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A Developmental Multimodal Model for Multiplication and Division

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This paper presents an analysis of young students' development of multiplication and division concepts based on a multimodal SOLO model. The analysis is drawn from two sources of data: a two-year longitudinal study of 70 Grade 2 to 3 students' solutions to 24 multiplicative word problems, and examples from a problem-centred teaching project with Grade 3 students. An increasingly complex range of counting, additive, and multiplicative strategies based on an equal-grouping structure demonstrated conceptual growth through ikonic and concrete symbolic modes. The solutions employed by students to solve any particular problem reflected the mathematical structure they imposed on it. A SOLO developmental model for multiplication and division is described in terms of developing structure and associated counting and calculation strategies.

Recent studies show that young children can develop multiplication and division concepts in the first years of schooling, highlighting the fact that current teaching practices may not be focussed on children's potential mathematical development (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Clark & Kamii, 1996; Hunting, Davis, & Pearn, 1996; Kouba, 1989; Mulligan & Mitchelmore, 1997). On the other hand, there is also growing evidence that once children reach the primary grades they are unable to solve problems involving multiplication and division or apply multiplicative number facts with meaning. In the upper grades, students find difficulty in using multiplicative reasoning in a range of contexts and in integrating their understanding of rational number with multiplication and division (Behr, Harel, Post, & Lesh, 1994; Bell, Greer, Grimison, & Mangan, 1989; Confrey & Smith, 1995).

Multiplicative reasoning is essential in the development of concepts and processes such as ratio and proportion, area and volume, probability, and data analysis. It is also clear that failure to develop multiplicative structures in the early years impedes the general mathematical development of students in secondary school, for example, in using algebra, functions, and graphs. A concomitant problem is that multiplicative concepts are often not well understood or well taught by teachers at primary and secondary level (Graeber, Tirosh, & Glover, 1989; Simon, 1993). It appears that difficulties faced by older students can be attributed, at least in part, to the lack of development of an equal-grouping structure in early concept formation (Mulligan & Mitchelmore, 1997; Steffe, 1994).

In analysing young children's intuitive models for multiplication and division problems, Mulligan and Mitchelmore (1997) found that the intuitive model employed to solve a particular problem did not necessarily reflect any specific problem feature but rather the mathematical structure that the student was able to impose on it. Students acquired increasingly sophisticated strategies based on an equal-grouping structure. Counting strategies were integrated into repeated addition and subtraction processes and then generalised into the binary operations of multiplication and division. Strategies used with concrete and sensory models were internalised and replicated at an abstract level with increasing sophistication.

Mulligan and Mitchelmore (1997) also found strong evidence of the use of multiple models. Students were not consistent in their use of intuitive models; they used different strategies depending on the way they imposed structure on the problem. This structure was influenced by their use of concrete materials, the problem type, the size and type of numbers used, and the student's known calculation strategies. The structural characteristics and counting strategies provide a basis for describing the development of multiplication and division concepts using a multimodal approach according to the Structure of Observed Learning Outcomes (SOLO) model (Biggs & Collis, 1982; Collis & Biggs, 1991). SOLO analysis has been applied extensively to a range of mathematical concepts such as volume measurement (Campbell, Watson, & Collis, 1992), early fraction development (Watson, Campbell, & Collis, 1993), formal mathematical thinking (Pegg & Coady, 1993), common and decimal fractions (Watson, Collis, & Campbell, 1995), data analysis (Reading & Pegg, 1996), algebra (Coady & Pegg, 1996), the concept of average (Callingham, 1997), and chance measurement (Watson, Collis, & Moritz, 1997). Earlier, analysis of an elementary multiplication problem used a SOLO mapping procedure to elucidate the transition from ikonic to concrete symbolic modes (Watson & Mulligan, 1990).

By analysing longitudinal and teaching project data of young students' solutions to multiplication and division problems and focusing on the relationship between different modes of functioning, we can identify more explicitly the development of underlying multiplicative structures. The context for this study will be set in terms of (a) previous research on early multiplication and division strategies, and (b) the results obtained from previous research using the SOLO model. A framework of responses will be presented and the elements of the framework will be integrated into a multimodal developmental model that provides a more coherent basis for developing a teaching, learning and assessment framework for multiplication and division concepts.

Research on Multiplication and Division

Early research on multiplication and division processes investigated generally the influence of linguistic aspects and the semantic structure of word problems on students' solutions (Nesher, 1988). Studies of secondary students found that mathematically equivalent problems of different semantic structures produced different solution strategies and varied widely in difficulty (Bell et al., 1989). Other researchers analysed models of multiplication and division (Fischbein, Deri, Nello, & Marino, 1985), and the notion of multiplicative structures (Vergnaud, 1988) gave new insights into understanding children's development of multiplicative reasoning. Multiplicative situations, such as equal grouping, comparison ("times as many"), arrays, and cartesian products, were classified according to the nature of the quantities involved and the relation among them (Greer, 1992; Schmidt & Weiser, 1995; Vergnaud, 1988). In this approach, multiplicative concepts and relationships are not viewed simply as isolated abstractions.

Studies investigating multiplication and division processes with younger children identified the development of sound problem-solving strategies from an early age and the importance of modelling and representation in this development (Anghileri, 1989; Carpenter et al., 1993; Clark & Kamii, 1996; Kouba, 1989; Mulligan, 1992; Steffe, 1994). Recent research on multiplicative reasoning has looked to the early development of multiplication, division, and fraction concepts through essential processes such as counting, partitioning, grouping, unitising, and "splitting" (Confrey, 1994; Hunting et al., 1996; Lamon, 1996; Steffe, 1994; Watson, Campbell, & Collis, in press). Although multiplicative reasoning does not emerge in instructional programs until the second or third grade, many researchers have been investigating the intuitive and informal development of these processes in an attempt to seek out new ways of addressing the difficulties experienced in learning and to formulate more valid assessment techniques than traditional multiple choice tests.

Classroom-based studies on teaching and learning multiplication and division have investigated the ways in which children devise and represent related problem situations and solve computations (Boero, Ferrari, & Ferrero, 1989; Carpenter et al., 1993; English, 1997; Lampert, 1990; Mulligan & Mitchelmore, 1996; Murray, Olivier, & Human, 1992; Nesher, 1988, 1992). New instructional approaches can encourage closer links between different modes of representation, but more in-depth research is needed to identify the internal mental structures that children use in developing an understanding of multiplicative structures. Examining children's intuitive recordings of multiplicative situations is one way of gaining further insight into this development.

