Mathematics Education Research Journal 2009, Vol. 21, No. 2, 76-95

Developing Mathematics Understanding and Abstraction: The case of Equivalence in the Elementary Years

Elizabeth Warren
Australian Catholic University

Tom J. Cooper
Queensland University of Technology

Generalising arithmetic structures is seen as a key to developing algebraic understanding. Many adolescent students begin secondary school with a poor understanding of the structure of arithmetic. This paper presents a theory for a teaching/learning trajectory designed to build mathematical understanding and abstraction in the elementary school context. The particular focus is on the use of models and representations to construct an understanding of equivalence. The results of a longitudinal intervention study with five elementary schools, following 220 students as they progressed from Year 2 to Year 6, informed the development of this theory. Data was gathered from multiple sources including interviews, videos of classroom teaching, and pre- and post- tests. Data reduction resulted in the development of nine conjectures representing a growth in integration of models and representations. These conjectures formed the basis of the theory.

From 2002 to 2006 we conducted a longitudinal intervention study, the *Early Algebraic Thinking Project* (EATP). This project followed the development of algebraic thinking of students in five elementary schools in Queensland, Australia, as they progressed from Years 2 to 6 (6 to 11 years old). The framework for the intervention was based on our knowledge and beliefs at the time. These included a structural view of mathematics (Sfard, 1991) and a cognitive perspective on learning (English & Halford, 1995; Hiebert & Carpenter, 1992). We also took into account students' difficulties with variables and the cognitive gap between arithmetic and algebra (Linchevski & Herscovics, 1996; Usiskin, 1988). Similar to the theoretical views of Dienes (1961), Skemp (1978) and Usiskin (1988), we viewed algebra as an abstract system in which interactions reflected the structure of arithmetic and where its primary importance lay in the way it represented these structures (e.g., balance principle, field and equivalence class properties) and generalised arithmetic. We did not consider aspects such as factorisation and simplification as the essence of algebra, but rather viewed algebra as a system characterised by indeterminacy of objects, an analytic nature of thinking and symbolic ways of designating objects (Radford, 2006).

Reflecting the foci of Scandura (1971), we considered algebra as consisting of two core approaches; relationships and change. In the first, operations can be either relational or static (e.g., $3 + 4 = 7$, three and four is seven) with equals as equivalence or "same value as". In the second, operations can be transformational or dynamic (e.g., $3 + 4 \Rightarrow 7$), three changes by adding four to give seven) with equals as a two way mapping from one side of an equation to the other (Linchevski, 1995). This perspective mirrors two of Kaput's (2006) three core strands: algebra as a study of structure and systems abstracted from computations and relations; and algebra as a study of functions, relations and joint variation. We utilised Malara and Navarra's (2003) distinction of arithmetic thinking focusing on product, and algebraic thinking focusing on process, in order to move from one to the other in classroom practice as the need arose. At times we needed arithmetic to support algebraic thinking (e.g., generalising the compensation principle of arithmetic) while at others we needed algebraic thinking to support arithmetic (e.g., adding 3 to 2 is the same process as adding 3 to 82, 3 to 1012, 30 to 20, and so on). We concurred with Mason's (2006) claim that the power of mathematics lies in the intertwining of algebraic and arithmetic thinking, each enhancing the other as students become numerate.

This paper uses one aspect of the development of the EATP students' abilities to generalise arithmetic structure to think algebraically, as indicated by their comprehension of equivalence of expressions and equations. The results from this intervention illustrate a theory of how structured sequences of models and representations effectively assist students to construct mathematical understanding and abstraction. In part, the paper is a reanalysis of the teaching sequences described in more detail in Warren (2008).

This paper has three sections. Firstly, it describes the basis of the 2002 - 2006 EATP intervention in terms of theoretical perspectives, teaching approaches, methodology, and overall findings. Secondly, it analyses the progress of the intervention to illustrate major conjectures with regard to using models and representations. And thirdly, it reflects on the conjectures in the light of new literature to delineate a theory that emerges from the EATP teaching.

Background

EATP: Theoretical Position on Models and Representations

At the time, our teaching of mathematical structures was based on the general consensus that mathematical ideas are presented externally (as concrete materials, pictures/diagrams, spoken words, and written symbols) and comprehended internally (in mental models or cognitive representations). From this perspective, mathematical understanding is exhibited by the number and strength of connections in the students' internal network of representations (Hiebert & Carpenter, 1992), and the development of an understanding of mathematical structure involves determining what is preserved and what is lost between specific structures which have some isomorphism (Gentner & Markman, 1994; Halford 1993).

Therefore, to access algebraic thinking, we constructed our intervention around models and representations. Models are ways of thinking about abstract concepts (e.g., balance for equivalence) and representations are various forms of the models (e.g., physical balances, balance diagrams, balance language, equivalence as balance). Given the paucity of literature concerning the development of algebraic thinking at the elementary level, Bruner's theory (1966) was utilised to assist us in selecting representations for the intervention lessons. Our selection broadly followed the enactive to iconic to symbolic sequence unless another imperative intervened. We considered mathematical development as cumulative rather than replacement and thus integrated various representations from different levels (e.g., an iconic picture of balance and enacting the number line). Based on our belief that no single model or system of representations would provide all of the insights, we used comparison of and transition between models and representations to support the emergence of algebraic thinking (e.g., using the balance and number-line models in unison). In this process we were influenced by three factors. The first was the four-step sequence proposed by Dreyfus (1991): (1) one representation, (2) more than one representation in parallel, (3) linking parallel representations, and (4) integrating representations. Second was Duval's (1999) argument that mathematics comprehension results from the coordination of at least two representational forms or registers: the multifunctional registers of natural language and figures/diagrams, and the mono-functional registers of notation systems (symbols) and graphs. Third was Duval's contention that learning involves moving from treatments to conversions to the coordination of registers. From our perspective, representations assist us in arriving at a mathematical certainty concerning the situation we are investigating. As Smith (2006) states, the representation becomes part of the knowledge of the learner; it is an integral component of the objectification process.

