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PROXIMINAL SUBSPACES OF 2-NORMED SPACES

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Abstract

In 1965 Gähler introduced 2*-normed spaces and since then, this topic have been intensively studied and developed. We shall introduce the notion of* 1*-type proximinal subspaces of* 2*-normed spaces and give some results in this field.*

Key words *b-proximinal subspaces,* 1*-type proximinal subspaces,* 2*-functionals*

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1 Introduction

The concept of linear 2-normed spaces has been investigated by S. Gähler[5] and has been developed extensively in different subjects by others (for example, [1]-[4] and [6]).

Let X be a linear space of dimension greater than 1 over K, where K is the real or complex number field, and let $\|.,.\|$ be a real-valued function on $X \times X$ satisfying the following conditions:

(i) $\|x,y\| = 0$ if and only if x and y are linearly dependent vectors.

(ii) $\|x,y\| = \|y,x\|$ for all $x, y \in X$.

(iii) $\|\lambda x, y\| = |\lambda|\ \|x,y\|$ for all $\lambda \in K$ and all $x, y \in X$.

(iv) $\|x+y, z\| \leq \|x,z\| + \|y,z\|$ for all $x, y, z \in X$.

Then $\|.,.\|$ is called a 2-norm on X and $(X, \|.,.\|)$ is called a linear 2-normed space. Some of the basic properties of 2-norms are that they are non-negative and $\|x, y+\alpha x\| = \|x,y\|$ for all $x, y \in X$ and all $\alpha \in K$.

Every 2-normed space is a locally convex topological vector space. In fact for a fixed $b \in X$, $p_b(x) = \|x, b\|$, $x \in X$, is a seminorm and the family $P = \{p_b : b \in X\}$ of seminorms generates a locally convex toplogy on X.

Let $(X, \|.,.\|)$ be a 2-normed space and W_1, W_2 two subspaces of X. A map $f : W_1 \times W_2 \longrightarrow K$ is called a bilinear 2-functional on $W_1 \times W_2$ whenever for all $x_1, x_2 \in W_1$, $y_1, y_2 \in W_2$ and all $\lambda_1, \lambda_2 \in K$:

(i) $f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_1, y_2) + f(x_2, y_1) + f(x_2, y_2)$;

(ii) $f(\lambda_1 x_1, \lambda_2 y_1) = \lambda_1 \lambda_2 f(x_1, y_1)$.

A bilinear 2-functional $f : W_1 \times W_2 \longrightarrow K$ is called bounded if there exists a non-negative real number M (called a Lipschitz constant for f) such that

$$|f(x, y)| \leq M\|x, y\|$$

for all $x \in W_1$ and all $y \in W_2$.

Also, the norm of a bilinear 2-functional f is defined by

$$\|f\| = \inf \{M \geq 0 : M \text{ is a Lipschitz constant for } f\}.$$

For a 2-normed space $(X, \|.,.\|)$, a subspace W of X and $b \in X$, we denote by $W_b^{\sharp}$ the Banach space of all bounded bilinear 2-functionals on $W \times < b >$.

We conclude this section by a known lemma needed in the proof of a main result.

Proposition 1.1.[2; Theorem 3.6]**.** *Let $(X, \|.,.\|)$ be a 2-normed space, W a subspace of X, $b \in X$ and $< b >$ the subspace of X generated by b. If $x_0 \in X$ is such that*

$$\delta = \inf\{\|x_0 - w, b\| : w \in W\} > 0,$$

then there exists a bounded bilinear functional $F : X\times < b > \longrightarrow K$ such that $F|_{W\times<b>} = 0$, $F(x_0, b) = 1$ and $\|F\| = \dfrac{1}{\delta}$.

2 Some Types of Proximinality in 2-normed Spaces

Definition 2.1. Let $(X, \|.,.\|)$ be a 2-normed space, W a subspace of X, $0 \neq b \in X$ and $< b >$ the subspace of X generated by b.

(i) $w_0 \in W$ is called a b-best approximation of $x \in X$ in W, if

$$\|x - w_0, b\| = \inf\{\|x - w, b\| : \quad w \in W\}.$$

The set of all b-best approximations of x in W is denoted by $P_W^b(x)$.

(ii) W is called b-proximinal if for every $x \in X \backslash (\overline{W} \backslash W)$, there exists $w_0 \in W$ such that

$$\|x - w_0, b\| = \inf\{\|x - w, b\| : \quad w \in W\},$$

where $\overline{W}$ denotes the closure of W in the seminormed space (X, p_b).

