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Calculus Students' Ability to Solve Geometric Related-Rates Problems

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This study assessed the ability of university students enrolled in an introductory calculus course to solve related-rates problems set in geometric contexts. Students completed a problem-solving test and a test of performance on the individual steps involved in solving such problems. Each step was characterised as primarily relying on procedural knowledge or conceptual understanding. Results indicated that overall performance on the geometric related-rates problems was poor. The poorest performance was on steps linked to conceptual understanding, specifically steps involving the translation of prose to geometric and symbolic representations. Overall performance was most strongly related to performance on the procedural steps.

University students' poor performance in beginning calculus courses has been well documented (Cipra, 1988; Culotta, 1992; Ferrini-Mundy & Gaudard, 1992; Ferrini-Mundy & Graham, 1991; Peterson, 1986). Reports that about half of all students enrolled in introductory calculus courses either fail or withdraw before completing the course provided the impetus for the calculus reform movement of the last fifteen years (Douglas, 1986; Steen, 1986; Tucker & Leitzel, 1995). Despite promising results from several studies comparing student performance in reformed and traditional classes (Bookman & Friedman, 1994; Meel, 1998; Park & Travers, 1996), the reform movement has suffered a backlash and many universities, especially in the United States, have returned to more traditional approaches to calculus instruction (Wilson, 1997). Even as the curricular pendulum swings, the reasons that students perform so poorly in calculus are still unclear (Thompson, 1994).

Although there are still many open questions about the teaching and learning of calculus, researchers studying student performance in higher-level mathematics, including calculus, have made several observations. For example, students tend to view mathematics as a collection of algorithmic procedures to be mastered (Dreyfus, 1990; Schoenfeld, 1994; Silver & Marshall, 1990). In general, students at the college level are proficient at performing algorithms, but they lack the ability to connect the algorithms to their underlying conceptual bases (Orton 1983a, 1983b). The inability to link the conceptual with the procedural is thought to be at the root of students' difficulties with higher-level mathematics (Dreyfus, 1990).

The focus of the present study is on university students' performance on geometric related-rates problems. The term *related-rates problem,* as used throughout this study, refers to the type of calculus problem that requires the determination of "the rate of change (with respect to time) of some variables based on their relationship to other variables whose rates of change are known" (Dick & Patton, 1992, p. 270). Although several reformed calculus texts exclude or de-emphasise related-rates problems (Davis, Porta, & Uhl, 1994; Hughes-Hallett, et al., 1994; Smith & Moore, 1996), these problems are standard fare in the traditional courses which are enjoying renewed popularity. The related-rates problems used in this study are restricted to those in a geometric context. In a *geometric related-rates problem,* the relationship between the variables, upon which the relationship between the rates is based, is described by a geometric equation. Students' difficulty with geometric related-rates problems has been noted by several authors (Balomenos, Ferrini-Mundy, & Dick, 1987; White & Mitchelmore, 1996). In addition, the multi-step, multi-faceted nature of these problems provides fertile ground for examining students' procedural knowledge and conceptual understanding of several important mathematical ideas that appear throughout the calculus course.

In the present study, a standard multi-step model for solving related-rates problems is identified. Each step in this model is then classified as procedural or conceptual. By analysing student performance on items corresponding to each step of the standard solution model, inferences are made about students' procedural knowledge and conceptual understanding.

Rationale

Solving Geometric Related-Rates Problems

By examining expositions and worked examples in several textbooks (Berkey, 1988; Dick & Patton, 1992; Feroe & Steinhorn, 1991; Lial, Miller, & Greenwell, 1993), interviewing mathematics faculty and graduate students, and examining student work, it was found that the solution of related-rates problems generally involves the same six steps. More complex problems may also require one additional step: the solution of an auxiliary problem—which in the case of geometric related-rates problems is related to the geometric context (e.g., proportions in similar triangles, use of the Pythagorean relationship). The seven steps are shown in Table 1; I will refer to them as the *standard solution model* for geometric related-rates problems. It should be noted that, although Steps 1-6 generally occur in sequence, Steps 1 and 2 may happen in either order and there may be frequent "backtracking". Step 7 may occur at any point.

