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# Teachers' Function Schemas and their Role in Modelling

Mohan Chinnappan
*University of Wollongong*

Mike Thomas
*The University of Auckland*

An important element in teaching is the quality of content and pedagogical knowledge that teachers use in the design and delivery of their lessons. In this paper we present a framework for investigating how this knowledge is structured and how it relates to the mathematical modelling process. The framework is then used to compare an experienced teacher's knowledge and teaching of functions with that of four trainee teachers. The data show that the experienced teacher has built up knowledge that is dominated by conceptual rather than procedural aspects of functions, whereas the prospective teachers have structures often lacking a strong conceptual base.

Much of the recent research on the learning of mathematics has focussed on students' ability to use previously-acquired knowledge in making progress with the solution of novel problems. An important development in this area has been appreciating that the quality of the knowledge that students acquire may have a significant influence on how well that knowledge is used in the search for solutions to problems. In a classroom setting, teachers play an active role not only in facilitating the acquisition of new knowledge by their students, but also in providing pedagogically valuable experiences that may assist in extending that knowledge into new territories.

A key element in the goals that teachers set for their lessons and the structuring of these lessons is their own understanding of both the subject matter and their students. Thus, the nature of a teacher's knowledge base underlying a particular mathematical topic and the teaching of that topic can be expected to exert a major influence on the quality of the understanding that students develop. While this point about the role of the teacher knowledge base has received considerable support in research findings (Ball & McDiarmid, 1990) and curriculum reform documents (National Council of Teachers of Mathematics, 2000) there is little information about the quality of teachers' subject-matter knowledge and how that knowledge could drive what students learn. In one study where the relationship of an experienced teacher's conceptions of function to his practice was examined, Lloyd and Wilson (1998) found that the "teacher's comprehensive and well-organised conceptions contribute to instruction characterised by emphases on conceptual connections, powerful representations, and meaningful discussion" (p. 270). In the present study, we continue the process of addressing this rather neglected issue by examining teacher knowledge of functions and how that knowledge is used in the modelling process.

## Schemas as Structured Mathematical Knowledge

Network theorists have advanced several frameworks in which to investigate concepts and their development. According to one view, conceptual growth and mathematical understanding can be interpreted in terms of conceptual nodes and relations between nodes (Anderson, 2000). As students' experiences with a concept or a set of concepts increase, they build more nodes and links resulting in 'chunks' that allow them to develop layers of mathematical understanding. Various attempts have been made to elucidate such cognitive structures. Among these, the notion of *schemas* has gained considerable support amongst researchers.

In the context of learning, schemas have been given a number of interpretations in the psychological literature. Skemp (1979) describes how we construct 'what we already know' by engaging in mental construction of reality by building and testing a schematic knowledge structure, where a schema is "a conceptual structure existing in its own right, independently of action" (p. 219). In the context of problem solving, Paas (1992, p. 429) describes how a schema "can be conceptualised as a cognitive structure that enables problem solvers to recognize problems as belonging to a particular category of problems that require particular operations to reach a solution". Sweller (1992, p. 47) agrees, defining a schema as "a cognitive construct that permits problem solvers to categorise problems according to the moves required to solve them." Because our existing schemas serve either to promote or restrict the association of new concepts, the quality of what an individual already knows is a key determinant of our ability to understand, or as Skemp (1979, p. 113) concludes, "our conceptual structures are a major factor of our progress".

What then do schemas comprise? Olive and Steffe (2002, p. 100) explain that "Schemas can be regarded as networks of connected concepts." Other descriptions include a 'connected collection of hierarchical relations' (Davis & Tall, 2002). Dubinsky and others (Dubinsky, 1991; Cottrill et al., 1996) use the acronym APOS to describe four components of Action, Process, Object, Schema in the building of mathematical knowledge. The chain of events, they suggest, develops as follows. Actions, when applied to objects become processes, which in turn become *encapsulated* as mental objects, and examples of these three link together to form cognitive structures or schemas. Thus, conceptual entities in mathematics often present themselves with two distinct but complementary faces: they may be viewed as dynamic processes or as static objects. To make a mathematical idea readily manipulable and applicable in other contexts, it must be available internally in a concise form, and the encapsulation of the process as an object is one way of accomplishing this. The relations that are constructed between the conceptual entities forming a schema could include similarities and dissimilarities between concepts, instances of a concept, procedures for using concepts for solving problems or affective factors related to those concepts.

According to Anderson (2000), two variables determine the quality of a schema: the spread of the network and the strength of the links between the various components of information located within the network. Olive and Steffe (2002) list three factors: the type of concepts making up the schema; the quantity of the connections between the concepts; and the quality of the connections. This last factor is based on the ideas of Skemp (1979) who describes the quality of the link as associative (an A-link) or conceptual (a C-link). In the former concepts are linked purely by association rather than by the conceptual relationship of the latter. A complex schema can be characterised as having a large network of ideas that are built around one or more core concepts. Further, the links between the various components in the network are robust, a feature which contributes both to access to concepts and to use of the schema in problem-solving and other situations. A well structured schema can also benefit students by helping them assimilate new incoming mathematical ideas because such a schema can be expected to have many conceptual points to link with. In this way, schemas provide a useful way to interpret the growth of mathematical knowledge and meanings.

