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Awareness of Pattern and Structure in Early Mathematical Development

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Recent educational research has turned increasing attention to the structural development of young students' mathematical thinking. Early algebra, multiplicative reasoning, and spatial structuring are three areas central to this research. There is increasing evidence that an awareness of mathematical structure is crucial to mathematical competence among young children. The purpose of this paper is to propose a new construct, Awareness of Mathematical Pattern and Structure (AMPS), which generalises across mathematical concepts, can be reliably measured, and is correlated with general mathematical understanding. We provide supporting evidence drawn from a study of 103 Grade 1 students.

Virtually all mathematics is based on pattern and structure. As Warren (2005) asserts, "The power of mathematics lies in relations and transformations which give rise to patterns and generalisations. Abstracting patterns is the basis of structural knowledge, the goal of mathematics learning" (p. 305).

There are many indications that an understanding of pattern and structure is important in early mathematics learning. For example, one of our earlier studies examined the structural characteristics of the representations of various numerical situations made by students from Grades 2-5 (Mulligan, Mitchelmore, Outhred, & Russell, 1997). Low achievers consistently produced poorly organised pictorial and iconic representations lacking in structure whereas high achievers used abstract notations with well-developed structures from the outset (Mulligan, 2002).

Other mathematics education research provides further evidence. Studies focused on imagery and arithmetic have indicated that students who recognise the structure of mathematical processes and representations acquire deep conceptual understanding (Pitta-Pantazzi, Gray & Christou, 2004; Gray, Pitta, & Tall, 2000; Thomas & Mulligan, 1995). Students with lower numerical achievement reported descriptive and idiosyncratic images; they focused on non-mathematical aspects and surface characteristics of visual cues. Visualisation skills, which frequently involve recognition of pattern and structure, are positively correlated with mathematical achievement (Arcavi, 2003; Booth & Thomas, 2000) and analogical reasoning (English, 2004). And psychological research has repeatedly shown that scores on mathematics achievement tests correlate moderately highly with scores on intelligence tests—which invariably include pattern recognition tasks.

Despite all this evidence, there have been remarkably few studies that have attempted to describe general characteristics of structural development in young children's mathematical understanding (Mulligan & Vergnaud, 2006). The purpose of this paper is to propose a new construct, *Awareness of Mathematical Pattern and Structure* (AMPS), which generalises across early mathematical concepts, can be reliably measured, and is correlated with mathematical understanding. It is our belief that a focus on AMPS could bring more coherence to our understanding of mathematical development.

Pattern and Structure in Early Childhood Mathematics

A mathematical *pattern* may be described as any predictable regularity, usually involving numerical, spatial or logical relationships. In early childhood, the patterns children experience include repeating patterns (e.g., ABABAB ...), spatial structural patterns (e.g., various geometrical shapes) and growing patterns (e.g., 2, 4, 6, 8, ...). Repeating patterns are particularly important, since they recur in measurement (which involves the iteration of identical spatial units) and multiplication (which involves the iteration of identical numerical units).

In every pattern, the various elements are organised in some regular fashion. For example, in the growing pattern of square numbers 0, 1, 4, 9, ..., the numbers increase by 1, 3, 5..., the sequence of odd numbers. In a circle, all the points on the circumference are the same distance from the centre. We define the way a mathematical pattern is organised as its *structure.* Mathematical structure is most often expressed in the form of a generalisation—a numerical, spatial or logical relationship which is always true in a certain domain.

As an example of pattern and structure in early mathematics learning, consider the rectangle shown in Figure 1. The pattern of 3 x 5 squares is obvious to adults, but not to young students (Outhred & Mitchelmore, 2000). They apparently do not perceive its implicit structure: three rows of five equally sized squares (or five columns of three) with their sides aligned vertically and horizontally. Repetition (of individual rows or columns) and spatial relationships (congruence, parallels and perpendiculars) are the essential structural features here.

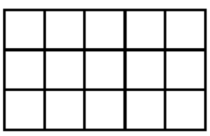
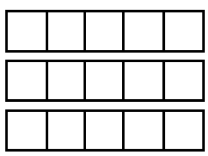
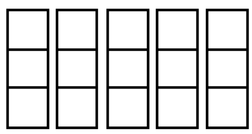
(a)  (b)  (c) 

Figure 1. Rectangular grid perceived as (a) 3 x 5, (b) 3 rows of 5, (c) 5 columns of 3.