Procedural Knowledge and Conceptual Understanding

Several authors (National Assessment of Educational Progress [NAEP], 1988; Cooney, Davis, & Henderson, 1975; Hiebert & Lefevre, 1986) have made a distinction between procedural knowledge and conceptual understanding. Procedural knowledge is characterised by the ability to note, select, and apply the appropriate concrete, numerical, or symbolic procedures required to solve a problem; and to verify and justify the correctness of these procedures. Conceptual understanding is characterised by the ability to identify examples and non-examples of a concept; to use, connect, and interpret various conceptual representations; to know, apply, distinguish, and integrate facts, definitions, and principles; and to interpret assumptions and relations in a mathematical setting (NAEP, 1988).

Table 1
Description and Classification of Steps in Solving Geometric Related-Rates Problems

Step	Description	Classification
1	Sketch the situation and label the sketch with variables or constants	Conceptual
2	Summarise the problem statement by defining the variables and rates involved in the problem (words to symbols translation) and identifying the given and requested information	Conceptual
3	Identify the relevant geometric equation	Procedural
4	Implicitly differentiate the geometric equation to transform a statement relating measurements to a statement relating rates	Procedural
5	Substitute specific values of the variables into the related-rates equation and solve for the desired rate	Procedural
6	Interpret and report results	Conceptual
7	Solve an auxiliary geometry problem	(varies)

White and Mitchelmore (1996) claim that procedural knowledge and conceptual understanding involve different types of concepts. *Abstract-general* concepts are formed by a generalising → synthesising → abstracting sequence (Dreyfus, 1991) or an interiorisation → condensation → reification process (Sfard, 1991). Such concepts are linked to one another to form conceptual understanding. By contrast, *abstract-apart* concepts are formed by learning symbolic manipulations without reference to their meaning. Students whose concepts are abstract-apart can only acquire procedural knowledge.

It was found that most of the steps in the standard solution model for geometric related-rates problems could be classified as procedural or conceptual on the basis of whether primarily procedural knowledge (involving abstract-apart concepts) or primarily conceptual understanding (involving abstract-general concepts) was required for its completion. The following explains the rationale for the classification included in Table 1.

Steps 1 and 6. The first and sixth steps both rely on students' ability to translate between verbal and symbolic representations of the quantities involved in the problem. These steps clearly require conceptual understanding. For example, White and Mitchelmore (1996) show that, in order to solve problems that require translation from prose to symbols, students must have an abstract-general concept of variable. In order to perform the translation, students must conceive of a variable as representing an element in the context and as retaining this meaning when mathematical processes are performed on it. Students whose concept of variable is abstract-apart can only manipulate symbols that are disengaged from the context.

By comparing student performance on isomorphic related-rates problems that required a varied amount of translation of prose to symbols, White and Mitchelmore (1996) concluded that many beginning calculus students have abstract-apart concepts of variable. Two of the common errors they report illustrate this. (1) An inappropriate use of a constant in place of a variable: Required to derive a rate at a particular instant, many students calculated the given rates at that instant—instead of deriving the related rate using variables and then finding its value at that instant. (2) A manipulation focus: Many students wrote down remembered formulae and then sought variables to substitute in them. Thompson (1994), in a study of student understanding of the rate concept in the context of the Fundamental Theorem of Calculus, also found that students tend to manipulate variables according to previously learned patterns of actions without attending to the meanings of the variables.

Step 2. The second step requires students to go beyond translation of individual variables in order to identify assumptions and construct representations of relations among variables and rates described in the problem statement. For example, the rate of change of a variable is represented by the derivative of that variable with respect to time. This step relies on students' conceptual understanding, because students must connect the symbolic representation for a derivative with the concept of derivative as a rate of change. In his seminal study of student understanding of differentiation and integration, Orton (1983b) discovered that one of the understandings that most often eluded students was the ability to interpret the derivative as a rate.

Perhaps one reason that students have so much difficulty interpreting derivatives as rates is that they are unable to distinguish between instantaneous and average rates of change. Schneider (1992) presented a related-rates problem to two classes of about 20 students, each of whom were in their second to last year of secondary school (about 16 years old). Schneider's study documented students' difficulties making the cognitive leap from average rate of change to instantaneous rate of change. The students who were able to make the leap did so by maintaining the distinction between the limiting value of an expression as a component variable approaches zero and the value of an expression when a component variable is equal to zero. Students who were unable to make the leap could not do so because they did not distinguish the limiting value from the exact value. Even among those who had completed three semesters of calculus, Thompson (1994) found that students could not understand covariation among rates because of their weak schemes for average and instantaneous rates of change. Schneider's and Thompson's findings give further credence to the categorisation of Step 2 as primarily involving conceptual understanding.