Models, particularly in their physical or concrete representational form, are endowed with two fundamental components, namely, translation and abstraction (Filloy & Sutherland, 1996). Translation encompasses moving from the state of things at a concrete level to the state of things at a more abstract level, with the model acting as an analogue for the more abstract. Abstraction is believed to begin with exploration and the use of processes or operations performed on lower-level mathematical constructs (English & Sharry, 1996; Sfard, 1991). However, as Filloy and Sutherland (1996) argue, models often hide what is meant to be taught and present problems when abstraction from the model is left to the pupil. Thus teacher intervention is a necessity if the development of detachment from the model to construction of the new abstract notion is to ensue. As we implemented the models and representations, we engaged in classroom inquiry-based discourse with the young students and continually explored new signs that would assist the students to extract the essence of the mathematics embedded in the exploration. We used Radford's (2003) notion of semiotic nodes. Gestures and language were seen as essential to this exploration as they revealed subtle shades of meaning that arose from the students' thinking (Tall, 2004). Thus, EATP was based on a socio-constructivist theory of learning, inquiry-based discourse and the simultaneous use of multi-representations to build new knowledge (Warren & Cooper, 2008).

Teaching Equivalence and Equations in the EATP

Past research has provided evidence that young students possess a narrow and restricted knowledge of the equals sign. They persistently interpret it as either a *syntactic indicator* (i.e., a symbol indicating where the answer should be written) or an operator sign (i.e., a stimulus to action or "to do something") (Behr, Erlwanger & Nicols, 1980; Carpenter, Franke, & Levi, 2003; Saenz-Ludlow & Walgamuth, 1998; Warren & Cooper, 2005). Instead it should be interpreted as quantitative "sameness" (i.e., both sides of an equation are the same and information can be obtained from either direction in a symmetrical fashion) (Kieran & Chalouh, 1992). This misunderstanding of the equals sign leads to many students believing that 2 + 3 = 5 + 2 not 3 + 2 (Saenz-Ludlow & Walgamuth, 1998; Warren & Cooper, 2005). This incorrect understanding of the equals sign appears to continue into secondary and tertiary education (Baroody & Ginsburg, 1983; Steinberg, Sleeman & Ktorza, 1991), affecting mathematics learning at these levels.

The balance and number-line models were chosen for the equivalence and equation component of EATP. Both models are based on measurement ideas: the balance modelling mass and the number line modelling length. The balance model predominated initially in the instructional sequences, with physical balance scales representing equivalence and weights representing numbers and operations. This model later progressed to diagrams of balances, with balance representing equivalence and numbers and operations representing themselves. The number-line model was used to demonstrate the identity and inverse principle for expressions. A typical activity involved students starting at a random point on the number line, walking forwards along the number line a certain distance to represent addition (e.g., 3 steps for adding 3), and walking back the same number of steps to reverse addition, thus returning to their starting position.

All models have advantages and limitations. While past research has indicated that the balance model is limited by its inability to model subtraction equations or unknowns as negative quantities (Aczel, 1998), we aspired to ameliorate this by having a diagram represent a "mathematical balance" which could include all operations and numbers. The advantage of the balance model is that (a) it considers both the right hand and left hand sides of equations, (b) it is not directional in any way (Pirie & Martin, 1997), and (c) it copes with the need to attend to the equation as an entity rather than an instruction to achieve a result. The advantage of the number line model is that it allows the modelling of both addition and subtraction situations. Its limitation lies in its simultaneous representation of unknowns as unknown lengths and numbers as known lengths, a visual that incorrectly encourages students to find the length of a unit from the known and apply this thinking to solve the unknown.

The language utilised during the teaching phase reflected the balance model. Expressions used in teaching included: "Equal is balanced, having the same value on each side of the balance scale." "Unequal is unbalanced, having different values on each side of the balance scale." The symbols used were '=' and '≠'. Equations were represented horizontally, often with more than one value following the equal sign.

Methodology

The methodology adopted for EATP was longitudinal and mixed-method using a design research approach, namely a sequence of teaching experiments that followed a cohort of students over a five-year period (Year

2 to Year 6). In line with this approach, during and between lessons hypotheses were conceived "on the fly" (Steffe & Thompson, 2000). Modifications in the design were responsive to observed actions and understanding of the teacher/researcher and the students. For example, many of the instructional tasks were generated prior to the teaching phase. During the lessons tasks were modified according to classroom discourse and interactions. New representations were introduced in order to challenge the students' thinking and to encourage them to justify their responses.

EATP was based on a re-conceptualisation of content and pedagogy for algebra in the elementary school. In particular, for this paper, it sought to identify the fundamental cognitive steps crucial for an understanding of equivalence and expressions.

Participants

The participants were a cohort of students and their teachers from five inner city, middle class Queensland schools. During the study, the cohort of students progressed from Year 2 to Year 6. In total, 220 - 270 students and 40 teachers participated in the study. All schools were following the Patterns and Algebra strand from the new Queensland Years 1-10 Mathematics Syllabus (Queensland Study Authority, 2004).

Procedure

All lessons were taught by one of the researchers. (For a description of the lesson components, please see Warren, 2008.) Although the teachers were well credentialised (all had 4-year training, in line with Queensland policy), the mathematics component of their training was limited and, like most elementary teachers in Queensland, they were not confident in teaching mathematics (Nisbet & Warren, 2000). In addition to this limitation, algebraic thinking is a new content area in the elementary classroom, requiring thinking and pedagogy that has not previously been explored in practice. Participating teachers in the research were accordingly unsure as to how to conduct lessons focusing on this new content area.

Data were collected from multiple sources, including videos of classrooms during the teaching phase. All lessons were videotaped using two cameras, one fixed on the teacher and the class as a whole and the other moving around the classroom, focusing on students' activity of interest. Data also included interviews with teachers and a randomly selected group of students, pre- and post-tests of algebraic thinking, field notes written by observing researchers, and artefacts (lesson plans, examples of students work). Detailed description of the EATP teaching utilised across Years 2 to 6 to develop algebraic thinking with regard to equivalence and equations is reported in Warren (2008).