## *Teacher Knowledge and Schema Induction*

When we examine mental schemas of teachers in any given content area, we become aware of what they see as the important links which they want students to build into their knowledge structures and towards which they try to structure the learning environment. However, discussions about teachers' content knowledge must also consider how that knowledge of functions could be translated into forms that are easily understood by students. This discrimination was made by Shulman (1986a, 1986b, 1987) in his analysis of mathematics teachers' content and pedagogical content knowledge. Content knowledge refers to mathematics concepts, conventions and procedures, while pedagogical content knowledge includes both understanding of students' difficulties about a mathematics topic, and also strategies that are adopted in teaching a particular topic of mathematics. Leinhardt (1989) has suggested the existence of similar links between teachers' subject-matter knowledge, their explanations, and the type of representations generated by them during teaching. It has been proposed that knowledge of functions and uses of functions can be seen as the content knowledge, but the *importance of emphasising* the uses of functions during teaching comprises the pedagogical content knowledge. In this paper, we are mainly concerned with describing teachers' content knowledge although it is possible to argue that some aspect of this knowledge can be construed as belonging to pedagogical content knowledge.

A schema-based analysis, therefore, suggests that teacher actions could promote the construction of powerful function schemas that would benefit student learning in two important ways. Firstly, students would better access prior knowledge and integrate that with incoming information. Secondly, students could be expected to deploy acquired knowledge flexibly during the process of problem analysis. Hence the question is, what is the nature of a teacher's knowledge that would promote the construction of sub-schemas whose content and links are pedagogically valuable?

The aim of this research study was to examine the above issue by characterising the schemas of teachers in the content area of function, in order to compare and contrast schemas of an experienced teacher with those of teachers who are new to the teaching of algebra and functions. While teachers' knowledge is influential in what students learn, the purpose of the current research was to analyse the nature and quality of teachers' knowledge that drives their instruction. We do this by, firstly, developing a macro-schema that draws out key relations and nodes. This hypothesised framework, we argue, provides more details about the translation and interaction between teachers' content and pedagogical content knowledge.

One major difference we have hypothesised has to do with encapsulation of processes as objects. Many individuals appear not to progress to the point where they can think in a ***proceptual*** (Gray & Tall, 1994) or ***versatile*** way (Hong & Thomas, 1998; Tall & Thomas, 1991; Thomas, 2002) about mathematical symbols, seeing them either as a process invoked by the symbol or as the concept represented by it. Instead, they are ***process–oriented*** (Thomas, 1994) in their thinking, constrained primarily to mathematical processes within a given representation system. For a teacher the absence of a conceptual, representation–free view may structure their thinking, causing them to over–stress procedural methods. In contrast the ***versatile*** teacher, with a global view of a concept crossing a number of representations, is able to see its components, or constituent processes, and relate these to the whole (Thomas & Hong, 2001).

Considering this in the context of function, how would the schemas and teaching approach of a teacher with a rich conceptual and inter-representational view of function diverge from those who have a primarily procedural, single representation-based perspective? The multidimensional nature of functions can present a particular challenge to prospective and new teachers when designing appropriate learning situations. Translating between various function representations is something which an experienced teacher might take for granted, but to do so one needs to have an overview of the way the definition of function relates to each representation, and how sub-concepts, such as independent and dependent variables, one-to-one, roots, discrete, continuous, and so on, are manifest in each representation (Hong, Thomas, & Kwon, 2000).

To illustrate this difference, consider the construction of a composite function $f \circ g$ from the functions $f$ and $g$ where

$$f(x) = x^2 \text{ and } g(x) = x+1, \text{ so that } f(g(x)) = (x+1)^2.$$

Here, one could easily teach students to carry out such a procedure without actually having the underlying concept of what a composite function is. However, our view is that it would be difficult for such a procedurally–oriented teacher to engage students from a conceptual perspective or to help them participate in mathematical modelling. What does this kind of teaching involve? We present in Figure 1 the

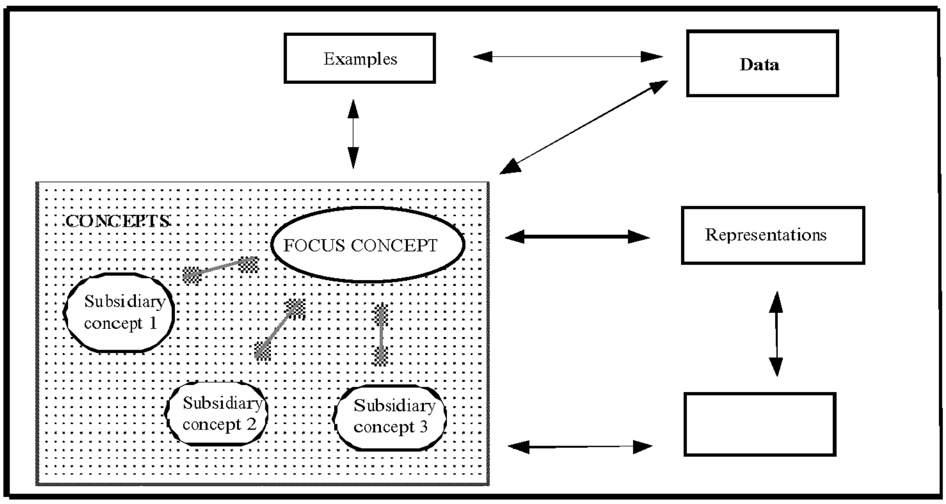



*Figure 1.* A structural framework for teachers' mathematical knowledge and modelling of a focus concept.

