}\left(\frac{1}{2\sqrt{2\pi}}\frac{d}{dp}\right)^k \Psi(p), \tag{1.3}$$

then we see that the coefficients $c_{j,k}\neq 0$ implies $4|(j+k)$.

Our investigation shows that (Theorem A below) replacement of the function $\Psi(p)$ in (1.1) by

$$\frac{\exp(2\pi i(p^2-p-1/16))+i\sqrt{2}\sin(\pi p)}{2\cos(2\pi p)} = -e^{-\pi i/8}\int_{0\swarrow w}\frac{e^{2\pi i(w-p)^2}dw}{2i\sin\pi w}$$

will also give an asymptotic expansion (with same coefficients $c_{j,k}$) of the remainder R_1 of Riemann-Siegel integral (in the form given by Levinson,[6], p. 387):

$$\exp(i\vartheta(t))\left[\sum_{n\le m} n^{-1/2-it}+\int_{m\swarrow z}\frac{\exp(\pi i z^2)z^{-1/2-it}dz}{2i\sin(\pi z)}\right]. \tag{1.1.1}$$

Our investigation also shows that, if we replace $\frac{1}{2i\sqrt{2t\pi}}\frac{d}{dp}$ by $u=\alpha\beta$, $\left(\frac{1}{2\sqrt{2\pi i}}\frac{d}{dp}\text{by}\beta\right)$ $c_{j,k}$ by $b_{j,k}e^{-\pi i(j+k)/4}$ and t_i by $(\alpha)^{-2}$, respectively, in (1.3) then we obtain a double (formal) series,

$$\begin{aligned} F(\alpha,\beta) &= \sum_{0\le j<\infty}\sum_{0\le k\le 3j} b_{j,k}\alpha^j\beta^k = \exp(\alpha^2/48-7\alpha^6/5760+\cdots) \\ &\times\exp((1/4)(\partial/\partial\beta)^2)\left[(1+\alpha\beta)^{-1/2}\exp(\alpha\beta^3/3-\alpha^2\beta^4/4+\alpha^3\beta^5/5-\cdots)\right], \end{aligned} \tag{1.4}$$

Theorem B, (4.2), (4.5), (4.6) below and remark in Appendix A), then the property mentioned above is in turn the following property of the function (1.4):

$$F(\alpha,\beta)-F(i\alpha,i\beta)\sim 0. \tag{1.5}$$

This note also gives a proof of (1.5) by showing the asymptotic functional equation

$$\begin{aligned} & e^{i\theta(t)} e^{\frac{1}{4it}\frac{\partial^2}{\partial u^2}} \{(1+u)^{-1/2} e^{it(log(1+u)-u+u^2/2)}\} \\ & e^{-i\theta(t)} e^{-\frac{1}{4it}\frac{\partial^2}{\partial u^2}} \left\{(1-u)^{-1/2} e^{-it(log(1-u)+u+u^2/2)}\right\} \sim 0 \end{aligned} \tag{1.6}$$

(the meaning of (1.5), (1.6) will make more accurate later), where (cf. [2], p. 154)

$$\vartheta(t) = \frac{t}{2}\log(\frac{t}{2\pi}) - \frac{t}{2} - \frac{\pi}{8} - \theta(t), \theta(t) \sim -\sum_{n=1}^{\infty} \frac{1}{2} \frac{(2^{2n-1}-1)B_n}{2n(2n-1)(2t)^{2n-1}} \tag{1.7}$$

(cf. Lemma 2 below), B_n being n-th Bernoulli number ([4], p. 3).

In appendix A we use (1.4) to obtain new datii of coefficients up to C_8. Note that (1.4) also guarentees directly that the coefficients $b_{j,k}$ are real and rational.

2 Asmptotic Expansion of Riemann-Siegel Integral

The Integrals in the seventh proof of functional equation of Riemann's zeta function $\zeta(s)$ in [1]. (b), (2.10.6) are called Riemann-Siegel integrals; for convenience we also use the form given by N. Levinson [6], p. 387:

$$\zeta(s) = \frac{e^{\frac{1}{2}\pi is}(2\pi)^s}{2\pi i} \int\limits_{w\nearrow 0} \frac{e^{\frac{iw^2}{(4\pi)}+\frac{1}{2}w} w^{-s} dw}{e^w - 1} + \frac{1}{\Gamma(s)(1+e^{-\pi is})} \int\limits_{z\searrow 0} \frac{e^{\frac{-iz^2}{(4\pi)}+\frac{1}{2}z} z^{s-1} dz}{e^z - 1}; \tag{2.1.1}$$

$$\frac{\Gamma(\frac{1}{2}s)\zeta(s)}{\pi^{\frac{1}{2}s}} = \frac{\Gamma(\frac{1}{2}s)}{\pi^{\frac{1}{2}s}} \int_{0\swarrow w} \frac{e^{\pi iw^2} w^{-s} dw}{2i\sin\pi w} + \pi^{\frac{1}{2}(s-1)}\Gamma(\frac{1-s}{2}) \int_{0\searrow z} \frac{e^{-\pi iz^2} z^{s-1} dz}{2i\sin\pi z}. \tag{2.1.2}$$

By Theorem of residue we have

$$\zeta(s) = \sum_{n=1}^{m} \frac{1}{n^s} + \sum_{n=1}^{m} \frac{\chi(s)}{n^{1-s}} + \int_{m\swarrow w} \frac{e^{\pi iw^2} w^{-s} dw}{2i\sin\pi w} + \int_{m\searrow z} \frac{\chi(s) e^{-\pi iz^2} z^{s-1} dz}{2i\sin\pi z}.$$

($\chi(s) = 2^s\pi^{s-1}\Gamma(1-s)\sin(\pi s/2), m = [\sqrt{(t/2\pi)}]$). In view of [1] (b), 4. 15 we see that

$$\begin{aligned} & \int_{m\swarrow w} \frac{e^{\pi iw^2} w^{-s} dw}{2i\sin\pi w} + \chi(s) \int_{m\searrow z} \frac{e^{-\pi iz^2} z^{s-1} dz}{2i\sin\pi z} \\ & = \frac{e^{-\pi is}\Gamma(1-s)}{2\pi i} \left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \right) \frac{w^{s-1} e^{-mw} dw}{e^w - 1}, \end{aligned} \tag{2.1.3}$$

where the straight lines C_1, C_2, C_3, C_4 join ∞, $c\eta + i\eta(1+c)$, $-c\eta + i\eta(1-c)$, $-c\eta - (2m+1)\pi i$, and ∞ ($0 < c < 1/2$, $\eta = \sqrt{(2\pi t)}$).