Results

As described in Warren (2008), the EATP findings on equivalence and equations indicated that: (i) early and middle years students can learn to understand the powerful mathematical structures if instruction is appropriate; (ii) connections between representations, conversions using different representations, and flexible movement between representations at opportune times enhance learning; (iii) a teaching focus on structure is highly effective for achieving mathematical goals. Five key aspects of equivalence and equation were highlighted: (i) equations as equivalence, (ii) the balance principle, (iii) the sign systems for unknowns (and variables), (iv) identity and inverse (for all operations), and (v) finding solutions and generalisations about real world problems involving more than one unknown. Both younger and older students were found to be capable of engaging in discussions involving simultaneous equations and principles associated with the equivalence class and field structures.

EATP assisted the students to gain a broader understanding of equality and arithmetic and showed that the intertwining of arithmetic thinking and algebraic thinking (as defined by Malara and Navarra) certainly had "pay offs" for both (Mason, 2006), with the students capable of searching for generalisations in computational contexts. The balance and number-line models were effective in the way sequences of representations were used to facilitate language and symbols. Students could act out with materials such as beam balances, cloth bags with objects and their accompanying pictures, walking games, paper strips and large number lines. Learning was also enhanced by creative representation-worksheet partnerships. These worksheets consisted of pictures and directions that reinforced understanding and highlighted principles. However, fundamental to the learning process was the role of the teacher assisting in the appropriate detachment and abstraction from the model to objectifying the mathematics inherent in the representation (Filloy & Sutherland, 1996). In particular, the results indicated that very young students can represent equivalence in equation form in un-numbered and numbered situations and they can generalise the equivalent class principles for equivalence. It was also evidenced that they are capable of generalising the balance principle for simple equations. The results also indicated that older students can represent equivalence with unknowns in equations form, generalise the balance principle for all operations, and use the balance principle to solve unknowns in linear equations, including equations with unknowns on both sides (see, Warren, 2008).

Conjectures and Findings

For the purpose of this paper, we re-analyse these descriptions to identify conjectures with respect to the use of models and representations.

Conjecture 1: Effective initial models/representations show underlying structure.

EATP began its development of equivalence and equations in Year 2 with unmeasured or unnumbered models (in line with Davydov, 1975; Dougherty, & Zilliox, 2003), using a physical balance and a collection of groceries. Students were asked to identify what was the same and what was different, a language building activity: *They weigh the same, they are the same shape, they are the same colour.* Students were then asked to identify two objects that had the same (equal) mass and different (not equal) mass. They checked their guesses by placing the objects on either side of a balance scale and verbally shared: *the mass of the pasta plus the rice is the same as (equal to) the mass of the salt plus the baked beans, and the mass of the beans plus the flour is different from (not equal to) the mass of the sugar and the pasta.* Cards marked with "=" and "≠" were placed on the balance scales so that equations could be easily read. The students wrote their findings as simple equations (see Figure 1).

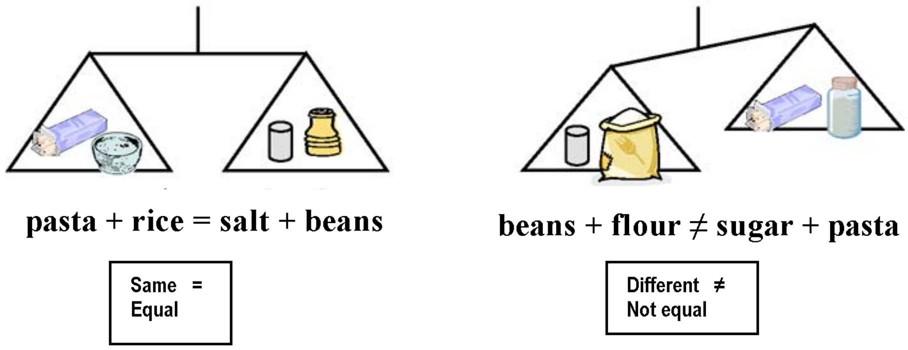


Figure 1: Mapping real world weight relationships

While these equations were not correct in terms of the mathematical symbol system, as it is a relationship between masses and not grocery types, they did serve as an effective model for revealing the underlying structure of equations. This model also supported informal discussions concerning the equivalent class properties. In particular, the symmetric property, for example, pasta + beans = sugar means that sugar = pasta + beans, was easily shown by rotating the physical balance through 180 degrees.

Conjecture 2: Effective sequences of models/representations must be nested.

EATP progressed to numbered situations in Year 2 by replacing the groceries with blue and green glass seashells all of the same mass (see Figure 2). As Warren (2008) describes, students' abilities to discuss and comprehend the equivalence class properties diminished because they believed, from previous arithmetic, that they should place the answer to an addition after the equals sign and "join" materials when adding (i.e., they saw equals as a *syntactic indicator* and an *operator sign*) in line with Saenz-Ludlow & Walgamuth (1998), Behr et al. (1980), Carpenter et al. (2003), and Warren (2006). For example, one student saw the result of 2 + 2 = 2 + 2 as 8 (by grouping all of the materials), while another student argued that 3 + 4 ≠ 6 + 1 because 3 + 4 ≠ 6.

In logical terms, equations of the form 3 + 4 = 7 are a subset (or are nested within) equations of the form 3 + 4 = 6 + 1, which in turn are a subset of the more general "pasta + rice = sugar + soap". One difficulty experienced here when working within regular classrooms, was that the researchers' activity with unnumbered and numbered equations *followed* the teachers' activity with arithmetic. In other words, the particular *preceded* the general. When the particular models and representations did not fit within the superstructure of the general, conflict developed between particular and generalised schemata.