organisational structure of teacher knowledge that we believe is relevant to teaching focussing on mathematical concepts and their development.

This approach attempts to capture the content and relations of a teacher's knowledge and the activation of this knowledge during the modelling process. The network of relations among the focus concept, subsidiary concepts, algorithms (or procedures), representations and examples illustrate structure, while the integration of data into examples in order to instantiate the focus concept constitutes the modelling (process). A focus concept will have associated subconcepts, along with algorithms for carrying out certain actions with them, or on them, and these algorithms will usually be based in a single representation. However the concept is not limited to one representation but will have a number of manifestations, each with its own associated algorithms. For example, a teacher might wish to provide a demonstration of functions in real–life situations. She could achieve this by asking the students to collect numerical values for two variables, say, the time $t$ seconds taken for a trolley to roll a certain distance $S$ down a given inclined slope. These variables form subsidiary concepts of independent and dependent variables, the values collected form the data, and the context of time/distance forms the example. Students could be required to tabulate the values for the two variables and generate a graph that shows a quadratic relationship between the variables in question. Thus the table and the graph constitute representations of the functional relationship. Estimation of the velocity of the trolley at a particular time, or the average velocity over a period require specific algorithms which could be executed in either the graphical or tabular representation.

In this episode the teacher is engaging in modelling, where real-world data are used via an example to build an understanding of the focus concept (quadratic function). In doing so, however, she is drawing on her knowledge of links among several concepts including the representations and algorithms. We argue that this latter activity provides an insight into the organisational quality of her schema related to content and pedagogical content knowledge (Shulman, 1986). In this approach, the emphasis is on a progression from the *formal aspects* to the *algorithmic components* associated with the focus concept (Fischbein, 1994). It should be noted that this process is usually cyclical. This analysis may be contrasted with a procedural perspective where a teacher starts with a symbolic representation for a function, seen as representing a procedure, and either operates on it, or uses it as an algorithm in the form of value in and value out. It stresses that teachers of mathematics need to have a broad view of mathematics and its learning. They should not be limited to seeing it as primarily a skills-based, algorithmic subject, nor should they be constrained to thinking in terms of a single representation. Rather, teaching mathematics should be seen as involving the student construction of concepts and the network of links between them and their sub-concepts (Vollrath, 1994) across a number of representations.

## Method and Results

This research employs a case study methodology, examining the conceptual structures with regard to function of individual teachers and their influence on their teaching. This paper describes the results in relation to one experienced teacher and four trainee teachers.

### *The Teachers*

A number of criteria were set up for defining whether a teacher could be categorised as experienced, and Margot[1], a secondary school teacher in Auckland, New Zealand, fulfilled each of these, having 31 years teaching experience, including 15 years using technology in mathematics teaching. She has been active in promoting the use of calculators and computers to other teachers in her school, and has attended professional development courses, including graduate study. In addition, she has run advisory courses on using graphic calculators and was recently seconded to a one-year appointment as a mathematics adviser. In contrast, we also interviewed four first year full-time graduate teacher trainees, Arlene, David, Moana, and Vincent, who had only taught mathematics on practicum and had never used technology in their teaching.

[1] All names are pseudonyms.

Each of the teachers took part in a non-structured, free recall interview, where they were asked to talk freely about functions and polynomials and how they teach (or would teach) them. The interviews were recorded on audiotape and afterwards transcribed for analysis. Later we were able to go into Margot's school and observe and videotape two lessons on function.

## *Margot's Schematic Structure*

Margot's interview established that she had a conceptual view of function underpinning her teaching. She saw function very much as a relationship between two variables and remarked that, when teaching about functions, what "you tend to concentrate on is the relationship between two variables, the fact that there is one variable affecting the outcome of another variable... So you're encouraging relationships between variables." Further, for her, this relationship was about change. She commented that "You could just say that there is a connection between these two variables, which one is causing the change, and what is the result of the change?" and "So what we tend to do is do this practical type work first where they're getting the idea of this variable changing and this one resulting." Her comments were often expressed in terms of practical examples, which she clearly saw as very important for her own understanding and her teaching, and as modelling the function concept. For example she talked about beakers of water, kangaroo jumps, pendulums, and costs. This linking of concepts and real world contexts was an intrinsic part of her schematic knowledge structures.