(iii) W is called 1-type proximinal if W is b-proximinal for all $0 \neq b \in X$, that is for every $0 \neq b \in X$ and every $x \in X \backslash (\overline{W} + < b >)$, where $\overline{W}$ denotes the closure of W in the seminormed space (X, p_b), there exists $w_0 \in W$ such that

$$\|x - w_0, b\| = \inf\{\|x - w, b\| : \quad w \in W\}.$$

Notices. (a) If $b \in W$ and dim $W = 1$, then $P_W^b(x) = W$ for all $x \in X$.

(b) If $x \in W$, then $P_W^b(x) = x + < b >$.

(c) If x is not in $W + < b >$, then $P_W^b(x) = \emptyset$ for all $x \in \overline{W} \backslash W$, where $\overline{W}$ denotes the closure of W in the seminormed space (X, p_b).

(d) If $x \in W + < b >$ and $x = w_1 + \lambda_1 b$, then $P_W^b(x) = \{w_1\}$.

(e) If $x \in < b >$, then $P_W^b(x) = W \bigcap < b >$.

(f) $P_W^b(x)$ is closed and convex in (X, p_b), for all $x \in X$.

Finally note that, W is b-proximinal if and only if $P_W^b(x) \neq \emptyset$ for all $x \in X \backslash (\overline{W} + < b >)$.

Theorem 2.2. *Let $(X, \|.,.\|)$ be a 2-normed space, W a subspace of X, $0 \neq b \in X$, $w_0 \in W$ and $< b >$ the subspace of X generated by b. Suppose that $x_0 \in X$ is such that*

$$\delta = \inf\{\|x_0 - w, b\| : w \in W\} > 0.$$

Then, $w_0 \in P_W^b(x_0)$ if and only if there exists $f \in X_b^\sharp$ such that $f|_{W \times <b>} = 0$, $f(x_0 - w_0, b) = \|x_0 - w_0, b\|$ and $\|f\| = 1$.

Proof. First suppose that there exists $f \in X_b^\sharp$ such that $f|_{W \times <b>} = 0$, $f(x_0 - w_0, b) = \|x_0 - w_0, b\|$ and $\|f\| = 1$. Then,

$$\begin{aligned} \|x_0 - w_0, b\| &= f(x_0 - w_0, b) = f(x_0, b) = f(x_0 - w, b) \\ &\leq \|f\| \; \|x_0 - w, b\| = \|x_0 - w, b\|, \end{aligned}$$

for all $w \in W$. Hence, $w_0 \in P_W^b(x_0)$. Conversely, let $w_0 \in P_W^b(x_0)$. Then,

$$\delta = \|x_0 - w_0, b\| = \inf\{\|x_0 - w, b\| : w \in W\} > 0.$$

By Proposition 1.1, there exists $g \in X_b^\sharp$ such that $g|_{W\times<b>} = 0$, $g(x_0, b) = 1$ and $\|f\| = \frac{1}{\delta}$. Now for $f = \delta g$ we have, $f|_{W\times<b>} = 0$, $f(x_0 - w_0, b) = \|x_0 - w_0, b\|$ and $\|f\| = 1$.

Corollary 2.3. *Let $(X, \|.,.\|)$ be a 2-normed space, W a subspace of X, $0 \neq b \in X$ and $< b >$ the subspace of X generated by b. Then, W is a b-proximinal subspace of X if and only if for every $x \in X\backslash(\overline{W}+ < b >)$, where $\overline{W}$ denotes the closure of W in the seminormed space (X, p_b), there exist $w_0 \in W$ and $f \in X_b^\sharp$ such that $f|_{W\times<b>} = 0$, $f(x - w_0, b) = \|x - w_0, b\|$ and $\|f\| = 1$.*

Lemma 2.4. *Let $(X, \|.,.\|)$ be a 2-normed space, W a subspace of X, $0 \neq b \in X$, $x \in X\backslash(\overline{W}+ < b >)$, where $\overline{W}$ denotes the closure of W in the seminormed space (X, p_b), and $< b >$ the subspace of X generated by b. Then, $M \subseteq P_W^b(x)$ if and only if there exists $f \in X_b^\sharp$ such that $f|_{W\times<b>} = 0$, $\|f\| = 1$ and $f(x_0 - m, b) = \|x_0 - m, b\|$ for all $m \in M$.*

Proof. Let $M \subseteq P_W^b(x)$ and fix $m_1 \in M$. By Theorem 2.2, there exists $f \in X_b^\sharp$ such that $f|_{W\times<b>} = 0$, $f(x_0 - m_1, b) = \|x_0 - m_1, b\|$ and $\|f\| = 1$. But, $f(x_0 - m, b) = \|x_0 - m_1, b\| = \|x_0 - m, b\|$, for all $m \in M$.