Awareness of the structure of such grid patterns can facilitate the learning of many mathematical concepts. For example, counting the squares as composite units leads to skip counting (e.g., 5, 10, 15 by fives and 3, 6, 9, 12, 15 by threes) and thence to multiplication as a binary operation (e.g., 3 times 5). Perceiving a composite unit of five or three squares is critical to the emergence of two-dimensional structure (R. Lehrer, personal communication). The calculation of volume is based on a similar structure, this time involving layers in three dimensions. An understanding of grid patterns can also help early learning of division and fractions.

Finding the structure of patterns is often regarded as *pre-algebraic thinking.* It is easy to see why. For example, the alternative decompositions shown in Figure 1 lead to the commutativity of multiplication, later expressed in the generalisation $ab = ba$. In the sequence of multiples of 3 (i.e., 3, 6, 9, 12, 15...), each number is simply 3 times its position in the sequence (e.g., the 4th number is 3×4), a relationship that may later be expressed as $t = 3n$. Again, children might calculate the number of squares around the perimeter of the rectangle as 5 + 5 + 3 + 3 and realise that the corners should not be counted twice (regardless of the size of the rectangle). This insight may later be expressed as $P = 2x + 2y - 4$.

The Role of Pattern and Structure in Early Mathematics Learning

The study of pattern and structure is embedded in a diverse range of studies of mathematical development in pre-school and the early years of schooling.

Number

Many studies have implicitly or explicitly examined the role of pattern and structure in young children's understanding of number concepts and processes such as counting, subitising, partitioning, and numeration (Wright, 1994; Young-Loveridge, 2002). In their studies on numeration, Cobb, Gravemeijer, Yackel, McClain, and Whitenack (1997) described first graders' coordination of units of 10 and 1 in terms of the structure of collections. Similarly, Thomas, Mulligan, and Goldin (2002) identified structural elements of the base ten-system (such as grouping, partitioning, and patterning) found within students' images and recordings of the numbers 1 to 100. In a study of partitioning, Hunting (2003) found that students' ability to change focus from counting individual items to identifying the structure of a group or unit was fundamental to the development of their number knowledge. Van Nes (2008) also found a strong link between developing number sense and spatial structuring in Kindergartners' finger patterns and subitising structures. Studies of partitioning and part-whole reasoning (Lamon, 1996; Young-Loveridge,

2002) indicate the importance of unitising and spatial structuring in developing fraction knowledge.

Extensive research on addition and subtraction concepts has highlighted young students' strategies in recognising the structure of word problems (Mulligan & Vergnaud, 2006) as well as structural relationships such as equivalence, associativity, and inversion (Warren & Cooper, 2003). Moreover, studies of multiplication and division have indicated that composite structure is central to multiplicative reasoning (Confrey & Smith, 1995; Steffe, 1994). In a longitudinal study of second graders' intuitive models of multiplication and division, Mulligan and Mitchelmore (1997) found that the intuitive model employed to solve a particular word problem did not necessarily reflect any specific problem feature but rather the mathematical structure that the student was able to impose on it. Many students used only additive strategies, but other students had acquired more sophisticated strategies based on an equal-groups structure and their calculation procedures reflected this structure.

English (1999) investigated ten-year-olds' structural understanding of combinatorial problems, another multiplicative field. While the majority could solve the problems, they often had difficulties explaining fully the two-dimensional structure of the problems and could rarely identify the cross-multiplication feature. Students' symbolic representations for the three-dimensional problems also suggested that they lacked a complete understanding of the combinatorial structure.

Measurement and Space

Several researchers have studied young students' understanding of *spatial structuring*, defined by Battista (1999) as:

> the mental operation of constructing an organization or form for an object or set of objects. It determines the object's nature, shape, or composition by identifying its spatial components, relating and combining these components, and establishing interrelationships between components and the new object. (p. 418).

For example, Battista, Clements, Arnoff, Battista, and Borrow (1998) and Outhred and Mitchelmore (2000) studied the development of spatial structuring in rectangular figures and arrays in the elementary grades. Both studies found that most students learn to construct the row-by-column structure of rectangular arrays by about Grade 4 and have by this time also acquired the equal-groups structure required for counting rows and layers in multiples.