Steps 3, 4 and 5. Since the third, fourth and fifth steps of the standard solution model involve the selection and application of symbolically based procedures, they may be characterised as calling on procedural knowledge. Orton (1983b) found that students had least difficulty using algorithms to compute derivatives, although he noted difficulties factoring and solving algebraic equations.

Step 7. Students' ability to solve auxiliary problems has not been previously examined in the context of ability to solve related-rates problems. Since the auxiliary problems vary widely, it is not possible to classify this step as procedural or conceptual. Like the related-rates problems themselves, some components of an auxiliary problem may require procedural knowledge whereas others require a conceptual understanding.

Balomenos, Ferrini-Mundy, and Dick (1987) suggested that one way to improve student performance on related-rates problems would be to have them gain more experience solving the geometry problems that are often embedded in them. The relationship between student performance on Step 7 and overall performance on related-rates problems should provides evidence on the importance of this step.

Statement of the Problem

This study aimed to characterise students' ability to solve geometric related-rates problems by identifying the conceptual and procedural knowledge required and, if possible, identifying which type of understanding is most closely related to successful performance. The specific questions guiding the research were as follows.

1. How do university students enrolled in an introductory calculus course perform on geometric related-rates problems?
2. How do such students perform on (a) the conceptual steps 1, 2 and 6, (b) the procedural steps 3, 4 and 5, and (c) the combined step 7 of the standard geometric related-rates model?
3. How is performance on the various steps of the model related to each other and to overall performance?

Method

In order to investigate the three research questions, a non-randomised performance study was designed. Students' ability to solve geometric related-rates problems was assessed using two written instruments. The first instrument, consisting of three open-ended geometric related-rates problems, was used to determine how students perform on geometric related-rates problems. The second instrument was used to assess students' ability to perform each step of the standard solution model independently. Although student performance on written tests is a limited source of information about their problem solving, it was felt that such tests could identify components which cause students the greatest difficulty. In doing so, it has been assumed that responses to written tests serve as reasonable indicators of students' conceptual and procedural understanding.

Sample

The sample for the study was selected from students enrolled in two introductory calculus courses (Calculus I) at a large, private university in an urban area in the northeast United States. The courses were designed for students majoring in mathematics, physics, or engineering, but were also open to students majoring in other subjects. Both courses were traditionally taught, and the same textbook (Berkey, 1988) was used in both classes.

There were 120 students in Course A. The majority of students in this course were enrolled in the College of Liberal Arts or the College of Engineering. Course B, with 40 students, was primarily designed for College of Engineering students who had performed relatively poorly on a mathematics placement test administered by the University. Both groups of students had similar mathematical preparation, as indicated by their standardised test scores (SAT).

Table 2
Selected Sample Characteristics ($n = 58$)

Characteristic	Value
Mean (SD) SAT-Mathematics score	607 (68)
Mean (SD) SAT-Verbal score	497 (118)
Mean (SD) number of university courses completed	0.2 (0.5)
Mean (SD) year at university	1.1 (0.4)
Percentage completed a previous calculus course	56%
Percentage female	69%

The 58 students (34 from Course A and 24 from Course B) who took both the test instruments form the sample for the present study. Table 2 presents summary statistics on some relevant characteristics of the sample. Analysis showed that this sample of students was reasonably representative of the 160 students enrolled in Courses A and B (Martin, 1997).

Instruments

Test 1. The first test instrument (Figure 1) was a mandatory quiz contributing to both course grades. It was designed to measure students' ability to solve standard geometric related-rates problems. The three problems were selected from Dick and Patton (1992) and Feroe and Steinhorn (1991) for their comparability to questions in the course text. In addition, the problems were chosen to vary in geometric context and the number of steps required for their solution. The content validity of Test 1 was verified by a panel of experts consisting of mathematics professors, a physics professor, and graduate students in mathematics-related fields.

Test 1 was administered immediately after instruction on related-rates problems, about a month after the courses began. It was taken by 154 students.