In this section we use the technique in [1], (b).4.16 to investigate the asymptotic expansion of

$$R_2 = \chi(s) \int_{m\searrow z} \frac{e^{-\pi iz^2} z^{s-1} dz}{2i\sin\pi z} = \frac{\Gamma(1-s)\sin\frac{\pi s}{2}}{\pi e^{\frac{1}{2}\pi is}} \int\limits_{w\nearrow 2\pi im} \frac{e^{\frac{i}{4\pi}w^2+\frac{1}{2}w} w^{s-1} dw}{e^w - 1}. \tag{2.1.4}$$

We need the following

Proposition 1.

$$
\begin{aligned}
&-\frac{e^{-2\pi i b^2+\pi i/8}}{2\pi i}\int\limits_{w\nearrow 0}\frac{e^{iw^2/(2\pi)+2bw+w/2}dw}{e^w-1}\\
&=-e^{\pi i/8}\int\limits_{0\searrow z}\frac{e^{-2\pi i(z-b)^2}dz}{2i\sin\pi z}=\frac{e^{-2\pi i(b^2-b-1/16)}-i\sqrt{2}\sin\pi b}{2\cos(2\pi b)}.
\end{aligned}
\tag{2.1.5}
$$

Proof. Let

$$
\phi(u,\tau)=\int\limits_{z\nearrow 0}\frac{e^{\frac{1}{4\pi}i\tau z^2+uz}}{e^z-1}\mathrm{d}z,\qquad (\mathrm{Re}\ (\tau)>0)
$$

where u,τ are complex parameters. In [7], p. 3, the case for $\tau=m/n$ has been worked out, i. e.

$$
\frac{1}{2\pi i}\phi(u,\frac{m}{n})=\frac{\sqrt{\frac{n}{m}}e^{-\frac{1}{4}\pi i}\sum\limits_{\mu=1}^{m}e^{\pi i\frac{n}{m}(u-\mu)^2}-\sum\limits_{\nu=0}^{n-1}e^{-\pi i\frac{m}{n}\nu^2-2\pi i\nu u}}{1-(-1)^{mn}e^{-2\pi i n u}}
\tag{2.1.6}
$$

$(m,n=1,2,3,\cdots)$. Moreover, if m,n are relatively prime and $e^{2\pi i u n}=(-1)^{mn}$, i. e. $u=\dfrac{l}{2n}$, $2|(l-mn)$, then the numerator of right side of (2.1.6) vanishes and we may obtain "Reciprocal relation" of Gauss sums $(2|(l-mn))$:

$$
\frac{1}{\sqrt{n}}\sum_{\nu(\mathrm{mod} n)}e^{-\pi i(\frac{m}{n}\nu^2+\frac{l}{n}\nu)}=\frac{1}{\sqrt{m}}e^{\frac{\pi i}{4}(\frac{l^2}{nm}-1)}\sum_{\mu(\mathrm{mod}\ m)}e^{\pi i(\frac{n}{m}\mu^2-\frac{l}{m}\mu)}.
\tag{2.1.7}
$$

Setting $\tau=2$, $u=1/2+2b$ in (2. 1. 6), we may obtain

$$
\frac{1}{2\pi i}\int\limits_{w\nearrow 0}\frac{e^{\frac{iw^2}{2\pi}+2bw+\frac{1}{2}w}dw}{e^w-1}=\frac{\sqrt{\frac{1}{2}}e^{-\pi i/4}(e^{\frac{\pi i}{2}(2b-\frac{1}{2})^2}+e^{\frac{\pi i}{2}(2b-\frac{3}{2})^2})-1}{1+e^{-4\pi i b}}
$$

$$
=\frac{\sqrt{1/2}e^{2\pi i b^2-\frac{\pi i}{8}}(e^{-\pi i b}-e^{-3\pi i b})-1}{2e^{-2\pi i b}\cos(2\pi b)}=\frac{i\sqrt{2}\sin\pi b(e^{2\pi i(b^2-\frac{1}{16})})-e^{2\pi i b}}{2\cos(2\pi b)}.
$$

Multiply both sides by $-\exp(-2\pi i(b^2-1/16))$ we obtain (2.1.5).

Note Setting $\tau=1$, $m=1,n=1,u=a$, we re-obtain [1],(b), (2. 10. 4).

We also need some properties of the special function $\phi(z)$ defined by ([1],(b), 4.16)

$$
\phi(z)=\exp\left\{(s-1)\log(1+z/\sqrt{t})-iz\sqrt{t}+iz^2/2\right\},
\tag{2.2}
$$

where $s=\sigma+it, 0<\sigma<1,\ t\to\infty$.

Proposition 2. *$\phi(z)$ is regular in the region $|\mathrm{Im}\log(\sqrt{t}+z)|<\pi$. If we write it as*

$$
\phi_N(z)+r_N(z)=\sum_{n=0}^{N-1}a_n z^n+r_N(z),\ \ (a_n=a_n(\sigma,t)=O(t^{-\frac{n}{2}+[\frac{n}{3}]})),
\tag{2.3}
$$

then

$$r_N(z) = O(|z|^N(5e/(2N\sqrt{t}))^{N/3}) \tag{2.3.1}$$

for $N < \dfrac{27t}{50}$, $|z| < \dfrac{20}{21}\left(\dfrac{2N\sqrt{t}}{5}\right)^{1/3}$*; and*

$$r_N(z) = O(\exp\{14|z|^2/29\}) \tag{2.3.2}$$

for $|z| < \sqrt{t}/2$.

Let $\eta = \sqrt{(2\pi t)}$ $(m = [\eta/(2\pi)])$. Since

$$e^{(s-1)\log(w/i\eta)} = e^{(\eta/2\pi)(w-i\eta)+(i/4\pi)(w-i\eta)^2}\phi((w-i\eta)/(i\sqrt{2\pi}))$$

and since

$$\frac{i}{4\pi}(w-i\eta)^2 + \frac{\eta}{2\pi}(w-i\eta) + \frac{i}{4\pi}w^2 = \frac{i}{2\pi}(w-i\eta)^2 - \frac{i\eta^2}{4\pi},$$

we have

$$\int\limits_{w\nearrow 2\pi im} \frac{w^{s-1}e^{iw^2/(4\pi)+w/2}}{e^w-1}\mathrm{d}w = e^{-i\eta^2/(4\pi)}(i\eta)^{s-1}$$
$$\times \int\limits_{w\nearrow 2\pi im} \frac{e^{(i/2\pi)(w-i\eta)^2+w/2}}{e^w-1}\left[\sum_{n=0}^{N-1} a_n\left(\frac{w-i\eta}{i\sqrt{(2\pi)}}\right)^n + r_N\left(\frac{w-i\eta}{i\sqrt{(2\pi)}}\right)\right]\mathrm{d}w;$$

and since along the integral path, the contribution of the part where

$$|w-i\eta| > \sqrt{t}/2,$$

has the estimation $O\{e^{-(A+\pi/2)t}\}$, we have

Corollary.

$$e^{-i\eta^2/(4\pi)}(i\eta)^{s-1}\int\limits_{w\nearrow 2\pi im}\frac{e^{(i/2\pi)(w-i\eta)^2+w/2}}{e^w-1}r_N\left(\frac{w-i\eta}{i\sqrt{2\pi}}\right)\mathrm{d}w$$
$$= O\left\{\eta^{\sigma-1}e^{-\frac{\pi}{2}t}\left(\frac{AN}{t}\right)^{\frac{N}{6}}\right\} + O\left\{e^{-(A+\frac{\pi}{2})t}\right\} = \eta^{\sigma-1}e^{-\frac{\pi}{2}t}\left\{O\left\{\left(\frac{AN}{t}\right)^{\frac{N}{6}}\right\} + O(e^{-At})\right\}, \tag{2.4.1}$$

with different small constant A.