Some curriculum projects have approached mathematical development through measurement rather than number concepts. The Measure Up project (Slovin & Dougherty, 2004) develops early mathematics through establishing measurement units and symbolic representations that focus on structural characteristics such as patterns and relationships, rather than superficial features of problem-solving situations.

Early Algebra and Modelling

Recent studies of early algebra (Carraher, Schliemann, Brizuela, & Earnest, 2006) have found that, given appropriate opportunities, young students can learn to make generalisations and develop abstract mathematical skills that reflect mathematical structure. Complementary studies support the notion that young students can develop functional thinking (Blanton & Kaput, 2005). Other studies with young students have highlighted the importance of structural relationships in the development of analogical reasoning (English, 2004).

Mathematical modelling provides rich opportunities for students to integrate their mathematical knowledge and use pattern and structure. English and Watters (2005) successfully focussed 8-year-olds' attention on structural characteristics such as patterns, interactions and relationships among elements of data rather than using superficial features in modelling problems.

The development of pattern and structure also features in studies of data modelling and statistical reasoning. In designing instruction to support learning through data exploration, Lehrer (2007) highlights the "challenges of imposing structure on data, of choosing displays to highlight aspects of structure, and of making judgments about phenomena in light of variability and uncertainty" (p. 23).

Development of a General Construct: Awareness of Mathematical Pattern and Structure (AMPS)

Each of the research areas reviewed above approaches the study of single topic-specific domains (although studies of mathematical modelling have tended to integrate several domains). An advantage of such an approach is the creation of an explicit knowledge base from which to formulate learning frameworks or developmental trajectories. A disadvantage is that it does not allow for the possibility that learners may develop common structural understandings. Given the similarities in the results for different areas, the question naturally arises: Is there such a general construct as *awareness of mathematical pattern and structure* (AMPS) that can be observed across a range of concept areas in early mathematics learning?

Many researchers have proposed structural theories that can be applied to mathematical development. For example, Piaget's stages of cognitive development (sensorimotor, preoperational, concrete operations, and formal operations) were derived from an analysis of the structure of students' responses (Piaget, 1970). The Structure of Observed Learning Outcomes (SOLO) taxonomy (Biggs & Collis, 1982), based on Piaget's theory, focuses explicitly on student responses. Our application of SOLO to young students' concepts of multiplication and division (Mulligan & Watson, 1998) drew our attention to the ikonic features and structural characteristics of mathematical responses. However, we found that these general theories did not provide sufficiently fine-grained categories for classifying mathematical structure in specific mathematical content areas.

In Goldin's (1998) model, mathematical representational systems develop through three broad stages:

1. An inventive/semiotic stage, in which characters or configurations are first given meaning
2. An extended stage of structural development
3. An autonomous stage, where the new system of representation can function flexibly in new contexts.

In collaboration with Goldin, we developed more explicit descriptions of the structural features of children's representations of the base-ten numeration system (Thomas et al., 2002). This analysis, which focussed on pictorial, ikonic, and symbolic characteristics of children's representations, would appear to be applicable to a wider range of mathematical concepts.

Mason (1996) believes that the roots of mathematical thinking lie in detecting sameness and difference, in making distinctions, in classifying and labelling, or simply in algorithm seeking. Our studies have led us to conjecture that young children who have learned to look for mathematical similarities and differences within and between patterns are likely to develop an understanding of the structure of those patterns. Moreover, they will tend to look for similarities and differences in new patterns and broaden their structural understanding accordingly. By contrast, students who tend not to notice salient features of structure (for whatever reason) are likely to focus on idiosyncratic, non-mathematical features in all situations. We thus consider AMPS to have two interdependent components: one cognitive (knowledge of structure) and one meta-cognitive (a tendency to seek and analyse patterns). Both are likely to be general features of how students perceive and react to their environment.

Multiplicative structures (from patterning in pre-school, through multiplication, numeration and fractions in middle school, to ratio and proportion, scales and trigonometry in secondary school) are central to the school curriculum. All involve processes such as repetition, grouping, partitioning and unitising, and all require spatial structuring in visualising and organising their structure. Multiplicative structures are therefore likely to feature strongly in any attempt to investigate AMPS in young children.