Example 2.5. Let $X = \mathbb{R}^2$, the plane, $W = \{(x, y) \in X : x = y\}$ and $\|(x_1, x_2), (y_1, y_2)\| = |x_1y_2 - x_2y_1|$ for all $(x_1, x_2), (y_1, y_2) \in X$. Let $b = (b_1, b_2) \in X\backslash\{(0, 0)\}$. Then, $\|.,.\|$ is a 2-norm on X, W is a b-proximinal subspace of X, $P_W^b(x) = W$ if $b \in W$ and $P_W^b(x) = \left\{(\frac{x_2b_1 - x_1b_2}{b_2 - b_1}, \frac{x_2b_1 - x_1b_2}{b_2 - b_1})\right\}$ if b is not in W.

Example 2.6. Let $X = \mathbb{R}^3$, $W = \{(0, x, 0) : x \in \mathbb{R}\}$ and

$$\|(x_1, x_2, x_3), (y_1, y_2, y_3)\| = |x_1y_2 - x_2y_1| + |x_2y_3 - x_3y_2| + |x_1y_3 - x_3y_1|$$

for all $(x_1, x_2, x_3), (y_1, y_2, y_3) \in X$. Let $b = (b_1, b_2, b_3) \in X\backslash W$. Then, $\|.,.\|$ is a 2-norm on X and W is a b-proximinal subspace of X. In fact, $P_W^b(x) = \left\{(0, \frac{x_2b_3 - x_3b_2}{b_3}, 0)\right\}$ if $b_1 = 0$ and $b_3 \neq 0$, $P_W^b(x) = \left\{(0, \frac{x_2b_1 - x_1b_2}{b_1}, 0)\right\}$ if $b_1 \neq 0$ and $b_3 = 0$, $P_W^b(x) = \left\{(0, \frac{x_2b_1 - x_1b_2}{b_1}, 0)\right\}$ if $b_1 \neq 0$, $b_3 \neq 0$ and $|b_1b_3x_3 + b_3^2x_1| \leq |b_1b_3x_1 - b_1^2x_3|$, and finally $P_W^b(x) = \left\{(0, \frac{x_2b_3 - x_3b_2}{b_3}, 0)\right\}$ if $b_1 \neq 0$, $b_3 \neq 0$ and $|b_1b_3x_3 + b_3^2x_1| \geq |b_1b_3x_1 - b_1^2x_3|$. Since $\dim W = 1$, $P_W^b(x) = W$ if $0 \neq b \in W$.

Theorem 2.7. *Let $(X, \|.,.\|)$ be a 2-normed space and W a subspace of X. Then, W is a 1-type proximinal subspace of X if and only if for every $0 \neq b \in X$ and every $x \in X\backslash(\overline{W}+ < b >)$, where $\overline{W}$ denotes the closure of W in the seminormed space (X, p_b), there exist $w_0 \in W$ and $f_b \in X_b^\sharp$ such that $f_b|_{W\times<b>} = 0$, $f_b(x - w_0, b) = \|x - w_0, b\|$ and $\|f_b\| = 1$.*

Proof. Note that $\delta = \inf\{\|x_0 - w, b\| : w \in W\} > 0$, whenever $x \in X\backslash(\overline{W}+ < b >)$. Hence, it is an immediate consequence of Theorem 2.2.

Definition 2.8. Let $(X, \|.,.\|)$ be a 2-normed space, E a subset of X and $0 \neq b \in X$. An element $x \in X$ is said to be b-orthogonal to an element $y \in X$, and we write $x \perp_b y$, if $\|x + \lambda y, b\| \geq \|x, b\|$ for every scalar λ. Also, an element $x \in X$ is said to be orthogonal to E, and we write $x \perp_b E$, if $x \perp_b y$ for all $y \in E$.

Lemma. 2.9. *Let $(X, \|.,.\|)$ be a 2-normed space, W a subspace of X, and $b \in X$. Then, $w_0 \in P_W^b(x)$ if and only if $x - w_0 \perp_b W$.*

Proof. Note that, $\|x - w_0 + \lambda w, b\| \geq \|x - w_0, b\|$, for all $w \in W$ and every scalar λ if and only if $w_0 \in P_W^b(x)$.