Margot emphasises throughout the one-to-one nature of a function, "one in, one out" clearly excluding a one-to-many relationship (many-to-one functions were not specifically mentioned). Furthermore, she is very conscious of the distinction between the subsidiary concepts discrete and continuous variables.

> One of the things that I find that causes confusion is the distinction between discrete and continuous. You take them away and you do, you know a quadratic patterning, your kangaroos jumping or whatever and that's a discrete pattern. And then all of a sudden you produce a parabola which is continuous, and I don't think myself at the moment that I'm yet very good at making the distinction there for them between the two, and I think a lot of them lose that.

What other subsidiary concepts did Margot have embedded in her overall conceptual view of function? Clearly from her comments *variable* is a primary subsidiary concept, but in any modelling episode there are others which emerge, as the example discussed later demonstrates.

## *The Student Teachers' Schemas*

The student teachers were asked in our interviews to talk about the definition of function as they saw it.

> Arlene: The two things I think that I understand when I talk about functions, is that spring to mind immediately out of the vertical line test because that's what I

learnt. OK so I know that to find the function you apply the vertical line test and you apply that because um ... you want the function to have only one $x$ value ...

David: To me uh ... definition of a function is a, maybe a special type of relation uh ... I understand that functions come under the heading of um ... relations. Um ... but uh ... I would say that we only get for a function, um.. we only get one value for the ... say an $x$, in a graph situation, for an $x$ versus $y$ we only get one. ... variable. Do you want me to draw a little graph? ... the algebra is the actual expression or working with the actual expression. And the function to me is the visualisation of that graph.

Moana: I got the idea that functions was a graph. Um ... it was a little curve or a straight line on a graph ... .I think it was in fifth or sixth form that they defined what function was and um they had this the vertical line test where they said OK, given for example $y$ equals, for example, $x$ cubed ... if you were to draw that on a graph by using the vertical line test and if it cuts the particular graph at one point on the $x$ axis if I've got this right, and then that is what you call a function.

One feature was immediately striking about these responses, and this was to dominate all the student teachers' interview comments. The student teachers had a strong tendency to think of functions graphically and in terms of process. Arlene, Moana and Vincent all specifically mentioned the process of the vertical line test on a graph, and David actually drew a sketch showing the test. Not only do they think of functions graphically but to Moana and David this relationship goes further. They think of them as actually being graphs. Moana also draws a clear distinction in her mind between functions and algebra: "Yes it's less time consuming and so they'll be able to concentrate on what the graph looks like, a function really looks like, rather than calculating. ... I think that what's really important is what they see what the actual function looks like rather than spend more time on algebra".

As one would expect, the richness of these student teachers' function schemas varied somewhat, with some displaying understanding of a large number of related sub-concepts while others were much more limited. However, for Arlene and Moana, understanding of these concepts was often mediated by a graphical or other visual representation. Arlene in her interview has a clearly expressed preference for pictures in mathematics, and *only* associates concepts such as stationary values, even and inverse functions with graphs, linking this viewpoint to her school experiences.

Arlene: To find where the function will be at a maximum value um ... and minimum values. I guess the stationary points um ... points of inflection, all interesting features of the graph and I'm pretty short of examples.

Once you understand the sine and cos and tan curves, once that comes to you, it's so nice ... it's a security thing, when all else fails you can go back to those and that's just fantastic to know that. And when you get completely stuck draw a little sketch and then something will come to you then you'll think, uh.

Similarly, Moana prefers to view the sub-concepts of increasing and decreasing functions, rates of change, roots, limits, maximum and turning points in terms of characteristics of graphs.

Moana: Whether it's increasing or decreasing all those other kind of terms that come in um ... uh ... what they say, the rate of change or you know the gradients, that

> kind, so they'll be able to look at that rather than the algebra itself. There is also the idea of limits for example, you can select two particular points on a graph on the function and as the two points get closer and closer together they form a gradient ... .
>
> Oh, just drawing my *x* and *y* axis now ... and I'm just doing any particular graph ... here. And then you see that these are the roots and also got the point of the maximum and I think that's important that students know that at that particular point that the gradient ... the gradient is zero and even that's our minimum, so differentiation comes in ... uh, there is also um ... so therefore turning points.

## *Modelling—An Illustrative Example*

We can illustrate Margot's approach to teaching via modelling, and the subsidiary concepts supporting it, by detailed reference to one example from her interview. She describes at some length one way to approach the teaching of rates of change with Year 12 students:

> What we then try to do again is to make the work as practical as possible, and last year what we did was we took the coil of rope, ... and we, I went out to [store name] and brought all this bits of string and they mark off. So this is bits of rope being wound onto a coil and they mark off with pen, and they get a table for the number of the coil and the length of the string.
>
> So what they're actually doing, is they're modelling rope being wound by machine or onto a spool or whatever. Worked beautifully, it was perfect, and then we gave them questions that, we asked them to graph it, so they were graphing, and then we asked them to estimate the rate at which the rope was going on between two integer values, so that they could work from the table.
>
> We then asked them to work out the rate at which it was going on [for] two interpolated values so they actually had to work from that curve, and then we asked them to work out the instantaneous rate at which it was going on, so that they had to have the idea of a tangent. But again, this is all functions because again we're looking at one variable resulting in a change and another variable and the resulting graph and how you interpret it, and when we modelled that on the graphics calculator, it was just beautiful.