Much of the recent research investigating young students' multiplication and division concepts has focused on the analysis of solution strategies. Broadly, strategies have been classified in two ways: calculation strategies and modelling strategies. *Calculation strategies* have been classified according to degree of abstractness by Kouba (1989) and Mulligan (1992). These involve increasingly sophisticated counting methods: direct or unitary counting, rhythmic and skip counting, additive strategies based on repeated addition, and multiplicative strategies. Some examples are given in Table 1. *Modelling strategies* primarily involve the use of physical objects (Kouba, 1989) such as counters or fingers, or the drawing of icons or tallies. Carpenter et al. (1993) reported the use of direct modelling by Kindergarten children as they learned to represent and solve simple multiplication and division word problems. All these modelling strategies have been reported by Kouba (1989) and Carpenter et al. (1993) as occurring in conjunction with all five calculation strategies in Table 1.

Mulligan and Mitchelmore (1997) analysed young children's intuitive models for multiplication and division, whereby an intuitive model was defined as an internal mental structure corresponding to a class of calculation strategies. The children's internal conceptual structures of multiplication and division were inferred from strategies such as grouping, partitioning, counting, and patterning. From the correct responses, three main intuitive models for multiplication and division were inferred: direct (unitary) counting, repeated addition and multiplicative operation. A fourth model, repeated subtraction, only occurred in division problems. All intuitive models were applied to both multiplication and

Table 1
Calculation Strategies for One-Step Whole Number Multiplicative Word Problems

Strategy	Definition
1. Direct counting	Physical materials are used to model the problem and the objects are simply counted without any obvious reference to the multiplicative structure.
2. Rhythmic counting	Counting follows the structure of the problem (e.g., "1, 2; 3, 4; 5, 6" or "6; 5, 4; 3, 2.") Simultaneously with counting, a second count may be kept of the number of groups as a "double count."
3. Skip counting	Counting is done in multiples (e.g., "2, 4, 6" or "6, 4, 2"), making it easier to keep count of the number of groups.
4. Additive calculation	Counting is replaced by calculations such as "2 + 2 = 4, 4 + 2 = 6" or "6 – 2 = 4, 4 – 2 = 2."
5. Multiplicative calculation	Calculations take the form of known facts (e.g., "3 times 2 is 6" or derivations from a known fact, e.g. "3 × 2 = 2 + 2 + 2.")

division problems of various semantic structures. Fundamental to processing a multiplicative situation effectively was the recognition of the appropriate equal-sized groups. The semantic structure of problems was found to influence performance, with some problem types being consistently more difficult than others, but the semantic structure of the problem did not necessarily correspond with the strategy used to solve it. This was in contrast to the models proposed by Fischbein et al. (1985).

Most students in the Mulligan and Mitchelmore (1997) study were not consistent in their intuitive models at any interview stage. Problem characteristics, such as semantic structure and the specific numbers or size of numbers in the problem, seemed to influence which intuitive model would be used. At each interview, there were some students who used the same intuitive model on all problems but there were others who used as many as three different models. On the other hand, there was a consistent progression of strategy development from interview to interview within each problem. Over all 12 multiplication problems, in only 3% of the cases did students successfully use a more primitive intuitive model to solve a problem that they had successfully solved at the previous interview.

A Developmental Model

Various models of the cognitive growth of mathematical concepts have been based on the development of structure and levels of abstraction in students' mathematical thinking (Collis & Biggs, 1991; Gray & Tall, 1994; Pirie & Keiren, 1992; Sfard, 1991). Intuitive and ikonic functioning is considered central to this development. In particular, the SOLO model has distinguished structural aspects of learning at five levels (Biggs & Collis, 1982, 1991; Collis & Biggs, 1991): prestructural (P), unistructural (U), multistructural (M), relational (R) and extended abstract (EA). Unistructural represents functioning which uses single elements; multistructural uses multiple elements, usually in sequence; relational implies multiple elements related to form a coherent argument; and extended abstract responses indicate a transition to the next higher mode. These structural levels are found within modes of increasing abstractness: sensorimotor, ikonic (IK), concrete symbolic (CS), formal-1, and formal-2. In an extension of their earlier SOLO model, Collis and Biggs (1991) indicate that unistructural, multistructural, and relational levels combine to form a learning cycle (U-M-R) which repeats within modes. Hence, researchers may seek to analyse responses for recursive development as multiple U-M-R cycles within modes.

The complex interaction of various structural elements within the development from informal knowledge to abstract functioning has formed the basis of analysis for many studies of mathematical concept formation. In studies of fraction knowledge (Watson et al., 1993), volume measurement (Campbell et al., 1992) and chance measurement (Watson et al., 1997), two U-M-R cycles have been identified in the concrete symbolic mode. For example, in their analysis of volume measurement, Campbell et al. (1992) showed two learning cycles within the concrete symbolic mode—with the relational aspects in the first cycle becoming unistructural elements in the second. The first relational level, for example, was distinguished by the recognition of volume in terms of the relationship $V = L \times B \times H$. In the first cycle, children used methods based on counting visible units and often lacked accurate and efficient counting procedures (e.g., omitting hidden units). Counting and repeated addition in the first cycle assisted in the recognition of the relationship $V = L \times B \times H$. The complexity of this relationship in comparison to the unistructural nature of individually counting cubes was what distinguished the relational level of development. Although the acquisition of counting, addition, and multiplication skills were necessary, they were not sufficient for the application of volume measurement methods beyond the first cycle of the concrete symbolic mode. In the second cycle there was a recognition of how the units relate to the whole figure in more complex problem-solving contexts. Not only was an account taken of parts that could not be seen but also the relationship $V = L \times B \times H$ was applied as a single entity to objects constructed of multiple parts.

Another critical dimension in the analysis of mathematical concepts using the SOLO model is the notion that students can be functioning in a number of modes and levels—depending, for example, on the problem situation, the sophistication of the concepts, and the quantities involved. Biggs and Collis (1991) argue that much thinking in the area of problem solving is multimodal. Thinking in the ikonic mode has been found to continue developing into the formal mode where it is used in solving novel mathematical problems by adolescents with high ability in mathematics (Collis, Watson, & Campbell, 1993). The relationship between ikonic functioning in high ability children and mathematical structure in representations of number has also been demonstrated (Brown & Presmeg, 1993; Thomas, Mulligan, & Goldin, 1996). There now appears to be growing acknowledgment of the notion of multimodal functioning, compared with more traditional stage-like models of learning (e.g., Campbell, Collis, & Watson, 1995; Watson et al., 1993; Watson & Collis, 1994). Watson et al. (1993), for example, describe, "the mutual interaction between ikonic and concrete symbolic development, with progress in one area feeding into and supporting progress in the other. Ikonic processes appear to provide a necessary support to concrete symbolic reasoning at some developmental points, particularly when understanding is limited" (p. 60). Development of higher modes can also serve to increase sophistication in ikonic functioning.