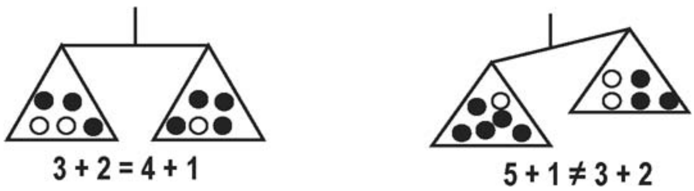
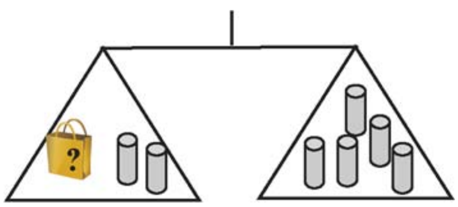


Figure 2: Mapping real world numerical relationships

Conjecture 3: Effective models easily extend to new components and expand to new applications.

In Year 3, numbered activities continued with physical balances, but with the shells being replaced with 125gram cans of baked beans and spaghetti. The aim was to introduce the balance principle and the unknown into the discussion about addition situations and to investigate solving for the unknown. For the balance principle, the focus of the lessons was on "keeping the scales balanced" and the balance generality: *if you add or subtract any number from one side of the balance scale, you need to add or subtract the same number from the other side to keep the scales balanced*. For the unknown, cans were secretly placed inside a cloth bag printed with question marks to enable equations with unknowns to be modelled with the balance. The balance principle was used with the bags to solve linear equations. All students found it easy to solve equations such as ? + 2 = 5 (see Figure 3).



? + 2 = 5

Figure 3: Mapping real world relationships to equations

A typical student discussion was as follows.

Oscar: If you take the bag from one side then you keep taking cans from the other until it is balanced. That is the unknown.

Jill: Take 2 cans from both sides so the unknown is 3.

Although students initially relied on number facts to solve for the unknown, most were able to justify results by using the balance principle (although some still wanted to add all of the numbers, resulting in ? = 7 in Figure 3). A few students could extend their knowledge and solve situations like ? + ? + 2 = ? + 5 even though solving this type of equation had not been taught. This problem solving indicated the ease in which the model and the physical representation could be extended to equations with unknowns (a new component) and expanded to solving these equations (a new application).

Conjecture 4: Effective sequences of models and representations move towards abstraction by retaining previous models' structures but allowing greater flexibility.

In the Year 4 EATP, the balance model representation changed from a physical balance to a movable dynamic diagram of a balance with magnetic numbers and magnetic shapes with question marks to represent the sides of the equations. This change was made to enable the model to include discussions about subtraction. The focus was on developing a new sort of balance, a "mathematics" balance, in which all operations were allowed (thus compensating for limitations in earlier representation). Figure 4 illustrates the new model. A typical student/teacher discourse was as follows.

Mat: You could add three to both sides.

T/R: How many would you have on this side (pointing to the LHS)?

Mat: Unknown.

T/R: How many on this side (pointing to the RHS)

Mat: 9.

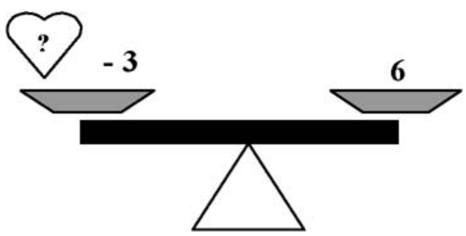


Figure 4. Materials used for subtraction situations

The diagram of the balance was constructed from magnetic strips to allow students to move the balance up and down as they added or subtracted numbers from each side. This maintained the metaphor of movement up and down indicating balance or imbalance (equal or not equal). It enabled the model to be extended beyond addition and allowed equations to be represented in a similar way to the physical model. This proved to be more flexible but was less real, a trade-off as we moved towards abstraction.

Conjecture 5: Use of multiple models enables complex procedures to be facilitated and student difficulties to be overcome.

Solution of linear equations using the balance principle is a complex activity with three difficulties. It requires students (i) to see the equals sign as a balance (and not to close prematurely); (ii) to understand inverse of operations in order to isolate the ?; and (iii) to be able to understand that balance and inverse require a *metaphor of opposites*. It encompasses what EATP calls a *compound difficulty* (Cooper & Warren, 2008). Using the balance strategy to solve equations such as ? + 8 = 12 requires the inverse operation (subtracting 8) to isolate the "?" and the same operation (subtracting 8 from both sides) to balance the equation. EATP tackled this *compound difficulty* by introducing new models.

First, to counter the propensity by some students to close on the equations (i.e., to add all the numbers), EATP undertook activities based on a same-value model to teach equals as equivalence. These activities assisted students to create addition and subtraction problems that involved comparing two situations to ascertain if they were equivalent (see Figure 5). The creation of the stories did not involve finding answers but rather focussed on whether the contexts were equivalent. This procedure appeared to assist students to move beyond the continual need to close addition and subtraction situations and to construct a new mental model that incorporated addition and subtraction situations that were equivalent. This finding supports our conjecture that using more than one model assists understanding of complex procedures. It also supports Conjecture 2 that models and representations are best developed in nested sequences. We conjecture that, for maximum effectiveness, these equivalence stories should precede the arithmetic stories such as "3 and 2 makes 5" (as equivalence stories include arithmetic stories).

Second, to build students' understanding of inverse, EATP students initially added and subtracted the same number of counters from groups of counters, again using a same-value/balance model. This procedure was not successful with some students and could not be applied to situations where the number being subtracted was larger than the number in the initial group. To continue building the idea of inverse, and to counter the compound difficulty, EATP introduced the students to a number-line model and the open number-line representation. This number-line representation allowed modeling of addition and subtraction (as moving backwards and forwards on the line) and unknowns (as a point identified by a question mark). This model was successful for most students (see Figure 6 for two typical responses). The first example illustrates the student's understanding of how subtracting 6 from and adding 6 to an unknown requires counting back and on 6 from an unknown point on the number line.