Research Questions

We aimed to find whether AMPS could be described and measured as a general construct that applies across a number of mathematical concepts and processes. In particular, we posed the following three research questions:

1. Can the structure of young students' responses to a wide variety of mathematical tasks be reliably classified into categories that are consistent across the range of tasks?
2. Do individuals demonstrate consistency in the structural categories shown in their responses?
3. If so, is the individual student's general level of structural development related to their mathematical achievement?

Although we believe that AMPS may be applicable across the age span, we restricted the initial investigation to Grade 1 students.

Method

The Assessment Instrument (PASA)

Our strategy was to select a wide range of tasks, reflecting the Grade 1 curriculum, which seemed to require primarily conceptual rather than procedural understanding. We gave preference to tasks which we felt, on the basis of previous research, were likely to show structural development in students' responses.

In selecting tasks, we consulted a wide range of empirical research on early mathematics learning and assessment (in particular, Clements & Sarama, 2007; Doig, 2005; Howell & Kemp, 2005; Wright, 1994). Several key processes were identified: subitising, unitising, partitioning, repetition, spatial structuring, multiplicative and proportional relationships, and transformation. We also included some of our earlier tasks on multiplicative reasoning (Mulligan & Wright, 2000) that had subsequently been incorporated into the statewide Schedule for Early Number Assessment (NSW Department of Education & Training, 2001).

Thirty-nine tasks involving key mathematical processes were devised. Each task required students to identify, visualise, represent, or replicate elements of pattern and structure. Table 1 lists the various tasks, categorised under Number, Measurement, and Space, in what we shall call the *Pattern and Structure Assessment* (PASA). It may be noted that some of the tasks extend beyond state curriculum expectations (e.g., constructing a pictograph). Such tasks were included in order to yield a wider range of responses than might otherwise be expected.

Table 1
Pattern and Structure Assessment (PASA) tasks

Number	Measurement	Space
Subitizing: quantify 2×3 and 3x3 arrays with and without partial screening	Length: demonstrate and describe length of model. Explain and use informal, equal-sized units to delineate length of stick	Patterns: model/draw self- generated patterns. Replicate self-generated patterns using other modes or invented symbols using model and from memory
Rote counting (oral): multiples of 2, 5 and 3 to 30 and beyond.	Length/fractions: partition in halves, thirds, quarters using continuous materials	Pattern/visual memory: reconstruct and draw triangular pattern of six dots; extend, symbolise and explain pattern
Perceptual counting: multiples of 2, 5 and 3 to 30 and beyond.	Length: draw features on an empty ruler.	2-dimensional Space: identify and visualise properties of 2D shapes, nets and boxes
Symbolic counting: represent multiples of 2, 5 and 3 on numeral tracks to 30 and beyond	Area/symmetry/unitising: visualise and calculate area of rectangles and triangles (sides 2, 3 and 4 units) using a single unit	
Base ten: use composite unit with currency (10 x 10c coins, 10 x $1.00 coins)	Area: complete drawing of units in 3 x 3 and 3 x 4 rectangular partial grids	Angles: identify and represent corners of a square using model
Partitioning: partition two 2 x 4 rectangular grids simultaneously in problem context	Volume: visualise and calculate area of boxes (4, 9, 12 units) using one unit with 2D net and box	Pictograph: complete partial horizontal pictograph from 2-way table using grid lines

Partition/ quotition: share and group 6÷2 and 6÷3 with and without models. Reformulate fair shares	Mass/weight: calculate and compare units of mass by comparing single and composite units (stack of 10 coins)	Pictograph: Construct horizontal pictograph aligned with 2-way table.
Combinatorial: find all possible combinations (2 x 3, 3 x 3, 3 x 4) and explain strategies	Time: draw 8 o'clock on an empty clock face	

Sample and Procedures

The PASA was administered in individual interviews to 103 first graders (55 girls and 48 boys), ranging from 5.5 to 6.7 years of age. Participants were drawn from nine state primary schools, chosen from six administrative districts in metropolitan Sydney. The sample represented a wide range of mathematical abilities as well as cultural, linguistic and socio-economic backgrounds. All interviews were videotaped.

Data Analysis

Responses were initially coded as correct, incorrect, or non-attempt. A composite score was then compiled for each individual, treating non-attempt as incorrect.