Lack of appropriate development through ikonic and concrete symbolic modes can adversely influence the transition to formal thinking for many students. Coady and Pegg (1996) distinguish algebraic reasoning at a non-formal or concrete symbolic level in responses that show students' reliance on numerical substitution and/or incorrect or inappropriate manipulative algebraic procedures. Over-emphasis on manipulation of symbols and procedures in the concrete symbolic mode may serve to prevent higher order cognitive growth at the formal level. Lack of formal thinking may be traced to difficulties in the development of mathematical structures and/or use of superficial procedures in the concrete symbolic and ikonic modes.

A developmental model describing the growth of multiplication and division processes needs to take account of the acquisition of an equal-grouping (composite) structure which is at the heart of multiplicative reasoning (Anghileri, 1989; Confrey, 1994; Kouba, 1989). A composite whole is a collection or group of individual items that must be viewed as one thing. For example, a child must view three items as "one three" in order for the unit "three" to be a countable unit. For a true understanding of multiplication and division the child needs eventually to coordinate groups of equal-sized groups and recognise the overall pattern relating composites of composites (e.g., "three sixes"). Steffe (1994) described this as a premultiplying scheme: "For a situation to be established as multiplicative, it is necessary at least to coordinate two composite units in such a way that one of the composite units is distributed over elements of the other composite unit" (p. 19). Other theorists emphasise the importance of this structure calling it unitising (Lamon, 1996) and re-initialising (Confrey, 1994). In addition, multiplication and division may require that quantities or their numerical referents be transformed as a result of the process. The quantity that is the product is a different type of quantity to the two like or unlike quantities that have been multiplied (e.g., combinations of t-shirts and jeans produce "outfits"). This description has the potential to fit well with the structure of updated U-M-R cycles within the concrete symbolic mode as incorporated in the SOLO model.

Once the initial strategies related to developing composites are developed and consolidated with repeated addition or repeated subtraction and sharing models, multiplicative reasoning must extend beyond these to a point where the commutativity of multiplication is recognised and the inverse relationship between multiplication and division is applied. The acquisition of multiplication and division as binary operations relies on the child's ability not only to develop composite structure and commutativity but also to recognise the relationship $m \times n$ as a composite unit m "operated upon" n times. This is quite different to a repeated addition notion of multiplication which is commonly used in teaching practice. Again a U-M-R cycle has the potential to describe the structure involved.

In describing the early development of multiplication and division concepts through SOLO levels and modes, it is essential to focus closely on composite structure which has its foundations in the ikonic mode and develops largely through the concrete symbolic mode. In the concrete symbolic mode, multiplicative concepts are experienced through mathematical notations and symbols and through written language; responses are usually evoked mentally rather than with concrete and sensory models. The basis of the SOLO analysis is the "observed learning outcome"; hence it is not appropriate to speculate on what children may have been thinking but only what they say, write or draw. In studying children's responses to multiplication and division problems prior to instruction, the transition to the concrete symbolic mode and the interaction between ikonic and concrete symbolic modes can be described. The SOLO model hence offers potential to provide a detailed structure for learning outcomes observed from student responses and to inform teachers of possible starting and finishing points as they plan classroom experiences for children.

Research Questions

The analysis reported in this paper seeks to describe and explain structural aspects of development in more depth than done previously, by reporting children's responses in terms of the SOLO model. The process of forming and modelling the composite structure of multiplication and division will be made more explicit through descriptions of children's responses in the ikonic and concrete symbolic modes and at the five levels of functioning: prestructural, unistructural, multistructural, relational and extended abstract.

The analysis attempts to throw light on a number of research questions:

1. How do children's responses to multiplicative problems show the development of composite structure in terms of the SOLO model with multimodal functioning?
2. How is the development of composite structure related to the development of counting and calculation strategies?
3. How does the ikonic mode support development in the concrete symbolic mode? Do children use both modes of functioning simultaneously?

Method

The analysis reported in this paper is based on data from two sources: a two-year longitudinal study of development of multiplication and division strategies with 70 Grade 2 students (Mulligan & Mitchelmore, 1997), and some examples of children's representations of multiplication and division word problems drawn from a problem-centred teaching project with ten Grade 3 students (Mulligan & Mitchelmore, 1996).

The longitudinal research was particularly focused on studying the overall changes in students' development of multiplication and division concepts prior to and during formal instruction. Seventy students were observed four times during Grades 2 and 3 as they solved the same set of 24 word problems. Clinical interviews were conducted in March/April and November/December in two successive years. At the time of the first interview, students had received teacher instruction in addition and subtraction but not in multiplication and division. Between the third and fourth interviews, all students were given instruction in basic multiplication facts with the 2-10 times tables but not in related division facts. The problems were chosen to represent five of the ten semantic structures identified by Greer (1992). The division problems were constructed as inverse problems. The twelve multiplication problems were classified as equal grouping, rate, comparison, array, and cartesian product. The twelve division problems were classified as partition, rate, quotition, comparison, and sub-grouping. The problems were paired, with one problem in each pair using small numbers and one using large numbers.

The problem-centred teaching project with ten Grade 3 students presented a closer view of children's representations of multiplicative situations as they interacted in a mini-classroom setting over a 10-week period. The students were not instructed in solving multiplication and division word problems in the regular classroom prior to or during the study. The researcher assumed the teaching role as a facilitator, providing problems for each student to solve in order of difficulty. Ten multiplication and division word problems were adapted from Greer (1992). Children were challenged to solve the problems in a variety of ways and to explain their solution processes. A range of concrete materials was made available for children to use in modelling the problem if they wished. They were required to record their solution as a drawing or diagram, and to explain how they solved the problem by writing a sentence. The researcher interviewed each child about similarities and differences between their solution strategies and representations of the problems during and after each teaching session. For further details of sample, procedures and problem types see Mulligan and Mitchelmore (1996, 1997).

Analysis of Results

The present analysis focuses on the transition from early ikonic functioning, based on concrete and sensory models, to the development of a composite structure of multiplication and division. Thus, all responses to the multiplication and division problems have been classified primarily for the development of an equal-grouping (composite) structure and corresponding calculation strategies using the SOLO taxonomy. While different problems may require different levels of response to be considered correct, what is of interest here is the range of responses at various levels, regardless of the task set or the correctness of the response.

Following the presentation of the model developed to explain the structure of students' responses related to early attempts at multiplication and division problems, examples will be given for each level. Multimodal functioning will be described as appropriate when it is seen in ikonic support for responses in the concrete symbolic mode. The responses of students in the longitudinal interviews will then be summarised over the four interviews to give an idea of the development occurring over time.

Table 2 presents a summary of SOLO levels of development from prestructural thinking through the ikonic and concrete symbolic modes—whereby, in the latter, two U-M-R cycles are identified. Each level is described in terms of the abstractness of the response, the emergence of composite structure, and the increasing sophistication of calculation strategies.