We have finally the sum

$$e^{-it/2}(i\eta)^{s-1}\sum_{n=0}^{N-1}\frac{a_n}{i^n(2\pi)^{n/2}}\int\limits_{w\nearrow 2\pi im}\frac{e^{(i/2\pi)(w-i\eta)^2+w/2}}{e^w-1}(w-i\eta)^n\mathrm{d}w. \tag{2.4.2}$$

The integral may be expressed as

$$(-1)^m \int_{w\nearrow 0} \exp\left\{\frac{i}{2\pi}(w+2m\pi i-i\eta)^2+\frac{w}{2}\right\}\frac{(w+2m\pi i-i\eta)^n}{e^w-1}\mathrm{d}w,$$

this is $n!$ times the coefficient of ξ^n in

$$\begin{aligned}
&(-1)^m \int_{w\nearrow 0} \exp\left\{\frac{i}{2\pi}(w+2m\pi i-i\eta)^2+\xi(w+2m\pi i-i\eta)+\frac{w}{2}\right\}\frac{dw}{e^w-1}\\
&= (-1)^m e^{\pi i\xi^2/2}\int_{w\nearrow 0} \exp\left\{\frac{i}{2\pi}(w+(2m-\xi)\pi i-i\eta)^2\right\}\frac{e^{\frac{1}{2}w}dw}{e^w-1}\\
&= (-1)^m e^{\pi i\xi^2/2}\int_{w\nearrow 0} \exp\left\{\frac{i}{2\pi}(w-(2p+\xi)\pi i)^2\right\}\frac{e^{\frac{1}{2}w}dw}{e^w-1}\\
&= \pi(-1)^m e^{\frac{1}{2}\pi i\xi^2}\int_{0\searrow z} \frac{e^{-2\pi i(z-p-\frac{1}{2}\xi)^2}dz}{\sin\pi z} = (-1)^{m-1}(2\pi i)e^{\frac{1}{2}\pi i\xi^2-\frac{1}{8}\pi i}\Psi_1(p+\frac{\xi}{2}),
\end{aligned}$$

where

$$\begin{aligned}
p &= \sqrt{\frac{t}{2\pi}}-\left[\sqrt{\frac{t}{2\pi}}\right]=\frac{\eta}{2\pi}-m, \qquad \Psi_1(b)=\frac{e^{-2\pi i(b^2-b-\frac{1}{16})}-i\sqrt{2}\sin(\pi b)}{2\cos(2\pi b)},\\
&= 2\pi(-1)^m e^{-5\pi i/8}\sum_{\mu=0}^{\infty}\Psi_1^{(\mu)}(p)\frac{\xi^\mu}{2^\mu\mu!}\sum_{\nu=0}^{\infty}\frac{(\pi i\xi^2)^\nu}{2^\nu\nu!}.
\end{aligned}$$

Hence we obtain

$$e^{\frac{1}{2}\pi i(s-1)}(2\pi t)^{\frac{1}{2}(s-1)}2\pi(-1)^m e^{-\frac{1}{2}it-\frac{5}{8}\pi i}\sum_{n=0}^{N-1}\sum_{\nu\le(n/2)}\frac{a_n n! i^{\nu-n}}{\nu!(n-2\nu)!2^n}(\frac{d/dp}{\sqrt{2\pi}})^{n-2\nu}\Psi_1(p). \qquad (2.4.3)$$

Since when $t\to\infty$, $2i\sin(\pi s/2)=-e^{-\pi is/2}(1+O\{e^{-At}\})$, denoting the last sum by S_N, we have the following result.

Theorem A. *If* $0\le\sigma\le 1, m=[\sqrt{(t/2\pi)}]$, *and* $N<At$, *where* A *is a sufficiently small constant, then the integral* R_2 *is*

$$(-1)^{m-1}e^{-\frac{1}{2}\pi i(s-1)-\frac{1}{2}it-\frac{1}{8}\pi i}(2\pi t)^{\frac{s-1}{2}}\Gamma(1-s)\{S_N+O\{(\frac{AN}{t})^{\frac{1}{6}N}\}+O(e^{-At})\}. \qquad (2.4.4)$$

Some similar discussion will lead to the asymptotic expansion of R_1, the first part of left hand side of (2.1.3), so we skip it.

Note If we take $\sigma=1/2$, then (1.3) is a in principle consequence of (2.4.4), except that $\Psi(p)$ is replaced by $\Psi_1(p)$ (resp., R by R_2).

3 Generating Function of $b_{j,k}$

Let $0 \le \sigma \le 1$, $x = \sqrt{t/(2\pi)}, b$ a small parameter,

$$J(1-s,b) := \int_{m\searrow z} e^{-\pi iz(z-2b)} z^{s-1}\mathrm{d}z, \qquad I(s,b) := \int_{m\nearrow w} e^{\pi iw(w-2b)} w^{-s}\mathrm{d}z. \tag{3.0}$$

In order to obtain the asympototic functional equation (1.5) or (1.6), we first replace the "kernel" $1/(2i\sin(\pi z))$, in the left hand side of (2.1.4), by $e^{2\pi ibz-\pi ib^2/2}$; or $e^{w/2}/(e^w-1)$ in the right by $e^{bw-\pi ib^2/2}$. We denote the new integral by R_2':

$$R_2' = \chi(s)e^{-\frac{\pi ib^2}{2}} J(1-s,b) = \frac{\Gamma(1-s)\sin\frac{\pi s}{2}}{\pi e^{\frac{\pi is}{2}}} \int_{w\nearrow 2\pi im} e^{\frac{iw^2}{4\pi}+bw-\frac{\pi ib^2}{2}} w^{s-1}\mathrm{d}w. \tag{3.1}$$

Owing to the properties of (2.2), we have an estimation like (2.4.1) for R_2'. So we finally obtain the sum

$$e^{-it/2}(i\eta)^{s-1}\sum_{n=0}^{N-1}\frac{a_n}{i^n(2\pi)^{n/2}} \int_{w\nearrow 2\pi im} e^{(i/2\pi)(w-i\eta)^2+bw-\pi ib^2/2}(w-i\eta)^n\mathrm{d}w. \tag{3.2}$$

The integral may be expressed as

$$\int_{w\nearrow 0} \exp\left\{\frac{i(w+2m\pi i-i\eta)^2}{2\pi}+b(w+2\pi im)-\frac{\pi ib^2}{2}\right\}(w+2m\pi i-i\eta)^n\mathrm{d}w, \tag{3.3}$$

this is $n!$ times the coefficient of ξ^n in

$$\begin{aligned}
&e^{ib\eta-\pi ib^2/2}\int_{w\nearrow 0}\exp\left\{\frac{i}{2\pi}(w+2m\pi i-i\eta)^2+(\xi+b)(w+2m\pi i-i\eta)\right\}\mathrm{d}w\\
&= e^{ib\eta-\pi ib^2/2+\pi i(\xi+b)^2/2}\int_{w\nearrow 0}\exp\left\{\frac{i}{2\pi}(w+(2m-\xi-b)\pi i-i\eta)^2\right\}\mathrm{d}w\\
&= e^{ib\eta+\frac{\pi i(\xi^2+2b\xi)}{2}}\int_{w\nearrow 0} e^{\frac{i}{2\pi}(w-(2p+\xi+b)\pi i)^2}\mathrm{d}w = \sqrt{2}\pi e^{bi\eta+\frac{\pi i(\xi^2+2b\xi)}{2}+\pi i/4}\\
&= \sqrt{2}\pi e^{\frac{\pi i\xi^2}{2}+\frac{\pi i}{4}+2\pi ibm}e^{2\pi ib(p+\frac{\xi}{2})} = \sqrt{2}\pi e^{2\pi ibm+\frac{\pi i}{4}}\sum_{\mu=0}^{\infty}\Phi^{(\mu)}(p)\frac{\xi^\mu}{2^\mu\mu!}\sum_{\nu=0}^{\infty}\frac{(\pi i\xi^2)^\nu}{2^\nu\nu!},
\end{aligned} \tag{3.4}$$

where $\Phi(\alpha)$ stands for $e^{2\pi ib\alpha}$. Hence we have

Theorem B. *If $0 \le \sigma \le 1, x = \sqrt{(t/2\pi)}, m = [x]$, and $N < At$, where A is a sufficiently small constant, then the integral $\sqrt{2}e^{\pi i/8}R_2'$ is*

$$e^{2\pi ibm}e^{-\frac{1}{2}\pi i(s-1)-\frac{1}{2}it-\frac{1}{8}\pi i}(2\pi t)^{\frac{s-1}{2}}\Gamma(1-s)\{S_N+O\{(\frac{AN}{t})^{\frac{1}{6}N}\}+O(e^{-At})\}, \tag{3.5}$$

where S_N denotes the sum in (2.4.4), except that $\Psi_1(\alpha)$ is replaced by $\Phi(\alpha)$.