Test 2. The second test was a ten-question instrument designed to measure students' ability to perform the seven steps in the standard solution model for geometric related-rates problems (see Table 1). Several questions contained multiple parts, each of which was considered a separate item, so that the total test consisted of 26 items. Each item addressed one and only one of the seven steps. (See Table 4 for the number of items addressing each step.)

Show all calculations. Answer every question with a full sentence.

1. Consider a balloon being inflated over time. Suppose the balloon is considered to be a perfect sphere and its radius is changing at the rate of 3 cm/sec. How fast is the volume of the balloon changing when the radius is 20 cm?
2. Two roads intersect at right angles. Two cars leave simultaneously from the intersection. The first car travels at a speed of 30 mph travelling due north. The other car travels at a speed of 40 mph travelling due east. How fast is the distance between them increasing 30 minutes later?
3. A water tank is in the shape of a right circular cone that has a radius of 5 feet and a height of 10 feet. It is positioned so that the cone points straight down. Water is being drained out of the tank at the rate of 2 cubic feet per minute. At what rate is the height of the water in the tank changing when there are 18π cubic feet of water in the tank?

Figure 1. Test 1.

Figures 2, 3, and 4 show some sample items from Test 2. In Figure 2, Question 8 addressed Step 6 of the standard solution model, Question 10a addressed Step 1, and Question 10b addressed Step 2. In Figure 3, Question 1 addressed Step 3, Question 4 addressed Step 4, and Question 7 addressed Step 5. Question 9 in Figure 4 addressed Step 7. The complete test is available in Martin (1997). Members of the expert panel confirmed its content validity.

Since Test 2 was fairly long, it was necessary to include a safeguard in the design so that the order of the questions would not affect student performance. Ten alternate forms of the test were constructed using a Latin Square Design and randomly assigned to students. An analysis of student performance on the ten versions of Test 2 indicated no significant differences among the ten forms.

The second test was administered within two days of Test 1. It was optional, but students were strongly encouraged to take it. Test 2 was taken by 58 students, all of whom had also taken Test 1.

Scoring

Test 1. Test 1 was scored by the researcher using a rubric based on solutions generated by the panel of experts. This rubric outlined the criteria for allocation of partial credit and was designed so that independent raters could score the test without requiring any additional instructions. The three problems were valued at 8, 10, and 12 points, respectively, for a total of 30 points. The second and third problems were allocated more points because the inclusion of auxiliary steps meant that there were more steps required for their completion.

8. A calculus student has left the following computations on the page under the problem below. Assuming that the computations are correct, write the answer to the problem in one or two sentences.

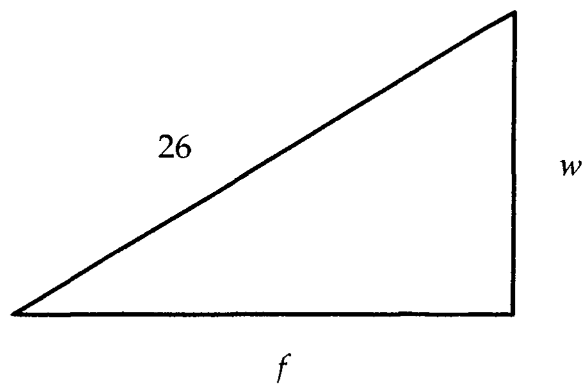
   A ladder 26 ft long leans against a vertical wall. If the lower end is being pulled away from the wall at the rate of 5 ft/sec, how fast is the top of the ladder moving when the lower end is 10 ft from the wall?

$$f^2 + w^2 = 26^2$$

$$2f\frac{df}{dt} + 2w\frac{dw}{dt} = 0$$

$$\frac{dw}{dt} = \frac{-f}{w}\frac{df}{dt}$$

$$\frac{dw}{dt} = \frac{-10}{24}(5) = \frac{-25}{12}$$



10. Consider the following problem. Do not perform any calculations.

    A woman standing on the bank of a river is reeling in a fish. The tip of her fishing rod is 5 ft above the water's surface at the bank's edge. How fast is the fish approaching shore, when there are 30 ft of line out from the tip of the rod and the woman is reeling in line at a rate of 3 in/sec?

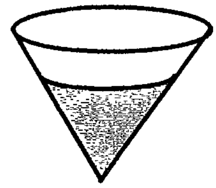
    (a) Draw a sketch that may be useful in visualising the situation and label all variables and known quantities.
    (b) Define (in words and symbols) all variables and rates that are relevant to the solution of the problem.