Although the prestructural imagistic mode is often regarded as part of ikonic functioning, it has been classified prior to ikonic mode here because it denotes those responses which have no structural features—that is, those which do not focus on specific mathematical elements related to the task presented. Prestructural imagistic responses can be useful precursors to the development of ikonic functioning and building of structure within ikonic mode. Images at the prestructural level may also be used in conjunction with higher order ikonic and concrete symbolic thinking. The U-M-R cycle within the ikonic mode is characterised by development moving toward the equal-grouping structure, where students rely on concrete and sensory models but still are not consistently conserving. The increasing complexity of structure will be shown in the examples which follow. The extended abstract level of the ikonic mode becomes the first level of the concrete symbolic mode.

Although children still visualise and use concrete support in the concrete symbolic mode, it is distinguished by responses that no longer rely on physical modelling and perceptual counting. This is consistent with prior analysis (Watson & Mulligan, 1990). Two U-M-R cycles are shown, where the first cycle is based on the development of composite units and the second based on the development of multiplication and division as operations deriving from these composite units, culminating in two relational levels denoted as R_{2a} and R_{2b}. Examples of responses at each mode and level are described in the following subsections.

Prestructural Responses

At the prestructural level, children typically elicited idiosyncratic responses that did not focus on specific mathematical elements or the structure of the problem. Prestructural responses to the multiplication array problem, for example, showed that children were influenced predominantly by their initial image of the situation given in the problem. Consider, for example, these responses to the problem "There are 4 lines of children with 3 children in each line. How many children are there altogether?":

P : We line up in twos.
P : Lots of children.
P : You can count the lines . . . I don't know how many.
P : If it's 4 lines . . . that's a lot.

Table 2
Summary of SOLO Levels for Responses to Multiplication and Division Problems

SOLO	Level	Strategies
P	Prestructural Imagistic	Forms idiosyncratic images of problems lacking in mathematical structure; generates non-mathematical solutions.
		Ikonic mode
U_1 (IK)	Pre-composites	Represents one element of the problem using concrete or sensory models; unable to conserve or form composites. Unitary counting may be used.
M_1 (IK)	Emergent Composites	Represents two elements of the problem independently using concrete or sensory models; unable to form composites. Unitary counting strategies may be used to calculate total.
R_1 (IK)	Initial Composites	Forms composites by equal grouping, sharing, or one-to-many grouping using concrete or sensory models; calculates total by unitary, rhythmic, skip, or additive counting, but still does not consistently conserve. Unable to generalise pattern of composites.
EA_1 (IK) - U_1 (CS)	Intermediate Composites	Forms composites by equal grouping, sharing, or one-to-many grouping using concrete or sensory models; consistently calculates total correctly by skip or double counting. Coordinates composites by relying on concrete model. Unable to generalise pattern of composites but consistently conserves.
		Concrete symbolic mode
U_1 (CS)	Intermediate Composites	Forms composites usually without concrete or sensory models; calculates solution by unitary counting or method which does not take advantage of composite structure. Unable to generalise pattern of composites.
M_1 (CS)	Composite Units	Forms composite units by equal grouping or sharing abstractly; calculates using composite structure with unitary or skip counting. Unable to coordinate composites; calculates number of composites separately.
R_1 (CS)	Coordinating Composite Units	Coordinates composites abstractly; calculates using composite structure with skip or double counting; generalises pattern of composites.

(Table 2 continued)

SOLO	Level	Strategies
U_2 (CS)	Repeated Addition and Subtraction	Uses composite units to calculate using repeated addition or subtraction. Identifies number of composites by adding on or subtracting the composite unit repeatedly. Unable to generalise pattern of composites.
M_2 (CS)	Repeated Addition and Subtraction	Uses composite units repeatedly and coordinates number of units as repeated addition and subtraction. Identifies number of composites as a pattern; able to generate symbolic notation (e.g., "3 + 3 + 3 + 3 = 12" or "12 – 3 – 3 – 3 – 3 = 0"). Derives multiplication and division facts from addition and subtraction facts.
R_{2a} (CS)	Multiplicative Operation	Coordinates two composite units simultaneously as a binary operation of multiplication or division where implicit sequence of multiples is treated as a single entity (e.g., "3, 6 times is 18"). Uses symbols × and ÷. Associativity and commutativity shown.
R_{2b} (CS)	Multiplicative Operation	Can recall and derive basic multiplication and division facts automatically; uses multiplication and division as inverse relationships. Generates number sentences in symbolic form from multiplication and division problems. Associativity and commutativity shown.

Ikonic Mode

In the ikonic mode, children began to form composites using concrete and sensory models but could not represent the structure in their minds. Images formed in the ikonic mode became critical to development in the concrete symbolic mode because they provided the corresponding structure upon which multiplicative operations were developed. If the child was unable to operate at a relational level in the ikonic mode, it was unlikely that development through the concrete symbolic mode would be based on composite structure.

Unistructural level. Children at the U_1 (IK) level repeated, or modelled with counters or fingers, only one element in the problem. For example, in the partition problem "There are 8 children and 2 tables in the classroom, how many children are seated at each table?", typical responses included the following:

U_1 (IK): 8 children at the table.

U_1 (IK): One (puts block), two (puts block) . . . two children.

U_1 (IK): 2 children.

If counting occurred it was usually perceptual counting and unrelated to the structure of the problem, such as "1, 2, 3, 4, 5, 6, 7, 8 . . . 8 children." There were attempts to model quantities in the problem by sharing or trial and error, but children were unable to form composites.

Multistructural level. M_1 (IK) responses were similar to those shown at the unistructural level, except that they focused on two elements of the problem. Although children could focus on more than one element, they were not yet able to form a composite unit (e.g., "three books is one three"). Incorrect responses often showed a reliance on addition and perceptual counting. For example, in the multiplication comparison problem "John has 3 books and Sue has 4 times as many. How many books does Sue have?", typical responses included the following:

M_1 (IK): That's 4 (models 4) and 3 (models 3).
M_1 (IK): 1, 2, 3, 4, 5, 6, 7, (counts) 7 altogether.
M_1 (IK): That's 4 and 3 makes forty-three.

Relational level. At the R_1 (IK) level, children attempted for the first time to integrate composite structure with their counting strategies by forming intermediate composite units through modelling and counting. In the quotition problem "There are 12 children with 4 at each table. How many tables?", for example, a child formed composites of 4 from 12 counters and expressed this as "4, 4, 4, . . . that's 3 tables." Within the ikonic mode this type of response may be seen as a generalisation, but initially it may not be complete. Compared with unistructural or multistructural responses, this level represents the development of multiplicative structure because the child forms and uses composites even though an error may occur in calculation.