4 Lemmata and Conclusion

In what follows b is small parameter, $t \to \infty$.

Lemma 1. *We have, as* $0 \le \sigma \le 1$, $t \to \infty$,

$$e^{\pi i/4+ib^2/(4\pi)} \int\limits_{0\searrow w} e^{-\pi i w^2} w^{-s} \begin{pmatrix} \mathrm{ch}(bw) \\ \mathrm{sh}(bw) \end{pmatrix} \mathrm{d}w = \int\limits_{0\nearrow z} e^{\pi i z^2} z^{s-1} \begin{pmatrix} \chi_1(s)\mathrm{ch}(bz) \\ \chi(s)\mathrm{sh}(bz) \end{pmatrix} \mathrm{d}z; \tag{4.1.1}$$

$$e^{\pi i/4+ib^2/(4\pi)} \int\limits_{0\searrow w} e^{-\pi i w^2} w^{s-1} \begin{pmatrix} \mathrm{ch}(bw) \\ \mathrm{sh}(bw) \end{pmatrix} \mathrm{d}w = \int\limits_{0\nearrow z} e^{\pi i z^2} z^{-s} \begin{pmatrix} \chi_1(1-s)\mathrm{ch}(bz) \\ \chi(1-s)\mathrm{sh}(bz) \end{pmatrix} \mathrm{d}z \tag{4.1.2}$$

($\chi_1(s) = 2^s\pi^{s-1}\Gamma(1-s)i\cos(\pi s/2) = \chi(s)(1+O(e^{-\pi t}))$).

Proof. Since we have

$$\int\limits_{w\nearrow 0} e^{iw^2/(4\pi)+iw(z+b)/(2\pi)}\mathrm{d}w = 2\pi e^{\pi i/4}e^{-i(z+b)^2/(4\pi)}.$$

As in [1] (b), 2. 10, multiplying both sides by $z^{s-1}(\sigma > 1)$, and then integrating from 0 to $\infty e^{-\pi i/4}$, we obtain

$$\int\limits_{w\nearrow 0} e^{iw^2/(4\pi)}\mathrm{d}w \int_0^{\infty e^{-\pi i/4}} e^{i(z+b)w/(2\pi)} z^{s-1}\mathrm{d}z = 2\pi e^{\pi i/4} \int_0^{\infty e^{-\pi i/4}} e^{-i(z+b)^2/(4\pi)} z^{s-1}\mathrm{d}z$$

(The inversion on the left-hand side is justified by absolute convergence; in fact $w = -c + \rho e^{\pi i/4}(c > 0)$, $z = re^{-\pi i/4}$ so that $\mathrm{Re}(izw) = -cr/\sqrt{2}$.)

Since

$$\int_0^{\infty e^{-\pi i/4}} e^{izw/(2\pi)} z^{s-1}\mathrm{d}z = e^{\pi i s/2}\Gamma(s)(\frac{w}{2\pi})^{-s},$$

we have

$$e^{\pi i s/2}\Gamma(s) \int\limits_{w\nearrow 0} e^{i(w^2+2bw)/(4\pi)}(\frac{w}{2\pi})^{-s}\mathrm{d}w = 2\pi e^{\pi i/4} \int_0^{\infty e^{-\pi i/4}} e^{-i(z+b)^2/(4\pi)} z^{s-1}\mathrm{d}z.$$

Hence we have

$$\begin{aligned} &e^{\pi i s/2}\Gamma(s) \int\limits_{w\nearrow 0} e^{iw^2/(4\pi)}(\frac{w}{2\pi})^{-s} \begin{pmatrix} 2\mathrm{ch}(ibw/2\pi) \\ 2\mathrm{sh}(ibw/2\pi) \end{pmatrix} \mathrm{d}w \\ &= 2\pi e^{\pi i/4} \int_0^{\infty e^{-\pi i/4}} e^{-i(z^2+b^2)/(4\pi)} z^{s-1} \begin{pmatrix} 2\mathrm{ch}(ibz/2\pi) \\ -2\mathrm{sh}(ibz/2\pi) \end{pmatrix} \mathrm{d}z; \end{aligned}$$

then by substitution $w = 2\pi i w_1$ and $z = -2\pi i z_1$, we have

$$2\pi i\Gamma(s)\int_{0\searrow w_1} e^{-\pi i w_1^2} w_1^{-s}\begin{pmatrix} 2\mathrm{ch}(-bw_1)\\ 2\mathrm{sh}(-bw_1)\end{pmatrix}\mathrm{d}w_1$$
$$= e^{-\pi is/2}2\pi e^{\pi i/4-ib^2/(4\pi)}(2\pi)^s\int_0^{\infty e^{\pi i/4}} e^{\pi i z_1^2} z_1^{s-1}\begin{pmatrix} 2\mathrm{ch}(bz_1)\\ -2\mathrm{sh}(bz_1)\end{pmatrix}\mathrm{d}z_1$$
$$= e^{-\pi is/2}2\pi e^{\pi i/4-ib^2/(4\pi)}(2\pi)^s\int_{0\nearrow z_1} e^{\pi i z_1^2} z_1^{s-1}\begin{pmatrix} 2\mathrm{ch}(bz_1)/(1-e^{-\pi is})\\ -2\mathrm{sh}(bz_1)/(1+e^{-\pi is})\end{pmatrix}\mathrm{d}z_1;$$

hence (4.1.1). Replace s by $1-s$ we obtain (4.1.2).

Replace b by $2\pi ib$, then we may have

$$\sqrt{2}(e^{\pi i(b^2/2-1/8)}I(s,-b)+\chi(s)e^{\pi i(-b^2/2+1/8)}J(1-s,b)) = O(e^{-At}). \tag{4.2.1}$$

Note that in (3.5), the multiplier before S_N is also $e^{2\pi im}x^{s-1}e^{-it/2-i\pi/8}\chi(s)(1+O(e^{-At}))$, dividing both sides of (4.2.1) by $e^{2\pi ibx}/\sqrt{x}$ we also have, under the condition in theorem B, and $\sigma = 1/2$,

$$e^{-2\pi ibp}\left[x^{-it}e^{\frac{it}{2}+\frac{\pi i}{8}}\widetilde{S}_N - \chi(1/2+it)x^{it}e^{-\frac{it}{2}-\frac{\pi i}{8}}S_N\right] = O\left\{\left(\frac{AN}{t}\right)^{\frac{N}{6}}\right\} + O(e^{-At}); \tag{4.2.2}$$

where

$$\widetilde{S}_N = \left[\sum_{n=0}^{N-1}\widetilde{a}_n(\frac{i}{\sqrt{2\pi}}\frac{\partial}{\partial\xi})^n e^{\pi i(-\frac{\xi^2}{2}+b\xi+2bp)}\right]_{\xi=0,\sigma=\frac{1}{2}}, \qquad \chi\left(\frac{1}{2}+it\right) = e^{-2i\vartheta(t)},$$

$\widetilde{a}_n$ is the conjugate of $a_n = a_n(1/2,t)$.