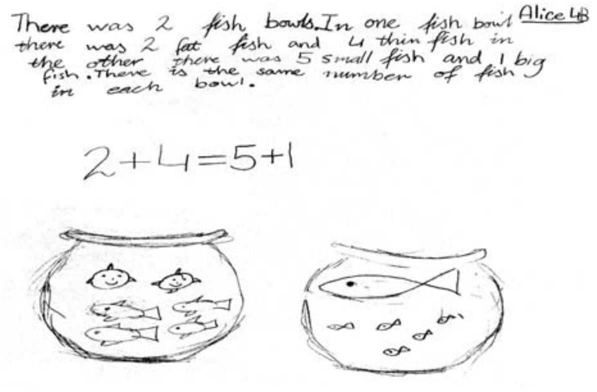


Figure 5: An equivalence story

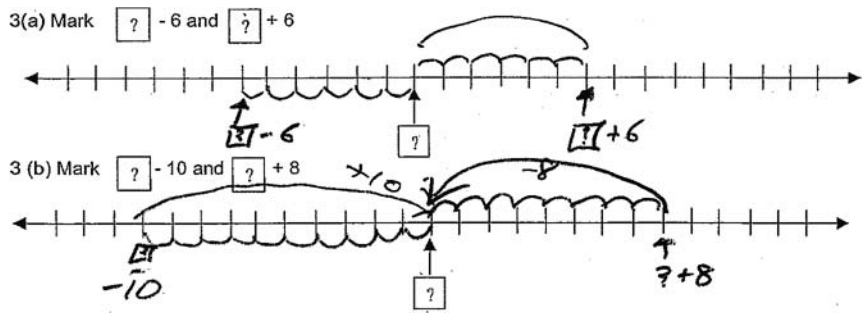


Figure 6. Examples of students' work with open number lines and unknowns

Conjecture 6: Complex situations and compound difficulties may require development of larger structures (superstructures) into which conflicts can be nested.

The compound difficulty, using the inverse operation for isolating the unknown and the same operation to solve for the unknown, was difficult for some students to understand. As a result, EATP introduced the term *expression* (not used in the Queensland syllabus), related it to equation (i.e., "an equation is an equivalence of expressions") and discussed how both expression and equation can be left unchanged. This placed the solving of linear equations for an unknown within a superstructure which included understanding the different ways in which equations and expressions remain unchanged (i.e., same operation to both sides of the equation and inverse operation for expressions).

This conjecture was also supported by activities looking at the co-variational relationship between two unknowns (Fuji, 2003). The equations ? + ??= 8 and ? + ? = 8 (where ? and ?? represent different unknowns) were modelled with a balance, using boxes labelled "?" and "??" for the unknowns and marbles to represent the numbers. Most students could begin to explore the co-variational relationships between two unknowns and also extend this thinking to situations involving rational numbers. However, some students could show numbers to solve ? = 4 and ?? = 4 but not ? + ?? = 8, while others believed that the unknown in ? + ?= 8 could be, for example, 3 and 5 as well as 4 and 4.

Conjecture 7: Often, for models to be effective, they need to be related to real world situations through language, acted out kinaesthetically and visualised.

The balance, set/same-value and number-line-models (respectively Figures 4, 5 and 6) were all initially developed in relation to real world situations, acted out kinaesthetically and then imagined visualised? Having the students act out being a balance with their bodies with plastic bags containing mass material hanging from their arms, was effective in the earlier years and in later years was utilised as an image we could draw upon (e.g., *What happens if we change something in our right hand? How do we get back into balance?*). Obviously, as described under Conjecture 5, the set/same value activity of Figure 5 was also in strong relation to real world situations and had an effect upon mental models.

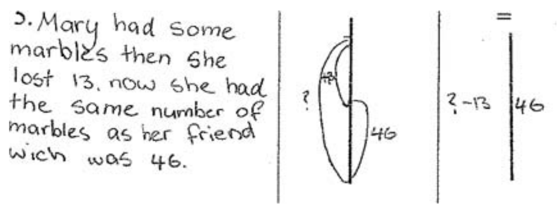
The number-line activity however was the most illuminating. Initially some students experienced difficulty with the Figure 6 activity. It was not until we drew an unmarked number line on the floor, had students physically stand on the 'unknown' spot, then walk three paces forward then three paces back that they began to realise that addition is the reverse/inverse of subtraction and subtraction is the reverse/inverse of addition. In these lessons, the EATP underestimated the importance of kinaesthetic movement and gestures in the development of mental models. As Radford (2006) succinctly claims, the perceptual act of noticing unfolds in a process mediated by multi-semiotic activity (e.g., spoken words, gestures, and drawings). The EATP experience showed that kinaesthetic movement is also an important element in the development of visual-mental models. In fact, at the completion of the lessons many students engaged in a process of walking up and down the number line until they were satisfied that they understood the relationship between addition and subtraction and the notion of identity.

Conjecture 8: Different models achieving the same outcome build deep understanding and abstraction.

In order to re-examine the compound difficulty students experienced with the balance and inverse principles, EATP decided to use the number-line model for solving linear equations for an unknown.

In the first stage, students were encouraged to write stories involving unknowns, model these stories with strips of paper, and write these stories as equations (see Figure 7 for two typical addition and subtraction stories and number-line illustrations). The success of this approach also supported Conjecture 7.

(a) An addition story



(b) A subtraction story

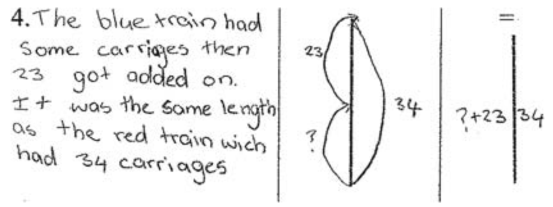


Figure 7. Two number line representations and accompanying stories for addition and subtraction.