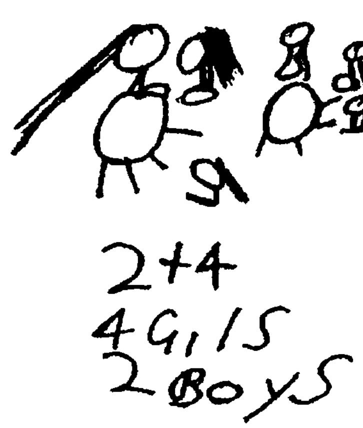

Figure 1. How Michelle, aged 7 years, represented two groups of three.

Figure 1 shows an ikonic relational response drawn from the teaching project (Mulligan & Mitchelmore, 1996), in which Michelle represented composite structure for the equal-grouping problem "There are 2 tables, 3 children at each table. How many children altogether?" She formed two groups of three but interpreted the situation as "4 girls and 2 boys," using unitary counting and addition to calculate an answer. This was meaningful for Michelle, but her calculation strategy and symbolisation of the problem did not take advantage of the composite structure of multiplication. Michelle used ikonic support to show a composite structure although this structure was not fully realised. In this case, Michelle's idiosyncratic interpretation of the situation was so strong that it restricted her representation and calculation strategies. She knew that she had a total of six children because she used unitary counting.

Extended abstract. At the extended abstract level of the ikonic mode, modelling and counting were still used and the total number of items was consistently calculated by skip or double counting. Children were able to coordinate composites using a visual stimulus such as a model or picture and could consistently conserve. This meant that they could consistently gain correct solutions from their model no matter how the model was arranged. They were still unable, however, to generalise the pattern of composites.

Concrete Symbolic Mode

The concrete symbolic mode was distinguished by responses that no longer relied solely on physical modelling and perceptual counting. Two U-M-R cycles were identified from both sets of data and these characterised two stages in the development of multiplication and division as binary operations. The first cycle was based on the development and coordination of abstract composite units. The second cycle built upon the composite units of the first cycle, using processes of repeated addition and repeated subtraction leading to multiplication and division as operations at two relational levels (R_{2a} and R_{2b}). Multimodal functioning in the form of ikonic support for responses will be noted throughout the examples.

First unistructural level. At this level, composite units of small quantities such as two, three, four or five items were formed mentally; but children still used counting methods reflecting the ikonic mode which did not take advantage of the composite structure. For example, in the partition problem referred to in the description of U_1 (IK) above, typical responses included the following:

U_1 (CS): That's 1, 2, 3, 4 and 1, 2, 3, 4 . . . 4 children at each table.

U_1 (CS): [Child shares by dealing mentally] 1 here, 1 there, 2 here, 2 there, 3 here, 3 there, 4 here, 4 there . . . 1, 2, 3, 4 . . . 4 at each table.

First multistructural level. M_1 (CS) was an advance on U_1 (CS) because children identified both the composite unit and the number of units as separate elements and used skip counting rather than unitary counting to calculate a solution. In partition and equal-grouping problems, students typically shared or constructed equal groups abstractly. For example, in the partition problem "There are 28 children and 4 tables in the classroom. How many children are seated at each table?", typical responses followed this pattern:

M_1 (CS): Divide it into 4 tables, half of 28 is 14 so half of 14 is 7, 7 at each table . . . now we had 4 tables so that's 7 and 7 and 7 and 7.

Most of the responses at this level showed that students focused on the number of units and the number of items in each unit separately. In the rate division problem "Peter bought 4 lollies for 20 cents. If each lolly cost the same, how much did one cost?", for example, some typical responses were as follows:

M_1 (CS): 5 for 1, 10 for 2, 15 for 3, 20 for 4.

M_1 (CS): 5, 10, 15 , 20 . . . 5 for 1.

First relational level. R_1 (CS) marked an important transition to multiplication and division as repeated addition and repeated subtraction. Children were able to coordinate composite units without the use of concrete material for the first time. This was accompanied by skip or double counting, where double counting signified an ability to generalise the pattern of composites. Consider, for example, the quotition problem "Twelve toys shared equally among the children. If they each had 3 toys how many children were there?" The following responses showed coordination of two composite units:

R_1 (CS): 12, 9, 6, 3 . . . that's 4 children.

R_1 (CS): 3 (1), 6 (2), 9 (3), 12 (4) . . . 4 children.

In the equal-grouping problem "There are 4 tables in the classroom and 7 children are seated at each table. How many children are there altogether?", the following responses showed children's ability to generalise the pattern of composites.

R_1 (CS): 7 at one table, 7, 7, 7 . . . that's 28.

R_1 (CS): 7 for 1, 14 for 2, 21 for 3, 28 for 4 . . . that's 7 four times.

Second unistructural level. At the U_2 (CS) level, in the second U-M-R cycle of the concrete symbolic mode, composite units and counting strategies employing a sequence of multiples were internalised as addition and subtraction. This was a limited strategy, however, because children were only able to use a small number of composites to add on or take away.

U_2 (CS): 3 plus 3 = 6.

U_2 (CS): 4 + 4 = 8.

U_2 (CS): 2 + 2 + 2 = 6.

Second multistructural level. Transition to M_2 (CS) was characterised by composite units being added or subtracted repeatedly, where the number of composites was coordinated as a pattern. Problems involving larger composite units such as four, six, or seven were solved using additive and subtractive strategies involving a sequence. Addition was largely employed to derive multiplication and division facts:

M_2 (CS): 3 + 3 are 6 and 3 are 9 and 3 are 12 . . . 4 threes are 12.

M_2 (CS): 21 - 7 makes 14 - 7 makes 7. . . that's 3 sevens in 21.

Figures 2 and 3 present some examples from the teaching project. In Figure 2, Natalie used composite units and repeated addition at the M_2 (CS) level to show the composite structure of the equal-grouping problem "There are 6 plates and 4 cakes on each plate. How many cakes altogether?" She represented the equivalent grouping problem ikonically, in a linear formation, and used repeated addition by adding on 4 successively. However, in her response to the comparison problem "Susan had 3 books and Jane had 5 times as many. How many books did Jane have?", shown in Figure 3, direct counting seemed to dominate her thinking despite her using formal notation 5 × 3 = 15. In this example, Natalie appeared to be using ikonic support to produce a response typical of concrete symbolic thinking where repeated addition and the symbolism of multiplication are used. Although Natalie used formal notation, she relied on forming and drawing equal groups, together with direct counting, to obtain an answer. In this case the use of multiplication notation was only a summarised form of repeated addition and did not mean that Natalie was using multiplication as a binary operation.