The emphasis on modelling is clear here. Not just in Margot's use of the word which indicates that she believes she is encouraging modelling, but in terms of the whole approach. Her aim is to take a real world 'practical' situation and represent it mathematically. She sees the practical situation as providing data which, directed by her rich conceptual structures, lead to modelling activity. In terms of our theoretical model, the focus concept of function is supported by, and related to, subsidiary concepts, each of which has a number of different representations. The ones she specifically mentions or alludes to for function are: average and instantaneous rate of change, interpolation, chord, tangent, gradient and variable. The major representations employed are symbolic, tabular, ordered pairs and graphical.

Margot also sees the value of technology in enabling manipulation of the mathematical concepts both within and between these different representations

(Kaput, 1992), but whether the technology is used or not, the transition between representations, preserving the conceptual structure of the mathematics, is a crucial one in her schemas, and she is able to accomplish these transitions. In this example, the variables are first symbolised (one representation) with the function being a relationship between the independent and dependent variables, and then there is a move between representations as the symbols enable access to a tabular representation. Working within this tabular representation the average rate of change is calculated from the two sets of values, using an algorithm or procedure. As a next step, Margot uses two values which have not been directly measured to stimulate the use of a graphical representation, with a co-ordinate or ordered pair representation of the data as the link between table and graph. Once in the graphical mode an algorithm to find the gradient of a chord from the use of interpolated values is employed (we note that although this can be done by linear interpolation from the table this was not mentioned here). Finally, also working within the graphical representation, the concept of instantaneous rate of change requiring the graphing of a tangent and an algorithm to find its gradient is introduced. This example does not involve algebraic symbolisation other than of the variables, since the situation has been adequately represented mathematically without recourse to this. The final step in the modelling process involves working within a particular representation to carry out algorithmic processes, in this case to calculate gradients or rates of change, namely between two points and at a point.

This example, which is lacking a symbolic, algebraic representation of function, is still completely about function for Margot, since "this is all functions because again we're looking at one variable resulting in a change and another variable". This fundamental conceptual mental construct, the linking of an independent and a dependent variable, runs through all her ideas on function.

When we isolate some of the key concepts which Margot is building into the modelling she is doing in the classroom what kind of rich relationships do we see? Table 1 presents an attempt to represent a macro view of this modelling example and its relationship to our theoretical model (note the items in each row do not necessarily correspond to each other).

The quality of some of the links in Margot's schemas may be observed in the comments she makes. One significant episode showing the linking between concepts and representations is the way that Margot connects the symbolic form of variable and function, with parameters and the inverse function.

> Well, where you have two variables where $x$ and $y$ are related to one another through a third variable often denoted by the letter $t$. So, for example, if you were to graph $x = t$, $y = t^2$ you would actually be graphing $y = x^2$. To get your inverse, all you then have to do is make $x = t^2$, $y = t$, and you get that mirror image. So that works really nicely.

Table 1
*A Macro View of Margot's Modelling Example and its Relationship with the Theoretical Model*

| Concepts | Physical Examples | Data | Representations | Algorithms |
|---|---|---|---|---|
| Function | Winding string on spool | Length of string | Symbolised variables | Calculate average rate of change (table) |
| Variable | Marking length | Number of coils | Table of values | Calculate gradient of a chord |
| Rate of change (average and instant) | Counting number of coils | | Table converted to ordered pairs | Calculate gradient of a tangent |
| Chord, Tangent | | | Graph | |
| Gradient | | | | |

We notice here that the use of the terms "graphing" and "mirror image" refer to a graphical representation, in the context of a manipulation within the symbolic representation whereby she combines $x = t$ and $y = t^2$ to get $y = x^2$. This demonstrates that her schemas are ***versatile*** enough to allow her to think and work between representations and that her schematic links between variable, function, parametric form, and inverse function are independent of the representations.

In contrast with Margot's inter-representational thinking, the teacher trainees seemed to lack the ability to relate some concepts across representational boundaries. For example, when discussing composite functions Arlene wanted to be able to understand what they mean in the graphical domain but was unable to, and said, "I know how you do it but I don't believe I could tell you why you do it or what the graph would look like, if you had for example, $x$ squared minus nine is your $\boldsymbol{f}$ function, um ... and your $g$ of $x$ [$g(x)$] was a three $x$ plus two. How they look on a graph is completely beyond me for a start. I wouldn't be able to tell you what they look like".

Sometimes a change from one symbolic representation to another can present cognitive obstacles, as illustrated by Arlene's remarks about the change from $f'(x)$ to $\dfrac{dy}{dx}$.

> Arlene: I couldn't do $\frac{dy}{dx}$, and as soon as I hit university it changed, and I don't know what made me change from $f'(x)$ because through school, … $f'(x)$ was, I mean, that's what I worked with. I liked that. And then all of sudden it became necessary to use this form although I got completely stumped in first year calculus with it. And it took me a long time to figure it out, but once I did it was great, like a revelation.