Next, students' responses to each of the 39 tasks were examined in order to identify their explicit structural features. These response categories were then ordered in terms of the degree of structure, and students' responses were coded accordingly. Where multiple responses were given, the student's most sophisticated and accurate response was coded. Videotapes, interview transcripts, interviewer notes and students' drawn and written recordings were all consulted in order to obtain the most valid coding.

Finally, the various categories for each task were examined to see if a common structural categorisation could be reliably applied.

Results

The Total PASA Score

Total PASA scores ranged from 3 to 33 out of 39, with a Cronbach alpha of 0.86. To investigate the validity of the total PASA score, the eight lowest-scoring and the eight highest-scoring students were identified.

Their teachers, who were not informed of the PASA scores, were then asked to comment on the student's level of mathematical achievement in school. Teachers used whatever assessment data they had at hand, which included the statewide Schedule for Early Number Assessment (SENA) (NSW Department of Education, 2001), their own classroom observations, and in some cases a standardised mathematics test. All the students who scored highest on the PASA total score were unambiguously classified as high achieving, and all the students who scored lowest were unambiguously classified as low achieving.

The total PASA score was therefore judged to be a valid and reliable measure of conceptual understanding of mathematics in Grade 1.

Structural Categorisation of Items

A total of 9.6% of the responses were non-attempts. The researchers were able to classify all other responses into the following four broad stages of structural development:

1. *Pre-structural stage (PRS)*. Representations lack any evidence of numerical or spatial structure. Most examples show idiosyncratic features.

2. *Emergent (inventive-semiotic) stage (ES)*. Representations show some relevant elements of the given structure, but their numerical or spatial structure is not represented.

3. *Partial structural stage (PS).* Representations show most relevant aspects of numerical or spatial structure, but the representation is incomplete.

4. *Stage of structural development (S)*. Representations correctly integrate numerical and spatial structural features.

These categories are very similar to those we identified in our earlier research on students' representations of base-ten numerals (Thomas et al., 2002).

Figure 2 illustrates the four stages for three selected PASA tasks: completing a 3 x 4 rectangular grid, drawing 8 o'clock on an empty clock face, and drawing a triangular pattern of dots from memory. (All drawings were made by different students.) The similarity in the structural characteristics of the drawings at each stage is apparent.

The researchers' coding for a random 20% of the students was reviewed by a research assistant; an inter-rater reliability of 94% was found. For each task, there were responses at all four stages. There was always a clear modal response which, however, varied from task to task. In other words, although the student responses to the various tasks could be reliably classified into categories that were consistent across tasks, the tasks were not consistent in the structural stage that they indicated. This was to be expected: Put simply, some tasks were easier than others.