Lemma 2. *We have another version of Binet first formula*

$$\log\Gamma(z+\frac{1}{2}) = z\log(z) - z + \frac{\log(2\pi)}{2} - \int_0^\infty\left(\frac{1}{u}-\frac{e^{u/2}}{e^u-1}\right)\frac{e^{-zu}}{u}\mathrm{d}u. \tag{4.3}$$

The proof of (4.3) is almost the same as that for $\log\Gamma(z)$ in [4], p. 124.

Note that $i\theta(t)$ is essential the integral above (with $z = it$).

Lemma 3. *Let $P(u), Q(u)$ be polynomials then the following two equations are equivalent:*

$$P\left(\frac{d}{d\xi}\right)e^{\frac{1}{2}a\xi^2} = e^{\frac{1}{2}a\xi^2}Q(a\xi), \quad Q(u) = e^{\frac{1}{2}a(\frac{d}{du})^2}P(u). \tag{4.4}$$

If $P(u) = u^k$ then the statement can be proved by induction in k.

Note that we may have a single identity (a, b, a_1 complex, $aa_1 \neq 0$) :

$$e^{-\frac{1}{2}ab^2} P(a_1 \frac{\partial}{\partial \xi}) e^{\frac{1}{2}a(b+\xi)^2}|_{\xi=0} = e^{\frac{1}{2a}\frac{\partial^2}{\partial b^2}} P(a_1 ab); \tag{4.5.1}$$

and that S_N and $\widetilde{S}_N$ in (4.2.2) are

$$S_N = \exp\left(\frac{1}{2\pi i}\frac{\partial^2}{\partial b^2}\right)\phi_N(b\sqrt{\pi/2}), \qquad \widetilde{S}_N = \exp\left(\frac{-1}{2\pi i}\frac{\partial^2}{\partial b^2}\right)\widetilde{\phi}_N(-b\sqrt{\pi/2}), \tag{4.5.2}$$

where (cf. (2.3))

$$\widetilde{\phi}(v) := \exp\{(-1/2 - it)\log(1 + v/\sqrt{t}) + iv\sqrt{t} - iv^2/2\}, \quad \widetilde{\phi}_N(z) = \sum_{n=0}^{N-1} \widetilde{a}_n v^n.$$

Hence the asymptotic functional equation (4.2.2) implies (cf. (1.7), lemma 2),

$$e^{i\theta(t)} \exp\left(\frac{1}{4i}\frac{\partial^2}{\partial v^2}\right)\phi_N(v) - e^{-i\theta(t)}\exp\left(-\frac{1}{4i}\frac{\partial^2}{\partial v^2}\right)\widetilde{\phi}_N(-v) = O\left\{\left(\frac{AN}{t}\right)^{\frac{1}{6}N}\right\} + O(e^{-At}). \tag{4.2.3}$$

Lemma 4. *Let*

$$b_n = b_n(\alpha) \sim \sum a_{j,n}\alpha^j$$

be the asymptotic Maclaurin series as $\alpha \to 0$, $n = 0, 1, \cdots, N-1$. *If*

$$P(v) = \sum_{n=0}^{N-1} b_n v^n = O(\alpha^M), \qquad M > 0$$

(*uniform in* $|v| < 1$), *then so does each coefficient* b_n.

To prove it by induction in N we only need the formula of difference quotient of high degree.

Proof of (1.6): For fixed s, two integrals in (4.2.1) are entire functions of b, and each coefficient of Maclaurin expansion, as a function in s, has the asymptotic property describe by Theorem B, i. e. can be regarded as the asymptotic Maclaurin series in $\sqrt{1/t}$. Let $b\sqrt{\pi/2} = v = u\sqrt{t}$, and

$$\exp\left(\frac{1}{4i}\frac{\partial^2}{\partial v^2}\right)\phi_N(v) = \sum_{n=0}^{N-1} a_n^{(N)} v^n, \qquad \exp\left(-\frac{1}{4i}\frac{\partial^2}{\partial v^2}\right)\widetilde{\phi}_N(-v) = \sum_{n=0}^{N-1} \widetilde{a}_n^{(N)}(-v)^n,$$

etc. then we only need to prove the statement that, fixed n ($n = 0, 1, \cdots$), for any $M > 0$, the coefficient of v^n in the left-hand side of (4.2.3), i. e. the difference

$$\exp(i\theta(t))a_n^{(N)} - \exp(-i\theta(t))(-1)^n \widetilde{a}_n^{(N)}, \tag{4.6}$$

is $O(t^{-M/2})$, if N sufficiently large.

If in addition we let $N < A_1 t^{\epsilon}$ in Theorem B for both $J(1-s,b)$ and $I(s,-b)$, then the O-term becomes $O(t^{-N(1-\epsilon)/6})$, hence if $N > 3M/(1-\epsilon)$ then the statement is true $(0<\varepsilon<1)$.

The proof of (1.6) is fulfiled.

Appendix A. New Datii About $c_{j,k}$

Rewriting (1.1) as

$$R \sim (-1)^{m-1}(\frac{t}{2\pi})^{-1/4} \sum_{0\le j, 0\le k\le 3j} c_{j,k} e^{\pi i(j+k)/4} (\sqrt{it})^{-j} \left(\frac{1}{2\sqrt{2i\pi}}\frac{d}{dp}\right)^{k} \Psi(p). \tag{1.3'}$$

By the aid of (1.4) and Dos Qbasic program we extend the datii of values of $b_{j,k}$, $c_{j,k} = (-1)^{(j+k)/4} b_{j,k}$:

$$\begin{aligned}
&c_{0,0}=1; c_{1,3}=-\frac{1}{3}, c_{2,2}=\frac{1}{2^2}, c_{2,6}=\frac{1}{2\cdot 3^2}, c_{3,1}=-\frac{1}{2^3}, c_{3,5}=-\frac{2}{5}, c_{3,9}=-\frac{1}{2\cdot 3^4};\\
&c_{4,0}=\frac{1}{2^5}, c_{4,4}=\frac{19}{2^5\cdot 3}, c_{4,8}=\frac{11}{2^3\cdot 3^2\cdot 5}, c_{4,12}=\frac{1}{2^3\cdot 3^5};\\
&c_{5,3}=-\frac{5}{2^3\cdot 3}, c_{5,7}=-\frac{17\cdot 53}{2^5\cdot 3^2\cdot 5\cdot 7}, c_{5,11}=-\frac{7}{2^2\cdot 3^4\cdot 5}, c_{5,15}=-\frac{1}{2^3\cdot 3^6\cdot 5};\\
&c_{6,2}=\frac{5}{2^5}, c_{6,6}=\frac{367}{2^7 3\cdot 5}, c_{6,10}=\frac{13\cdot 1453}{2^6\cdot 3^4\cdot 5^2\cdot 7}, c_{6,14}=\frac{17}{2^5\cdot 3^5\cdot 5}, c_{6,18}=\frac{1}{2^4\cdot 3^8\cdot 5};\\
&c_{7,1}=-\frac{5}{2^6}, c_{7,5}=-\frac{11\cdot 37}{2^8\cdot 5}, c_{7,9}=-\frac{61\cdot 109}{2^5\cdot 3^4\cdot 5\cdot 7}, c_{7,13}=-\frac{2131}{2^4\cdot 3^5\cdot 5^2\cdot 7},\\
&c_{7,17}=-\frac{1}{2^3\cdot 3^6\cdot 5}, c_{7,21}=-\frac{1}{2^4\cdot 3^9\cdot 5\cdot 7};\\
&c_{8,0}=\frac{41}{2^{11}}, c_{8,4}=\frac{7\cdot 61}{2^{10}}, c_{8,8}=\frac{5281}{2^{12}\cdot 3^2\cdot 7}, c_{8,12}=\frac{88651}{2^7\cdot 3^5\cdot 5^2\cdot 7},\\
&c_{8,16}=\frac{19\cdot 587}{2^8\cdot 3^6\cdot 5^2\cdot 7}, c_{8,20}=\frac{23}{2^6\cdot 3^8\cdot 5^2}, c_{8,24}=\frac{1}{2^7\cdot 3^{10}\cdot 5\cdot 7}.
\end{aligned}$$