This notation difficulty seems to indicate a lack of understanding of the concept underlying it, suggesting that the learning of $f'(x)$ may have been primarily procedural.

There were other general comments about the desire for linking algebra and graphs, such as that of Moana.

> Moana: … maybe adding functions would be the same as adding the polynomials, and you can relate that to adding functions as part of algebra and you can present adding functions in terms of graphs, for example, graph *y* equals *x* squared [$y = x^2$] and *y* equals *x* squared plus one [$y = x^2 + 1$]. Adding those two functions they see how it relates visually on graph. And then also um ... in algebrawise, what you come out with. So the outcome in algebra and the outcome with graphing it.

However, it is manifest that she does not have a clear conceptual view of functions here, distinguishing a false dichotomy between polynomials in the algebraic domain and functions in the graphical. For Vincent the graphical representation in his mind is an obstacle to understanding, preventing him from separating out the independent variable in the algebraic form. He spoke of how "say we've got the relation being a circle, yet we can perform an operation on that circle say sine of the *x* value. And that comes up a function. You can convert the relation into a function and I found that unusual". David on the other hand was able to relate some ideas across representations. He spoke of the link between solving a quadratic equation by factorisation and by using the graphical intercepts, and the value of this for students.

> David: *x* squared plus three *x* plus two [$x^2 + 3x + 2$] is our expanded form. O.K. so on a graph, um ... our intercepts can be found by this factorised form of that ... obviously I've first of all started with a factorised form and then expanded it so that it was easy for me to come up with it. But um … when we're looking for a cutting point on the *x* axis, these factorised forms ... so *x* equals minus two will give us *y* value of zero so we'll have a minus two and minus one … . And then ... so therefore I think that the graphing solution is a good example of why the factorising is so important so if you can show the two things at once for a student it is a good link in your mind.

This is a good example of how the trainee teacher David seemed to be much further along in the process of constructing his pedagogical content knowledge than the other three students. Already he is thinking about the advantages for the students of the manner in which he will teach, but this was lacking in the responses of the other three trainee teachers.

A limited representational perspective influenced the modelling ability of the student teachers. They found the idea of applications of functions difficult to describe, being limited to the ones they were taught at school, primarily in calculus, and often in the context of growth and decay functions described using differential equations. Moana talks of population growth, and two of the others comment:

> Arlene: To be honest my main experience of functions has been in the classroom. I haven't had any real world experience with the functions … . I'm sure that there are many, many applications but just to give you just a few examples of, I suppose let's take a weta [NZ insect] population, that's dependent on many factors and when it levels out, or when some of those factors will influence the population numbers, either greater or less than, and possibly interested in that seasonal and all that sort of stuff so you could possibly do a [unclear adjective] function for that.

> Vincent: To me, when I ask or think about applications of functions and polynomials, like where in everyday life can we use them to model situations things like that. Um … like say we've got … like … . Like … uh … I think we, at [Year 13] level when we had the um … differential equations, the growth curves, and things like that, bacteria. I think it's that area of the maths was good because it gave you practical applications for the underlying maths.

In terms of our model of teachers' mathematical knowledge we would have to say that these teachers, unlike Margot, will have to work hard to bridge the gap between their conceptions and the idea of mathematical modelling, since their knowledge of applications is very limited.

## The Modelling Approach in the Classroom

For Margot, on the other hand, modelling is the essence of what she is trying to do in her teaching, and the transition between representations, preserving the conceptual structure of the mathematics, is a crucial one in her schemas. She describes her view of modelling, and its importance in these terms:

> I think we could bring modelling … down with the use of the graphic calculators … once we're confident ourselves in the use of the equipment … .Taking a practical situation … and fitting an equation to it, fitting a graph to it.

We note again that her comment on modelling is not limited to a single representation, but she talks of both equations and graphs as integral parts of the process, part of her knowledge structure.

Margot was asked to prepare 1 or 2 lessons on "how you would teach linear functions leading to solutions of equations." Her clear emphasis was again seen in one of the questions which she gave herself to address in her lesson plan: "Do they understand what is meant by a mathematical model?" She chose to set this modelling of a linear function in the context of a 'Mexican Wave', so popular at stadium sports events. Her two lessons involved the students in modelling a Mexican wave. Margot's conceptual schemas gave her a clear idea of the key concept of function and the important subsidiary concepts, such as variable, linearity, interpolation and extrapolation which she was to use in the lesson.

The class performed six 'Mexican' waves, with 3, 7, 10, 11, 14 and 21 students in turn participating, while one student with a stopwatch timed how many seconds they took and then recorded it on the board. These data values were then made into a table (see Table 2) and students were asked to use the table of values to:

- estimate the time for a wave with 5 people
- estimate the time for a wave with 30 people
- estimate the number of people for a wave taking 5.5 minutes,

in two different ways, writing their working and reasoning on the worksheet provided.