Figure 2. Sample conceptual items from Test 2.

For each problem, points were awarded for each of Steps 3-7 of the standard solution model (except for Problem 1, which contained no Step 7). Steps 1 and 2 were omitted because they could be performed mentally without written evidence of their completion. The various steps are, of course, chained (i.e., the correct completion of a step depends on the correct completion of previous steps). The full number of points was awarded for each step if the appropriate procedure was performed correctly, even if incorrect information was incorporated from previous steps.

To determine the reliability of Test 1 scores, another rater independently scored a quarter of the test papers. The correlation between the two raters' total scores was 0.95. The correlations between the scores given by the two raters for Problems 1, 2, and 3 were 0.89, 0.90, and 0.90, respectively. These correlations indicated satisfactory inter-rater reliability. To measure the internal consistency of Test 1, Cronbach's alpha was used. The value of 0.63 indicated a moderate internal consistency which was acceptable for the purposes at hand.

1. Consider a conical water cup, with a small hole through which water leaks from the bottom. See sketch.



Suppose the rate at which the radius of the water in the cup is decreasing is known.

(a) Record a geometric formula that would be useful in finding the rate at which the volume is changing.

4. Implicitly differentiate the following equations with respect to time.

(a) $x^3+y^3+z^3= w^3$
(b) $V = x^2y$
(c) $A = s^2$

7. Suppose $n = 3$ cm, $\frac{dn}{dt} = 4$ cm/sec, $\frac{dV}{dt} = 5$ cm^3/sec. Find $\frac{dA}{dt}$ in each of the following situations. (Be sure to include units in answer.)

(a) $\frac{dA}{dt} = 2\pi n\frac{dn}{dt}$
(b) $\frac{dV}{dt} = \frac{1}{3}n\frac{dA}{dt} + \frac{1}{3}\frac{dn}{dt}4\pi n^2$

Figure 3. Sample procedural items from Test 2.

9. Two roads intersect at right angles. A car leaves the intersection headed eastbound and travels at 40 mph. Two hours later, another car leaves the same intersection heading northbound at 30 mph. How far is each car from the intersection and from the other car, 5 hours after the eastbound car leaves? See sketch.

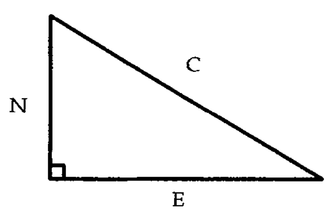


Figure 4. Sample auxiliary problem from Test 2.

Test 2. Test 2 items were scored as correct or incorrect by the researcher. Because there was no partial credit, no scorer reliability check was carried out. Cronbach alphas were 0.73 for the test as a whole, 0.57 for the total score on the conceptual cluster (the 6 items related to Steps 1, 2, and 6) and 0.55 for the total score on the procedural cluster (the 18 items related to Steps 3, 4, and 5).

Results

Table 3 shows how students performed Test 1 as a whole and on each of the three questions individually. In each case, the mean and standard deviation is reported as a percentage of the maximum possible score. The difference between the mean scores of students enrolled in the two courses was not significant ($p > 0.05$).

Table 3
Mean and Standard Deviations of Subscores and Total Score on Test 1 (n = 58)

Question (context)	Mean score	Standard deviation
1. (Sphere)	55%	25%
2. (Triangle)	52%	35%
3. (Cone)	27%	25%
Total score	43%	22%

Test 2 results are summarised in Table 4. Again, means and standard deviations are reported as percentages of the maximum possible score for each cluster of items. The mean score for the conceptual steps (Steps 1, 2, and 6) varied from 17% to 47%, giving an overall (unweighted) mean of 30%. The mean score for the procedural steps (Steps 3, 4, and 5) varied from 41% to 78%, with an unweighted overall mean of 54%.

Table 4
Mean and Standard Deviations of Subscores on Test 2 (n = 58)

Step	No. of items	Mean score	Standard deviation
1. Sketch and label	2	17%	29%
2. Summarise information	3	26%	23%
3. Identify equation	11	78%	21%
4. Implicitly differentiate	5	44%	36%
5. Substitute and solve	2	41%	33%
6. Interpret results	1	47%	50%
7. Solve auxiliary problem	2	53%	38%

Table 5 shows how the subscores on Test 2 were related to each other and to the total score on Test 1. The Test 1 total score was significantly related to the Test 2 subscores for Step 2 (one of the conceptual steps); Steps 3, 4, and 5 (the three procedural steps); and Step 7. All correlations among Test 2 subscores were positive and about half of them were statistically significant.