Remark. The least common multiple of the denominators M above is $2^{12}\cdot 3^{10}\cdot 5^2\cdot 7 = 42326323200$. First find non zero coefficients $b_{j,k}^{(1)}$ in the product of power series

$$M\cdot(1-(1/2)\alpha\beta+(3/8)\alpha^2\beta^2-\cdots)\times(1+\alpha\beta^3[(1/3)-(1/4)\alpha\beta+\cdots]$$

$$+(1/2!)\alpha^2\beta^6[(1/9)-(1/6)\alpha\beta-\cdots]-\cdots) = \sum_{0\le j\le 8, 0\le k\le 3j} b_{j,k}^{(1)}\alpha^j\beta^k+\cdots \tag{A.2}$$

(Note that for $\sigma=\frac{1}{2}$ in (2.3), $\varphi_{25}(z)=\sum_{k=0}^{24} a_k z^k, Ma_k=\sum_{\frac{k}{3}\le j\le k} b_{j,k}^{(1)} e^{\frac{\pi i(k-j)}{4}}\cdot t^{-\frac{j}{2}}$). Then for fixed j, find $b_{j,k}^{(2)} (0\le k\le 3j)$ by the formula

$$b_{j,k}^{(2)} = b_{j,k}^{(1)} + \frac{k(k+1)}{4} b_{j,k+2}^{(1)} + \frac{k(k+1)(k+2)(k+3)}{32} b_{j,k+4}^{(1)} + \cdots$$

(finite sum, note that S_{25} in (3.5) with $\sigma = \frac{1}{2}, b = \beta\sqrt{2/\pi i}$, were

$$\frac{1}{M}\sum_{k=0}^{24}\sum_{k/3\le j<\infty} b_{j,k}^{(2)}\alpha^j\beta^k, \tag{A.2.1}$$

if we had abandoned all non zero $b_{j,k}^{(1)}$ with $K > 24$ in (A.1)); at last for fixed $k(k = 0, \cdots, 24)$, find final $b_{j,k}(k/3 \le j \le 8)$ in

$$(1 + \alpha^2/48 + \alpha^4/4608 - \cdots) \times \sum_{k/3\le j\le 8} b_{j,k}^{(2)}\alpha^j = \sum_{k/3\le j\le 8} M \cdot b_{j,k}\alpha^j + \cdots . \tag{A.3}$$

Note that $b_{j,k}$ only depends on $b_{l,n}^{(1)}, l \le j, n \ge k$.

Appendix B. About $L(s, f)$

In order to apply the above we also discuss Dirichlet series briefly. Let $m > 1$ be a fixed odd integer, Dirichlet Series $L(s, f)$ be analytic continuation of $\sum_{n=1}^{\infty}\{f(n)/n^s\}(Re(s) > 1)$, $f(n)$ being a bounded (complex valued) function in integer n such that $f(n') = f(n)(m|(n' - n))$. (Such functions $f(n)$ form a space of dimension m). It is well known that the following Fourier transform and its inversion are complex linearly:

$$\widetilde{f}(k) = \frac{1}{\sqrt{m}}\sum_{n=0}^{m-1} f(n)e^{-2\pi ink/m}, \qquad f(n) = \frac{1}{\sqrt{m}}\sum_{k=0}^{m-1}\widetilde{f}(k)e^{2\pi ink/m}; \tag{B.1.1}$$

and we may define the "canonical extention of $f(n)$" as follows:

$$F_f(z) = \frac{1}{\sqrt{m}}\sum_{k=(1-m)/2}^{(m-1)/2}\widetilde{f}(k)e^{2\pi izk/m} = \sum_{n=(1-m)/2}^{(m-1)/2} f(n)\frac{\sin(\pi(z-n))}{m\sin(\pi(z-n)/m)}. \tag{B.1.2}$$

We give examples of functions $f(n)$ such that $\widetilde{f}(n) = \bar{f}(n)$.

Example 1. Let $X(n)(\mathrm{mod}(m))$ be a (even) primitive character. Since

$$\tau(X)\bar{X}(n) = \sum_{k=1}^{m} X(k)e^{-\frac{2\pi ink}{m}}, \qquad \Big(\tau(X) = \sum_{n,\mathrm{mod}\ m} X(n)e^{\frac{2\pi in}{m}} = \sqrt{m}e^{i\alpha}\Big)$$

(cf. [3], p. 92, (5)) we see that $\widetilde{X}(n) = \bar{X}(n)e^{i\alpha}$ and $X(n)e^{-\frac{i\alpha}{2}}$ is such an example.

Example 2. Let $e_h(\nu) = (-1)^\nu e^{\frac{-\pi i\nu^2+2\pi ih\nu}{m}}$, $\bar{e}_h(\mu) = (-1)^\mu e^{\frac{\pi i\mu^2-2\pi ih\mu}{m}}$. In (2.1.7), if we exchange the letters m(resp. μ) and n(resp. ν) we have

$$\frac{1}{\sqrt{m}}\sum_{\mu(\mathrm{mod}\ m)} e^{-\pi i(\frac{n}{m}\mu^2+\frac{l}{m}\mu)} = \frac{1}{\sqrt{n}}e^{\frac{\pi i}{4}(\frac{l^2}{nm}-1)}\sum_{\nu(\mathrm{mod} n)} e^{\pi i(\frac{m}{n}\nu^2-\frac{l}{n}\nu)} \tag{2.1.7$'$}$$

$(2|(l-mn))$, then let $n=1, l=m-2h+2k$, we obtain

$$\sqrt{1/m}\sum_{\mu=1}^{m} e_h(\mu)e^{-\frac{2\pi ik\mu}{m}} = e^{\frac{\pi i}{4}(\frac{(m-2h+2k)^2}{m}-1)} = e^{\frac{\pi i(m-1)}{4}}(-1)^{k-h}e^{\frac{\pi i(k-h)^2}{m}},$$

hence the Fourier transform of $e_h(\mu)$ is $(-1)^h e^{\frac{\pi i}{4}(m-1)+\frac{\pi ih^2}{m}}\bar{e_h}(k)$. As in example 1, $\epsilon_h e_h$ satisfies $\widetilde{f}(n)=\bar{f}(n)$, where $\epsilon_h = e^{-\pi i(h/2+\frac{m-1}{8}+\frac{h^2}{2m})}$.