Table 2
*The Mexican Wave Data*

| Number of students (*N*) | Time in seconds (*t*) |
|---|---|
| 3 | 2.16 |
| 7 | 4.44 |
| 10 | 5.13 |
| 11 | 6.31 |
| 14 | 8.53 |
| 21 | 15.75 |

The aim of these questions was to pave the way for a discussion of the subsidiary concepts of interpolation, extrapolation and inverses. Among the interesting methods employed by the students to get the answer for the time taken for 5 students was: to take half the time for 10; to find the time for one student from each row of the table, find the mean of these and multiply it by 5; and the following linear interpolation method of student P:

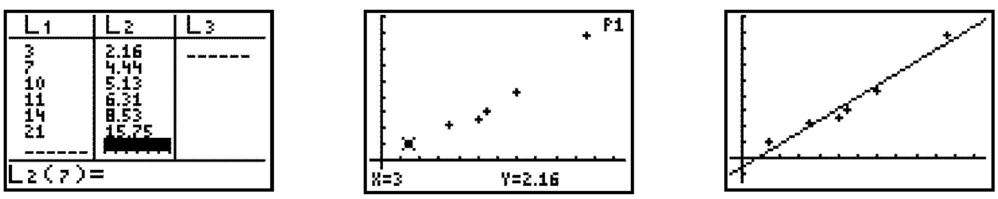
P: Okay, from 3 to 7 there are 4 numbers. So I divided by 4 and then I timesed it by 2 to get the number 5, to get 5. And then the answer is that to the number to 3 people and that's what I got ... and that equalled 2.28. And then I divided that by 4 ... and that equals 0.57. And then I timesed that answer so, 0.57 timesed by 2 which is 1.14 and then added that to this time here [i.e., 2.16], which got 3.3.

This first lesson introduced the basic idea of function in context. In the second lesson the process of modelling was particularly highlighted, and the graphic calculator provided the catalyst for an inter-representational approach.

Each student was given a TI-83 graphic calculator to use. Margot's view of the values on the whiteboard at this point was clear: "Okay, what we've got here is data, and we're going to enter them into our calculators". It was her conceptual schemas which structured the format in which the data were analysed in the modelling. Since she had a clear vision of dealing with a (one-to-one) function, the data were first entered into the calculator as table values using the list function in the STAT menu with two variables. Next came a change of representation to a graph. Margot explicitly linked the symbolic representation of the variables to their meanings in words, "On the *x*-axis I've got people and on the *y*-axis I've got time."

Once the graph was drawn Margot was keen to emphasise the idea of discrete data. The graphic calculator enabled this in a clear and meaningful way. Margot says "Press TRACE. Can you see the little flashing cursor there? And it says *x* is 3 and *y* is 2.16. There's your first point. Press your right hand arrow. Look it skips up the points. And every time it skips up the points it gives you the co-ordinates." In this way she effectively introduces in a visual manner the idea of discrete data points, at the same time emphasising both the link to the symbols *x* and *y* and the link to the co-ordinate (or ordered pair) representation, which is implied but not made explicit on the screen.

Margot's next aim is to introduce the linearity of the functional relationship. She asks "Tell me about that data" and receives replies such as: "It's going up." "Not joined." "It's not steady." and "It's a squiggly" and one student says, "One of those points is way out of line." Margot had deliberately not introduced the idea of linearity immediately, but had waited to see what the students made of the graphical data pattern. Once the idea of a linear relationship was out in the open the calculator was used to find the equation of, and draw, the regression line, $y=0.75x-1.22$. Figure 2 shows the calculator screens that the students were able to see and work with.



giving an estimate different from the ones they got from the table. Finally the class was encouraged to use the model to answer the original questions posed in the previous lesson, assisting them to see the value of it for finding approximate answers to those questions which require interpolation, or extrapolation, from the data. Of course, it is possible to critique the details of this approach. Margot did not address the issue of moving from discrete data to a continuous function (although she was aware of the problem as stated earlier), and neither did she encourage discussion on whether the line should pass through (0, 0) (hers did not). However, such details, while important, are not our focus in this discussion which is concerned primarily with her modelling perspective using the idea of function.

## Discussion

In this paper we have introduced a framework for studying teachers' conceptual knowledge in a given domain in relation to mathematical modelling. This framework considers the role of a teacher's schematic knowledge and the importance of having high quality, strong connections between the concept under focus, subsidiary concepts and their various representations. It is our contention that it is the richness and robustness of the structure of teachers' schemas which are a primary influence on whether their teaching of the focus concept is procedural or conceptual. We cannot be sure that Margot is typical of experienced teachers, and in many ways she probably is not, but the data presented here, with its limitations, raises the question of whether experienced teachers may have more links to subsidiary concepts, and place greater emphasis on these. Our hypothesis is that they do, and that such links may enable them to move between representations more easily, keeping the focus concept intact, while gaining the advantages that each has to offer. In contrast, an emphasis on procedural aspects of the focus concept would make it more difficult to move across representations and hence tend to anchor the focus concept in a single representation, for example, the graphical or symbolic.