Table 5
Correlations between Total Test 1 Score and Test 2 Subscores

	T1	T2.1	T2.2	T2.3	T2.4	T2.5	T2.6
T2.1	0.15						
T2.2	0.31*	0.26*					
T2.3	0.39**	0.30*	0.28*				
T2.4	0.63**	0.34**	0.34**	0.43**			
T2.5	0.38**	0.15	0.18	0.19	0.23		
T2.6	0.14	0.30*	0.32*	0.01	0.21	0.16	
T2.7	0.49**	0.23	0.40**	0.48**	0.44**	0.21	0.11

Note: T1 = Total score on Test 1, T2.1= Subscore 1 on Test 2, and so on.
*$p < 0.05$ **$p < 0.01$

Discussion

Overall Performance

Even though the students had recently studied the material, the average percentage score was below 60% for all of the geometric related-rates problems on Test 1. The students had the least difficulty with Question 1 (the sphere problem) and the most difficulty with Question 3 (the cone problem). There are several factors which might have contributed to this difference.

An important factor seems to have been the degree of convergent and divergent thinking required to analyse the geometric context. Question 1 requires only identification of a geometric formula, differentiation, substitution, and algebraic manipulation. It does not require any reasoning about the sphere other than correctly identifying and linking variables. By contrast, Questions 2 and 3 incorporate auxiliary steps which require reasoning within the geometrical context of each problem. In Question 2, however, the problem statement (which includes references to rates, time, distance, and right-angled triangles) cues all the important ideas related to the solution of the auxiliary problem. On the other hand, the statement of Question 3 would lead one to sketch a cone—whereas the auxiliary problem requires recognition of similar triangles obtained by sectioning the cone.

Another factor contributing to problem difficulty may have been the number of steps required to solve the auxiliary problem: Question 1 has no auxiliary problem, and the auxiliary problem in Question 2 requires fewer steps than the auxiliary problem in Question 3 with fewer choices at each step. Finally, students may simply have been more familiar with the distance-rate-time relationship and the Pythagorean theorem required in Question 2 than with similar triangle relationships and the formula for the volume of a cone in Question 3.

Performance on Conceptual Steps

Performance on Test 2 was lowest for two of the conceptual steps, Steps 1 and 2 (see Table 4). This result echoes findings of Dreyfus (1990), Orton (1983a, 1983b), and White and Mitchelmore (1996).

To complete Step 1, students must sketch a geometric figure that represents the relationships described in the problem statement and label the figure with variables and constants as appropriate. Students' difficulty with this step (17% correct) reinforces Dreyfus' (1990) conclusion that when students solve calculus problems, visualisation is rare and is disconnected from algebraic representations.

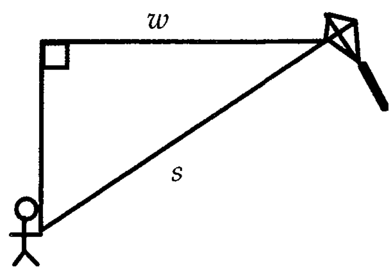
To complete Step 2, students must identify the given and requested information and verbally and symbolically represent it. In order to perform this step, students must engage in a focusing $\rightarrow$ identifying $\rightarrow$ synthesising $\rightarrow$ devising process. First, students must focus on relevant sections of the statement of the problem. That is, they must realise that a phrase such as "the rate at which the radius is changing" can be thought of as a coherent whole and that it has significance in determining what is given and what is requested in the problem. Next, a key phrase that indicates the nature of the mathematical object, such as "rate of change", must be identified. In order to translate from a verbal to a symbolic representation, students must then synthesise the key phrase, generate a mental image of the mathematical object, and retrieve a set of standard symbolic representations for the concept. Finally, students must devise a symbolic representation that is appropriate for the specific rate requested in the problem, such as $\frac{dr}{dt}$.