Remark 1. The functions $F_h(n)=\frac{1}{2}\epsilon_h(e_h(n)+e_{-h}(n))$, $h=0,1,\cdots,(m-1)/2$ form a base of subspace of functions $F(n)$ that $F(-n)=F(n)$. Moreover, if in addition that $\widetilde{F}(n)=\bar{F}(n)$, then $F(n)$ must be a real linear combination of $F_h(n)$.

By a formula of Hurwitz ([4], p. 129-130, cf. [1].b, p. 37) we know that

$$\zeta(s,a) = 2^s\pi^{s-1}\Gamma(1-s)\sum_{n=1}^{\infty} n^{s-1}\sin\left(\frac{\pi s}{2}+2n\pi a\right), \qquad \sigma<0<a\le 1$$

($\zeta(s,a)$ the analytic continuation of $\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}$, and $\zeta(s,1)=\zeta(s)$). Hence if $f(n)=f(-n)$ then we have

$$\begin{aligned} L(s,f) &= \frac{1}{m^s}\sum_{l=1}^{m} f(l)\zeta(s,\frac{l}{m}) = \frac{2^s\pi^{s-1}\Gamma(1-s)\sin(\frac{\pi s}{2})}{m^{s-\frac{1}{2}}}\sum_{n=1}^{\infty} n^{1-s}\widetilde{f}(n) \qquad (\sigma<0);\\ L(s,f) &= \frac{2^s\pi^{s-1}\Gamma(1-s)\sin(\frac{\pi s}{2})}{m^{s-\frac{1}{2}}}L(1-s,\widetilde{f}), \quad \sigma<0, \end{aligned} \tag{B.2.1}$$

similarly if $f(n)=-f(-n)$ then we have

$$L(s,f) = \frac{2^s\pi^{s-1}\Gamma(1-s)i\cos(\frac{\pi s}{2})}{m^{s-\frac{1}{2}}}L(1-s,\widetilde{f}). \tag{B.2.2}$$

Corollary of Remark 1. If F, F_h are as in Remark 1, then (B.2.1) for F is a real linear combination of equations (B.2.1) for F_h, $h=0,1,\cdots,(m-1)/2$.

Theorem 1. *Let* $\chi(s)=2^s\pi^{s-1}\Gamma(1-s)\sin(\pi s/2)$, $f_h(n)=(-1)^n e^{-\frac{\pi in^2}{m}}\cos(2\pi hn/m)$ *and* $\chi_h(s)=\frac{\chi(s)}{m^{s-\frac{1}{2}}}\epsilon_h^{-2}$, *(cf. example 2), then* $\widetilde{f_h}(n)=\bar{f}_h(n)\epsilon_h^{-2}$, *and*

$$L(s,f_h) = \sum_{k=1}^{m}\frac{f_h(k)}{m^s}\zeta(s,\frac{k}{m}) = \chi_h(s)L(1-s,\bar{f}_h) \tag{B.3.1}$$

$$= \int_{0\searrow w}\frac{e^{-\pi iw^2/m}w^{-s}\cos(\frac{2\pi hw}{m})dw}{2i\sin(\pi w)} + \chi_h(s)\int_{0\swarrow z}\frac{e^{\frac{\pi iz^2}{m}}\cos(\frac{2\pi hz}{m})z^{s-1}dz}{2i\sin(\pi z)}. \tag{B.3.2}$$

Proof. If in (2.1.6) we take $\tau=1/m$ then we have

$$\frac{1}{2\pi i}\phi(u,\frac{1}{m}) = \frac{1}{2\pi i}\int_{w\nearrow 0}\frac{e^{\frac{iw^2}{4m\pi}+uw}}{e^w-1}\mathrm{d}w = \frac{\sqrt{m}e^{\frac{-\pi i}{4}+\pi im(u-1)^2} - \sum_{\nu=0}^{m-1}e^{-2\pi i\nu(u+\frac{\nu}{2m})}}{1+e^{-2\pi imu}}.$$

If $u = 1/2 + iz/(2m\pi)$ then the above result takes the form

$$\frac{1}{2\pi i}\int\limits_{w\nearrow 0}\frac{e^{\frac{iw^2}{4m\pi}+\frac{izw}{2m\pi}+\frac{1}{2}w}}{e^w-1}\mathrm{d}w = \frac{-\sqrt{m}e^{\frac{\pi i(m-1)}{4}-\frac{iz^2}{4m\pi}+\frac{1}{2}z}+\sum_{\nu=0}^{m-1}(-1)^\nu e^{\frac{\nu z}{m}-\frac{\pi i\nu^2}{m}}}{e^z-1}. \quad \text{(B.4)}$$

Replacing z by $z+2\pi ih$ ($h=0,\pm1,\cdots,\pm(m-1)/2$), we obtain

$$\frac{1}{2\pi i}\int\limits_{w\nearrow 0}\frac{e^{\frac{iw^2}{4m\pi}+\frac{izw}{2m\pi}+(\frac{1}{2}-\frac{h}{m})w}}{e^w-1}\mathrm{d}w$$
$$= \frac{-\sqrt{m}(-1)^h e^{\frac{\pi i(m-1)}{4}-\frac{iz^2}{4m\pi}+\frac{\pi ih^2}{m}+\frac{h}{m}z+\frac{1}{2}z}+\sum_{\nu=0}^{m-1}(-1)^\nu e^{\frac{\nu}{m}z-\frac{\pi i\nu^2}{m}+\frac{2\pi ih\nu}{m}}}{e^z-1}. \quad \text{(B.4.}h\text{)}$$

As in [1].b, p. 27, multiplying both sides by $z^{s-1}(\sigma>1)$, and integrating from 0 to $\infty e^{-\pi i/4}$, using

$$\int_0^{\infty e^{-\frac{\pi i}{4}}}\frac{z^{s-1}e^{\frac{\nu}{m}z}dz}{e^z-1} = \Gamma(s)\zeta(s,1-\frac{\nu}{m});$$
$$\int_0^{\infty e^{-\frac{\pi i}{4}}} e^{\frac{i}{2m\pi}zw}z^{s-1}\mathrm{d}z = e^{\frac{\pi i}{2}s}\int_0^\infty e^{-\frac{1}{2m\pi}yw}y^{s-1}dy = e^{\frac{\pi i}{2}s}(\frac{w}{2m\pi})^{-s}\Gamma(s),$$

we also obtain

$$\int\limits_{w\nearrow 0}\frac{e^{\frac{iw^2}{4m\pi}+(\frac{1}{2}-\frac{h}{m})w}}{2\pi i(e^w-1)}\mathrm{d}w\int_0^{\infty e^{-\frac{\pi i}{4}}} e^{\frac{izw}{2\pi m}}z^{s-1}\mathrm{d}z = \Gamma(s)e^{\frac{\pi i}{2}s}(2\pi m)^s\frac{1}{2\pi i}\int\limits_{w\nearrow 0}\frac{e^{\frac{iw^2}{4m\pi}+(\frac{1}{2}-\frac{h}{m})w}w^{-s}}{e^w-1}\mathrm{d}w$$
$$= -(-1)^h\sqrt{m}e^{\frac{\pi i(m-1)}{4}+\frac{\pi ih^2}{m}}\int_0^{\infty e^{-\frac{\pi i}{4}}}\frac{e^{-\frac{iz^2}{4m\pi}+(\frac{1}{2}+\frac{h}{m})z}z^{s-1}}{e^z-1}\mathrm{d}z$$
$$+\Gamma(s)\sum_{\nu=0}^{m-1}(-1)^\nu e^{-\frac{\pi i\nu^2}{m}}e^{\frac{2\pi i\nu}{m}h}\zeta(s,\frac{m-\nu}{m}). \quad \text{(B.5.h)}$$