When it comes to constructing modelling ability a view of function solely as a graph or as an algebraic formula is a hindrance. It is relatively easy to think up examples which give rise to pairs of values of variables which can later be approximated by a specific, exact algebraic function. However, it is much harder to start with a specific function and then try to produce an example which fits it. In addition a procedural perspective of function based on working with formulas is likely to cause problems in seeing a set of values of two variables as a function, further limiting the potential for modelling.

A key aspect of the student teachers' responses was their fixation with visual representation of functions with little concern about the links to their symbolic equivalents, and the potential effect for student learning. This finding is consistent with recent studies on functions which have shown that students tended to experience difficulty in interpreting graphical and symbolic representation of functions (Leinhardt, Zaslavsky, & Stein, 1990; Mitchelmore & Cavanagh, 2000). We argue that the continued emphasis on visualising functions without drawing out the underlying algebra will do little to change the above deficiency in students' understanding of functions.

In spite of the dominance of their school mathematics experience in their thinking, the student teachers we have interviewed do seem to be engaged in the initial stages of the process of accommodation of their function schema as they prepare to teach mathematics in school. As they examine their content knowledge and begin to construct pedagogical content knowledge of function, they are coming to recognise, as they did in the interview, that they have gaps in their function schema. However they are confident that they can fill these gaps once they are teaching. One way that they have already begun this process of development is by learning from colleagues in the schools where they are on teaching practicum. It is clear that this is a two-fold process including both learning ways to teach function (pedagogical content knowledge) and increasing their subject-matter knowledge by increasing the network of information embedded in their function schema. Student teacher David referred to two specific areas where he has benefited from his observation of an experienced teacher: the linking of graphical and algebraic representations of functions; and the utilisation of computers in order to construct these links in a dynamic manner.

The difference between procedural and conceptual approaches to function may be illustrated by considering the teaching of a graphical solution to the problem: "Find graphically where $y = x^2 - 3$ is zero". A teacher, such as Margot, who has a rich schema for function with many subsidiary concepts, is likely to keep alive the concept of a one–to–one (or many–to–one) relationship between variables as the representations are traversed. The table of values, the set of co–ordinates and the graph are each simply viewed as another representation of the symbolically presented functional relationship which assigns a value $x^2 - 3$ to a value $x$, in a one–to–one manner. The question (which could be answered in any representation) then becomes 'Which value of $x$ produces the value 0?' In contrast, teachers who have less rich schemas may concentrate on a procedural approach which lacks this underlying linkage. A sequence of conceptually unconnected procedures, in or between representations, is the result. Thus, students may calculate values of $y$ given certain values of $x$. They may transfer these to a graph by a matching procedure which aligns the first number to the $x$–axis and the second to the $y$–axis, giving a sequence of points, and then join these up. The final procedure, in the graphical mode, sees them read the value where the curve crosses the $x$–axis. While the solutions may be the same, the conceptual knowledge built is quite different.

One interesting feature of Margot's action was that she was able to shift between representations without drawing out potential conceptual links between the representations and the concept in question. There seem to be a number of occasions in mathematics teaching when teachers may not explicitly make the link between different representations of a conceptual idea (and indeed some where it may not be desirable to do so). For example, when solving $ax + b = cx + d$ in the algebraic representation, how often do teachers emphasise the link to what corresponds in the graphical representation to adding k$x$ to both sides? Similarly, we may teach a matrix method to solve A$x$ = $\lambda x$ to find eigenvalues, but how often is a graphical meaning for the solutions (when the vector space is, say $R^2$ or $R^3$) linked to the algebra? Our contention is that only those teachers whose conceptual structures are rich and contain the explicit links will be able to make these connections for students.

The present study has highlighted the importance of analysing potential relationships that might exist between teachers' knowledge and that constructed by their students. The results of the study support the findings of Lloyd and Wilson (1998), indicating that experienced teachers tend to activate and use highly organised knowledge schemas which could help students develop knowledge structures with similar quality. Chinnappan (1998) showed that even when students had built up a reasonable number of schemas in the domain of geometry, they were not able to construct rich representations of the given problem. This weakness in their solution process could be attributed to students applying schemas that had more procedural than conceptual information. We suggest that teachers need to build up and draw more on conceptually-dominated schemas of the type revealed by our experienced teacher in order to promote a more flexible approach to mathematical learning and problem solving by students.

One other lesson arising from our analysis is that when teachers begin their first teaching post they clearly have a continuing need to be supported by their colleagues. Experienced teachers should take the time to collaborate with new teachers on how they might present concepts such as function to their classes. Three areas in particular where this study suggests that this would be particularly advantageous with regard to function are: (a) inter-representational approaches to the concept of function, (b) use of computers and graphic calculators to promote representational links for functional concepts, and (c) realistic contexts giving rise to examples for modelling of functions.

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## Authors

Mohan Chinnappan, Faculty of Education, The University of Wollongong, Wollongong, NSW 2522, Australia. E-mail: <mohan@uow.edu.au>.

Michael O. J. Thomas, Department of Mathematics, The University of Auckland, PB92019, Auckland, New Zealand. E-mail: <moj.thomas@auckland.ac.nz>.