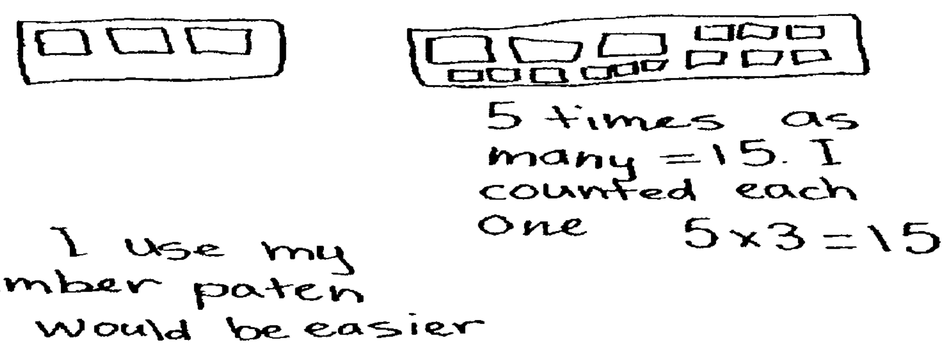

Figure 2. Natalie's response to equal-grouping problem.

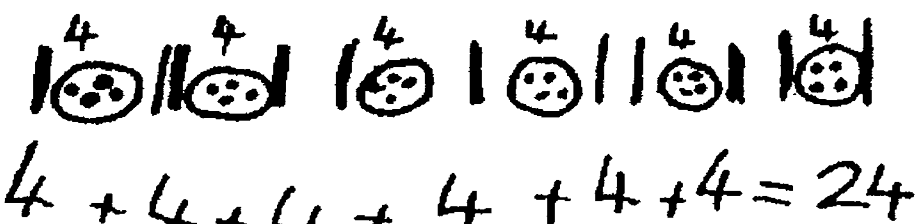

Figure 3. Natalie's response to comparison problem.

Second relational level. In R_2 (CS), there were two sublevels of performance observed. At the R_{2a} (CS) level, multiplication and division appeared explicitly as binary operations for the first time. The final term of an implicit sequence of composite units was treated as a single entity. Responses at the M_2 (CS) level were based on repeated addition but R_{2a} (CS) responses were distinctly different from these because the composite unit was "operated upon." For example, in the comparison problem "She has 3 books and Sue has 5 times as many," responses included the following:

R_{2a} (CS): 3 . . . 5 times is 15.

R_{2a} (CS): 5 threes are 15.

In contrast, at the M_2 (CS) level composites were restricted to a repeated addition model based on grouping such as "3 lots of 5 makes 15," no matter what the problem structure was. A relational understanding of multiplication and division was shown particularly through problem situations (such as array problems) that gave rise to the use of composites as factors—children reasoning, for example, "3 by 4, that's 12" rather than "3 + 3 + 3 + 3 =12."

Figure 4 shows how Samantha demonstrated multiplication as an operation by replicating one group of three pencils nine times. It was observed that she did this without using direct counting or repeated addition to obtain an answer. Samantha, like Natalie, formed one group initially, "to be timesed." This may be significant in that many children who were developing multiplication as an operation eventually formed one group to represent the multiplier. This is an R_{2a} (CS) response with ikonic support.

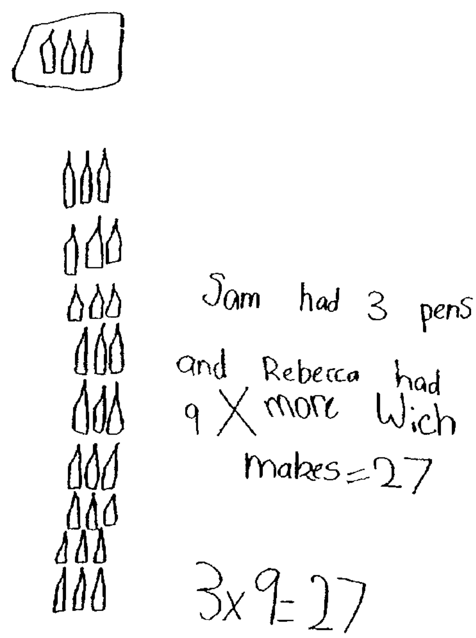

Figure 4. Samantha's response to comparison problem.

At the R_{2b} (CS) level, responses were characterised by the automatic recall and application of basic multiplication and division facts. Unknown facts could be derived from known facts, where children used multiplication and division as inverse processes to derive a solution. Children responding at this level solved a range of multiplicative problems including structures of equal grouping, comparison, array, cartesian product, and ratio. Typical responses included the following.

R_{2b} (CS): 6 times 4 is 24 so 12 times 4 is 48.

R_{2b} (CS): 72 divided by 8 . . . 9 eights are 72.

R_{2b} (CS): 3 by 7 is 21, so 42 divided by 7 must be 6.

R_{2b} (CS) responses also showed understanding of commutativity in that the order in which multiplications are performed does not affect the final product. This appeared to be the consolidation typical of movement into another U-M-R cycle necessary for more complex problem solving. The new cycle would be accompanied by understanding the commutativity of multiplication and the inverse relationship between multiplication and division, rather than rote learning of corresponding multiplication and division facts.

Analysis of Responses to Multiplication and Division Problems by Response Mode and Level

Table 3 and Table 4 show the overall distribution of responses to multiplication and division problems across the various SOLO levels at each interview in the longitudinal study. In these tables, data have been pooled from the 12 multiplication or division problems. Concrete symbolic responses have been grouped in three categories to correspond with the prior analysis of intuitive models (Mulligan & Mitchelmore, 1997): U_1, M_1, R_1 (CS) responses use a direct counting model without concrete modelling, U_2 (CS) and M_2 (CS) responses use repeated addition (or subtraction) without concrete modelling, and R_{2a} (CS) and R_{2b} (CS) responses use multiplicative operations.

Table 3 shows that prestructural solutions were quite common in Year 2 but rare in Year 3. Prestructural responses were universally incorrect. The most common mode at interview 1 was the ikonic mode; by comparison, the most frequent mode for the last three interviews was the concrete symbolic mode (combining all categories).

Table 3
Percentage Distribution of Responses to Multiplication Problems Across SOLO Levels, by Interview

SOLO level	Interview			
	1	2	3	4
P	20	16	6	5
IK	46	37	41	31
U_1, M_1, R_1 (CS)	9	11	10	9
U_2, M_2 (CS)	22	31	29	26
R_{2a}, R_{2b} (CS)	3	5	14	29

Note. n = 70, 68, 62, 60 for interviews 1 to 4, respectively.

Table 4
Percentage Distribution of Responses to Division Problems Across SOLO Levels, by Interview

SOLO Level	Interview			
	1	2	3	4
P	24	21	11	6
IK	44	37	47	29
U_1, M_1, R_1 (CS)	12	13	11	10
U_2, M_2 (CS)	17	25	21	29
R_{2a}, R_{2b} (CS)	3	4	10	26

Note. n = 70, 68, 62, 60 for interviews 1 to 4, respectively.