Task (a) Solution (b)

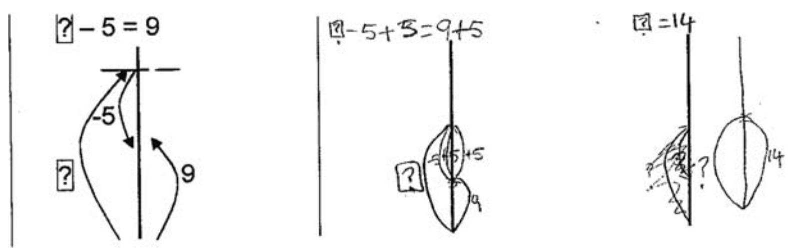


Figure 8. Use of number line to solve an equation; (a) a typical task, (b) a student's solution.

Most students could successfully model addition and subtraction problems using the number-line model. It appeared to integrate well with previous work with the balance model, facilitating correct application of inverse and balance to solve linear equations with unknowns (see Figure 8). It proved useful as an iconic representation for the 'melding' of the two conflicting principles (i.e., applying the opposite to isolate the unknown and the same to keep the equation balanced) suggesting that integrating models results in a greater understanding of concepts. It appeared to remove difficulties even when the balance model was reused in the next year.

Conjecture 9: Effective models have sequences of representations that move from the physical to the abstract.

In Year 6, EATP utilised the balance model and appropriate representations to solve complex linear equations where the unknown occurred on both sides (the didactic cut). For example, students were capable of modelling equations such as 5 x ? + 22 = 7 x ? – 2 using the magnetic diagram of a balance, shapes with question marks, and operations and numerals (see Figure 9) and able to solve for the unknown using inverse and balance principles.

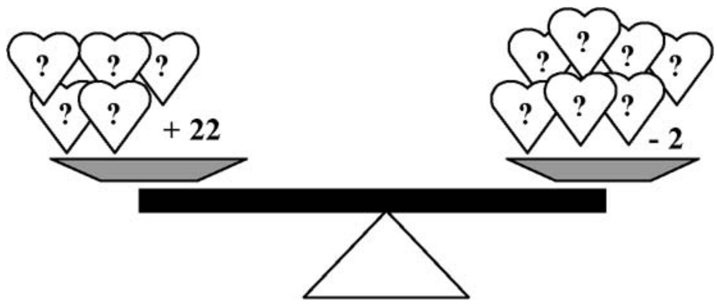


Figure 9. Representation of a complex linear equation

Although there was some difficulty in extending the balance principle to such complex equations, students successfully moved from this representation to a representation where 5 x ? + 22 and 7 x ? – 2 were placed on either side of a drawing of a balance, and finally to one where 5 x ? + 22 = 7 x ? – 2 was given in horizontal, symbolic equation form. This activity appeared to develop a mental model of the balance principle that assisted students in seeing arithmetic in a way that allowed them not only to solve algebraic equations but also to justify their actions. This result also supported Conjecture 3 that effective models allow extension and expansion (and compensate for limitations in earlier representations).

Emerging theory

The reanalysis of the descriptions of equivalence and equation activities from Warren (2008) has resulted in nine conjectures. Because of the novelty of the material being taught, there was little research to inform EATP's practice, and so a combination of what worked in other mathematical concepts plus analysis of the mathematical structures behind equivalence and the equations formed a backbone for the conjectures. Those that seemed to work appear in this paper and their implications for teaching and learning theory are discussed in the next section. Thus the following section is truly conjectural, and is designed to explore theoretical possibilities. However, prior to sharing our hypothesising, the relationship between learning and teaching and mathematics' structure and abstraction is explored.

Learning, Learning-Teaching, Abstraction, and Generalisation

The literature presents two differing perspectives on the ontology of student learning, namely the learning trajectory and the learning-teaching trajectory. While both perspectives have many commonalities, the main differences lie in their emphasis on the act of teaching in the learning process, and the prescriptiveness of the resultant curriculum. From the first perspective, learning consists of a series of natural developmental progressions identified in empirically-based models of children's thinking and learning (Clements, 2007). Teaching is secondary to the act of learning and consists of the implementation of instructional tasks designed to engender this development. The resultant curriculum consists of diagnostics tests, learning hierarchies and purposely-selected instructional tasks.

In contrast, the learning-teaching trajectory has three interwoven meanings each of equal importance: (1) a learning trajectory that gives an overview of the learning process of students; (2) a teaching trajectory that describes how teaching can most effectively connect with and stimulate the learning process; and (3) a subject matter outline, indicating which core elements of the mathematical curriculum should be taught (Van den Heuvel-Panhuizen, 2008). It is believed that the learning-teaching trajectory provides a mental education map that can help teachers make didactical decisions as they interact with students' learning and instructional tasks. It serves as a guide at the meta-level. The resultant curriculum tends to be more open and flexible, with teachers choosing and adapting activities in order to enhance student learning.

It is the second perspective that has greatest resonance with the research presented in this paper. EATP was designed to identify not only key transitions in student learning in the domain of early algebra, but also to identify particular teaching actions that support these transitions. As such, it was based on a belief that the act of teaching is as important as the act of learning, and that learning is not necessarily a step-by-step process progressing through hierarchical levels.

As argued in Cooper and Warren (2008), mathematical structure and abstraction is based on students' ability to generalise from particular examples to general rules and from real-world situations to abstract representations. This is particularly applicable to the balance principle. Initially the principle is discussed in terms of removing two cans from each side of a physical balance but eventually becomes a general rule encompassing all numbers and operations. For such generalisation EATP reflects English and Halford's (1995) *mapping instruction* approach which focuses on teaching to identify similarities between isomorphic procedures (e.g., what is the same in the processes for "34 –16" and "3 weeks 4 days subtract 1 week 6 days"). This approach is the basis behind integrating the different models and representations when solving linear equations with unknowns (Conjectures 5 and 8); the commonality between the models encompasses the kernel of the mental model that is the outcome of the teaching.