In order to engage in this process, one must have an abstract-general (White & Mitchelmore, 1996) concept of variable. Meaningful translation from prose to symbols requires that the abstract symbols themselves must be intimately tied to the concepts they represent (Harel & Kaput, 1991). In addition, the ability to represent rate of change of a variable, say radius, with a derivative notation, say $\frac{dr}{dt}$, requires that the variable be thought of as an entity which can be acted upon by the differentiation process. Based on student performance (mean score 26%), we can surmise that few of the students in the present study had an abstract-general understanding of variable in this context.

Specifically, when students were asked to identify whether quantities or rates mentioned in a problem statement varied or remained constant, they were often unable to do so correctly. For example, when asked to identify variables or rates that remain constant over time in Problem 5(a), an item linked to Step 2, 25 students (43%) incorrectly answered s, $s = 200$ ft, or 100 ft (see Figure 5). In effect, these students were claiming that a quantity that varied over time (the amount of string let out or the height of the kite above her hand) was constant, and the values of this "constant" was the value of the variable at the *instant* in question. This inability to distinguish between variables and constants was one of the prominent error patterns noted by White and Mitchelmore (1996).

5. Consider the following problem.

   A girl is flying a kite in a wind that is blowing east at a rate of 50 ft/min. How fast must the string be let out at the instant when the kite is 100 ft above her hand and there is already 200 ft of string let out? See sketch.



Choose from among s, w, $\frac{ds}{dt}$, or $\frac{dw}{dt}$ to answer the following questions. Variables or rates may be used more than once.

(a) Which variable(s) or rate(s) retains a constant value for any time, t? Give the constant value(s).

Figure 5. Test 2 item in which students must identify constants.

Students' responses to Problem 5(a) highlighted some other difficulties as well. Although a literal interpretation of the problem would render the physical situation impossible (the kite would fall if its eastward motion were the same as the speed of the wind), none of the students seemed bothered by this. The sketch provided for students labelled the horizontal distance between the person and the kite as w, implying the horizontal movement was due to the wind. Although the students were not all able to correctly identify $\frac{dw}{dt}$ as a constant, none complained of a physical impossibility. Specifically, 22 students called $\frac{dw}{dt}$ a constant and 23 called it a variable. About half of each of those groups attached the value of 50 ft/min to the symbolic representation $\frac{dw}{dt}$. The other half only used the rate symbol, without any numerical values. Of the 7 students who attached the value 50 ft/min to a symbol other than $\frac{dw}{dt}$, 6 of those said it was the value of w and only one said it was the value of $\frac{ds}{dt}$. Since the students were not questioned about their responses to this problem, it is not clear why they accepted, without question, the physical contradictions in the setting. Perhaps they lacked an understanding of the mathematical or physical relationships in the problem. Alternatively, if they had sufficient background knowledge, perhaps they lacked the desire or experience to evaluate their solutions from a sense-making perspective.

The low mean (47%) for the third conceptual step, Step 6, supports Orton's

(1983b) finding that students have difficulty interpreting derivatives. The fact that students performed better on this step than they did on Steps 1 and 2 suggests that translation from symbols to prose is easier than translation from prose to symbols. When translating symbols to prose, it is not necessary to identify a key phrase or excerpt of text. When one examines a list of algebraic steps that culminate in a solution, it is clear that the last line (e.g., $\frac{dw}{dt} = \frac{-25}{12}$ in Problem 8, Figure 2) is the important information on which to focus. The only translation that is required is the translation of the derivative notation (i.e., $\frac{dw}{dt}$ represents the rate at which the ladder is sliding up the wall). This step also requires the interpretation of the sign of the numerical value and an assignment of units based on the context of the problem. An examination of the errors associated with this problem revealed that students' difficulties were often due to factors other than an inability to translate the derivative notation. The most frequently occurring error was an incorrect interpretation of the sign of the answer; another frequent error was the failure to report units or the reporting of incorrect units. In fact, 66% of the students were able to correctly translate a symbolic representation of the derivative to a verbal representation. This finding indicates that decoding symbols may be much easier for students than encoding symbols.

Performance on Procedural Steps

Performance on Test 2 was strongest (78% correct) for one of the procedural steps, Step 3. It could be argued that this step is the least intellectually demanding of the procedural steps, reflecting only an ability to memorise formulas. However, if the students cannot recall basic geometric formulas, they cannot hope to solve geometric related-rates problems correctly. Performance on Steps 4 and 5 (44% and 41%, respectively) suggests an even lower ceiling on the number of problems that students could possibly solve correctly. Being able to differentiate implicitly and to substitute and solve are especially critical in a computation-intensive, traditional calculus course where no technological aids are available.