Table 4 shows the distribution of SOLO levels in responses to the division problems at each interview stage. Because performance on division tasks was slightly lower than for multiplication problems, prestructural solutions (all incorrect) accounted for slightly more responses than for multiplication; but they also decreased rapidly. The pattern of ikonic and concrete symbolic mode responses was virtually the same as for the multiplication problems. A substantial number of concrete symbolic responses used repeated addition, with progression to a multiplicative operation model at interview 4.

The change in the distribution of responses across SOLO levels from interview to interview corresponds to the development of the equal-grouping structure and associated calculation strategies. By tracing the paths of responses of individual students it was possible to identify those who progressed from ikonic to concrete symbolic mode during the period of the interviews. Some general patterns are reported here. For multiplication problems, of the 46% of children using ikonic mode at interview 1, about half (56%) gave ikonic responses at interview 4. The other children (44%) responded in the first cycle of concrete symbolic mode or the U_2 level of the second cycle of the concrete symbolic mode. Similarly, 25% of the children who responded at ikonic level (41%) at interview 3 gave responses at the U_2 (CS) or M_2 (CS) levels at interview 4. For division problems, of the 44% of children giving ikonic responses at interview 1, 35% of these progressed to the U_2 (CS) or M_2 (CS) levels by interview 4. The increase in ikonic responses to 47% at interview 3 can be attributed to those students moving from prestructural responses to the ikonic mode. Further, on all multiplication and division problems, in only 3% of the cases did students successfully use a less sophisticated strategy to solve a problem that they had successfully solved at the previous interview and only 2% of students failed to solve the same problem again.

The following examples show how two individual students progressed over the four interviews. The first example shows how Nathan moved from prestructural to ikonic mode and then developed through to the unistructural level of the second cycle of concrete symbolic mode. He made the following responses to the quotition problem "12 toys are shared equally among the children. If they each had 3 toys how many children were there?" at successive interviews:

P: They have to share the toys.
U_1 (IK): They get three toys. That's how many they get each.
U_1 (CS): [Child shares by dealing mentally, keeping track with fingers] 1, 2, 3 toys, 4, 5, 6 toys, 7, 8, 9 toys, 10, 11, 12 toys.
U_2 (CS): That's 3 each so 3 +3 = 6 + 3 = 9 + 3 =12. So that's 1, 2, 3, 4 threes.

Nathan modelled the problem with counters in the U_1 (IK) mode and used the same model abstractly at the U_2 (CS) level. He visualised three toys in each group but used unitary counting to represent and calculate the number of composite units. He used his fingers to help keep track of the number of items.

In the second example, Sarah progressed from the extended abstract level of the ikonic mode through to the relational level (R_{2a}) of the second cycle of concrete symbolic mode. Her responses at the four interviews to the comparison problem "Sue has 3 books, and Jane has 5 times as many. How many books does Jane have?" were as follows:

EA (IK): [Child models 3 books first, then models another 5 groups of 3 counters] That's 3, 6, 9, 12, 15 books altogether [counts books in multiples of 3].
M_1 (CS): There's 3 books for Sue so 5 times would be 3, 6, 9, 12, 15 . . . 1, 2, 3, 4, 5.
U_2 (CS): That's 3 + 3 = 6; 3 + 3 = 6 and 3 makes 9, 12, 15.
R_{2a} (CS): 3, 5 times is 15.

Sarah showed a more sophisticated development of the composite structure of multiplication than Nathan as she used her knowledge of the multiple sequence of threes early in the ikonic mode to assist her in coordinating composite units. The progression shown by Nathan and Sarah highlight the differences in students' development of composite structure, their ability to model the problems abstractly, and the increasing level of sophistication of their calculation strategies.

Discussion

The formation of images at the prestructural and ikonic level appeared critical to the development of the equal-grouping (composite) structure. Images initially lacking in structure at the prestructural level became more organised mathematical elements in the ikonic mode; random drawings of ikons gave way to drawings showing the structure of the groups, albeit unequal. Children initially giving prestructural responses (20% at interview 1) became less reliant on their physical or imagined reality and began to focus on numerical aspects of the problem in the ikonic mode. This related to their ability to interpret the semantic structure of the problem and their ability to represent equal-sized groups through physical or concrete models.

The SOLO analysis also distinguished the transition from prestructural and U_1 (IK) and M_1 (IK) levels, to the R_1 (IK) level, by the development of counting strategies that eventually took advantage of composite structure. At the R_1 (IK) level, the formation of composites with a concrete or physical model such as fingers required the child to establish a group of single items as one unit in order for them to be considered a countable unit (e.g., "three ones as one three"). Children whose counting was limited to the initial forward number sequence were incapable of doing this. Forming and counting composite units required a modification to the counting sequence using a numerical pattern or spatial pattern. This provided a basis for a pattern of multiples which took into account the number in each group as well as the counting pattern (e.g., "2, 4, 6" and "3 groups of 2"). Responses in the ikonic mode were often based on partitioning a number of items into equal groups using a skip counting sequence. In some cases, only one group or row of items was used repeatedly as the composite unit to represent and calculate the total number of items. Steffe (1994) referred to this as a figurative composite unit because the child uses the unit mentally as a "tangible" model whose elements can be visualised and counted.

At the R_1 (IK) level children were still unable to generalise the pattern of composites, even though increasingly appropriate composite counting strategies developed. The calculation of the total number of items, or the number of groups, was assisted by the concrete or physical model which provided the structure of composites without the child having to use them abstractly. The transition to the EA (IK) level and U_1 (CS) level was marked by the ability to conserve consistently; the equal-grouping structure could be recognised no matter what the arrangement of items or situation presented.

The development of efficient counting (skip and double counting) and calculation strategies has been accepted as integral to developing composite structure. Coordinating composite units (e.g., "three threes as a unit of nine") depends on the ability to move beyond counting based on a unistructural notion and to use a pattern of multiples as a double count ("1, 2, 3 (one), 4, 5, 6 (two)," etc.) mentally. While the development of direct counting and ikonic functioning precedes development of composite structure, there exists a complex interrelationship between counting and composite structure in the concrete symbolic mode. This has been described in this analysis as two cycles. The use of skip and double-counting procedures gives rise to more efficient processes that take advantage of the equal-grouping structure in the second cycle where repeated addition (or subtraction) is generalised as an operation.

The development of repeated addition or repeated subtraction in the second cycle of the concrete symbolic mode does not constitute a full conceptual understanding of multiplication or division. It is not until the relational level that the development of multiplication and the related division process is distinguished as the distribution of one composite unit across elements of another composite unit (e.g., "six, three times means $6 \times 3 = 18$"), thus generalising the structure of composites. Critical to developing a relational understanding of multiplication is the ability to see multiplication and division in an inverse relationship and to explain commutativity (e.g., $6 \times 9 = 9 \times 6$). Children who are able to recall multiplication and division number facts without being able to explain and represent the composite structure are not yet functioning at the relational level.