Corollary 2.10. *Let $(X, \|.,.\|)$ be a 2-normed space and W a subspace of X. Then, W is a 1-type proximinal subspace of X if and only if for every $0 \neq b \in X$ and every $x \in X \backslash (\overline{W} + < b >)$, where $\overline{W}$ denotes the closure of W in the seminormed space (X, p_b), there exists $w_0 \in W$ such that $x - w_0 \perp_b W$.*

Lemma 2.11. *Let $(X, \|.,.\|)$ be a 2-normed space, W a subspace of X, $0 \neq b \in X$ and $\overline{W}$ denotes the closure of W in the seminormed space (X, p_b). Then the following are equivalent:*

(a) *W is b-proximinal.*

(b) *$W + < b >$ is closed in (X, p_b) and for every $x \in X \backslash (\overline{W} + < b >)$, there exists an element $0 \neq y_0 \in W_x = W \bigoplus < x >$ such that $y_0 \perp_b W$.*

(c) *$W + < b >$ is closed in (X, p_b) and for every $x \in X \backslash (\overline{W} + < b >)$, every $\varphi \in (W_x)_b^\sharp$ with the property*

$$W = \{y \in W_x : \varphi(y) = 0\}$$

has at least one maximal element, that is, $z \in W_x \backslash \{0\}$ such that $\varphi(z, b) = \|\varphi\| \|z, b\|$.

Proof. (a)⇒(b). Let $\{g_n + \lambda_n b\}_{n \geq 1}$ be a sequence in $W + < b >$ and $g_n + \lambda_n b \longrightarrow x_0$ for some $x_0 \in X$. Choose $g_0 \in P_W^b(x_0)$. Then, $\|x_0 - g_0, b\| \leq \|x_0 - g_n, b\| = \|x_0 - g_n - \lambda_n b, b\| \longrightarrow 0$. Hence, $x_0 \in W + < b >$. Now, for every $x \in X \backslash (\overline{W} + < b >)$, take $g_0 \in P_W^b(x)$. Then, $0 \neq y_0 = x - g_0 \in W_x$ and $y_0 \perp_b W$.

(b) ⇒(c). For every $x \in X \backslash (\overline{W} + < b >)$, there exists an element $0 \neq y_0 \in W_x$ such that $y_0 \perp_b W$. Then, $0 \in P_W^b(y_0)$. Thus by Theorem 2.2, there exists $\psi \in (W_x)_b^\sharp$ such that $\|\psi\| = 1$, $\psi|_{W \times <b>} = 0$, $\psi(y_0, b) = \|y_0, b\|$. Let now $\varphi \in (W_x)_b^\sharp \backslash \{0\}$ be arbitrary with the property $W = \{y \in W_x : \varphi(y) = 0\}$ has at least one maximal element. Then, there exists a non-zero scalar λ such that $\varphi = \lambda \psi$. Hence,

$$\varphi(\bar{\lambda} y_0, b) = (\lambda \psi)(\bar{\lambda} y_0, b) = |\lambda|^2 \psi(y_0, b) = |\lambda|^2 \|y_0, b\| = \|\lambda \psi\| \|\bar{\lambda} y_0, b\| = \|\varphi\| \|\bar{\lambda} y_0, b\|.$$

Therefore, $\bar{\lambda} y_0$ is a maximal element of φ.

(c)⇒(a). For every $x \in X \backslash (\overline{W}+ < b >)$, choose $\varphi \in (W_x)^\sharp_b$ such that $W = \{y \in W_x : \varphi(y) = 0\}$ and $0 \neq z \in W_x$ such that $\varphi(z,b) = \|\varphi\|\|z,b\|$. Put $\psi = \frac{\varphi}{\|\varphi\|}$. Then, $\|\psi\| = 1$, $\psi|_{W\times<b>} = 0$, $\psi(z,b) = \|z,b\|$. By Theorem 2.2, $0 \in P^b_{W_x}(z)$. Now, put

$$w_0 = x - \frac{\varphi(x,b)}{\varphi(z,b)}z.$$

Note that $w_0 \in W$, because $\varphi(w_0,b) = 0$. Also, $\frac{\varphi(z,b)}{\varphi(x,b)}(w - w_0) \in W$ for all $w \in W$. On the other hand, $\|x - w_0, b\| = |\frac{\varphi(x,b)}{\varphi(z,b)}|\|z,b\| \leq |\frac{\varphi(x,b)}{\varphi(z,b)}|\|z - \frac{\varphi(z,b)}{\varphi(x,b)}(w - w_0), b\| = \|x - w, b\|$. Therefore, $w_0 \in P^b_W(x)$.

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