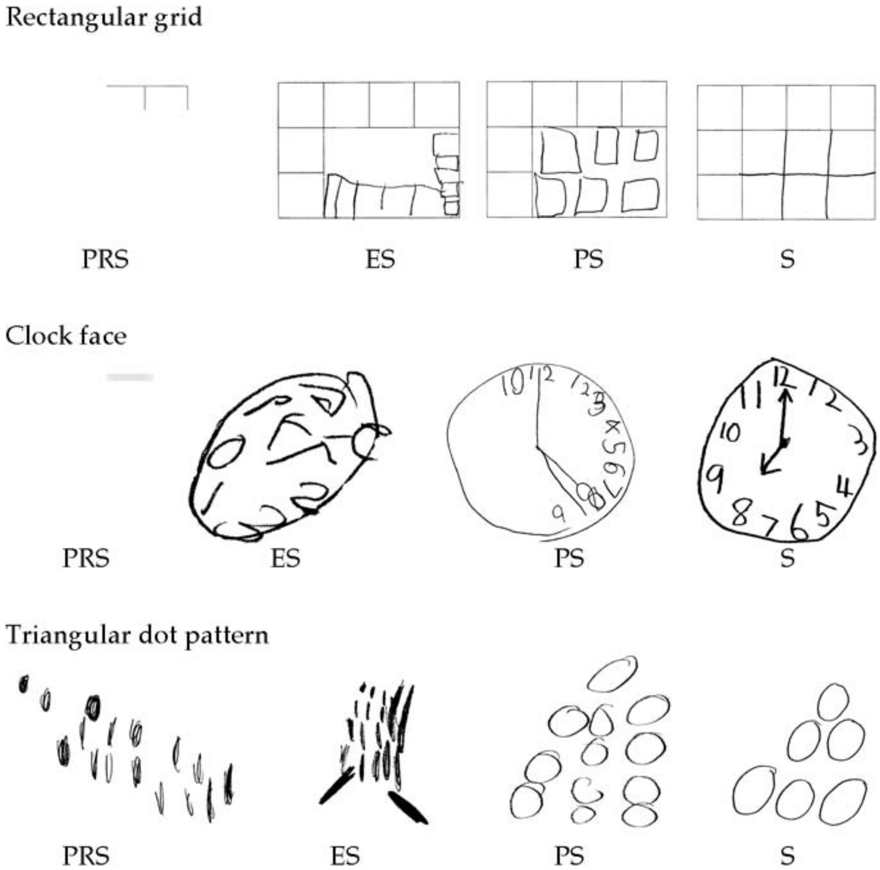


Figure 2. Typical Grade 1 students' responses at four structural stages.

Structural Consistency Within Students

Each student's responses did not show the same structural stage across all 39 tasks. This was not unexpected, given the variable level of difficulty of the tasks. However, for each student, there was a clear mode in the structural stages of their responses: The frequency of the modal stage was at least twice the frequency of each other stage. Moreover, except for two students, the modal frequency was at least 50%. Each student could therefore be unambiguously and confidently assigned to a single broad stage of structural development. These broad stages were distributed as follows: PRS: 11%, ES: 38%, PS: 27% and S: 24%.

It was noticeable that the responses from the ES students showed greater variability across tasks than in the other groups. The quality and type of structural features shown were not consistent; for example, equal sized units were used differently in area and length tasks. This variability may well indicate the existence of sub-categories representing different forms of emergent structure that are content dependent.

Relation of Structural Level to General Understanding

Figure 3 is a box and whisker plot showing the relation between students' overall stage of structural development and their total PASA score.

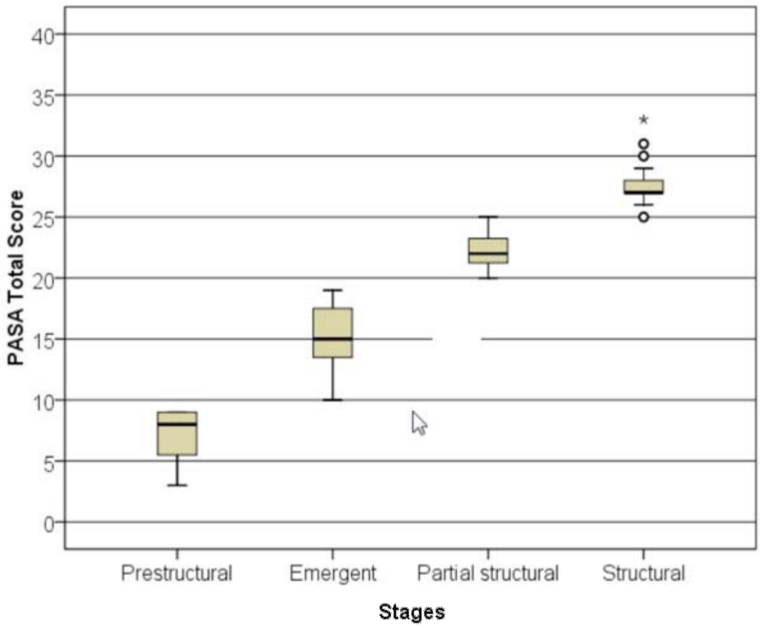


Figure 3. Range of PASA total scores by stage of structural development

There was a remarkable correlation between the two: With only one exception, the total PASA scores within each of the four groups of students fell into discrete ranges: PRS: 3-9, ES: 10-19, PS: 20-25, S: 26-33. This represents an almost perfect correlation between students' conceptual understanding of mathematics (as measured by PASA) and their stage of structural development.

Discussion

Our results show convincingly that student responses to a wide range of conceptual tasks suitable for 5-6 year olds can be reliably categorised according to their mathematical structure (Research Question 1). About 90% of responses from 103 students to 39 representative tasks could be assigned to one of four categories (pre-structural, emergent structural, pre-structural and structural) which represent significant development in students' understanding of mathematical pattern and structure.

Moreover, we have found that, although individual students react differently to different tasks, they tend to show the same structural stage in their responses to the majority of tasks (Research Question 2). This finding is the first indication that there may indeed be a general student characteristic that we have called Awareness of Mathematical Pattern and Structure (AMPS) which applies across a wide range of early mathematical concepts.