This observation highlights the importance of children acquiring the binary structure of multiplication, where the product represents one or more units distributed simultaneously over one or more composites (equal-sized units). Furthermore, the situation is more complex than for addition and subtraction because, in multiplication and division, cases arise where two quantities are combined to form a third quantity that is unlike either of them. Generalising the combination process is critical to fully understanding the operations of multiplication and division.

In terms of the first two research questions addressing the development of composite structure, the data showed that this emerged from prestructural and ikonic mode responses through the development of increasingly sophisticated counting strategies. The development from unitary perceptual counting, skip counting, and double counting through to repeated addition and subtraction without direct modelling was based on the formation and coordination of composites. The data showed a complex interdependence between the existence of ikonic level support and the development of counting and calculation strategies. Skip- and double-counting strategies closely reflected composite structure with and without modelling at the ikonic and concrete symbolic modes respectively, and each served to support the development of repeated addition and multiplication and division as operations.

In relation to the third question concerning the role of ikonic support in the concrete symbolic mode, it appears that ikonic functioning continued to be used through the concrete symbolic mode because ikonic images became connected with increasingly complex composite units and notational representations of multiplication and division. Repeated addition, for example, gave rise to the symbolism of multiplication and division provided that the child could see the overall pattern of composites and move beyond using repeated addition for calculation. In the teaching project, ikonic support was used in a variety of ways by children to explain their methods of solution; in many cases, it was necessary to gain a solution.

It may also be asked if children develop composite structure in the concrete symbolic mode but do not use this structure mentally because it is less cognitively demanding to revert to an ikonic mode of functioning. For example, it is easier to share 12 concrete items into groups of 4 to establish the number in each group than to keep track mentally of the distribution of items and number of groups simultaneously. In the SOLO model, the former process belongs to the relational level in the first cycle of the concrete symbolic mode—whereas the latter belongs to the second cycle. Children may also visualise ikonic support to assist the way they impose structure on multiplicative situations. Even when operating in the concrete symbolic mode, the use of repeated addition notation such as 3 + 3 + 3 + 3 may be built upon images of groups of 3 at the ikonic level. It is often difficult to judge the level of ikonic support if it is not overtly displayed in a response. It is suggested, however, that for responses whose main outcome is a concrete symbolic elaboration, then an aspect of "abstract" ikonic support is most likely to be a genuine multimodal contribution—it would not be possible for the ikonic mode on its own to support that level of abstractness.

Evidence of multimodal functioning was more clearly shown through children's representations and explanations of responses in the teaching project. Because children were encouraged to record their representations, regardless of whether they were in the ikonic or concrete symbolic modes, the interaction between modes could be investigated. In cases where children were operating mainly in the second cycle of the concrete symbolic mode, ikonic drawings represented more sophisticated models using notational and symbolic systems that had been translated "down" to the ikonic level. While this was unnecessary for a solution, recordings showed the level of structure in the child's thinking more clearly.

Implications

In terms of instruction, students may need ikonic support in modelling some semantic structures so that they can apply counting strategies successfully to them. However, once children have established composite structure and associated skip and double counting strategies in the ikonic mode, concrete and physical models of support can be gradually removed in order to encourage a visualised structure of composites in the concrete symbolic mode. Even if children revert to less sophisticated unitary counting methods to calculate a solution, the move to concrete symbolic thinking is necessary for the development of multiplication and division as operations. Ikonic support can still be used, but children might be encouraged to visualise these images rather than to revert to concrete or pictorial models. Evidence from children's drawings in the teaching project showed in many cases that ikonic support was unnecessary to gain a solution but was useful in showing how ikonic images influenced the solution process.

Beyond unitary counting and sharing in the concrete symbolic mode children can be encouraged to use the equal-grouping structure to develop more efficient strategies involving repeated addition. Steffe (1994) and Confrey (1994) both argue that we should teach alternative constructs for multiplication (i.e., repeated addition and coordinating composite units) and abandon any single interpretation for all children. When repeated addition becomes generalised, the idea of multiplication and division as operations emerges at the relational level of the second concrete symbolic cycle. One implication of this is that acquisition and retrieval of basic multiplication and division facts must be based on the child explaining the relationship between composites (e.g., "6 . . . 3 times is 18" symbolised as $6 \times 3 = 18$ and "3 groups of 6 are 18" symbolised as 3×6 is 18). Ikonic support may be used to distinguish the difference in composite structure.

In considering the appropriateness of the SOLO model for structuring young students' outcomes on mathematical tasks, it is useful to compare the results of this study with those of Watson et al. (in press) who analysed the responses of 30 children from pre-Grade 1 to Grade 4 on two hands-on tasks involving fractions. In problem-solving contexts, children were asked to divide a pancake fairly among three dolls and to determine one-half and one-third of twelve marbles. The responses reflecting ikonic and first cycle concrete symbolic understanding parallel well the structure observed in the present study. In the ikonic mode, a single idea of sharing was realised with concrete material being split; but there was no recognition of the distribution of shares to all recipients. This parallels the inability to form composites at the U_1 (IK) level in the present study. At the M_1 (IK) level, the idea of fractional sharing included all who should receive shares; but there was no attempt at equality of shares. These responses correspond to those in the current study that took more elements of a problem into account but did not form composites. At the R_1 (IK) level in both studies, children used more complex but idiosyncratic methods, lacking the ability to conserve.

In the first cycle of the concrete symbolic mode, U_1 (CS) responses for fractional sharing involved the conservation of an equal number of parts given to recipients, regardless of the size of the parts of the fractions involved. The inability to go beyond giving the same numerical value (of pieces) to each recipient is similar to the use of unitary counting in the current study where more complex patterns were not recognised. At the M_1 (CS) level different fractions were distinguished, although not appropriately, and fair sharing was attempted in terms of the amount of substance as well as the number of pieces. Sharing was done laboriously, in a similar fashion to the forming of composites by children in the present study. At the R_1 (CS) level in both studies, patterns associated with geometry or counting were consolidated in the problems presented to the students. It would be of interest in future research to present the same children with tasks based on both topics in order to further elucidate the structural relationship of the two areas of the curriculum.

SOLO analysis of children's responses to multiplication and division problems highlights the importance of assessing critical elements of developing composite structure and associated counting and calculation strategies. The detail provided throughout three cycles in two modes of functioning can assist teachers and curriculum planners to be sure they cater for all possibilities of students' functioning. Lack of appreciation of the structure may be detrimental to the understanding of some children. These aspects have been integrated into a Learning Framework in Number (NSW Department of Education and Training, 1998) as part of a classroom-based assessment project currently operating in Kindergarten and Grade 1 classrooms in New South Wales public schools. Further research will investigate young children's development of composite structure in relation to other key aspects of developing number knowledge.

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