Replace h by $(-h)$ we obtain (B.5.$(-h)$)), adding (B.5.h) and (B.5.$(-h)$), using

$$\int_0^{\infty e^{-\frac{i\pi}{4}}}\frac{ch(\frac{hz}{m})e^{\frac{-iz^2}{4m\pi}+\frac{1}{2}z}}{e^z-1}z^{s-1}\mathrm{d}z = \frac{1}{1+e^{is\pi}}\int\limits_{0\searrow z}\frac{ch(\frac{hz}{m})e^{\frac{-iz^2}{4m\pi}+\frac{1}{2}z}}{e^z-1}z^{s-1}\mathrm{d}z,$$

and using $f_h(m-k)=f_h(k)$ we at last obtain

$$\Gamma(s)e^{\frac{\pi is}{2}}(2\pi m)^s\frac{1}{2\pi i}\int\limits_{w\nearrow 0}\frac{2ch(\frac{h}{m}w)e^{\frac{iw^2}{4m\pi}+\frac{1}{2}w}w^{-s}}{e^w-1}\mathrm{d}w = -(-1)^h\sqrt{m}e^{\frac{\pi i(m-1)}{4}+\frac{\pi ih^2}{m}}$$
$$\times\int\limits_{0\searrow z}\frac{2ch(\frac{h}{m}z)e^{-\frac{iz^2}{4m\pi}+\frac{1}{2}z}z^{s-1}}{(1+e^{\pi is})(e^z-1)}\mathrm{d}z+\Gamma(s)\sum_{\nu=0}^{m-1}(-1)^\nu e^{-\frac{\pi i\nu^2}{m}}2\cos(\frac{2\pi h\nu}{m})\zeta(s,\frac{m-\nu}{m}).$$

Dividing both sides by $2\Gamma(s)m^s$, and then let $w = 2\pi i w_1$, $z = 2\pi i z_1$, we finally obtain

$$L(s, f_h) = \int_{0\searrow w_1} \frac{e^{\frac{-\pi i w_1^2}{m}} w_1^{-s} \cos(\frac{2\pi h w_1}{m})}{2i\sin(\pi w_1)} \mathrm{d}w_1 + \chi_h(s) \int_{0\swarrow z_1} \frac{e^{\frac{\pi i z_1^2}{m}} z_1^{s-1} \cos(\frac{2\pi h z_1}{m})}{2i\sin(\pi z_1)} \mathrm{d}z_1,$$

and hence (B.3.2). (B.3.1) is consequence of (B.3.2).

We now search Riemann- Siegel integral representation for $L(s, X)$, which is a linear combination of equations (B.3.2), where $X(n)$ is as in example 1, with $X(-1) = 1$. Let

$$mc_h(X) = \sum_{n,\mathrm{mod}(m)} X(n)(-1)^n e^{\frac{\pi i n^2}{m}} \cos(\frac{2\pi h n}{m}); \qquad X(n) = \sum_{h,\mathrm{mod}(m)} c_h(X) f_h(n),$$

so that $\sqrt{m}c_h(X)$ (in h) is the Fourier transform of $f(n) = (-1)^n e^{\frac{\pi i n^2}{m}} X(n)$ (in n), and (B. 1. 2) implies that the canonical extention of $f(n)$ is

$$X^{\#}(w) := \sum_{h=(1-m)/2}^{(m-1)/2} c_h(X)\cos(\frac{2\pi h w}{m}) = \sum_{n=(1-m)/2}^{(m-1)/2} \frac{(-1)^n e^{\pi i n^2/m} X(n) \sin\pi(w-n)}{m \sin\frac{\pi(w-n)}{m}};$$

By Principle of superposition (for (B. 1. 1) and (B. 2. 1)) we also see that $\sqrt{m}c_h(X)\epsilon_h^{-2}$ (in h) is the Fourier transform of $(-1)^n e^{-\frac{\pi i n^2}{m}} \bar{X}(n)\tau(X) = \bar{f}(n)\tau(X)$, and we have

$$\sum_{h=(1-m)/2}^{(m-1)/2} c_h(X)(-1)^h e^{\frac{\pi i(m-1)}{4} + \frac{\pi i h^2}{m}} \cos(\frac{2\pi h z}{m}) = \tau(X) \sum_{n=(1-m)/2}^{(m-1)/2} \bar{X}(n) \frac{e^{-\frac{\pi i n^2}{m}} \sin\pi z}{m\sin\frac{\pi(z-n)}{m}}$$

$$= \tau(X)\bar{X}^{\#}(z) \qquad (\tau(X)\tau(\bar{X}) = m);$$

$$L(s, X) = \int_{0\searrow\widetilde{w}} \frac{e^{-\pi i \widetilde{w}^2/m} X^{\#}(\widetilde{w})\widetilde{w}^{-s} d\widetilde{w}}{2i\sin(\pi\widetilde{w})} + \frac{\chi(s)\tau(X)}{m^s} \int_{0\swarrow\widetilde{z}} \frac{e^{\pi i \widetilde{z}^2/m} \bar{X}^{\#}(\widetilde{z})\widetilde{z}^{s-1} d\widetilde{z}}{2i\sin(\pi\widetilde{z})}$$

$$= \int_{0\searrow\widetilde{w}} \sum_{|n|<m/2} \frac{X(n) e^{-\pi i(\widetilde{w}^2 - n^2)/m} \widetilde{w}^{-s} d\widetilde{w}}{2mi\sin(\pi(\widetilde{w}-n)/m)} + \frac{\chi(s)\tau(X)}{m^s} \times \cdots$$

$$= \sum_{|n|<m/2} X(n) \int_{(-n/m)\searrow w} \frac{e^{-\pi i w(mw+2n)}(mw+n)^{-s} dw}{2i\sin(\pi w)} + \cdots,$$

(in the last step, for each n, let $\widetilde{w} = mw + n$, $\widetilde{z} = mz + n$).

Since the only zeros of denominator are integers, the path of integration can be changed as $w \searrow [(1 - n/m)]$ ($[x]$ being integral part of x); and since the simultaneous replacement of w by $w \pm 1$ and of n by $n \pm (-m)$ do not change the integrand, so we may change n to $(n + m)$ for those n that $n \le 0$ (resp. w to $w - 1$) and obtain the uniform path $w \searrow 0$ for $n = 1, 2, \cdots, m$, and finally we obtain

Theorem 2. *For primitive (even) character $X(n)(mod(m))$,*

$$L(s,X)=\sum_{k=1}^{m}\{\int_{w\searrow 0}X(k)e^{-\pi iw(mw+2k)}(mw+k)^{-s}\frac{dw}{2i\sin(\pi w)}$$

$$+\frac{\chi(s)\tau(X)}{m^s}\int_{z\swarrow 0}\bar{X}(k)e^{\pi iz(mz+2k)}(mz+k)^{s-1}\frac{dz}{2i\sin(\pi z)}\}. \qquad (B.6)$$

Similar facts hold for odd functions $g_h(n)=f_h(n)i\tan(2\pi hn/m)$ and odd primitive character $X(n)$.

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