There is also good evidence that AMPS may be associated, as predicted, with mathematical achievement (Research Question 3). In this study, there was an almost perfect correlation between students' general mathematical understanding (as measured by the total PASA score) and the structural stage most frequently shown in their responses. This is, of course, a limited finding because the same instrument was used to measure both variables. However, independent evidence is provided by the fact that eight students who gave predominantly structural responses were judged by their teachers as highly competent at mathematics and eight who gave predominantly pre-structural responses were judged to be mathematically weak.

Furthermore, in a follow-up study (Mulligan, Mitchelmore & Prescott, 2005), we found that the eight high-achievers made such progress in the following school year that a new category, Advanced Structural, had to be constructed to represent their increased AMPS. Conversely, the eight low-achievers still tended to focus on non-mathematical features of special interest to them and several remained in the pre-structural stage.

We may conjecture several mechanisms that could explain why students with high levels of AMPS would have developed mathematical thinking more than those with low levels of AMPS:

- Students with high levels of AMPS would become knowledgeable about spatial structures such as rectangular arrays that incorporate equal sized units at an earlier age.
- Through their awareness of these structures (in particular, equal grouping and unitising), they would more easily learn basic properties of number, space and measurement.
- They would have a tendency to look for patterns and to explore similarities and differences between them. They would hence learn new structures more easily (Mason, 1996).
- They would be aware of what constitutes a *mathematical* pattern and would hence ignore irrelevant features when learning new concepts.

Taken together, the findings of this study demonstrate that AMPS is a construct that could provide new insights into early mathematics learning. Unlike other, more general theories of structural development, AMPS promises to provide a unified lens through which to view mathematical development—a lens that focuses on deep understanding rather than procedural skills. In particular, research on the proposed learning mechanisms could lead to valuable insights into how children develop early algebraic thinking.

Implications for Curriculum and Assessment

We can also use the AMPS lens to examine the early mathematics curriculum, its pedagogy, and its assessment in a new light.

The questions that naturally arise from the study just described are: Can AMPS be taught? Would student achievement in mathematics be improved as a result? We have some reason to believe, on the basis of further several exploratory studies, that the answer to both questions is 'Yes'. For example, we have found that a pre-school intervention focused on patterning can lead to a significant improvement in mathematical outcomes at the end of the following year (Papic & Mulligan, 2007). A year-long professional development program across Years K-6 of a NSW primary school, in which various teaching tasks designed to teach AMPS were developed, led to what teachers judged were substantial improvements in mathematical achievement (Mulligan, Prescott, Papic & Mitchelmore, 2006). From this experience, a Pattern and Structure Mathematics Awareness Program (PASMAP) was constructed and tested with 10 low-achieving Kindergarten students during 15 weekly teaching episodes; again the students showed impressive growth (Mulligan, Mitchelmore, Marston, Highfield, & Kemp, 2008). We are currently engaged in a systematic evaluation study of an entire Kindergarten curriculum based around PASMAP, sited in two schools in Brisbane and two in Sydney (Mulligan, English, & Mitchelmore, 2008), which should provide definitive answers to the two questions posed at the start of this paragraph.

There seems to be a recent search for greater integration in the mathematics curriculum. For example, some research groups have advocated equal importance of geometric and measurement concepts that can support number concepts (van Nes, 2008), and some argue that measurement, which may incorporate geometry and spatial reasoning, can provide an alternative pathway to mathematics learning (Slovin & Dougherty, 2004; Lehrer & Lesh, 2003). Others advocate an approach to early mathematics learning focused on algebraic thinking, mathematical modelling, and data exploration that often cuts across other curriculum areas (Carraher et al., 2006). AMPS could provide the necessary unifying construct and a common framework within which to organise teaching and learning experiences as well as the assessment of mathematical understanding.

Despite the recent inclusion of "patterns and algebra" in elementary mathematics curricula around the world, we know of no school syllabus that includes pattern and structure as a fundamental organising framework. Curriculum and assessment generally consider parallel content strands (usually number, space, measurement, data, and patterns and algebra) and do not encourage teachers to seek important connections between different concepts and processes. An alternative structure, with AMPS as the core and the various topic strands built around it, could prove much more effective.

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