These results appear contrary to Orton's (1983b) conclusion that the procedural aspects of differentiation are well understood. This study is not directly comparable to Orton's, since his observations were made on the basis of students' performances on tasks involving only explicitly defined functions. However, the students in this study had at least a week of practice with implicit differentiation just before the two tests were given. In addition, given their mathematical backgrounds, we can assume that these students all had extensive experience with algebraic equations in preparation for calculus. The fact that the average score on the three procedural steps was still only just over 50% must raise serious concern.

Relation between Procedural Knowledge, Conceptual Understanding, and Problem Solving Performance

The present study confirms what others (such as Dreyfus, 1990; Orton, 1983a, 1983b; and White and Mitchelmore, 1996) have found, that students have greater difficulty with conceptual steps than they have with procedural steps in problem solving. Many students may have been working with variables on a strictly procedural level, with the symbols disconnected from the concepts they represent. As an illustration, consider the case of Mary, who recorded a valid implicit differentiation for all three equations in Problem 4 (see Figure 3). In addition, Mary was able to correctly substitute into and solve both equations in Problem 7 (also shown in Figure 3). However, on Problem 5(a) (shown in Figure 5) Mary claimed that the variable quantities were constant and that the constant rate was variable. This inconsistency in her performance indicates an abstract-apart understanding of variables and rates.

From studies that document students' weak conceptual ability, many have concluded that, if students only understood what they were doing, they would perform better (Dreyfus, 1990). If that were true, then we would expect substantial correlations between performance on conceptual steps and performance on geometric related-rates problems. However, in this study, performance on only one of the three conceptual steps was significantly correlated with performance on Test 1, and the correlation was quite weak. In contrast, performance on all three procedural steps and the auxiliary step was significantly correlated with performance on Test 1 and the correlations were all higher.

How do we account for this counterintuitive result? It cannot be dismissed as a statistical artefact, because scores on the conceptual steps did not have a lower reliability or lower variation than scores on the procedural steps (see Table 4). However, it may have been due to the omission of Steps 1 and 2 from the scoring rubric for Test 1. To explore this possibility, Test 1 was re-scored allocating points to the first two steps (wherever there was any written evidence). This change led to no significant change in the total score, expressed as a percentage of the maximum possible score ($p > 0.10$), and correlations with the subscores on Test 2 only changed slightly. In particular, the correlation between scores on Test 1 and on Subscore 2 on Test 2 increased to 0.36 and became statistically significant at a higher level of significance ($p < 0.01$). There is, then, some evidence that students who are better able to summarise a problem statement tend to be better at solving geometric related-rates problems.

Conclusions

The present study has demonstrated that calculus students are poor at solving geometric related-rates problems, particularly those that require the completion of auxiliary steps. Proponents of the "back to traditional" approach to calculus instruction should note that even in classes primarily populated by mathematics, science, and engineering students, the traditional instructional approach led to disappointingly poor performance on steps linked to procedural knowledge as well as on steps primarily relying upon conceptual understanding.

If it is true that "[symbolic] notations help provide the basis for conceptual presence" (Harel and Kaput, 1991, p. 89), then it is critical for students to be confident, competent users of symbolic representations. However, it is also critical for students to be able to make connections among verbal, symbolic, and graphical representations. It is important for calculus students to engage in both the processes of decoding and encoding symbolic representations, recognising that these two skills may represent different levels of conceptual understanding.

The students in this study were unable to translate previous success in geometry and algebra to success in calculus. It is no longer appropriate to assume that success in prior mathematics courses will predict success in calculus. It is time to insist that both the conceptual and procedural domains and the links between them receive considerably more attention throughout the school curriculum.

In this study, we found lower correlations between conceptual understanding and problem-solving performance than between procedural understanding and problem-solving performance. One possible explanation is that it is entirely possible to be "successful" in traditional calculus without understanding what you are doing. Anti-reformists might counter that the danger of conceptually focused reformed curricula is that it is possible to be "successful" in reformed calculus despite weak algebraic or procedural skills. Perhaps both perspectives have some truth to them. Although the debate over traditional versus reformed calculus lingers on, it is clear that we must first determine what success in calculus ought to be, then design curriculum and assessment to match that vision.

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