Modern research by Radford (2003, 2006) suggests that generalisation emerges from factual (gesture and rhythm driven) and contextual (language driven) activities as well as symbolic activities, requires students to perceive the particular and use this to conceive the general; and involves two components, grasping and expressing. This complexity is evident in Conjecture 7 where a return to real world and kinaesthetic activities were required for generalisations to be comprehended. Expressing generalisations tends to follow a sequence from quasi-variable in terms of a variety of numbers (Fuji & Stephenson, 2001) to language to symbols. This progression was evident within the sequence of activities across the five years, and in the comprehension of the balance and inverse principles within Years 3 and 4.

Theoretical Framework

The nine conjectures identified in the development of equivalence and equations knowledge reflected a growth in the abstraction of ideas and complexity of tasks that was inversely related to the reality (physicality) of representations. They also represented a growth in the integration of models and representations. From a reappraisal of their relationships, the following emerges as a basis of a theory for a teaching/learning trajectory designed to build abstraction.

Theory hypothesis 1: Translation to abstraction occurs not within a model or representation but across models and representations that follow a structured sequence. This hypothesis is a consequence of all conjectures. There appears to be no "magic bullet"; abstraction is built from model to model and representation to representation.

Theory hypothesis 2: Effective models and representations show the underlying structure of the mathematical ideas and easily extend to new components and expand to new applications. This hypothesis encompasses Conjectures 1 and 3. It begins the teaching trajectory by providing the following criteria for determining effective models: (i) strong isomorphism between the desired internal mental model and the initial external model that covers the important aspects of the mental model, (ii) lack of distractors that draw attention away from isomorphisms; and (iii) many options in terms of representations that enable the model to extend to new components (such as variables) and expand to new applications (such as finding solutions to problems). Both the balance and number-line models have these attributes; the number-line model is stronger in representing inverse relations; whilst the balance model provides a more powerful portrayal of equivalence.

Theory hypothesis 3: In an effective sequence, models and representations develop in three ways: (i) increased flexibility, following the general sequence concrete to dynamic diagram to static diagram to symbols; (ii) decreased overt structure, following the general sequence of structure in action, to structure alluded to in visuals, to structure visualized in the mind; (iii) increased coverage, where later representations compensate for limitations in earlier representations; and (iv) connectedness to reality, always relating the form of the representation to real world instances. This hypothesis encompasses Conjectures 4 and 7. The balance model is particularly powerful in terms of its sequence of increased flexibility as it moves from physical to diagrammatic representations.

Theory hypothesis 4: Sequencing should ensure consecutive steps are nested. This hypothesis encompasses Conjecture 2 and is particularly important. Difficulties and conflicts arise if later models and representations are not subsets of earlier ones. This problem was most clearly evidenced by the closure created by teachers giving prominence to arithmetic computation before equivalence was taught. This hypothesis is also important because it implies that the engagement with unnumbered situations before numbered enables students to effectively attend to mathematical structure, thus reinforcing Hypothesis 2.

Theory hypothesis 5: Complex procedures can be facilitated by integrating more than one model; however, such integrations can give rise to compound difficulties which require the development of superstructures. This hypothesis encompasses Conjectures 5, 6 and 8. It is most relevant to learners in their later years. It is best evidenced by the way balance and number line-models were used together at the point of solving linear equations with an unknown. The notion of superstructures is not well developed in the literature, especially with regard to integrating models to develop deep understanding of concepts.

Theory hypothesis 6: Abstraction is facilitated by comparing different representations of the same mental model to identify commonalities that encompass the kernel of the mental model. This hypothesis is an extension of Conjecture 9. It reflects the success of using the number line and balance models for the same purpose (solving the equation), particularly in terms of the extension to variables on both sides of the equation and simultaneous equations. It also implies that effective structured sequences of models and representations are dual, built around at least two models that act as a spine for the development of the mathematical idea.

Conclusions

The hypotheses described above offer promise as the beginning of a theory about the use of models and representations in learning-teaching trajectories for abstraction and generalisation. They are supported by our data showing the development of functional thinking and equivalence in EATP. The role of superstructures cannot be underestimated. These were particularly evident as we grappled with the students' compound difficulty. In the later years, EATP found it more effective to introduce functional thinking before equivalence and equations within each year. It appeared that function activity built a strong superstructure around the inverse and identity principles, which assisted in the solution of linear equations with one unknown and prevented conflict between inverse and balance and the development of compound difficulties in the solution process.

References

Aczel, J. (1998). Learning algebraic strategies using a computerised balance model. In

A. Oliver & K. Newstead (Eds.), *Proceedings of the 22nd conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp.1-16). Stellenbosch, South Africa: University of Stellenbosch.

Baroody, A. J., & Ginsburg, H. P. (1983). The effects of instruction on children's understanding of the equals sign. *Elementary School Journal, 84*, 199-212.

Behr, M., Erlwanger, S., & Nichols, E. (1980). How children view the equals sign. *Mathematics Teaching, 92*, 13-18.

Bruner J. S., (1966). *Towards a theory of instruction*. New York: Norton.

Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth NH: Heinemann.

Clements, D., (2007). Curriculum research: Towards a framework for a research based curriculum. *Journal for Research in Mathematics Education, 30*, 35-70.

Cooper, T. J., & Warren, E. (2008). The effect of different representations on years 3 to 5 students' ability to generalise. *Zentralblatt fur Didaktik der Mathematik, 40*(1), 23-37.

Davydov, V. V. (1975). The psychological characteristics of the "prenumeral" period of mathematics instruction. In L. P. Steffe (Ed.), *Children's capacity for learning mathematics* (Vol VII, pp. 109-205). Chicago: University of Chicago.

Dienes, Z. P. (1961). *On abstraction and generalisation*. Boston: Harvard University Press.

Dougherty, B., & Zilliox, J. (2003). Voyaging from theory and practice in teaching and learning: A view from Hawaii. In Pateman, N., Dougherty, B., & Zilliox, J. (Eds.), *Proceedings of the 27th conference of the International Group for the Psychology of Mathematics Education* (Vol.1, pp. 31-46). Hawaii: University of Hawaii.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.) *Advanced mathematical thinking* (pp.25-41). Dordtrecht, The Netherlands: Kluwer.

Duval, R. (1999). Representations, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st conference of the North American chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 3–26) Publisher.

English, L,, & Halford, G. (1995). *Mathematics education: Models and processes*. Hillsdale NJ: Lawrence Erlbaum.

English, L. D., & Sharry, P. (1996). Analogical reasoning and the development of algebraic abstraction. *Educational Studies in Mathematics, 30*, 135-157.

Filloy, E., & Sutherland, R. (1996). Designing curricula for teaching and learning algebra. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International Handbook of Mathematics Education* (Vol. 1, pp. 139-160). Dordrecht, The Netherlands: Kluwer.

Fuji, T. (2003). Probing students' understanding of variables through cognitive conflict: Is the concept of a variable so difficult for students to understand? In N. Pateman, G. Dougherty & J. Zilliox (Eds.), *Proceedings of the 27th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 47-65). Hawaii: University of Hawaii.

Fujii, T., & Stephens, M. (2001). Fostering understanding of algebraic generalisation through numerical expressions: The role of the quasi-variables. In H. Chick, K. Stacey, J. Vincent, & J.Vincent (Eds.), *The future of the teaching and learning of algebra. Proceedings of the 12th ICMI study conference* (Vol. 1, pp. 258-64). Melbourne:

Gentner, D., & Markman , A. B. (1994). Structural alignment in comparison: No difference without similarity. *Psychological Science, 5*(3), 152-158.

Halford, G. S. (1993). *Children's understanding: The development of mental models*. Hillsdale NJ: Lawrence Erlbaum.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 65-97). New York: MacMillan.

Kaput, J. (2006). What is algebra? What is algebraic reasoning? In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 5-17). Mahwah NJ: Lawrence Erlbaum.

Kieran, K., & Chalouh, L. (1992). Prealgebra: The transition from arithmetic to algebra. In T. D. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 179-198). New York: Macmillan.

Linchevski, L. (1995). Algebra with number and arithmetics with letters: A definition of pre-algebra. *Journal of Mathematical Behaviour*, *14*, 69-85.

Linchevski, L., & Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra: Operating on equations in the context of equations. *Educational Studies in Mathematics*, *30*, 36-65.

Malara, N., & Navarra, G. (2003). *ArAl Project: Arithmetic pathways towards favouring pre-algebraic thinking*. Bologana, Italy: Pitagora Editrice.

Mason, J. (2006). Making use of children's powers to produce algebraic thinking. In J.Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 57-94). Mahwah NJ: Lawrence Erlbaum.

Nisbet, S., & Warren, E. (2000). Primary school teachers' beliefs relating to teaching and assessing mathematics and factors that influence these beliefs. *Mathematics Teacher Education and Development*, *2*, 34-47.

Pirie, S. E. B., & Martin, L. (1997). The equation, the whole equation, and nothing but the equation! One approach to the teaching of linear equations. *Educational Studies in Mathematics*, *34*, 159-181.

Queensland Study Authority. (2004). *Mathematics Years 1-10 syllabus.* Brisbane: Author: Retrieved 10 May 2009 www.qsa.qld.edu.au

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalisation. *Mathematical Thinking and Learning*, *5*, 37-70.

Radford, L. (2006). Algebraic thinking and the generalisation of patterns: A semiotic perspective. Paper presented at the North America Chapter of 28th annual meeting of the International Group for the Psychology of Mathematics Education, Merida, Mexico.

Saenz-Ludlow, A., & Walgamuth, C. (1998) Third graders' interpretation of equality and the equal symbol. *Educational Studies in Mathematics*, *35*, 153-187.

Scandura, J. M. (1971). *Mathematics: Concrete behavioural foundations*. New York: Harper & Row.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.

Skemp, R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher*, *26*(3), 9-15.

Smith, E. (2006). Representational thinking as a framework for introducing functions in the elementary curriculum. In J. Kaput, D., Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 133-160). Mahwah NJ: Lawrence Erlbaum.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-306). Mahwah NJ: Lawrence Erlbaum.

Steinberg, R., Sleeman, D. & Ktorza, D. (1991). Algebra students' knowledge of equivalence of equations. *Journal for Research in Mathematics Education*, *22*, 112-121.

Tall. D. (2004). Thinking through the three worlds of mathematics. In M. Hoines & A. Fuglestad (Eds.), *Proceedings of the 28th conference of the International group for the Psychology of Mathematics Education*, (Vol. 4, pp. 281-288). Bergen, Norway: Bergen University College.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. Coxford

& A. P. Schulte (Eds.), *Ideas of algebra, K-12* (pp. 8-19). Reston: NCTM.

Van den Heuvel-Panhuizen. M. (2008). *Realistic Mathematics Education as work in progress.* Retrieved 7th May 2008 from http://www.fi.uu.nl/publicaties/literatuur/4966.pdf

Warren E. (2006). Learning comparative mathematical language in the elementary school: A longitudinal study. *Educational Studies in Mathematics, 62*, 169-189.

Warren, E., (2008, May). *Early algebraic thinking: The case of equivalence in an early algebraic context.* Keynote address to the conference on Future Curricular Trends in School Algebra and Geometry. The University of Chicago and The Field Museum, Chicago IL.

Warren, E., & Cooper T. (2005). Young children's ability to use the balance strategy to solve for unknowns. *Mathematics Education Research Journal, 17(1)*, 58-72.

Warren, E., & Cooper T. J. (2008). Generalising the pattern rule for visual growth patterns: Actions that support 8 year olds' thinking. *Educational Studies in Mathematics, 67*, 171-185.

Authors

Elizabeth Warren, Australian Catholic University, Brisbane, Queensland, Australia. <e.warren@acu.edu.au>

Tom Cooper, Queensland University of Technology, Brisbane, Queensland, Australia. <tom.cooper@qut.edu.au>